02-680 Module 5

Essentials of Mathematics and Statistics

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Vectors

A vector is a quantity with both magnitude and direction. It is either written as $\vec{v}$, or V.

- An n-dimensional vector x is defined as an element of the **Cartesian product** of n copies of the real numbers:
$$x \in \mathbb{R}^n$$
- This means that $x = (x_1, x_2, \ldots, x_n)$, and if we want to reference the i-th element of x we will write x_i.
- (or sometimes $x[i]$. This is true of tuples as well)

For example, $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, and $\langle \frac{1}{2}, \frac{1}{2}\rangle$ are all vectors of length 2 where elements are from $\mathbb{Q}$ (or we would probably say $\mathbb{R}$).

- For the vector $x = \langle \frac{1}{2}, \frac{3}{2}\rangle$, we have $x_1 = \frac{1}{2}$, and $x_2 = \frac{3}{2}$

Vectors are sometimes contrasted with scalars, which are just numbers:

- That is, a **scalar** is an element of $\mathbb{R}$

Vectors are also sometimes written in parenthesis, so we may see an n-vector x written as $x = (x_1, x_2, \ldots, x_n)$

Vector Arithmetic

Sum of Vectors

The **sum** of two vectors $x, y \in \mathbb{R}^n$, written $x + y$, is a vector $z \in \mathbb{R}^n$

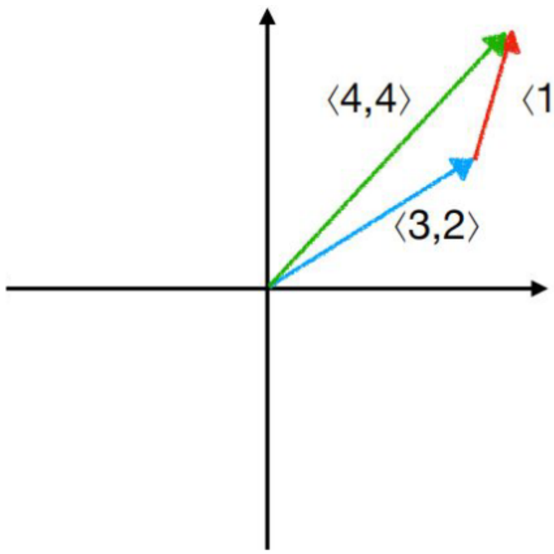
- where for every index $i \in \{1, 2, \ldots, n\}$ we have $z_i = x_i + y_i$
- Note: the sum of two vectors with different sizes is meaningless

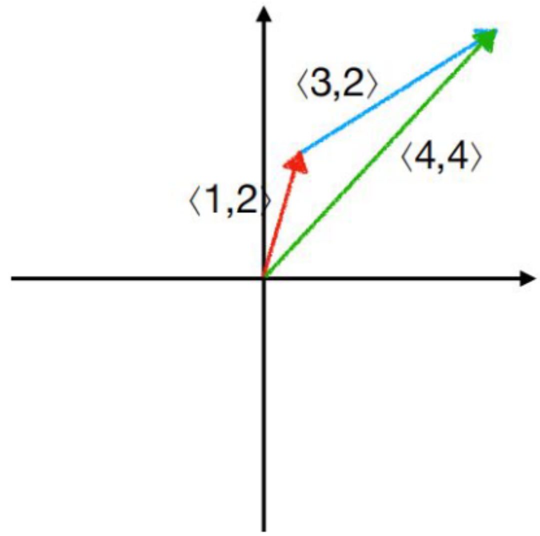
Examples:

- $\langle 3, 4\rangle + \langle 2, 1\rangle = \langle 5, 5\rangle$
- $\langle 1.1, 2.2, 3.3\rangle + \langle 4.4, 5.5, 6.6\rangle = \langle 5.5, 7.7, 9.9\rangle$
- $\langle 0, 1, 2\rangle + \langle 3, 4\rangle$ does not make sense!
- Vector addition operations are element-wise.

For $n = \{1, 2, 3\}$, we can visualize vectors.

- We usually draw them as arrows.
- This can help us interpret operations like addition.





Note that the order does not matter since addition is commutative.

Scalar Product

Given a vector $x \in \mathbb{R}^n$, and a real number $a \in \mathbb{R}$ the **scalar product**

- is a vector $z \in \mathbb{R}^n$,
- written $z = ax$,
- where for every index $i \in \{1, 2, \ldots, n\}$ we have $z_i = ax_i$.

Examples:

- $2\langle 3, 4\rangle = \langle 6, 8\rangle$
- $4.4\langle 1.1, 2.2, 3.3\rangle = \langle 4.84, 9.68, 14.52\rangle$

Dot Product

Given two vectors, $x, y \in \mathbb{R}^n$, the **dot product**, denoted $x \cdot y$

- is a value in $\mathbb{R}$ calculated as $\sum_{i=1}^{n} x_i y_i$
- that is, it is the sum of corresponding components.

Example:

$$\langle 1, 2, 3\rangle \cdot \langle 4, 5, 6\rangle = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

Intuitively, it measures the extent to which the two vectors point in the "same direction."

- Consequently, a unit vector dot product with itself will always return 1. (Because they point in the same direction).
- Dot product is maximized if the two vectors are pointing in the same direction.
- If vectors point in opposite directions (e.g., angle between them is obtuse), the dot product is negative.
- Dot product of two perpendicular vectors is always 0.

