Five Dimensional Algebraic Tori

immediate

January 29, 2018

Abstract

The rationality problem for algebraic tori is well known. It is known that any algebraic torus is unirational over its field of definition. The first purpose of this work is to solve rationality problem for 5 dimensional stably rational algebraic tori with an indecomposable character lattice. In order to do so, we have studied the associated character lattices of the mentioned algebraic tori. For each character lattice L, either we see the lattice as an associated lattice to a root system (of which rationality of its corresponding algebraic torus is known) or we find a reduced component of L so that we can relate rationality of the associated algebraic torus to lower dimensions. Using these two main methods from [**?**], we solve rationality problem in some cases.

1 Introduction

An interesting problem in algebraic geometry is the rationality problem, i.e. for a given algebraic variety, determine if it is rational or not. Naturally this is a difficult problem to solve. Hence it makes sense to consider some relaxed notions of rationality.

An algebraic F-variety X is called rational if there exists a birational map form X to $\mathbb{A}^n$ for some n. Algebraically X is rational if the rarional function field of X is F-rational, i.e. it is a purely transcendental extension of F. X is called stably rational if $X \times \mathbb{A}^n$ is rational. Similarly one can say X is stably rational if a purely transcendental extension of the rational function field of X is rational. Finally X is called unirational if the rational function field of X is a subfield of a purely transcendental extension of F. Geometrically this means there exist a dominant rational map from $\mathbb{A}^n$ to X for some n.

It can be seen that rationality implies stable rationality and it implies unirationality. It is well-known that the converses of these implications do not hold in general.

Assume X is a quasi projective F-variety. A F-variety Y is called an F-form of X if $X \otimes_k \bar{F} \cong Y$. A d-dimensoinal algebraic F-torus is a F-form of $\mathbb{G}_m^d$.

In the case of algebraic tori, we know that all of them are unirational. Moreover there exists a famous conjecture by Voskeresenskii, which states that, any stably rational algebraic torus is rational. To the best of the authors knowledge the conjucture is still open.

There is a one-to-one coresspondence between the isomorphism classes of n dimensional algebraic F-tori and representations of $G_F = Gal(F_s/F)$ into $GL(n, \mathbb{Z})$, where F_s is the seperable closure of F. It is understood that there exist a finite Galois extension of F, say K, such that $T \otimes_F K \cong \mathbb{G}_m^n$. The smalles such Galois extensoin (resp. Galois group) is called the splitting field (resp. splitting group) of T.

The isomorphism classes of n dimensional algebraic F-tori are in bijection with conjugacy classes of finite subgroups of $GL(n, \mathbb{Z})$. We know that the number of these finite subgroups up to conjugacy is finite. If $G \leq GL(n, \mathbb{Z})$ then the standard lattice, $L =< e_i : 1 \leq i \leq n >_{\mathbb{Z}}$ where $e_i = [\delta_{1i}]$, with the action (right multiplication) of G defines a $\mathbb{Z}G$ lattice L_G. Assume K/F is a finite Galois extension with $G = Gal(K/F)$. Now $T_G = K(L_G)^G$ is and algebraic torus. Note that conjugate groups correspond to isomorphic lattices and hence are associated to isomorphic algebraic tori. Conversley for a given n-dimensional algebraic torus T, which is defined over F and split by K, its character group is a $\mathbb{Z}G$ lattice of rank n and hence associates to a conjugacy class of subgroups of $GL(n, \mathbb{Z})$.

In the case of algebraic tori, we know that all of them are unirational. Moreover there exists a famous conjecture by Voskeresenskii, which states that, any stably rational algebraic torus is rational. To the best of the authors knowledge the conjucture is still open. During the 60's and afterwards there was a substantional amount of work in rationality problem for algebraic tori, which is mainly done by Voskeresenskii, Endo, Miyata, Saltman, Sansuc and colliet-thelene. Their approach to address the problem was the correspondence between algebraic tori and G-lattices. Their efforts led to characterization of stably rational and retract rational algebraic tori based on their character lattices.

Voskeresenskii proved that any two dimensional algebraic torus is rational. In early 90's Kunyavski classified three dimensional algebraic tori up to rationality. He showed that except for 15 three dimensional algebraic tori which are not retract rational, the rest of them are rational.

In 2012 Hoshi and Yamasaki published a paper [**?**] in which they classified algebraic tori of dimensions 4 and 5 up to stable rationality. Their classification is based on computing the flasque classes of algebraic tori in GAP. They showed that in rank 5, there are exactly 311 indecomposable G-lattices which are stably rational. More precisely they showed their stable rationality by finding the maximal groups and proved stable rationality of their subgroups. The following table presents the maximal groups they found.

Number	CARAT ID	G	$\#G$
1	$(5,942,1)$	$\mathrm{Imf}(5,1,1)$	3840
2	$(5,953,4)$	S_6	720
3	$(5,726,4)$	$C_2^4 \rtimes S_4$	384
4	$(5,919,4)$	$C_2 \times S_5$	240
5	$(5,801,3)$	$C_2 \times (S_3^2 \rtimes C_2)$	144
6	$(5,655,4)$	$D_8^2 \rtimes C_2$	128
7	$(5,911,4)$	S_5	120
8	$(5,946,2)$	S_5	120
9	$(5,946,4)$	S_5	120
10	$(5,947,2)$	S_5 ,	120
11	$(5,337,12)$	$D_8 \times S_3$	48
12	$(5,341,6)$	$D_8 \times S_3$	48
13	$(5,531,13)$	$C_2 \times S_4$	48
14	$(5,533,8)$	$C_2 \times S_4$	48
15	$(5,623,4)$	$C_2 \times S_4$	48
16	$(5,245,12)$	$C_2^2 \times S_3$	24
17	$(5,81,42)$	$C_2 \times D_8$	16
18	$(5,81,48)$	$C_2 \times D_8$	16

Table 1: The maximal 18 groups in the 311 cases found by Hoshi and Yamasaki in [**?**].

In 2015 Lemire [**?**], proved that, except for possibly ten of the 4 dimensional stably rational algebraic tori found by Hoshi and Yamasaki, all of them are rational. The rationality of the ten exceptional cases is still unknown. The author did not use any computer based arguments except for finding generating sets of groups and lattices of subgroups in GAP. The rationality results we are presenting are based on the ideas used in [**?**]. We present algorithms which may be applied to character lattices of algebraic tori, in order to investigate their rationality. These algorithms provide machinery to reduce the rationality problem in a specific dimension to lower dimensions.

From now on we will call the groups mentioned in Table **??** respectively G_1 to G_{18}. L_G represents the corresponding G-lattice to a finite subgroup of $\mathrm{GL}(n,\mathbb{Z})$, G, as defined in Definition 1. When we say a group or a lattice is rational we mean their corresponding algebraic torus is rational. By a decomposable (matrix) group, we mean its corresponding lattice is decomposable. We say G' is the dual group of G if G' is the corresponding group to the dual of L_G.

In this chapter we will investigate the rationality of $G_1,\dots,G_{18}$. In some cases, we prove that the group is hereditarily rational. There are two main methods that we will use, both of which were used in [**?**]. The first method is reducing the rationality of a five dimensional torus to rationality in lower dimensions. The second one is to see them as lattices of which the rationality is known.

2 GAP: Carat and CrystCat

GAP [**?**] stands for Groups, Algorithms, Programming, and is a computer algebra system for computations in discrete algebra with emphasis on computations in group theory. GAP is an open source system which is accessible directly or in SAGE [**?**]. GAP provides various packages for computations in matrix groups and representation theory. For our purposes we need Carat and CrystCat packages of GAP.

The GAP package Carat provides functions of the stand-alone programs of *CARAT*, which is a package for the computations related to crystallographic groups. Carat contains the catalog of all conjugacy classes of finite subgroups of $\mathrm{GL}(n,\mathbb{Z})$ for n up to six. More precisely the Carat package gives access to all $\mathbb{Q}$-classes and $\mathbb{Z}$-classes and maximal classes over $\mathbb{Z}$ (for the number of these classes see Table 2).

Remark. The $\mathbb{Q}$-classes are conjugacy classes over $\mathbb{Q}$. We note that some $\mathbb{Z}$-classes may belong to the same conjugacy class over the rationals.

n	# conjugacy classes of finite subgroups of $\mathrm{GL}(n,\mathbb{Z})$	# conjugacy classes of maximal finite subgroups of $\mathrm{GL}(n,\mathbb{Z})$	# conjugacy classes of finite subgroups over $\mathbb{Q}$
1	2	1	2
2	13	2	10
3	73	4	32
4	710	9	227
5	6079	17	955
6	85308	39	7103

Table 2: Numbers of conjugacy classes which are accessible in Carat.

It is worth mentioning that Carat contains information about crystallographic groups which we will not use. The CrystCat Package in GAP also provides a catalog of crystallographic groups up to dimension 4. The catalog mostly covers the data in [**?**]. CrystCat and Carat are complement of each other.

The GAP ID, (n, m, l, k) of a finite subgroup G of $\mathrm{GL}(n,\mathbb{Z})$ means that G is of rank n and belongs to k-th $\mathbb{Z}$ class of the l-th $\mathbb{Q}$-class of the m-th crystal system. This works for $2 \le n \le 4$. Hoshi and Yamasaki wrote a GAP code using the Carat package to have easy access to the j-th $\mathbb{Z}$-class of the i-th $\mathbb{Q}$-class group of rank n. They called this Carat ID. The GAP scripts written by Hoshi and Yamasaki are available from

http://www.math.h.kyoto-u.ac.jp/ yamasaki/Algorithm/

The algorithms introduced in the next section are implemented in GAP (needs some functions from the codes written by Hoshi and Yamasaki) and the code is available from

https://github.com/armin-jamshidpey/Algebraic-Tori

Since the actions of matrix groups in GAP are considered from right, throughout this chapter we work with row vectors instead of columns. One may also use the columns by considering the dual groups.

3 Basic Results

In this section we present some important results about rationality problem for algebraic tori. Before presenting the results we need the following definition to avoid repeating the same assumptions.

Definition 1. If G is a finite subgroup of $\mathrm{GL}(n,\mathbb{Z})$, then the corresponding lattice to G which is denoted by L_G is the rank n lattice generated by the standard basis, i.e. $L_G = \langle e_i : i = 1, \ldots, n \rangle_{\mathbb{Z}}$ where $(e_i)_j = \delta_{ij}$. The action of G on L_G is given by multiplication from right on the e_i's. Moreover, if $G \cong \mathrm{Gal}(K/F)$ for some finite Galois extension K/F then $K[L_G] \cong K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, that is the Laurent polynomial ring, and $K(L_G)$ which is the quotient field of $K[L_G]$ is isomorphic to $K(x_1, \ldots, x_n)$ (x_i's are algebraically independent over K) are equipped with an action of G as

- G acts as Galois group on K
- $\forall g \in G,\ \ g(x_i) = \prod_{j=1}^{j=n} x_j^{a_{ij}}$ where a_{ij}'s are given by $g(e_i) = \sum_{j=1}^{j=n} a_{ij} e_j$.

Also T_G is the corresponding algebraic torus to L_G i.e. T_G is an algebraic torus defined on F which splits over K, with character lattice L_G.

By the duality explained before, $K(L_G)^G$ is the rational function field of T_G. From now on we work with finite subgroups of $\mathrm{GL}(n,\mathbb{Z})$ (up to conjugacy) and when we consider their corresponding lattice (or algebraic torus), L_G (T_G), we mean the lattice (algebraic torus) defined in Definition 1.

Theorem 2. *[**?**] (No Name Lemma) Let M be a permutation G-lattice and F be a G-field. Then $F(M)^G$ is rational over F^G.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a $\mathbb{Z}$-basis which is permuted by G and $V = \sum_{i=1}^n F x_i$ be a F vector space. Applying Proposition **??** to V, we find $y_1, \ldots, y_n \in V^G$ such that $V = \sum_{i=1}^n F y_i$. This implies $F(M) = F(x_1, \ldots, x_n) = F(y_1, \ldots, y_n)$. Hence

$$F(M)^G = F(y_1, \ldots, y_n)^G = F^G(y_1, \ldots, y_n).$$

□

Note that if $G = \mathrm{Gal}(F/K)$, then $F^G = K$. For another version of the No Name Lemma see [**?**].

Theorem 3. *[**?**, Proposition 9.5.1] Assume L is a sign permutation G-lattice and F is a G-field. Then $F(M)^G$ is rational over F^G.*

The rationality problem for algebraic tori of dimension one was concrete. For dimension two Voskresenskii used a geometric method to prove the below result.

Theorem 4. *[**?**] Any 2 dimensional algebraic torus over k, is k-rational.*

We talked about the duality between category of algebraic tori and category of G-lattice. Having the duality in hand one may ask how to interpret the notions from one side to the other side. One of the important results about translating the facts about the rationality of algebraic tori, into the language of G-lattices is given below.

