02-680 Module 5

Essentials of Mathematics and Statistics

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Vectors

A vector is a quantity with both magnitude and direction. It is either written as $\vec{v}$, or V.

- An n-dimensional vector x is defined as an element of the **Cartesian product** of n copies of the real numbers:
$$x \in \mathbb{R}^n$$
- This means that $x = (x_1, x_2, \ldots, x_n)$, and if we want to reference the i-th element of x we will write x_i.
- (or sometimes $x[i]$. This is true of tuples as well)

For example, $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, and $\langle \frac{1}{2}, \frac{1}{2} \rangle$ are all vectors of length 2 where elements are from $\mathbb{Q}$ (or we would probably say $\mathbb{R}$).

- For the vector $x = \langle \frac{1}{2}, \frac{3}{2} \rangle$, we have $x_1 = \frac{1}{2}$, and $x_2 = \frac{3}{2}$

Vectors are sometimes contrasted with scalars, which are just numbers:

- That is, a **scalar** is an element of $\mathbb{R}$

Vectors are also sometimes written in parenthesis, so we may see an n-vector x written as $x = (x_1, x_2, \ldots, x_n)$

Vector Arithmetic

Sum of Vectors

The **sum** of two vectors $x, y \in \mathbb{R}^n$, written $x + y$, is a vector $z \in \mathbb{R}^n$

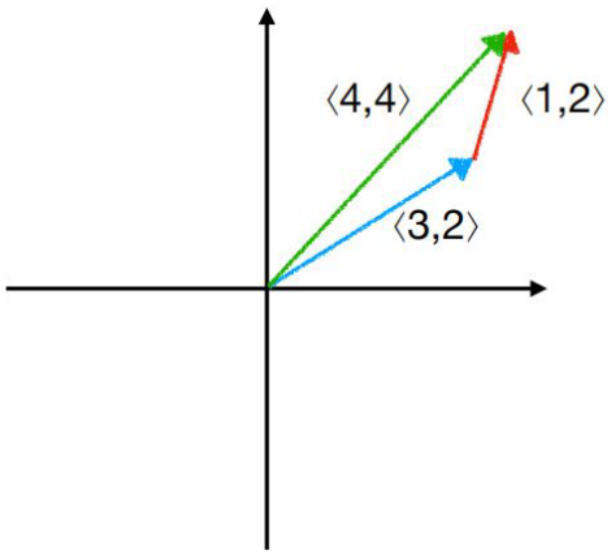
- where for every index $i \in \{1, 2, \ldots, n\}$ we have $z_i = x_i + y_i$
- Note: the sum of two vectors with different sizes is meaningless

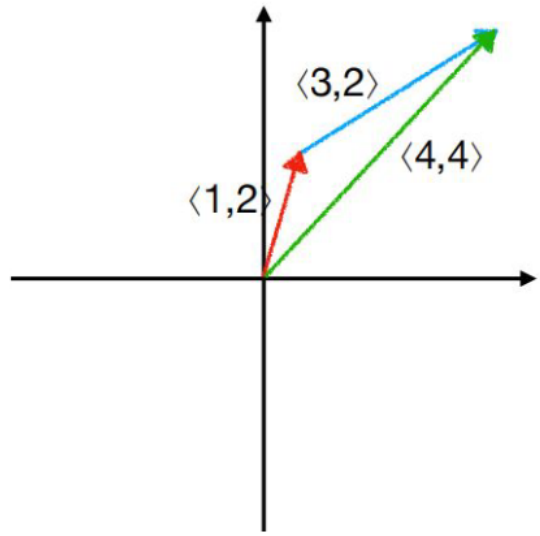
Examples:

- $\langle 3, 4 \rangle + \langle 2, 1 \rangle = \langle 5, 5 \rangle$
- $\langle 1.1, 2.2, 3.3 \rangle + \langle 4.4, 5.5, 6.6 \rangle = \langle 5.5, 7.7, 9.9 \rangle$
- $\langle 0, 1, 2 \rangle + \langle 3, 4 \rangle$ does not make sense!
- Vector addition operations are element-wise.

For $n = \{1, 2, 3\}$, we can visualize vectors.

- We usually draw them as arrows.
- This can help us interpret operations like addition.





Note that the order does not matter since addition is commutative.

Scalar Product

Given a vector $x \in \mathbb{R}^n$, and a real number $a \in \mathbb{R}$ the **scalar product**

- is a vector $z \in \mathbb{R}^n$,
- written $z = ax$,
- where for every index $i \in \{1, 2, \ldots, n\}$ we have $z_i = ax_i$.

Examples:

- $2\langle 3, 4\rangle = \langle 6, 8\rangle$
- $4.4\langle 1.1, 2.2, 3.3\rangle = \langle 4.84, 9.68, 14.52\rangle$