02-712 Week 11

Biological Modeling and Simulation

Aidan Jan

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Stochastic Processes

Suppose we have a state space and transition probabilities. Let $S = \{0, 1\}$ Let p represent the probability of transitioning from $0 \to 1$, and q represent the probability of transitioning from $1 \to 0$.

Let us define the starting probabilities

- $P(x_0 = 0) = \pi_0(0)$
- $P(x_0 = 1) = \pi_1(0)$

Now, suppose we want to calculate the rate of change of the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx$$

We want to find r. How do we do this? We first write the equation to find the probability of some given state, t that the state is zero.

$$P(x_t = 0) = (1-p)P(x_{t-1} = 0) + qP(x_{t-1} = 0)$$

Simplifying,

$$\begin{aligned} &= (1-p)P(x_{t-1} = 0) + q(1 - P(x_{t-1} = 0)) \\ &= (1-p-q)P(x_{t-1} = 0) \end{aligned}$$

We can convert this recursive equation to the following:

$$P(x_t = 0) = (1-p-q)^t \pi_0(0) + q\left(\sum_{j=0}^{t-1} (1-p-q)^j\right)$$

We can simplify this:

$$\begin{aligned} &= \frac{q}{p+q} + (1-p-q)^t \left(\pi_0(\infty) - \frac{q}{p+q}\right) \\ &= \frac{q}{p+q} \end{aligned}$$

The right term goes to zero since at large t (e.g., the model has been running for a long time), the $(1-p-q)^t$ term approaches zero.

We know that $P(x_t = 1) = 1 - P(x_t = 0)$, so therefore,

$$P(x_t = 1) = \frac{p}{p+q}$$

We can generate the **stationary distribution** from these results, which describes the stochastic distribution after a long time of running the model.

$$\left\{\frac{q}{p+q}, \frac{p}{p+q}\right\}$$

The Pure Birth Model

In this model, we have a birth rate and a starting population, and that's it.

- No death
- Birth rate is b for all bacterial
- Birth rate is independent across individuals

Let $m(t)$ be the function that represents birth. We are given the rate of birth,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = bm(t)$$

to start. Therefore, $m(t) = e^{bt}$. Suppose we want to find the population at some state, when the population is n. At each time step, since this model is stochastic, there are two possible outcomes:

- Either one birth can occur (and therefore, $x_t = mx_{t-\Delta t} = m - 1$)
- No births can occur ($x_t = mx_{t-\Delta t} = m - 1$)

We can write

$$P_m(t + \Delta t) = P(x_{t+\Delta t} = m) = P_{m-1}(t) \cdot b\Delta t(m-1) + P_m(t)(1 - b\Delta tm)$$

We know from the rate of birth, $\frac{\mathrm{d}P_m(t)}{\mathrm{d}t} = \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = b(m-1)P_{m-1}(t) - bmP_m(t)$