Theorem 5. *[**?**] Let M and M' be two G-lattices and F/K be a finite Galois extension with Galois Group G. Then $[M]^{fl} = [M']^{fl}$ if and only if $F(M)^G$ and $F(M')^G$ are stably isomorphic.*

The next two results give us necessary and sufficient conditions for stable rationality and retract rationality in terms of G-lattices. Having these two criteria gives us some control over the birational classification of algebraic tori of small dimension.

Theorem 6. *[**?**, Theorem 9.5.4] Let M be a G-lattice and F/K be a finite Galois extension with Galois group G. $[M]^{fl}$ is invertible if and only if $F(M)^G$ is retract K-rational.*

Theorem 7. *[**?**, Theorem 1.6]Let M be a G-lattice and F/K be a finite Galois extension with Galois group G. $[M]^{fl} = 0$ if and only if $F(M)^G$ is stably K-rational.*

The above two theorems can be used to see any stably rational algebraic torus is retract rational.

The following two theorems classifies algebraic tori of dimension 4 and 5 up to stable rationality. In [**?**] the authors gave a complete classification of mentioned tori, however they did not say anything about rationality of tori of dimension 4 and 5. The main idea of their work was to investigate the last 3 results above, by means of computer algebra system, GAP.

Theorem 8. *[**?**, Theorem 1.9] Let F/K be a finite Galois extension with Galois Group $G \leqslant \mathrm{GL}(4,\mathbb{Z})$. Assume G acts on $L = F(x_1, x_2, x_3, x_4)$ as above. (For tables of the below subgroups see [**?**, Page 4])*
(i) L^G is stably K-rational if G is (up to conjugacy) one of a list of 487 subgroups of $\mathrm{GL}(4,\mathbb{Z})$.
(ii) L^G is not stably but retract K-rational if G is (up to conjugacy) one of a list of 7 subgroups of $\mathrm{GL}(4,\mathbb{Z})$.
(iii) L^G is not retract K-rational if G is (up to conjugacy) one of a list of 216 subgroups of $\mathrm{GL}(4,\mathbb{Z})$.

In 2015, Lemire showed that except for possibly ten, all stably rational groups found by Hoshi and Yamasaki are rational (see [**?**]).

Theorem 9. *[**?**, Theorem 1.12] Let F/K be a finite Galois extension with Galois Group $G \leqslant \mathrm{GL}(5,\mathbb{Z})$. Assume G acts on $L = F(x_1, x_2, x_3, x_4, x_5)$ as above. (for tables of below subgroups see [**?**, Pages 134-144])*
(i) L^G is stably K-rational if G is (up to conjugacy) one of a list 3051 subgroups of $\mathrm{GL}(5,\mathbb{Z})$.
(ii) L^G is not stably but retract K-rational if G is (up to conjugacy) one of a list 25 subgroups of $\mathrm{GL}(5,\mathbb{Z})$.
(iii) L^G is not retract K-rational if G is (up to conjugacy) one of a list of 3003 subgroups of $\mathrm{GL}(5,\mathbb{Z})$.

There are examples of varieties which are stably rational but not rational. So in general being stably rational is not the same as being rational. However, there is a conjecture about the equivalence of stable rationality and rationality for algebraic tori.

Conjecture. [**?**, Section 2.6.1] Any stably rational algebraic torus is rational.

According to Theorem 9 we know all stably rational algebraic tori of dimension 5. However, the theorem does not say anything about rationality of those tori. An interesting problem is to find all rational tori between 3051 mentioned tori in the theorem.

We call $G \leq \mathrm{GL}(n,\mathbb{Z})$ irreducible (resp. indecomposable), if the corresponding lattice to G be irreducible (resp. indecomposable).

4 Families of Rational Algebraic Tori

In the next chapter, we investigate on rationality of stably rational algebraic tori of dimension 5. We will try to reduce their rationality to the rationality of some well understood algebraic torus. In this small section we present some families of algebraic tori which are rational, so that we can relate our algebraic tori to one of these families. It is already mentioned that every n dimensional algebraic torus has a corresponding finite subgroup of $\mathrm{GL}(n,\mathbb{Z})$. In order to study the rationality of algebraic tori, we study its corresponding group. We would rather to consider maximal groups and prove rationality for their subgroups, instead of proving it case by case. The following definitions are borrowed from [**?**].

Definition 10. Let L be a G-lattice for $G \leq \mathrm{GL}(n,\mathbb{Z})$. If all algebraic tori with character lattice $L \downarrow^G_H$ and splitting group H are rational, for any subgroup $H \leq G$, then we call L hereditarily rational.

Definition 11. If T is an algebraic torus and L is its corresponding lattice, then T is called hereditarily rational if L is hereditarily rational.

By Theorem 2, a quasi-split torus is rational. For a permutation G-lattice L and any subgroup $H \leq G$, since $L \downarrow^G_H$ is a permutation lattice, the corresponding torus to $L \downarrow^G_H$ is rational. In other words a quasi split torus is hereditarily rational. Simillarly by Theorem 3 and above argument for (a sign permutation lattice), we conclude that any algebraic torus with a sign permutation character lattice is hereditarily rational.

In particular this is true for an algebraic torus with character lattice the root lattice $\mathbb{Z}B_n$ as an $W(B_n)$-lattice. It is also known that any rank n sign permutation lattice, is isomorphic to the restriction of $\mathbb{Z}B_n$ to a subgroup of $W(B_n)$.

Proposition 12. *[**?**, Proposition 1.5] Suppose P is a permutation projective G-lattice and G is the Galois group of a finite Galois extension, K/F. If*

$$0 \longrightarrow M \longrightarrow L \longrightarrow P \longrightarrow 0$$

is an exact sequence of G-lattices, then the fields $K(L)^G$ and $K(M \oplus P)^G$ are isomorphic over F.

One can use the above proposition and Theorem 2 to conclude the following theorem.

Theorem 13. *[?, Proposition 1.6] Suppose P is a permutation G-lattice and G is the Galois group of a finite Galois extension, K/F. If*

$$0 \longrightarrow M \longrightarrow L \longrightarrow P \longrightarrow 0$$

is an exact sequence of G-lattices, then $K(L)^G$ is rational over $K(M)^G$.

An important corollary of the above theorem will be used frequently in chapter 3, in order to prove the rationality of algebraic tori.

Corollary 14. *Suppose P is a permutation G-lattice and G is the Galois group of a finite Galois extension, K/F. If*

$$0 \longrightarrow M \longrightarrow L \longrightarrow P \longrightarrow 0$$

is an exact sequence of G-lattices and $K(M)^G$ is rational over F, then $K(L)^G$ is rational over F.

In [?, Section 2.4.8] the author has shown that any algebraic torus with an augmentation ideal lattice is hereditarily rational. More precisely let T be an algebraic torus defined over F and splits over K and $G = \mathrm{Gal}(K/F)$. Assume the character lattice of T is I_X (the kernel of the augmentation map), and $\mathbb{Z}[X]$ is a G-permuattion lattice, where

$$0 \longrightarrow I_X \longrightarrow \mathbb{Z}[X] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{1}$$

is an exact sequence and $\varepsilon : \mathbb{Z}[X] \to \mathbb{Z}$, $x \to 1$ is the augmentation map. The exact sequence (1) corresponds to the exact sequence of F algebraic tori

$$0 \longrightarrow \mathbb{G}_m \longrightarrow R_{K_1/F}(\mathbb{G}_m) \times \cdots \times R_{K_t/F}(\mathbb{G}_m) \longrightarrow T \longrightarrow 0$$

where K_i/F (for $i = 1, \ldots, t$) are intermediate fields of K/F and K/K_i is Galois. Now $T = \prod_{i=1}^{t} R_{K_i/F}(\mathbb{G}_m)/\mathbb{G}_m$ and is rational. We note that for any subgroup H of G, $I_X \downarrow_H^G$ is also an augmentation ideal. Hence an algebraic tori with augmentation ideal character lattice is hereditarily rational.

It is worth mentioning that passing to dual lattices in 1 we get

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[X] \xrightarrow{\varepsilon} J_X \longrightarrow 0$$

where $J_X = I_X^*$ is called Chevalley module. The corresponding algebraic torus to J_X has interesting properties and is called norm one torus. Chevalley was the first one who discovered that norm one torus is not necessarily rational.

The following lemma was used in [?] to show that a given G-lattice is isomorphic to $J_{G/H}$.

Lemma 15. *[?, Remark 4.1] Let L be a G-lattice. If there exist $x \in L$ such that,*

- $\langle G.x \rangle_{\mathbb{Z}} = L$
- $\mathrm{Stab}_G(x) = H$
- $\sum_{g \in G} gx = 0$,

then $L \cong J_{G/H}$.

5 Reduction Algorithms

Assume

$$0 \to M \to L_G \to N \to 0$$

is a short exact sequence of G-lattices such that N is a permutation projective G-lattice. If K/F is a finite Galois extension with $G \cong \mathrm{Gal}(K/F)$, then by Theorem (13), $K(L_G)^G$ is rational over $K(M)^G$. Thus, rationality of $K(M)^G$ over F implies rationality of $K(L_G)^G$ over F.

Suppose L_G is an indecomposable G-lattice. In this section, we present methods to examine the possibility of existence of such a short exact sequence for L_G, with N a permutation G-lattice.

Although sign permutation lattices are not permutation projective, constructing a short exact sequence of G-lattices

$$0 \to M \to L_G \to N \to 0$$

where N is a rank one sign permutation G-lattice might help to determine rationality of the associated algebraic torus to L_G. Note that existence of such a sequence does not directly imply rationality. However, under some conditions the rationality may be concluded.

The goal of this section is to provide tools to get exact sequences mentioned above for a given indecomposable G-lattice. The idea behind all of the methods is a simple fact which we explain briefly here.

A lattice L_G, is reducible as a G-lattice if and only if $\mathbb{Q}L_G = L_G \otimes_{\mathbb{Z}} \mathbb{Q}$ has a proper $\mathbb{Q}[G]$-submodule W of dimension $0 < m < n$.

Let L_G be a G-lattice of rank n and W is an m dimensional proper $\mathbb{Q}[G]$-submodule of $\mathbb{Q}L_G$. Then $L_G \cap W$ is a sublattice of L_G of rank m such that $\mathbb{Q}(L_G \cap W) = W$. Then

$$0 \to L_G \cap W \to L_G \to L_G/(L_G \cap W) \to 0$$

is a short exact sequence of G-lattices. Note that this implies in particular that $L_G/(L_G \cap W)$ is torsion free so that a $\mathbb{Z}$-basis of $L_G \cap W$ can be extended to a $\mathbb{Z}$-basis of L_G.

In the next paragraphs we are specifically looking for an $n-1$ dimensional proper $\mathbb{Q}[G]$-submodule of $\mathbb{Q}L_G$.

If we start with the dual lattice L_G^*, and we are able to find a rank 1 permutation sublattice of L_G^*, we get

$$0 \to \mathbb{Z} \to L_G^* \to M \to 0,$$

where $M = L_G^*/\mathbb{Z}$ is of rank $n-1$. Then by dualizing the sequence we have

$$0 \to M^* \to L_G \to \mathbb{Z} \to 0$$

as desired.

Now, we explain how to find a permutation rank one sublattice of L_G^*. In order to get a one dimensional $\mathbb{Q}[G]$-submodule of $\mathbb{Q}L_G^*$, we use the eigenspaces of the transposes of a generating set of G. Let $\{\sigma_1, \dots, \sigma_m\}$ be the transposes of a generating set of G and let $G^* = \langle \sigma_1, \dots, \sigma_m \rangle$. Suppose E_{1,σ_i} is the left nullspace of $\sigma_i - I$ over $\mathbb{Q}$. We define

$$E_1 = E_{1,\sigma_1} \cap \cdots \cap E_{1,\sigma_n}.$$

Note that G^* acts trivially on E_1. If $E_1 \neq 0$ then we can choose a nonzero vector $u \in E_1$. Let $u = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}) \in E_1$ such that $\gcd(a_i, b_i) = 1$. If $m = \operatorname{lcm}(b_1, \dots, b_n)$ then $mu = (a'_1, \dots, a'_n) \in \mathbb{Z}^n$. If $\gcd(a'_1, \dots, a'_n) = d$ then $v = \frac{m}{d}u \in L_G \cap E_1$ and the gcd of its entries is 1.

As a consequence, we can extend $\{v\}$ to a $\mathbb{Z}$-basis of L_G^*. A general algorithm to do this extension is given by Magliveras et al in [**?**]; GAP also has a function which does the job. In most of the cases that we will see in the next section, v had a ± 1 as an entry, which makes the basis extension so simple: if v_j is ± 1 then $\{e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_n, v\}$ forms a $\mathbb{Z}$-basis for L_G^*.

