Source field reproduction

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We may model the environment as a sphere from where plane waves arrive at the microphones. The pressure field can be described as

$$p(\boldsymbol{x},t) = \int \frac{d\omega}{2\pi} d\Omega A(\omega, \hat{\boldsymbol{k}}) e^{-i\omega t + i\hat{\boldsymbol{k}}k\boldsymbol{x}}, \tag{1}$$

where $\omega = kc$, and $\hat{\boldsymbol{k}}$ is a unit vector pointing to the spacial angle Ω. The partial amplitudes $A(\omega, \hat{\boldsymbol{k}})$ are complex numbers.

To simplify notation, we change to a 3D momentum integration and write

$$p(\boldsymbol{x},t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} A(\boldsymbol{k}) e^{-ikct + i\boldsymbol{k}\boldsymbol{x}}. \tag{2}$$

The observations are at points $\boldsymbol{x}_n$ $(n = 1, 2, \dots N)$, with the results $p_n(t)$. So for the source density we have a "loss function"

$$L = \sum_{n=1}^{N} \int dt \left(p_n(t) - p(\boldsymbol{x}_n, t)\right)^2. \tag{3}$$

This should have a minimum.

Although this form is not enough to determine $A(\boldsymbol{k})$, but we may add some assumptions to make it unique. We may increase the time integral in the $\sim A^2$ term to infinite, and we may assume that the n sum provides alignment of directions. With this assumption, using the reality of the pressure field, we can write

$$\sum_{n=1}^{N} \int dt p(\boldsymbol{x}_n, t)^2 = \sum_{n=1}^{N} \int dt \left| \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} A(\boldsymbol{k}) e^{-ikct + i\boldsymbol{k}\boldsymbol{x}_n} \right|^2 = \tag{4}$$

$$= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} A(\boldsymbol{k}) A^*(\boldsymbol{\ell}) \sum_{n=1}^{N} \int dt e^{-i(k-\ell)ct + i(\boldsymbol{k}-\boldsymbol{\ell})\boldsymbol{x}_n} = \tag{5}$$

$$= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} |A(\boldsymbol{k})|^2. \tag{6}$$

The mixed term reads

$$\sum_{n=1}^{N} \int dt (p_n^*(t) p(\boldsymbol{x}_n, t) + p_n(t) p^*(\boldsymbol{x}_n, t), \tag{7}$$

then the first term reads

$$\sum_{n=1}^{N} \int dt \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} A(\boldsymbol{k}) p_n^*(t) e^{-ikct+i\boldsymbol{k}\boldsymbol{x}_n} = \frac{d^3\boldsymbol{k}}{(2\pi)^3} A(\boldsymbol{k}) P^*(\boldsymbol{k}), \tag{8}$$

where

$$P(\boldsymbol{k}) = \sum_{n=1}^{N} \int dt\, e^{ikct-i\boldsymbol{k}\boldsymbol{x}_n} p_n(t). \tag{9}$$

is the (discrete) Fourier transform of the sources.

Omitting the all-observed p_n^2 term

$$L \to \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left[|A(\boldsymbol{k})|^2 - P^*(\boldsymbol{k}) A(\boldsymbol{k}) - P(\boldsymbol{k}) A^*(\boldsymbol{k}) \right]. \tag{10}$$

Therefore, in this limit, we have a deterministic answer for the source field as

$$A(\boldsymbol{k}) = P(\boldsymbol{k}). \tag{11}$$

For numerical calculations we can use the Fourier transformed form

$$P(\omega, \hat{\boldsymbol{k}}) = \sum_{n=1}^{N} e^{-i\omega,\hat{\boldsymbol{k}}\boldsymbol{x}_n/c} p_n(\omega). \tag{12}$$

We single out a bunch of unit vectors $\{\hat{\boldsymbol{k}}_1, \hat{\boldsymbol{k}}_2, \dots, \hat{\boldsymbol{k}}_K\}$, and introduce a matrix

$$M_{an} = \hat{\boldsymbol{k}}_a \boldsymbol{x}_n / c \tag{13}$$

then we can write in matrix notation

$$P(\omega) = e^{-i\omega M} p(\omega) \tag{14}$$

We diagonalize the M matrix as

$$M = U^{-1} D U, \tag{15}$$

where D is diagonal. Then

$$P(\omega) = U^{-1} e^{-i\omega D} U\, p(\omega). \tag{16}$$

For numerical calculations we have to compute the U matrix once. We can work with discrete Fourier modes $\omega = \{\omega_1, \dots \omega_N\}$, then we have to calculate the

$$Z_{ia} = e^{-i\omega_i \lambda_a} \tag{17}$$

again only once in the complete run. What we have to update is the pressure values.

Or, alternatively, we can use the matrix

$$Q_{ian} = e^{-i\omega_i \hat{\boldsymbol{k}}_a \boldsymbol{x}_n / c}, \tag{18}$$

and we store it – it does not occupy such a large memory. Then

$$P_i = Q_i p_i, \tag{19}$$

without the summation over i.

This is of course not exact, since neither the n sum, nor the time integral (that is not infinite) are restrictive enough. But we can say, the above formula gives an unbiased and consistent estimate of the source density, in the statistical sense, i.e. in the limit of $N \to \infty$ and infinite time integration regime it provides the true answer.

Although the result can be plotted for individual ω values, it may be better to provide the direction dependent power spectrum of the source density. This is

$$W(\hat{\boldsymbol{k}}) = \int \frac{dk}{2\pi} |A(\boldsymbol{k})|^2. \tag{20}$$

Substituting the above formulae into this expression we find

$$W(\hat{\boldsymbol{k}}) = \sum_{n,m=1}^{N} \int dt\, p_n(t + \frac{\hat{\boldsymbol{k}}\boldsymbol{x}_n}{c}) p_m(t + \frac{\hat{\boldsymbol{k}}\boldsymbol{x}_m}{c}), \tag{21}$$

so it comes from the correlation function of the sources.