Vector Norm

The norm of a vector $x \in \mathbb{R}^n$ is defined as $||x|| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} (x_i)^2}$.

- This can be thought of as the magnitude (or length) of a vector.
- Informally, ℓ_p norm of a vector is a measure of its size.
- Here, $p \in \mathbb{Z}^{\geq 1}$.
- All norms, written as a function $||\_||_p : \mathbb{R}^n \to \mathbb{R}$

In general, the formal definition of ℓ_p norm of a vector is:

$$||x||_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$$

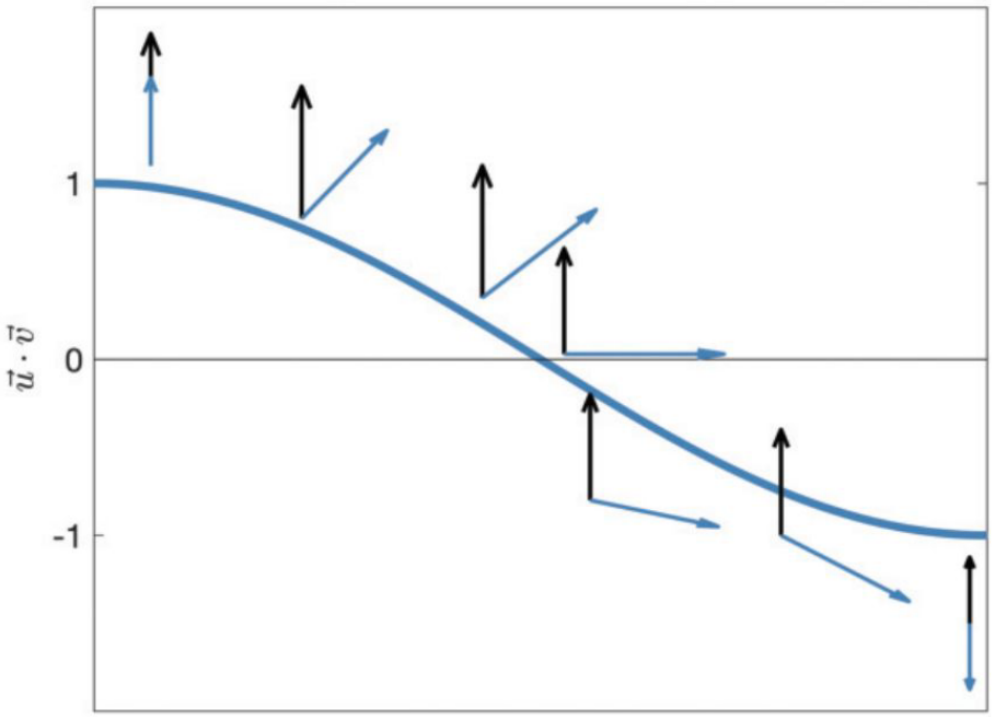
The typical norm we use is therefore $||x||_2$. **Norm Properties:**

- Non-negativity: The norm of any vector is always non-negative. Norm represents length of a vector, and lengths cannot be negative.
- Absolute Homogeneity (or Positive Scalability): If a vector is scaled by a scalar c, its norm also scales by $|c|$.
- Triangle Inequality: $\forall x, y \in \mathbb{R}^n : ||x + y||_p \leq ||x||_p + ||y||_p$, the norm of the sum of two vectors is **no greater** than the sum of their individual norms.
  - This is like saying "taking a direct route is never longer than going through an intermediate stop." It guarantees that the norm behaves like a **distance** in geometry.
- Definiteness: $||x||_p = 0 \Leftrightarrow x = 0$. The only vector with a norm of zero is the zero vector.

Orthogonal Vectors

If the dot product of vectors $\vec{u}$ and $\vec{v}$ is 0, then the two vectors are perpendiular, or **orthogonal**.

- If two vectors are generally pointing in the **same direction**, then their dot product is positive. If they are generally pointing in opposite directions, the dot product is negative.
- The dot product gets larger as the two vectors get closer to parallel and smaller as the two vectors get closer to anti-parallel.



If two orthogonal vectors are the edges of a triangle, $||x+y||^2 = ||x||^2 + ||y||^2$

Vector Length

Often times, called the ℓ_2 or **Euclidian-norm** or a vector.

- If we think of the vector as an arrow, we can say the length of the vector (arrow) is the same as the hypotenuse right triangle with each leg having the same length as each one of the elements.
- In that case, if we know that vector $x = \langle x_1, x_2 \rangle \in \mathbb{R}^2$, the length is $\sqrt{x_1^2 + x_2^2}$

Manhattan / Taxicab Distance

The ℓ_1-norm (Manhattan / Taxicab distance) is represented by

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

The ℓ_1 norm of a vector is the **sum of the absolute values** of its components. It's called the Manhattan or Taxicab distance because it measures distance by summing movements along grid lines - like a taxi driving in a city.

The ℓ_∞ Norm

The ℓ_∞ norm is represented by:

$$||x||_\infty = \max_{x_i \in x}(|x_i|)$$

For example, if $x = [3, -4, 2]$, then $||x||_\infty = \max(|3|, |-4|, |2|) = 4$.

- The ℓ_∞ norm of a vector is the maximum absolute value of its components.