Since it is possible to extend v to a basis for the lattice L_G^*, there exists a change of basis matrix T in $\mathrm{GL}(n, \mathbb{Z})$ such that

$$T\sigma_i T^{-1} = \left[\begin{array}{c|c} \delta_i & * \\ \hline 0 & 1 \end{array}\right]$$

for some $\delta_i \in \mathrm{GL}(n-1, \mathbb{Z})$. Since we consider the finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$ up to conjugacy, we can work with $G' = TG^*T^{-1}$. By considering the first $n-1$ vectors of the standard basis of the G'-lattice $L_{G'}$ (which is isomorphic to L_G^*), can form the G'-lattice M such that $L_{G'}/\mathbb{Z} = M$ and we get

$$0 \longrightarrow \mathbb{Z} \longrightarrow L_{G'} \longrightarrow M \longrightarrow 0.$$

By dualizing the sequence we get

$$0 \longrightarrow M^* \longrightarrow L_{G'}^* \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Note that $L_{G'}^*$ is isomorphic to L_G.

The explained method above is presented as an algorithm here.

Algorithm 1 Fixed Point Algorithm

Input: A finite subgroup G of $\mathrm{GL}(n,\mathbb{Z})$, given by its generators $\{\sigma_1,\ldots,\sigma_m\}$.

Output: A matrix $T \in \mathrm{GL}(n,\mathbb{Z})$ such that $T\sigma_i^t T^{-1} = \left[\begin{array}{c|c} \delta_i & * \\ \hline 0 & 1 \end{array}\right]$ where $\delta_i \in \mathrm{GL}(n-1,\mathbb{Z})$, and sublattices M, N such that $0 \longrightarrow N \longrightarrow L_G^* \longrightarrow M \longrightarrow 0$ is an exact sequence of lattices.

```
E ← [ σ_1^t − I | σ_2^t − I | ··· | σ_n^t − I ]
W ← LeftNullspace(E)
if W is not zero then
        choose a nonzero v ∈ W
        if v ∉ Z^n
                find c ∈ Z s.t cv ∈ Z^n and gcd(cv) = 1
                v ← cv
        end if
        apply the algorithm in [?] to extend v to a basis B = {β_1, ..., β_{n−1}, v} for L_G
        T ← [β_1  ···  β_{n−1}  v]^t
        N ← Zv
        M ← L/N
        return M, N, T
        end if
else
        return fail
end if
```

Remark. If the algorithm returns a matrix T then for $\sigma \in \{\sigma_1^t,\ldots,\sigma_m^t\}$,

$$T\sigma T^{-1} = \sigma'$$

where

$$\sigma' = \left[\begin{array}{c|c} \delta & * \\ \hline 0 & 1 \end{array}\right].$$

for some $\delta \in \mathrm{GL}(n-1,\mathbb{Z})$. More precisely

$$T\sigma = \sigma' T$$

and the last row of $T\sigma$ is nothing but $v\sigma = v$. This implies that the last row of $\sigma' = [0 \ \ldots \ 0 \ 1]$.

Example 16. Let $G \leq \mathrm{GL}(4,\mathbb{Z})$ be generated by

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}.$$

The transposes are

$$\sigma = \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \text{ and } \tau = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then the $E_{1,\sigma}$ is the left nullspace of

$$\sigma - I_4 = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -2 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

One can verify that

$$\{\begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}\}$$

is a basis for $E_{1,\sigma}$. Similarly $E_{1,\tau}$ is the left nullspace of

$$\tau - I_4 = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

and it is generated by $\begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}$.

It is not hard to see $\begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix} \in E_{1,\sigma}$. Thus

$$E_1 = E_{1,\sigma} \cap E_{1,\tau} = \langle \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix} \rangle$$

and $\begin{bmatrix}0 & 0 & -1 & 1\end{bmatrix} \in L$. Now

$$\{\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & -1 & 1\end{bmatrix}\}$$

forms a $\mathbb{Z}$-basis for L. The change of basis matrix T is

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}$$

Hence

$$T\sigma T^{-1} = \left[\begin{array}{ccc|c} 0 & -1 & 2 & 1\\ -1 & -1 & 2 & 1\\ 0 & -1 & 1 & 0\\ \hline 0 & 0 & 0 & 1\end{array}\right]$$

$$T\sigma T^{-1} = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0\\ 1 & 0 & -2 & -1\\ 0 & 0 & -1 & -1\\ \hline 0 & 0 & 0 & 1\end{array}\right]$$

Now by dualizing we get

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0\\ -1 & -1 & -1 & 0\\ 2 & 2 & 1 & 0\\ \hline 1 & 1 & 0 & 1\end{array}\right] \text{ and } \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & -2 & -1 & 0\\ \hline 0 & -1 & -1 & 1\end{array}\right]$$

as a generating set for a conjugate of G. By defining $M \subset L_G$ generated by e_1, e_2 and e_3 the following exact sequence will be obtained

$$0 \longrightarrow M \longrightarrow L_G \longrightarrow \mathbb{Z} \longrightarrow 0$$

We call the above process the fixed point algorithm. One can generalize it as follows. Assume $E_{\lambda,\sigma}$ be the left kernel of $\sigma - \lambda I$ over the rationals. Now define $E_{\pm 1,\sigma}$ to be the set $\{E_{1,\sigma}, E_{-1,\sigma}\}$. Assume $E_{\pm 1,G}$ is the Cartesian product of $E_{\pm 1,\sigma}$ for all $\sigma \in G^*$, i.e.

$$E_{\pm 1,G} = E_{\pm 1,\sigma_1} \times \cdots \times E_{\pm 1,\sigma_n}.$$

If there exist $A \in E_{\pm 1,G}$ such that $W = \bigcap_{B \in A} B \neq 0$, then we can find a nonzero $v \in L_G^* \cap W$. As we have seen in the fixed point algorithm, we can extend a multiple of v to a $\mathbb{Z}$-basis for L_G^*. Then we can get a change of basis matrix T, such that

$$T\sigma_i T^{-1} = \left[\begin{array}{c|c} \delta_i & * \\ \hline 0 & \pm 1\end{array}\right]$$

for some $\delta_i \in \mathrm{GL}(n-1, \mathbb{Z})$. Thus by a similar argument we can use the new representative of the conjugacy class of G^* to form an equivalent lattice and similarly by choosing the first $n-1$ elements of the standard basis of $L_{G'}$ we can produce

$$0 \longrightarrow \mathbb{Z}^- \longrightarrow L_{G'} \longrightarrow M \longrightarrow 0.$$

By dualizing the sequence we get

$$0 \longrightarrow M^* \longrightarrow L_{G'}^* \longrightarrow \mathbb{Z}^- \longrightarrow 0.$$

Again note that $L_{G'}^*$ is isomorphic to L_G.

This process will be called the sign fixed point algorithm and it is presented as an algorithm here.

Algorithm 2 Sign Fixed Point Algorithm

Input: A finite subgroup G of $\mathrm{GL}(n,\mathbb{Z})$, given by its generators $\{\sigma_1,\ldots,\sigma_m\}$.

Output: A matrix $T \in \mathrm{GL}(n,\mathbb{Z})$ such that $T\sigma_i^t T^{-1} = \left[\begin{array}{c|c} \delta_i & * \\ \hline 0 & 1 \end{array}\right]$ where $\delta_i \in \mathrm{GL}(n-1,\mathbb{Z})$, and sublattices M, N such that $0 \longrightarrow N \longrightarrow L_G^* \longrightarrow M \longrightarrow 0$ is an exact sequence of lattices.

1: for g in $\{\sigma_1^t,\ldots,\sigma_m^t\}$ do
  $E_g \leftarrow$ the set of left nullspaces of $g \pm I$ over $\mathbb{Q}$
 end do
2: $E \leftarrow E_{\sigma_1^t} \times E_{\sigma_2^t} \times \cdots \times E_{\sigma_m^t}$
3: $W \leftarrow 0$
4: while $W = 0$ and $E \neq \emptyset$ do
  $A \leftarrow$ a random element of E
  $W \leftarrow \bigcap_{a\in A} a$
  $E \leftarrow E \setminus A$
 end do
5: if W is not zero then
  choose a nonzero $v \in W$
  if $v \notin \mathbb{Z}^n$
   find $c \in \mathbb{Z}$ s.t $cv \in \mathbb{Z}^n$ and $\gcd(cv) = 1$
   $v \leftarrow cv$
  end if
  apply the algorithm in [**?**] to extend v to get a basis $B = \{\beta_1,\ldots,\beta_{n-1}, v\}$ for L
  $T \leftarrow \begin{bmatrix}\beta_1 & \cdots & \beta_{n-1} & v\end{bmatrix}^t$
  $N \leftarrow \mathbb{Z}v$
  $M \leftarrow L/N$
  return M, N, T
  end if
 else
  return fail
 end if

Example 17. Let $G \leq \mathrm{GL}(4,\mathbb{Z})$ be generated by

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

The transposes are

$$\sigma = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \text{ and } \tau = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Then

$$\sigma - I_4 = \begin{bmatrix} -2 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\sigma + I_4 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\tau - I_4 = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\tau + I_4 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

By computing the left nullspaces we get

$$E_{1,\sigma} = \langle [0,0,1,0], [1,1,0,1] \rangle$$
$$E_{-1,\sigma} = \langle [[2,0,1,0], [1,-1,0,1]] \rangle$$
$$E_{1,\tau} = \langle [-1,-1,-1,1] \rangle$$
$$E_{-1,\tau} = \langle [1,-1,1,1] \rangle$$

So

$$E_{1,\sigma} \cap E_{1,\tau} = 0$$
$$E_{1,\sigma} \cap E_{-1,\tau} = 0$$
$$E_{-1,\sigma} \cap E_{1,\tau} = \langle [1,1,1,-1] \rangle$$
$$E_{-1,\sigma} \cap E_{-1,\tau} = 0$$

Let $W = E_{-1,\sigma} \cap E_{1,\tau}$. So $[1,1,1,-1] \in L \cap W$ is extendable to a basis for L by vectors

$$[1,0,0,0]\,,\ [0,1,0,0] \ \text{ and } \ [0,0,1,0]$$

and the corresponding transformation is

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\sigma' = T\sigma T^{-1} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\tau' = T\tau T^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now by dualizing we get

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ \hline -1 & 0 & 1 & -1 \end{array}\right] \text{ and } \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \end{array}\right]$$

as a generating set for a conjugate of G. By defining $M \subset L_G$ generated by e_1, e_2 and e_3 the following exact sequence will be obtained

$$0 \longrightarrow M \longrightarrow L_G \longrightarrow \mathbb{Z}^- \longrightarrow 0.$$

In general to get a $\mathbb{Q}[G]$-submodule W of $\mathbb{Q}L$, where $L = L_G$ is a G-lattice, one can use the decomposition of $\mathbb{Q}L$ (provided that it is decomposable). It is not always easy to find the decomposition of a $\mathbb{Q}[G]$-module. The most well-known tool for module decomposition is the meataxe algorithm. The algorithm was first introduced by Parker in [**?**] in order to check irreducibility of a finite dimensional module over a finite field and finding explicit submodules in case of reducibility. Later on Parker extended the idea of the meataxe algorithm to characteristic zero (see [**?**]). His algorithm can be used to decompose an integral representation of a finite group. In [**?**], the authors provided machinery which enables us to decompose $\mathbb{Q}[G]$-modules up to dimension 200. So there are algorithms which give the decomposition over $\mathbb{Q}$. We invite the reader to see [**?**] and [**?**] for more details.

Assume $G \leq \mathrm{GL}(n,\mathbb{Z})$ is finite and $\mathbb{Q}L = L \otimes_{\mathbb{Z}} \mathbb{Q}$ is the $\mathbb{Q}$-vector corresponding space to L. If $\mathbb{Q}L$ is a decomposable $\mathbb{Q}[G]$-module, then there exists a change of basis matrix such that generators of the $\mathbb{Q}$-class of G can be written as block diagonal matrices

$$T\sigma_i T^{-1} = \left[\begin{array}{c|c} \delta_i & 0 \\ \hline 0 & \gamma_i \end{array}\right]$$

where $\delta_i \in \mathrm{GL}(m,\mathbb{Q})$ and $\gamma_i \in \mathrm{GL}(m',\mathbb{Q})$ for some $m, m' \in \mathbb{Z}$. Let $\{e_1,\ldots,e_m,e_{m+1},\ldots,e_{m+m'}\}$ be the standard basis for $\mathbb{Q}L$. The $\mathbb{Q}M$ and $\mathbb{Q}N$ generated respectively by $\{e_1,\ldots,e_m\}$ and $\{e_{m+1},\ldots,e_{m+m'}\}$ are invariant (set wise) under the action of G and $\mathbb{Q}L = \mathbb{Q}M \oplus \mathbb{Q}N$. Now $T^{-1}(\mathbb{Q}M)$ is a G-stable subspace and

$$M = L \cap T^{-1}(\mathbb{Q}M)$$

is G stable. Then we get

$$0 \longrightarrow M \longrightarrow L \longrightarrow L/M \longrightarrow 0$$

as an exact sequence of lattices.

