# 02-680 Module 3
## Essentials of Mathematics and Statistics

### Aidan Jan

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## Ordered Collections

Recall that a set is a well-defined collection of distinct objects.

- No duplicates

- No order

- Suitable for representing "which" elements are in a group.

However, in many contexts, order matters. For example, a set like {latitude, longitude} cannot distinguish between $(31.2, 121.4)$ and $(121.4, 31.2)$. These are different locations!

### Sequences

A sequence, also known as a list or tuple, is an **ordered collection** of objects. (typically called components or entries)

- When the number of objects in the collection is 2, 3, 4, or $n$

- the sequence is called an (ordered pair), triple, quadruple, or $n$-tuple.

- In some conventions they may be written using **angle brackets** $\langle \rangle$, but parenthesis also work.

Tuples may be used for representing positions, color, etc.

### Tuples

When these have small cardinality (length) we can use terms like

- Ordered pair

- Triple

- Quadruple

- or more generically an "$n$-tuple"

## Cartesian Product

A very useful way to **construct** set of tuples is using the **cartesian product** operator, which in essence creates all possible pairs of elements from two sets.

$$S \times T = \{\langle x, y \rangle | x \in S \land y \in T\}$$

As an example, let's remember the first two sets from our examples before:

$$A = \{\text{"Welcome"}, \text{"to"}, \text{"02-680"}\}$$
$$B = \{x^2 | x = 2 \lor x = 3\} = \{4, 9\}$$

In this case, the **cartesian product** is

$$A \times B = \{\langle\text{"Welcome"}, 4\rangle, \langle\text{"to"}, 4\rangle, \langle\text{"02-680"}, 4\rangle, \langle\text{"Welcome"}, 9\rangle, \langle\text{"to"}, 9\rangle, \langle\text{"02-680"}, 9\rangle\}$$

It doesn't have to be different sets in the cartesian product though, we can have the product with a set and itself. In fact, this is performed so often it has its own notation:

$$B \times B = B^2 = \{\langle 4, 4\rangle, \langle 4, 9\rangle, \langle 9, 4\rangle, \langle 9, 9\rangle\}$$

This notation also generalizes, so $S^3 = S \times S \times S$, etc.

- Note that $S^2$ will contain all the tuples of length 2, $S^3$ will contain all the tuples of length 3, etc.

- If a tuple is raised to the zeroth power, it results in an empty tuple.

## Sets of Tuples with Varying Lengths

Sometimes we want to have a set of tuples of different lengths. Remember sets don't need to be over objects of the same type.

Let $B = \{4, 9\}$. In this case, $B^2$ represents all the 2-tuples over $B$, and $B^3$ represents all the 3-tuples over $B$.

- Therefore, the union $B^2 \cup B^3$ will contain both, even though the tuples are not the same length.

### Kleene Star

If we wanted to enumerate all binary numbers up to 8 digits (while omitting leading 0's), how do we express this set?

- We can use a cartesian product and a union to define the set:

$$\{1\} \times \bigcup_{i=1}^{7} \{0, 1\}^i$$

The $\{1\}$ ensures the first digit is 1. The union represents the other digits. However, sometimes we want the set of all tuples of **any length**, then we use the **Klenne Star**, denoted by $*$ to generate all finite-length tuples formed from elements of a set.

- For some set $S$, we define $S^* = \bigcup_{i=0}^{\infty} S^i$.

- $S^0 = \langle\rangle$ (the empty tuple) for any $S$, and in the case of $\Sigma^0$, we often call it the **empty string**.

## Strings as Tuples Over an Alphabet

In certain contexts, sequences of elements from the same set are called **strings**. For a set $\Sigma$ called an alphabet, a string over $\Sigma$ is an element of $\Sigma^n$, for some nonnegative integer $n$.

In other words, a string is any element of $\bigcup_{n \in Z_{\geq 0}} \Sigma^n$. The length of a string $x \in \Sigma^n$ is $n$.

- For example, the set of all 5-letter English words is a subset of:

$$\{A, B, \ldots, Z\}^5$$

**Notation Conventions for Strings**

When writing strings, we usually omit angle brackets and commas (unlike regular tuples.)

**Empty String**

Strings are just sets of characters in an alphabet. They can have any non-negative length, including zero. For an alphabet $\Sigma$:

$$\Sigma^0 \{\epsilon\}$$

Here, $\epsilon$ is the empty string:

- It contains zero elements

- It is unique (only one such string exists)

| String | Belongs to | Explanation |
|---|---|---|
| ABRACADABRA | $\{A, B, \ldots, Z\}^{11}$ | A string of 11 capital letters |
| 11010011 | $\{0, 1\}^8$ | A binary string of length 8 |