The above idea can be turned into an algorithm. In order to do so, one need to compute the decomposition of $\mathbb{Q}L$ (meataxe or any other algorithm can be applied). If in the previous step the change of basis, namely T, to get the decomposition is not computed, it should be done next. The next step is to choose a component of the decomposition, say $\mathbb{Q}M$ and a basis of it. After that, $M = T^{-1}(\mathbb{Q}M) \cap L$ is a sublattice of L. The last step is to extend a basis of M to L (see [**?**] for an algorithm).

Here is an example which shows the above idea in practice. The $\mathbb{Q}$-class of the group is presented from the list of $\mathbb{Q}$-classes in rank 4 provided in [**?**].

Example 18. Consider the group G generated by

$$\sigma = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \text{ and } \tau = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The $\mathbb{Q}$-class of G is accessible in the list of rank 4 groups in [**?**] by the name cryst4[149]. The generators of the $\mathbb{Q}$-class are

$$\sigma' = \left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \text{ and } \tau' = \left[\begin{array}{ccc|cc} 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right].$$

Considering the orders of matrices we can figure out there exist an invertible integral matrix T such that

$$T\sigma T^{-1} = \sigma'$$

$$T\tau T^{-1} = \tau'.$$

Assume 25 indeterminates $t_{00}, \ldots, t_{44}$ and the matrix

$$T = \begin{bmatrix} t_{00} & t_{01} & t_{02} & t_{03} & t_{04} \\ t_{10} & t_{11} & t_{12} & t_{13} & t_{14} \\ t_{20} & t_{21} & t_{22} & t_{23} & t_{24} \\ t_{30} & t_{31} & t_{32} & t_{33} & t_{34} \\ t_{40} & t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix}.$$

Then

$$T \cdot \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot T$$

$$T \cdot \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \cdot T.$$

The transformation T can be found by solving (and replacing parameters) the linear system obtained from above equations as

$$T = \begin{bmatrix} -1 & 1 & -2 & 1 & 1 \\ -1 & 2 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Define $\mathbb{Q}M = \langle e_1, e_2, e_3 \rangle_{\mathbb{Q}}$, since

$$T^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

we have

$$T^{-1}(e_1) = \left[-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right]$$

$$T^{-1}(e_2) = \left[-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right]$$

$$T^{-1}(e_3) = \left[-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right]$$

Hence $T^{-1}(\mathbb{Q}M) = \langle\left[-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right], \left[-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right], \left[-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right]\rangle_{\mathbb{Q}}$ Now

$$M = L \cap T^{-1}(\mathbb{Q}M) = \langle\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix}\rangle_{\mathbb{Z}}$$

By extending the above basis of M to a basis of L (by adding $[0, 0, 1, 0, 0]$ and $[0, 0, 0, 0, 1]$) and forming the change of basis matrix we get

$$S = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \left[\begin{array}{ccc|cc} 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & -1 \end{array}\right]$$

Now we can get the following exact sequence of lattices

$$0 \longrightarrow M \longrightarrow L \longrightarrow L/M \longrightarrow 0.$$

6 Rationality Problem for 5 Dimensional Indecomposable Stably Rational Algebraic Tori

Up to now we have presented some algorithms to reduce a lattice and form exact sequences of specific type. In this section we apply those algorithms to the 18 indecomposable lattices which are maximal in the set of indecomposable stably rational rank 5 lattices found in [**?**].

In some cases instead of applying any algorithm we interpret the lattice as a root lattice of a root system. The general idea is to identify the given lattice as a lattice in one of the hereditarily rational families of lattices. We also try to reduce some lattices to one of those families.

There are also cases where our reduction does not provide enough information to decide about the rationality. In these cases we reduce the lattice and provide some information which may help to decide about their rationality. There are also irreducible lattices among the 18 maximal ones. A partial lattice of maximal subgroups of the irreducible cases is provided so that our algorithms work for maximal subgroups.

Throughout this section for a finite subgroup $G \leq \mathrm{GL}(n, \mathbb{Z})$, the corresponding algebraic torus and the corresponding lattice (see Definition 1) is denoted respectively by T_G and L_G. When we say a group is rational or a lattice is rational we mean the corresponding algebraic torus is rational.

6.1 Case G_1

G_1 is one of the 7 maximal indecomposable finite subgroups of $\mathrm{GL}(5, \mathbb{Z})$. This is the automorphism group of root system B_5. So we can recognize the lattice as $(\mathbb{Z}(B_5), \mathrm{Aut}(B_5))$. This lattice is hereditarily rational (see [**?**]).

Alternatively by looking at the generators of G_1

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

One can also see that the corresponding lattice is sign permutation which implies rationality of L_{G_1}

6.2 Case G_2

This is a group isomorphic to S_6. Following [?] we show that the dual lattice is isomorphic to Chevalley module J_{S_6/S_5}. The group G_2 is generated by

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

The dual lattice corresponds to the group generated by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

One can verify that e_1 is a cyclic generator of $L^*_{G_2}$.

Now we have to make sure that the stabilizer subgroup of e_1 is isomorphic to S_5.

```
gap> GD2:= GroupByGenerators([A,B]);
<matrix group with 2 generators>

gap> S:= Stabilizer(GD2 ,e1);
<matrix group with 6 generators>

gap> StructureDescription(S);
"S5"
```

The last step is to check if $\sum_{g\in H} e_1.g = 0$

```
gap> n:= [0,0,0,0,0];
0
gap> for g in GD2 do n:= n + (e1*g); od;
gap> n;
[ 0, 0, 0, 0, 0 ]
```

This shows that the lattice is isomorphic to J_{S_6/S_5} and its dual is isomorphic to the augmentation ideal I_{S_6/S_5}. This implies that G_2 is hereditarily rational.

6.3 Case G_3

By applying Algorithm 1 to the dual lattice, we get the change of basis matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Now changing the basis and dualizing gives us the group

$$\left[\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

This yields the following exact sequence of lattices

$$0 \longrightarrow M \longrightarrow L_{G_3} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where M corresponds to [4, 32, 21, 1]. In [?] the author has proved the corresponding lattice to [4, 32, 21, 1] is $\mathbb{Z}B_4$ which is hereditarily rational. This proves that T_{G_3} is hereditarily rational.

Alternatively one can see from the above generators of H, that M is a sign permutation lattice which is hereditarily rational.

6.4 Case G_4

Algorithm 2 produces the change of basis matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 2 \end{bmatrix}$$

With the above transformation we can see the new representative for G_4 is generated by

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & -1 \end{array}\right]$$

Now we can produce

$$0 \longrightarrow M_{G_4} \longrightarrow L_{G_4} \longrightarrow \mathbb{Z}^- \longrightarrow 0,$$

where M_{G_4} corresponds to [4,31,7,1]. One can verify that $M_{G_4}^*$ is the Chevalley module J_{S_5/S_4} (apply Lemma Nicole with $e_1 = [1, 0, 0, 0]$), thus its dual lattice is hereditarily rational. However, since L_{G_4}/M_{G_4} is sign permutation, we can not conclude rationality of G_4.

6.5 Case G_5

Algorithm 2 produces the change of basis matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

With the above transformation we can see the new generators for G_5 are given by

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

Now we can produce

$$0 \longrightarrow M_{G_5} \longrightarrow L_{G_5} \longrightarrow \mathbb{Z}^- \longrightarrow 0.$$

The corresponding group to M_{G_5} has GAP ID [4,29,9,2].

6.6 Case G_6

Algorithm 2 produces the change of basis matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

With the above transformation we can see the new generators for G_6 are given by

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now we can produce

$$0 \longrightarrow M_{G_6} \longrightarrow L_{G_6} \longrightarrow \mathbb{Z}^- \longrightarrow 0.$$

M_{G_6} is a sign permutation lattice and therefore hereditarily rational. However, since L_{G_6}/M_{G_6} is sign permutation, we can not conclude rationality of G_6.

6.7 Case G_7

Algorithm 1 produces the change of basis matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 2 \end{bmatrix}$$

With the above transformation we can see the new representative for G_7 is generated by

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & -2 & -2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ \hline 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

One can verify that its dual is the Chevalley module J_{S_5/S_4} (apply Lemma Nicole with $e_1 = [1, 0, 0, 0]$). Alternatively in [**?**] the author has proved that [4,31,7,1] is hereditarily rational. Using GAP we can verify that [4,31,4,1] (up to conjugacy) is a subgroup of [4,31,7,1]. This proves rationality of T_{G_7}.

6.8 Cases G_8, G_9 and G_{10}

It is well known that a representation ρ is absolutely irreducible if and only if $\langle\chi_\rho, \chi_\rho\rangle = 1$ where χ_ρ is the corresponding character to ρ. Since the groups are of small order, 120, we can test if $\langle\chi_\rho, \chi_\rho\rangle$ is one or not. We recall

$$\langle\chi_\rho, \chi_\rho\rangle = \frac{1}{n}\sum_{\sigma\in G}(tr(\sigma)^2)$$

where $G \leq \mathrm{GL}(n, \mathbb{Q})$ and $|G| = n$.

```
gap> n:= 0;
0
gap> for g in AsList(G8) do
> n:= n+ (1/Size(G8))*Trace(g)^2;
> od;
gap> n;
1

gap> n:= 0;
0
gap> for g in AsList(G9) do
> n:= n+ (1/Size(G9)) *Trace(g)^2;
> od;
gap> n;
1

gap> n:= 0;
0
gap> for g in AsList(G10) do
> n:= n+ (1/Size(G10))*Trace(g)^2;
> od;
gap> n;
1
```

Since all 3 groups are irreducible, we consider their maximal subgroups up to conjugacy and check if their maximal subgroups are reducible. In case that a maximal subgroup is irreducible we consider its maximal subgroups again and continue this process.

Figures 1, 2 and 3 respectively present lattices of subgroups of G_8, G_9 and G_{10}. These are not the complete lattices of the mentioned groups. We just considered the lattices up to the level where algorithm one returns an output. Appendix (a) is devoted to show that the shaded groups are hereditarily rational.

6.9 Case G_{11}

Algorithm 2 produces the change of basis matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & -2 & 2 \end{bmatrix}$$

With the above transformation we can see a new representative for G_{11} is generated by

$$\left[\begin{array}{cccc|c} -1 & -1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -3 & 0 \\ \hline 0 & 1 & 1 & 1 & -1 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 2 & 1 & 2 & 2 & 0 \\ -2 & 0 & -3 & -2 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

Now we can produce

$$0 \longrightarrow M_{G_{11}} \longrightarrow L_{G_{11}} \longrightarrow \mathbb{Z}^- \longrightarrow 0.$$

$M_{G_{11}}$ has GAP ID [4,20,20,4]. A generating set for [4,20,20,4] is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

As we can see its corresponding lattice decompose into 2 rank two lattices which are hereditarily rational.

6.10 Case G_{12}

Algorithm 1 provides the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & -2 & 2 \end{bmatrix}$$

as our desired change of basis which gives us a new representative for the conjugacy class of G_{12}, namely the group generated by

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & -3 & 0 & -2 & 0 \\ \hline 0 & 1 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & -2 & -2 & 0 \\ -2 & -2 & -1 & -2 & 0 \\ 2 & 2 & 2 & 3 & 0 \\ \hline -1 & -1 & -1 & -1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

Now we get the following exact sequence for $L_{G_{12}}$

$$0 \longrightarrow M_{G_{12}} \longrightarrow L_{G_{12}} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $M_{G_{12}}$ has gap ID [4,20,17,2]. A generating set of [4,20,17,2] is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

and its corresponding lattice is decomposable to rank 2 lattices which we know their rationality. Thus, $M_{G_{12}}$ is hereditarily rational and this implies that $T_{G_{12}}$ is hereditarily rational.

6.11 Case G_{13}

By algorithm 1 we have the following transformation matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

which provides the new generators of the representative of the conjugacy class of G_{13} as

$$\left[\begin{array}{cccc|c} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 2 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & -1 & 1 \end{array}\right]$$

Now we can get

$$0 \longrightarrow M_{G_{13}} \longrightarrow L_{G_{13}} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

On the other hand GAP returns the GAP ID of $M_{G_{13}}$ In [**?**] the author has proved that [4,25,9,2] is hereditarily rational which means $T_{G_{13}}$ is hereditarily rational.

6.12 Case G_{14}

By algorithm 1 we have the following transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

which provides the new representative of the conjugacy class of G_{14} as

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

Now we can get

$$0 \longrightarrow M_{G_{14}} \longrightarrow L_{G_{14}} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

On the other hand GAP returns the GAP ID of $M_{G_{14}}$ as [4, 25, 8, 5]. In [**?**] it is shown that the subgroups of [4,25,8,5] are rational except for possibly 8 subgroups

$$[4, 6, 2, 11], [4, 12, 4, 13], [4, 13, 2, 6], [4, 13, 3, 6], [4, 13, 7, 12], [4, 24, 4, 6], [4, 25, 4, 5], [4, 25, 8, 5].$$

Each of the above groups corresponds to a subgroup of G_{14} and except for them, the rest are rational.

6.13 Case G_{15}

The group G_{15} is isomorphic to $C_2 \times S_4$ and is generated by

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is the G-lattice discussed in Example 18. So the information is being recalled from that example.

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which provides a new representative of the conjugacy class of G_{15} as the group generated by

$$\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \text{ and } \left[\begin{array}{ccc|cc} 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & -1 \end{array}\right].$$

Now we get

$$0 \longrightarrow M \longrightarrow L_{G_{15}} \longrightarrow L/M \longrightarrow 0.$$

One can see that L/M is a rank 2 lattice which is the corresponding lattice to the group, H generated by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

H is isomorphic to S_3 and its corresponding lattice is not even sign permutation. We can consider maximal subgroups of G_{15} as we did for the irreducible lattices.

6.14 Case G_{16}.

Algorithm 1 we have the following transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

which provides the new generators of the representative of the conjugacy class of G_{16} as

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & -1 & -1 & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -1 & -1 & 0 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now we can get

$$0 \longrightarrow M_{G_{16}} \longrightarrow L_{G_{16}} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

On the other hand GAP returns the GAP ID of $M_{G_{16}}$ as [4, 14, 10, 2]. In [**?**] it is shown that [4, 31, 7, 1] is hereditarily rational. Using GAP one can verify that [4,14,10,2] (up to conjugacy) is a subgroup of [4,31,7,1] which means $T_{G_{16}}$ is hereditarily rational.

6.15 Case G_{17}

Algorithm 1 we have the following transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

which provides the new generators of the representative of the conjugacy class of G_{17} as

$$\left[\begin{array}{cccc|c} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ \hline -1 & 0 & -1 & 0 & 1 \end{array}\right] \left[\begin{array}{cccc|c} -2 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 3 & 0 & 0 & 2 & 0 \\ \hline -3 & 0 & 0 & -1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ \hline 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

we can get

$$0 \longrightarrow M_{G_{17}} \longrightarrow L_{G_{17}} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

On the other hand GAP returns the GAP ID of $M_{G_{17}}$ as [4, 13, 7, 12]. In [**?**] it is shown that the subgroups of [4,13,7,12] are rational except for possibly 5 subgroups

$$[4, 6, 2, 11], [4, 12, 4, 13], [4, 13, 2, 6], [4, 13, 3, 6], [4, 13, 7, 12]$$

Each of the above groups corresponds to a subgroup of G_{17} and except for them, the rest are rational.

6.16 Case G_{18}

Algorithm 1 we have the following transformation matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & -1 \end{bmatrix}$$

which provides the new generators of the representative of the conjugacy class of G_{18} as

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ \hline -1 & 1 & 1 & -1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

Now we can get

$$0 \longrightarrow M_{G_{18}} \longrightarrow L_{G_{18}} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

On the other hand GAP returns the GAP ID of $M_{G_{18}}$ as [4, 13, 7, 12]. In [**?**] it is shown that the subgroups of [4,13,7,12] are rational except for possibly 5 subgroups

$$[4,6,2,11], [4,12,4,13], [4,13,2,6], [4,13,3,6], [4,13,7,12]$$

Each of the above groups corresponds to a subgroup of G_{18} and except for them, the rest are rational.

6.17 Conclusion

The 18 maximal indecomposable stably rational lattices found in [**?**] are divided into 4 families. The first family are the ones interpreted as lattices of root systems which are hereditarily rational. The second family contains all lattices on which Algorithm 1 will not fail. The third family contains lattices on which Algorithm 2 does not fail while Algorithm 1 fails. The last family contains all lattices on which either, both Algorithms 1 and 2 fail but still the general idea of reduction works, or they are irreducible.

All lattices of the first family are hereditarily rational. The second family contains lattices of which after reduction, rationality of the reduced component is unknown. By arguments in the previous sections we have proved the following theorem.

Theorem 19. *The groups presented in Table 3 are hereditarily rational.*

CARAT ID	Group Structure	$\#G$	Description.
$(5,942,1)$	$\mathrm{Imf}(5,1,1)$	3840	The root lattice of B_5
$(5,953,4)$	S_6	720	The root lattice of A_5
$(5,726,4)$	$C_2^4 \rtimes S_4$	384	reduced component $[4,32,21,1]$
$(5,911,4)$	S_5	120	reduced component $[4,31,4,1]$
$(5,341,6)$	$D_8 \times S_3$	48	reduced component $[4,20,17,2]$
$(5,531,13)$	$C_2 \times S_4$	48	reduced component $[4,25,9,2]$
$(5,245,12)$	$C_2^2 \times S_3$	24	reduced component $[4,14,10,2]$

Table 3: Hereditarily rational groups among the maximal 18 groups found in [**?**].

The exceptional cases of the second family are presented in Table 4. In each case the reduced component is stably rational as proved in [**?**]. Their rationality is unknown yet. In [**?**] the author has proved that subgroups of $[4,25,8,5]$ are rational except for possibly

$$[4,6,2,11], [4,12,4,13], [4,13,2,6], [4,13,3,6], [4,13,7,12], [4,24,4,6], [4,25,4,5], [4,25,8,5].$$

There will be precisely one subgroup of G_{14} with each of the dimension 4 reduced components in the above list, so except for possibly those subgroups of G_{14} the rest are rational. The exceptional cases are presented in Table 5.

From the above list

$$[4,6,2,11], [4,12,4,13], [4,13,2,6], [4,13,3,6], [4,13,7,12]$$

are subgroups of $[4,13,7,12]$ which means we have the same problem for cases G_{17} and G_{18}. Hence except for possibly the subgroups of G_{17} and G_{18} associated to above list their rest of subgroups are rational. For the exceptional cases see Table 6 and Table 7.

CARAT ID	Group Structure	$\#G$	Description.
$(5,533,8)$	$C_2 \times S_4$	48	reduced component $[4,25,8,5]$
$(5,81,42)$	$C_2 \times D_8$	16	reduced component $[4,13,7,12]$
$(5,81,48)$	$C_2 \times D_8$	16	reduced component $[4,13,7,12]$

Table 4: Groups among 18 maximals which are reduced but rationality of rank 4 sublattice is unknown

CARAT ID	Group Structure	#G	Description.
$(5, 32, 52)$	$C_2 \times C_2 \times C_2$	8	reduced component [4, 6, 2, 11]
$(5, 99, 53)$	D_8	8	reduced component [4, 12, 4, 13]
$(5, 103, 22)$	$C_4 \times C_2$	8	reduced component [4, 13, 2, 6]
$(5, 98, 22)$	D_8	8	reduced component [4, 13, 3, 6]
$(5, 81, 50)$	$C_2 \times D_8$	16	reduced component [4, 13, 7, 12]
$(5, 522, 15)$	S_4	24	reduced component [4, 24, 4, 6]
$(5, 521, 15)$	S_4	24	reduced component [4, 25, 4, 5]
$(5, 533, 8)$	$C_2 \times S_4$	48	reduced component [4, 25, 8, 5]

Table 5: Subgroups of G_{14} that have associated tori which are stably rational but whose rationality is unknown.

CARAT ID	Group Structure	#G	Description.
$(5, 32, 49)$	$C_2 \times C_2 \times C_2$	8	reduced component [4, 6, 2, 11]
$(5, 99, 52)$	D_8	8	reduced component [4, 12, 4, 13]
$(5, 103, 16)$	$C_4 \times C_2$	8	reduced component [4, 13, 2, 6]
$(5, 98, 16)$	D_8	8	reduced component [4, 13, 3, 6]
$(5, 81, 42)$	$C_2 \times D_8$	16	reduced component [4, 13, 7, 12]

Table 6: Subgroups of G_{17} that have associated tori which are stably rational but whose rationality is unknown.

CARAT ID	Group Structure	#G	Description.
$(5, 32, 46)$	$C_2 \times C_2 \times C_2$	8	reduced component [4, 6, 2, 11]
$(5, 99, 54)$	D_8	8	reduced component [4, 12, 4, 13]
$(5, 103, 24)$	$C_4 \times C_2$	8	reduced component [4, 13, 2, 6]
$(5, 98, 24)$	D_8	8	reduced component [4, 13, 3, 6]
$(5, 81, 48)$	$C_2 \times D_8$	16	reduced component [4, 13, 7, 12]

Table 7: Subgroups of G_{18} that have associated tori which are stably rational but whose rationality is unknown.

Theorem 20. *All subgroups of G_{14}, G_{17} and G_{18} are rational except for possibly the subgroups in Table 5, Table 6 and Table 7.*

There are 4 cases, namely G_4, G_5, G_6 and G_{11}, in which after the reduction we get a rank one sign permutation lattice (more information is given in Table 8). The same also happened in some subgroups of irreducible lattices (see Table 9). It is possible that these groups are hereditarily rational, but we do not currently have a proof. One possible approach to prove rationality in these cases may be the following argument.

Assume L is a lattice in the third family, that is, there exists an exact sequence of lattices such that

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z}^- \longrightarrow 0$$

Moreover assume M is a hereditarily rational. Using a flasque resolution of L we have

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] & L \arrow[r] \arrow[d] & \mathbb{Z}^- \arrow[r] & 0 \\
 & & P \arrow[d] & & \\
 & & Q \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Suppose K/F is a finite Galois extension and $G = \mathrm{Gal}(K/F)$. We note that by the No Name Lemma we obtain $K(P)^G = K(y_1, \dots, y_t)$ and $K(M)^G = F(x_1, \dots, x_{n-1})$ is implied by rationality of M. Now if there exist a permutation lattice Q' such that $P \subset M \oplus Q'$ (or even if $L \subset M \oplus Q'$) then $K(L)^G \subset K(M)(Q')^G$ which implies unirationality of $K(L)^G$ over $K(M)^G$. Now

Lüroth's theorem implies rationality of $K(L)^G$ over F.

$$\begin{tikzcd}
K(M \oplus Q')^G = K(M)^G(z_1, \dots, z_s) \arrow[d, dash] \\
K(P)^G = K(y_1, \dots, y_t) \arrow[d, dash] \\
K(L)^G = K(M)(\mathbb{Z}^-)^G \arrow[d, dash] \\
K(M)^G = F(x_1, \dots, x_{n-1}) \arrow[d, dash] \\
F
\end{tikzcd}$$

The following table summarizes the information about reduction of lattices in the third family.

CARAT ID	G	$\#G$	Description.
$(5, 919, 4)$	$C_2 \times S_5$	240	reduced component $[4, 31, 7, 1]$
$(5, 801, 3)$	$C_2 \times (S_3^2 \rtimes C_2)$	144	reduced component $[4, 29, 9, 2]$
$(5, 655, 4)$	$D_8^2 \rtimes C_2$	128	reduced component $[4, 32, 17, 1]$
$(5, 337, 12)$	$D_8 \times S_3$	48	reduced component $[4, 20, 20, 4]$

Table 8: The groups corresponding to maximal stably rational tori of dimension 5 whose associated lattices are indecomposable and have a rank 1 sign quotient.

For the last family we have considered their maximal subgroups and we could not decide about the rationality of the groups presented in the following table.

Theorem 21. *All groups in Table 10 are hereditarily rational. That is, all subgroups of G_8, G_9, G_{10} and G_{15} except for possibly the subgroups in Table 9 are hereditarily rational.*

A proof of Theorem 21 is provided in Appendix A.

Carat ID	G	$\#G$	Description
$[5, 173, 4]$	S_3	6	reduced comp. $[4, 17, 1, 1]$, rank 1 sign perm. quot.
$[5, 391, 4]$	D_{12}	12	reduced comp. $[4, 21, 3, 1]$ rank 1 sign perm. quot.
$[5, 461, 4]$	$C_2^2 \times S_3$	24	reduced comp.$[3, 6, 7, 1]$, quot $[2, 4, 4, 1]$
$[5, 580, 4]$	A_4	12	reduced comp.$[3, 7, 1, 1]$, quot $[2, 4, 1, 1]$
$[5, 606, 4]$	$C_2 \times A_4$	24	reduced comp. $[3, 7, 2, 1]$, quot $[2, 4, 1, 1]$
$[5, 607, 4]$	S_4	24	reduced comp. $[3, 7, 4, 1]$, quot $[2, 4, 2, 1]$
$[5, 607, 9]$	S_4	24	reduced comp. $[3, 7, 4, 1]$, quot $[2, 4, 2, 2]$
$[5, 608, 4]$	S_4	24	reduced comp.t $[3, 7, 3, 1]$, quot $[2, 4, 2, 1]$
$[5, 917, 3]$	$C_5 \rtimes C_4$	20	reduced comp. $[4, 31, 1, 1]$ rank 1 sign perm. quot.
$[5, 917, 4]$	$C_5 \rtimes C_4$	20	reduced comp. $[4, 31, 1, 1]$ rank 1 sign perm. quot.
$[5, 623, 4]$	$C_2 \times S_4$	48	reduced comp. $[3, 7, 5, 1]$, quot $[2, 4, 2, 1]$
$[5, 952, 2]$	A_5	60	absolutely irreducible
$[5, 952, 4]$	A_5	60	absolutely irreducible
$[5, 946, 2]$	S_5	120	absolutely irreducible
$[5, 946, 4]$	S_5	120	absolutely irreducible
$[5, 947, 2]$	S_5	120	absolutely irreducible

Table 9: Subgroups of G_8, G_9, G_{10} and G_{15} that have associated tori which are stably rational but whose rationality is unknown.

The following table present the reduced components of the subgroups mentioned in Theorem 21.

Number	CARAT ID	G	$\#G$	Description.
1	$[5,6,3]$	C_2	2	[4, 2, 2, 2]
2	$[5,18,28]$	$C_2\times C_2$	4	[4, 4, 3, 4]
3	$[5,19,14]$	$C_2\times C_2$	4	[4, 5, 1, 10]
4	$[5,22,14]$	$C_2\times C_2\times C_2$	8	[4, 6, 1, 9]
5	$[5,57,8]$	C_4	4	[4, 7, 1, 2]
6	$[5,81,54]$	$C_2\times D_8$	16	[4, 13, 7, 5]
7	$[5,98,28]$	D_8	8	[4, 13, 3, 3]
8	$[5,99,57]$	D_8	8	[4, 12, 4, 7]
9	$[5,164,2]$	C_3	3	[4, 11, 1, 1]
10	$[5,174,2]$	S_3	6	[4, 17, 1, 3]
11	$[5,174,5]$	S_3	6	[4, 17, 1, 2]
12	$[5,389,4]$	D_{12}	12	[4, 21, 3, 2]
13	$[5,901,3]$	D_{10}	10	[4, 27, 3, 1]
14	$[5,918,4]$	$C_5\rtimes C_4$	20	[4, 31, 1, 2]

Table 10: Hereditarily rational subgroups of G_8, G_9, G_{10} and G_{15}.

Remark. The union of the set of groups in Table 9 and the set of subgroups of the groups in Table 10, is the set of all subgroups of G_8, G_9, G_{10} and G_{15}.

Appendix B presents tables of conjugacy classes of indecomposable subgroups of $\mathrm{GL}(5,\mathbb{Z})$ which correspond to stably rational tori of dimension 5 from Hoshi and Yamasaki's list. From this list, those stably rational tori of dimension 5 whose rationality is unknown are listed.

A Proof of Theorem 21 This chapter is devoted to a concrete proof of Theorem 10. We use simillar methods to the ones we already applied in chapter 3. After reducing the lattices we compare the sublattices with the rational ones introduced in [**?**] and [**?**].

A (5,6,3)

The group is generated by

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding lattice is sign permutation. This implies rationality of the corresponding torus.

B (5,18,28)

The group is generated by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding lattice is a sign permutation lattice. Thus it is hereditarily rational.

C (5,19,14)

Algorithm 2 produces the change of basis matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

With the above transformation we can see the new representative is generated by

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

we can produce

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The corresponding group to M has GAP ID [4,5,1,10] and also can be generated by

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right] \text{ and } \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]$$

So M decomposes into a direct sum of a rank one sign permutation lattice (which is hereditarily rational) and a rank 3 lattice given by a group, H, generated by

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & -1 \end{array}\right] \text{ and } \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Looking at the generators of H tells us we can form

$$0 \longrightarrow \mathbb{Z}^{-} \longrightarrow L_H \longrightarrow P \longrightarrow 0$$

where P is given by the group generated by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since P is a permutation lattice, by Corollary 14 we conclude that [4,5,1,10] is hereditarily rational. This implies our desired result which is hereditarily rationality of (5,19,14).

D (5,22,14)

The group is generated by

$$\left[\begin{array}{cccc|c} 1&0&0&0&-1\\ 0&1&0&0&1\\ 0&0&1&0&1\\ 0&0&0&1&-1\\ \hline 0&0&0&0&-1 \end{array}\right], \left[\begin{array}{cccc|c} 1&0&0&0&0\\ 0&0&0&1&-1\\ 0&0&1&0&0\\ 0&1&0&0&1\\ \hline 0&0&0&0&1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 0&0&1&0&1\\ 0&1&0&0&0\\ 1&0&0&0&-1\\ 0&0&0&1&0\\ \hline 0&0&0&0&1 \end{array}\right]$$

Now we define P to be the lattice corresponding to, H, generated by

$$\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0 \end{bmatrix} \text{ and } \begin{bmatrix} 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1 \end{bmatrix},$$

We can see the corresponding lattice to (5,22,14), L, fits into the following exact sequence

$$0 \longrightarrow \mathbb{Z}^- \longrightarrow L \longrightarrow P \longrightarrow 0.$$

and since P is permutation, by Corollary 14 we can conclude that L is hereditarily rational.

E (5,57,8)

The group is generated by

$$\begin{bmatrix} -1&0&0&0&0\\ 0&0&0&1&0\\ 0&-1&0&0&0\\ 0&0&0&0&1\\ 0&0&-1&0&0 \end{bmatrix}$$

and the corresponding lattice is a sign permutation lattice which is hereditarily rational.

F (5,81,54)

The group is generated by

$$\begin{bmatrix} 1&0&1&0&1\\ 0&1&0&0&0\\ 0&0&0&0&-1\\ 0&0&1&1&1\\ 0&0&-1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0&0\\ -1&0&0&1&0\\ 1&0&0&0&0\\ 0&1&1&0&0\\ -1&0&-1&0&-1 \end{bmatrix} \text{ and } \begin{bmatrix} 0&0&-1&-1&-1\\ 0&1&0&0&0\\ 0&0&1&0&0\\ -1&0&-1&0&-1\\ 0&0&0&0&1 \end{bmatrix}$$

The dual group is generated by

$$\begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 1&0&0&1&-1\\ 0&0&0&1&0\\ 1&0&-1&1&0 \end{bmatrix}, \begin{bmatrix} 0&-1&1&0&-1\\ 0&0&0&1&0\\ 1&0&0&1&-1\\ 0&1&0&0&0\\ 0&0&0&0&-1 \end{bmatrix} \text{ and } \begin{bmatrix} 0&0&0&-1&0\\ 0&1&0&0&0\\ -1&0&1&-1&0\\ -1&0&0&0&0\\ -1&0&0&-1&1 \end{bmatrix}$$

Algorithm (2) produces the change of basis matrix

$$\begin{bmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&-2&1&-1&-1 \end{bmatrix}$$

With the above transformation we can see the new representative is generated by

$$\left[\begin{array}{cccc|c} 1&2&0&2&0\\ 0&-1&0&-2&0\\ 0&2&1&2&0\\ 0&0&0&1&0\\ \hline 0&1&0&1&1 \end{array}\right], \left[\begin{array}{cccc|c} 0&2&1&0&0\\ 0&-1&0&0&0\\ 1&2&0&0&0\\ 0&0&0&-1&0\\ \hline 0&1&0&0&1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 1&-2&-2&-2&0\\ 0&2&1&1&0\\ 0&-2&-1&-2&0\\ 0&-1&-1&0&0\\ \hline 0&-1&-1&-1&1 \end{array}\right]$$

Now by considering M to be the corresponding lattice to

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -2 & -2 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & -1 & -2 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

we can produce

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The generators of [4,13,7,5] (another representative of the corresponding conjugacy class to M) are

$$\left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \text{ and } \left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

The generators of rank 3 lattice are

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

the CrystCatZClass of the former group is [3,4,6,4] which is rational by [**?**] and its subgroups are [3, 1, 1, 1], [3, 2, 1, 2], [3, 2, 2, 2], [3, 3, 1, 4], [3, 3, 2, 4], [3, 4, 2, 2] and [3, 4, 6, 4] where all of them are rational. This implies that (5, 81, 54) is hereditarily rational.

G (5,98,28)

The group is generated by

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 0 \end{bmatrix}$$

The dual group is generated by

$$\begin{bmatrix} 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Algorithm (1) produces the change of basis matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & -2 & -1 & 1 & -2 \end{bmatrix}$$

With the above transformation we can see the new representative is generated by

$$\left[\begin{array}{cccc|c} -6 & -5 & 0 & -2 & 0 \\ 5 & 4 & 0 & 2 & 0 \\ -3 & -3 & 1 & 0 & 0 \\ 5 & 5 & 0 & 1 & 0 \\ \hline 3 & 2 & 0 & 1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 5 & 6 & 0 & 0 & 0 \\ -4 & -5 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ -3 & -4 & -1 & 0 & 0 \\ \hline -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

Now by considering M to be the corresponding lattice to

$$\begin{bmatrix} -6 & -5 & 0 & -2 \\ 5 & 4 & 0 & 2 \\ -3 & -3 & 1 & 0 \\ 5 & 5 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 5 & 6 & 0 & 0 \\ -4 & -5 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ -3 & -4 & -1 & 0 \end{bmatrix}$$

we can produce

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The generators of [4,13,3,3] are

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]$$

The generators of rank 3 lattice are

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

the GAP ID of the former group is [3,4,6,4] which is hereditarily rational by the argument given in the previous case. So (5,98,28) is hereditarily rational.

H (5,99,57)

The group is generated by

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

The dual group is generated by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Algorithm (1) produces the change of basis matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & -1 & -1 & 1 \end{bmatrix}$$

With the above transformation we can see the new representative is generated by

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -2 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 1 & -1 & 0 & -2 & 0 \\ \hline 0 & 1 & 0 & 1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -2 & -2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ -1 & -2 & -1 & -1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Now by considering M to be the corresponding lattice to

$$\begin{bmatrix} 1 & -2 & 0 & -2 \\ -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & -1 & 0 & -2 \end{bmatrix} \text{ and } \begin{bmatrix} -2 & -2 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & -2 & -1 & -1 \end{bmatrix}$$

we can produce

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The generators of [4,12,4,7] are

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

The generators of rank 3 lattice are

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

the GAP ID of the former group is [3,4,6,4] which is hereditarily rational by argument given in the previous case. So (5,99,57) is hereditarily rational.

I (5,164,2)

The group is generated by

$$\left[\begin{array}{cc|ccc} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right]$$

The corresponding lattice decomposes into a rank 2 lattice which is hereditarily rational and a rank 3 sign permutation lattice which is also hereditarily rational. Hence (5,164,2) is hereditarily rational.

J (5,174,2)

The group is generated by

$$\begin{bmatrix} 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The dual group is generated by

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Algorithm (1) produces the change of basis matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 \end{bmatrix}$$

With the above transformation we can see the new representative is generated by

$$\left[\begin{array}{cccc|c} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Now by considering M to be the corresponding lattice to

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

we can produce

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The generators of [4,17,13] are

$$\left[\begin{array}{cc|cc} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right] \text{ and } \left[\begin{array}{cc|cc} -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

and the lattice decomposes into rank 2 lattices which we know they are hereditarily rational. This implies that (5,174,2) is hereditarily rational.

K (5,174,5)

The group is generated by

$$\left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \text{ and } \left[\begin{array}{cc|ccc} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

and the lattice decomposes into a rank 2 lattice and a rank 3 sign permutation lattice, both of which are hereditarily rational. This implies that (5,174,5) is hereditarily rational.

L (5,389,4)

The group is generated by

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dual group is generated by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Algorithm (1) produces the change of basis matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

With the above transformation we can see the new representative is generated by

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

Now by considering M to be the corresponding lattice to

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

we can produce

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z} \longrightarrow 0.$$

the above group is [4,21,3,2] which has the following subgroups [4, 1, 1, 1], [4, 3, 1, 3], [4, 5, 1, 1], [4, 11, 1, 1], [4, 17, 1, 2], [4, 17, 1, 3], [4, 21, 1, 1] and [4, 21, 3, 2] where all of them are rational.

M (5,901,3)

The group is generated by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The dual group is generated by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Algorithm (1) produces the change of basis matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

With the above transformation we can see the new representative is generated by

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array}\right] \text{ and } \left[\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now by considering M to be the corresponding lattice to

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

we can produce

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The lattice M corresponds to [4,27,3,1] with subgroups [4, 1, 1, 1], [4, 3, 1, 3], [4, 27, 1, 1], [4, 27, 3, 1] where all of them are rational. So (5,901,3) is hereditarily rational.

N (5,918,4)

The group is generated by

$$\begin{bmatrix} 0 & -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The dual group is generated by

$$\begin{bmatrix} 0 & -1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

Algorithm (1) produces the change of basis matrix

$$\begin{bmatrix} 0 & 1 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

With the above transformation we can see the new representative is generated by

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ -1 & 0 & -2 & -1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Now by considering M to be the corresponding lattice to

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ -1 & 0 & -2 & -1 \end{bmatrix}$$

we can produce

$$0 \longrightarrow M \longrightarrow L \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The lattice M corresponds to [4,31,1,2] which is a subgroup of $[4, 31, 7, 1]$. In [**?**] it is shown that $[4, 31, 7, 1]$ is hereditarily rational and so is (5,918,4).

Carat ID	Carat ID	Carat ID	Carat ID	Carat ID	Carat ID
(5, 18, 27)	(5, 19, 14)	(5, 22, 14)	(5, 23, 27)	(5, 28, 8)	(5, 28, 16)
(5, 28, 27)	(5, 30, 8)	(5, 30, 16)	(5, 30, 29)	(5, 30, 33)	(5, 31, 41)
(5, 31, 49)	(5, 32, 13)	(5, 32, 31)	(5, 32, 40)	(5, 32, 46)	(5, 32, 49)
(5, 32, 52)	(5, 32, 55)	(5, 32, 56)	(5, 32, 59)	(5, 37, 5)	(5, 37, 7)
(5, 38, 10)	(5, 39, 7)	(5, 74, 9)	(5, 74, 18)	(5, 75, 9)	(5, 75, 18)
(5, 75, 28)	(5, 75, 37)	(5, 75, 41)	(5, 78, 37)	(5, 78, 41)	(5, 81, 12)
(5, 81, 25)	(5, 81, 38)	(5, 81, 42)	(5, 81, 48)	(5, 81, 49)	(5, 81, 50)
(5, 81, 51)	(5, 81, 54)	(5, 82, 12)	(5, 82, 25)	(5, 84, 9)	(5, 84, 18)
(5, 92, 9)	(5, 92, 16)	(5, 98, 12)	(5, 98, 16)	(5, 98, 19)	(5, 98, 22)
(5, 98, 24)	(5, 98, 25)	(5, 98, 28)	(5, 99, 12)	(5, 99, 22)	(5, 99, 29)
(5, 99, 41)	(5, 99, 45)	(5, 99, 51)	(5, 99, 52)	(5, 99, 53)	(5, 99, 54)
(5, 99, 57)	(5, 100, 22)	(5, 100, 29)	(5, 101, 9)	(5, 101, 16)	(5, 101, 22)
(5, 102, 16)	(5, 102, 22)	(5, 103, 12)	(5, 103, 16)	(5, 103, 19)	(5, 103, 22)
(5, 103, 24)	(5, 103, 25)	(5, 103, 28)	(5, 106, 5)	(5, 108, 5)	(5, 108, 11)
(5, 108, 15)	(5, 110, 11)	(5, 110, 15)	(5, 114, 4)	(5, 117, 4)	(5, 117, 8)
(5, 117, 13)	(5, 117, 18)	(5, 118, 4)	(5, 118, 8)	(5, 120, 18)	(5, 120, 22)
(5, 121, 4)	(5, 121, 8)	(5, 121, 12)	(5, 122, 11)	(5, 126, 5)	(5, 131, 15)
(5, 131, 19)	(5, 133, 4)	(5, 133, 8)	(5, 133, 13)	(5, 133, 18)	(5, 137, 4)
(5, 138, 5)	(5, 138, 11)	(5, 156, 4)	(5, 160, 8)	(5, 161, 4)	(5, 161, 8)
(5, 162, 4)	(5, 162, 8)	(5, 222, 9)	(5, 222, 10)	(5, 223, 11)	(5, 223, 12)
(5, 224, 10)	(5, 225, 11)	(5, 225, 12)	(5, 226, 11)	(5, 226, 12)	(5, 226, 23)
(5, 226, 24)	(5, 227, 12)	(5, 230, 14)	(5, 230, 15)	(5, 231, 11)	(5, 231, 12)
(5, 232, 15)	(5, 242, 10)	(5, 243, 9)	(5, 243, 10)	(5, 245, 11)	(5, 245, 12)
(5, 267, 4)	(5, 268, 4)	(5, 271, 4)	(5, 272, 4)	(5, 275, 8)	(5, 277, 4)
(5, 292, 4)	(5, 293, 4)	(5, 302, 8)	(5, 304, 8)	(5, 306, 8)	(5, 307, 6)
(5, 308, 8)	(5, 308, 12)	(5, 337, 12)	(5, 341, 6)	(5, 389, 4)	(5, 390, 4)
(5, 391, 4)	(5, 391, 8)	(5, 392, 4)	(5, 404, 4)	(5, 409, 4)	(5, 410, 6)
(5, 413, 5)	(5, 414, 7)	(5, 416, 4)	(5, 424, 6)	(5, 426, 5)	(5, 434, 4)
(5, 435, 5)	(5, 436, 4)	(5, 461, 4)	(5, 462, 4)	(5, 465, 6)	(5, 501, 4)
(5, 519, 9)	(5, 519, 14)	(5, 520, 5)	(5, 520, 8)	(5, 520, 14)	(5, 521, 5)
(5, 521, 8)	(5, 521, 14)	(5, 521, 15)	(5, 522, 9)	(5, 522, 14)	(5, 522, 15)
(5, 524, 7)	(5, 529, 7)	(5, 529, 17)	(5, 531, 7)	(5, 531, 13)	(5, 531, 16)
(5, 533, 7)	(5, 533, 8)	(5, 537, 7)	(5, 538, 7)	(5, 541, 7)	(5, 580, 4)
(5, 606, 4)	(5, 607, 4)	(5, 607, 9)	(5, 608, 4)	(5, 608, 9)	(5, 623, 4)
(5, 623, 9)	(5, 640, 2)	(5, 641, 2)	(5, 644, 2)	(5, 645, 2)	(5, 655, 4)
(5, 656, 4)	(5, 665, 2)	(5, 666, 2)	(5, 668, 4)	(5, 670, 5)	(5, 671, 5)
(5, 672, 4)	(5, 674, 4)	(5, 684, 5)	(5, 687, 5)	(5, 689, 4)	(5, 696, 5)
(5, 700, 4)	(5, 703, 4)	(5, 704, 7)	(5, 704, 11)	(5, 705, 4)	(5, 705, 9)
(5, 705, 11)	(5, 706, 4)	(5, 706, 14)	(5, 707, 4)	(5, 709, 7)	(5, 710, 4)
(5, 711, 4)	(5, 713, 4)	(5, 715, 4)	(5, 726, 4)	(5, 742, 4)	(5, 750, 4)
(5, 750, 8)	(5, 753, 4)	(5, 754, 4)	(5, 754, 8)	(5, 756, 4)	(5, 758, 4)
(5, 760, 4)	(5, 760, 8)	(5, 762, 4)	(5, 763, 7)	(5, 763, 11)	(5, 773, 2)
(5, 774, 2)	(5, 785, 5)	(5, 801, 3)	(5, 822, 2)	(5, 823, 2)	(5, 846, 2)
(5, 852, 3)	(5, 853, 3)	(5, 854, 4)	(5, 855, 4)	(5, 856, 2)	(5, 869, 4)
(5, 870, 3)	(5, 889, 3)	(5, 890, 3)	(5, 891, 2)	(5, 892, 2)	(5, 900, 2)
(5, 901, 3)	(5, 902, 2)	(5, 904, 3)	(5, 909, 2)	(5, 910, 3)	(5, 910, 4)
(5, 911, 3)	(5, 911, 4)	(5, 912, 3)	(5, 912, 4)	(5, 917, 3)	(5, 917, 4)
(5, 918, 3)	(5, 918, 4)	(5, 919, 3)	(5, 919, 4)	(5, 926, 5)	(5, 926, 6)
(5, 931, 3)	(5, 931, 4)	(5, 933, 1)	(5, 934, 1)	(5, 935, 1)	(5, 936, 1)
(5, 937, 1)	(5, 938, 1)	(5, 939, 1)	(5, 940, 1)	(5, 941, 1)	(5, 942, 1)
(5, 943, 1)	(5, 944, 1)	(5, 945, 1)	(5, 946, 2)	(5, 946, 4)	(5, 947, 2)
(5, 947, 4)	(5, 951, 4)	(5, 952, 2)	(5, 952, 4)	(5, 953, 4)	

Table 11: The 311 indecomposable stably rational 5 dimensional algebraic tori with an indecomposable character lattice.

Carat ID	Carat ID	Carat ID	Carat ID	Carat ID	Carat ID
(5, 31, 41)	(5, 31, 49)	(5, 32, 46)	(5, 32, 49)	(5, 32, 52)	(5, 38, 10)
(5, 39, 7)	(5, 78, 37)	(5, 78, 41)	(5, 81, 42)	(5, 81, 48)	(5, 81, 50)
(5, 98, 16)	(5, 98, 22)	(5, 98, 24)	(5, 99, 52)	(5, 99, 53)	(5, 99, 54)
(5, 100, 22)	(5, 100, 29)	(5, 102, 16)	(5, 102, 22)	(5, 103, 16)	(5, 103, 22)
(5, 103, 24)	(5, 110, 11)	(5, 110, 15)	(5, 118, 4)	(5, 118, 8)	(5, 120, 18)
(5, 120, 22)	(5, 122, 11)	(5, 131, 15)	(5, 131, 19)	(5, 160, 8)	(5, 162, 4)
(5, 162, 8)	(5, 224, 10)	(5, 227, 12)	(5, 232, 15)	(5, 242, 10)	(5, 267, 4)
(5, 271, 4)	(5, 275, 8)	(5, 292, 4)	(5, 302, 8)	(5, 304, 8)	(5, 306, 8)
(5, 337, 12)	(5, 390, 4)	(5, 391, 4)	(5, 392, 4)	(5, 404, 4)	(5, 410, 6)
(5, 414, 7)	(5, 416, 4)	(5, 424, 6)	(5, 426, 5)	(5, 434, 4)	(5, 435, 5)
(5, 465, 6)	(5, 521, 15)	(5, 522, 15)	(5, 533, 8)	(5, 607, 4)	(5, 608, 4)
(5, 623, 4)	(5, 641, 2)	(5, 645, 2)	(5, 655, 4)	(5, 666, 2)	(5, 670, 5)
(5, 671, 5)	(5, 674, 4)	(5, 703, 4)	(5, 704, 7)	(5, 704, 11)	(5, 706, 4)
(5, 706, 14)	(5, 709, 7)	(5, 710, 4)	(5, 715, 4)	(5, 753, 4)	(5, 754, 4)
(5, 754, 8)	(5, 758, 4)	(5, 763, 7)	(5, 763, 11)	(5, 774, 2)	(5, 801, 3)
(5, 822, 2)	(5, 846, 2)	(5, 852, 3)	(5, 854, 4)	(5, 856, 2)	(5, 869, 4)
(5, 870, 3)	(5, 889, 3)	(5, 890, 3)	(5, 891, 2)	(5, 910, 4)	(5, 912, 4)
(5, 917, 4)	(5, 919, 4)	(5, 926, 6)	(5, 946, 2)	(5, 946, 4)	(5, 947, 2)
(5, 952, 2)					

Table 12: The cases among the 311 groups whose rationality is unknown (109 cases).

Carat ID	Carat ID	Carat ID	Carat ID	Carat ID	Carat ID
(5, 31, 41)	(5, 31, 49)	(5, 32, 46)	(5, 32, 49)	(5, 32, 52)	(5, 38, 10)
(5, 39, 7)	(5, 78, 37)	(5, 78, 41)	(5, 81, 42)	(5, 81, 48)	(5, 81, 50)
(5, 98, 16)	(5, 98, 22)	5, 98, 24)	(5, 99, 52)	(5, 99, 53)	(5, 99, 54)
(5, 100, 22)	(5, 100, 29)	(5, 102, 16)	(5, 102, 22)	(5, 103, 16)	(5, 103, 22)
(5, 103, 24)	(5, 110, 11)	(5, 110, 15)	(5, 118, 4)	(5, 118, 8)	(5, 120, 18)
(5, 120, 22)	(5, 122, 11)	(5, 131, 15)	(5, 131, 19)	(5, 160, 8)	(5, 162, 4)
(5, 162, 8)	(5, 224, 10)	(5, 227, 12)	(5, 232, 15)	(5, 242, 10)	(5, 267, 4)
(5, 271, 4)	(5, 275, 8)	(5, 292, 4)	(5, 302, 8)	(5, 304, 8)	(5, 306, 8)
(5, 337, 12)	(5, 390, 4)	(5, 391, 4)	(5, 392, 4)	(5, 404, 4)	(5, 410, 6)
(5, 414, 7)	(5, 416, 4)	(5, 424, 6)	(5, 426, 5)	(5, 434, 4)	(5, 435, 5)
(5, 465, 6)	(5, 521, 15)	(5, 522, 15)	(5, 533, 8)	(5, 641, 2)	(5, 645, 2)
(5, 655, 4)	(5, 666, 2)	(5, 670, 5)	(5, 671, 5)	(5, 674, 4)	(5, 703, 4)
(5, 704, 7)	(5, 704, 11)	(5, 706, 4)	(5, 706, 14)	(5, 709, 7)	(5, 710, 4)
(5, 715, 4)	(5, 753, 4)	(5, 754, 4)	(5, 754, 8)	(5, 758, 4)	(5, 763, 7)
(5, 763, 11)	(5, 774, 2)	(5, 801, 3)	(5, 822, 2)	(5, 846, 2)	(5, 852, 3)
(5, 854, 4)	(5, 856, 2)	(5, 869, 4)	(5, 870, 3)	(5, 889, 3)	(5, 890, 3)
(5, 891, 2)	(5, 910, 4)	(5, 912, 4)	(5, 917, 4)	(5, 919, 4)	(5, 926, 6)

Table 13: The groups in the previous table on which Algorithm (2) works (102 cases).

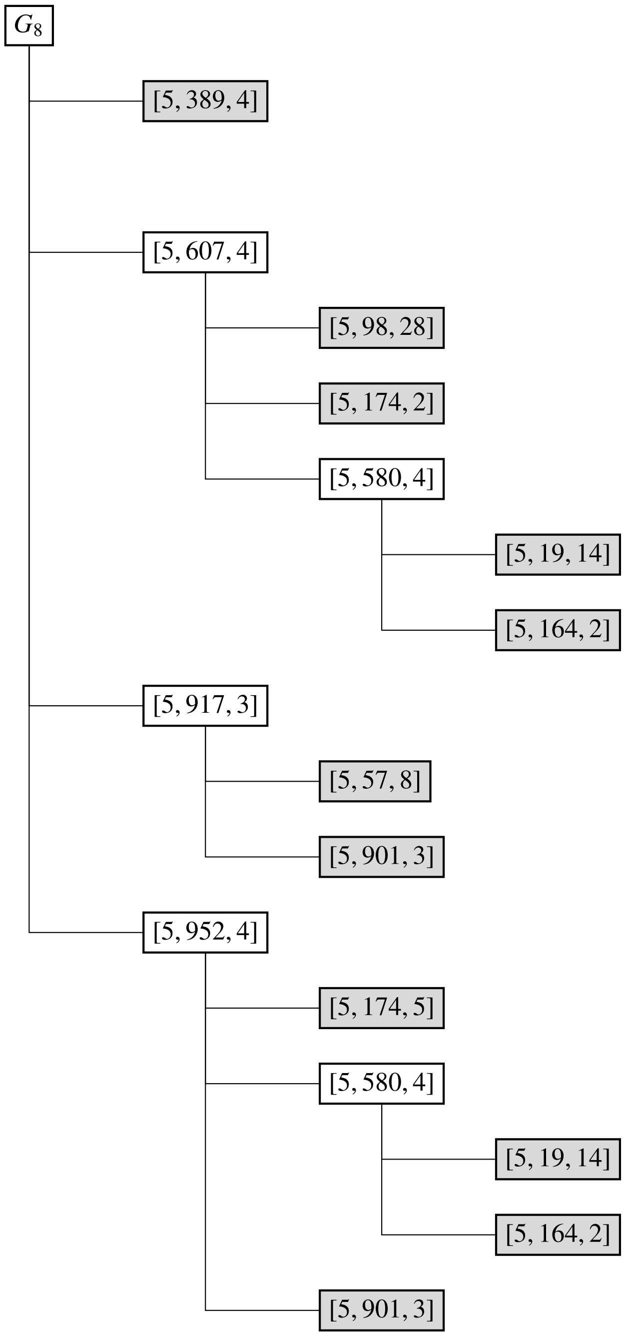

Figure 1: Conjugacy classes of subgroups of G_8. Algorithm (1) works for the gray ones.

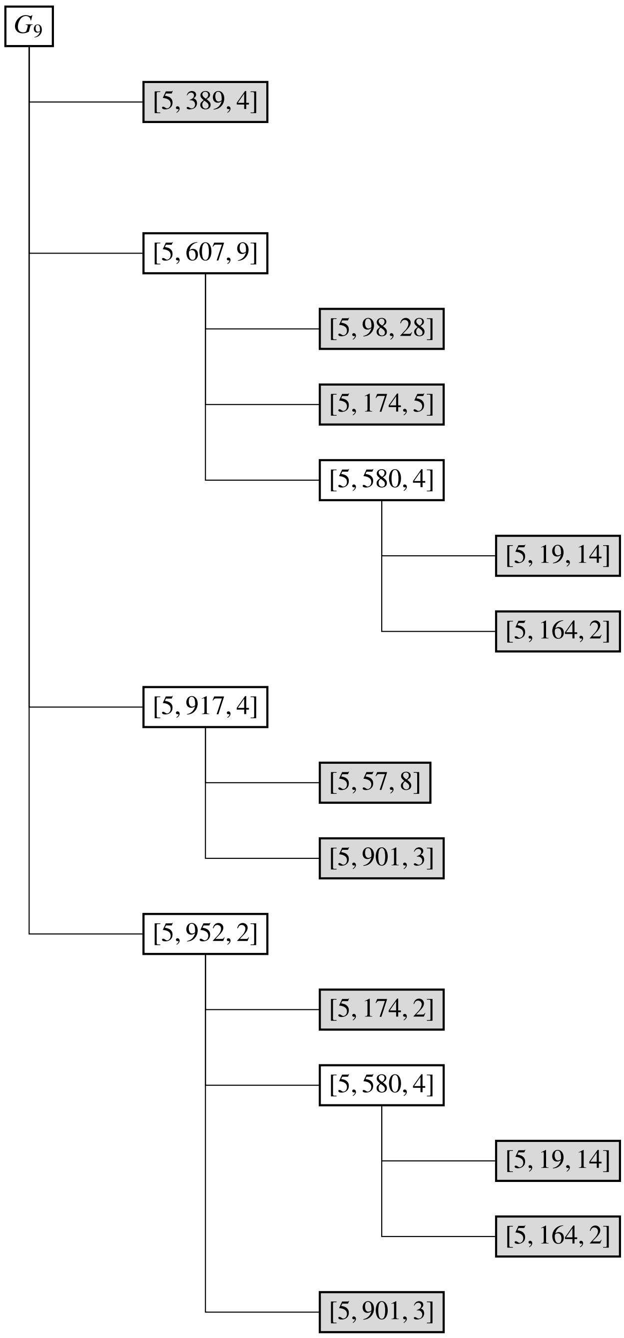

Figure 2: Conjugacy classes of subgroups of G_9. Algorithm (1) works for the gray ones.

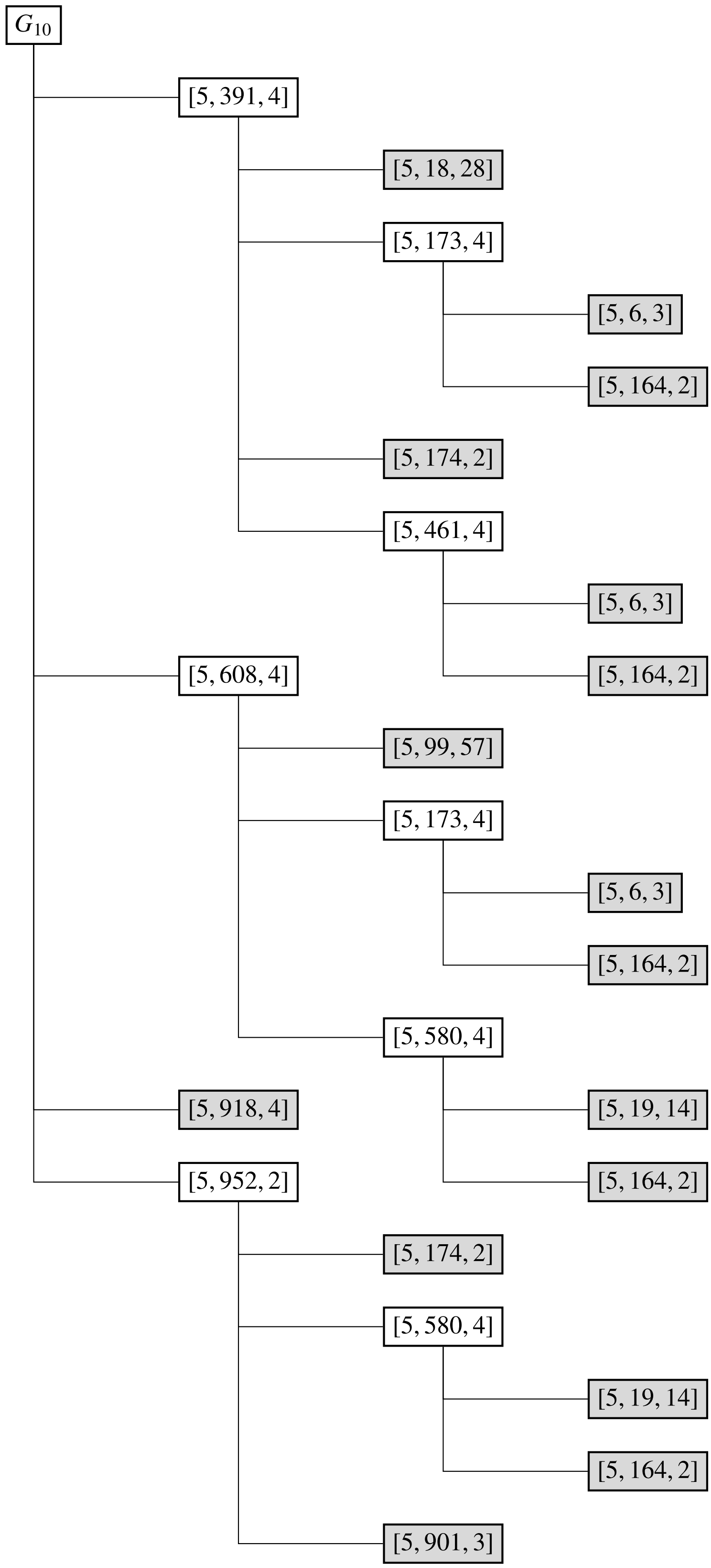

Figure 3: Conjugacy classes of subgroups of G_{10}. Algorithm (1) works for the gray ones.

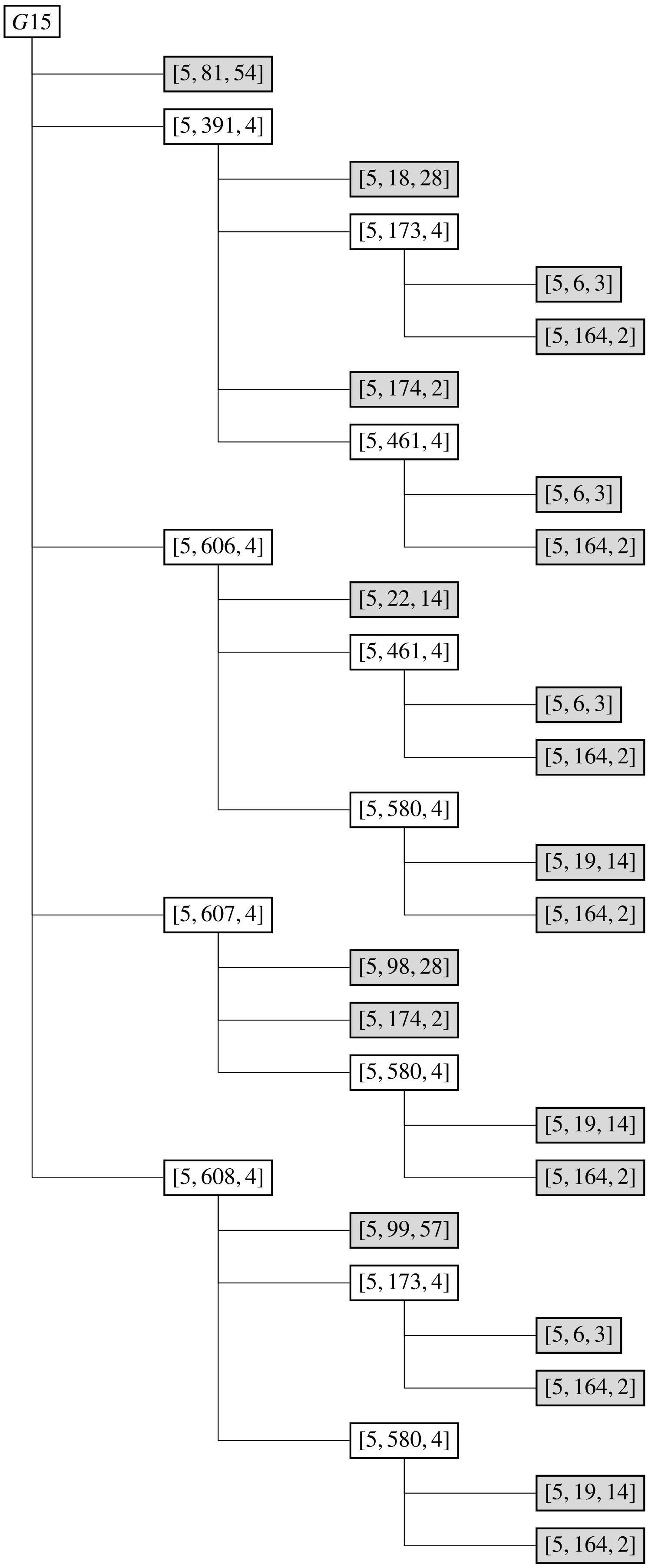

Figure 4: Conjugacy classes of subgroups of G_{15}. Algorithm (1) works for the gray ones.