02-680 Module 2

Essentials of Mathematics and Statistics

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Number Sets

Booleans

A boolean is a data type that takes two values: True and False.

- Sometimes we use 1 and 0 instead
- Can also be thought of as yes/no, on/off, etc.
- can be represented using 1 bit.

Integers

An integer is a number with **no fractional** part.

- We use the symbol $\mathbb{Z}$ to represent the set of **all possible integers**
  - There is a lot of them (infinitely many!), so unlike booleans we can't just list all integers.
  - Can be positive or negative or zero. (As opposed to **natural numbers**, represented by $\mathbb{N}$, and includes only positive integers)
  - The number of bits needed to represent an integer increases with the value.

Rational Numbers

A rational number is one that can be represented by a **ratio** between two integers. That is, it can be written in form $\frac{m}{n}$, where $m, n \in \mathbb{Z}$.

- We use the symbol $\mathbb{Q}$ to represent the set of all **rational numbers**.
- Real numbers that are not rational are called **irrational numbers**.

Real Numbers

A real number is one that can be used to **measure a continuous one-dimensional quantity** such as a length, duration, or temperature.

- Includes all integers, rational numbers, and all numbers "between" rational numbers.
- We use the symbol $\mathbb{R}$ to represent the set of **all real numbers**.
- Typically we need more bits to store real-valued numbers
  - proportional to the value and precision
  - many representations

Complex Numbers

Complex numbers were invented in order to allow for taking the square root of negative numbers

- Each complex number has a **real** and **imaginary** part, and is written as $a + bi$, where $a, b \in \mathbb{R}$
- We denote the set of all **complex numbers** as $\mathbb{C}$.
- Note that $\mathbb{R}$ is a subset of $\mathbb{C}$, where all values of b are 0.

Set Theory

Why does set theory matter?

Suppose we define a property like:

"Genes that are significantly upregulated under condition A"

Then we collect all the genes that match this property. This collection is a **set**.

We may also define a set by listing its members. Single-cell transcriptomics may identify:

$$\text{Cell type set} = \{\text{T cell, B cell, Macrophage, Fibroblast}\}$$

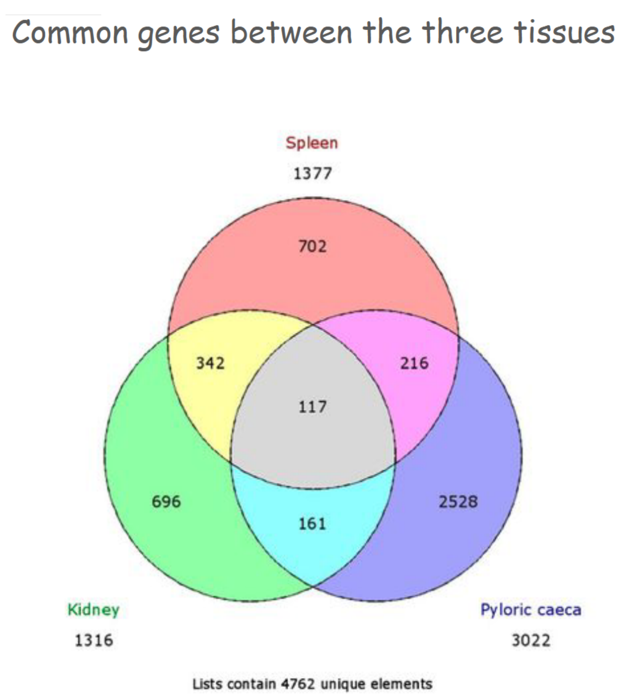
A **sample space** is the **collection** of all possible outcomes of an experiment.

- The sample space of an experiment can be thought of as a set, or collection, of different possible outcomes; and each outcome can be thought of as a point, or an element, in the sample space.

Venn Diagrams

Venn diagrams can be used to compare sets. Each circle represents a set, and members common to multiple sets are included in the overlap of the circles.

Below is an example using differentially expressed genes (DEGs):



What is a Set?

A **set** is an **unordered collection** of objects. We have already been talking about these in the abstract:

- Set of all integers $\mathbb{Z}$
- Set of all real numbers $\mathbb{R}$

We write a set using braces "{" and "}"

- use capital letters as names (when applicable)
- if we wanted to define the set of all algebraic operators we may use $O = \{+, -, \cdot, \div\}$
- or maybe the set of all prime numbers: $P = \{2, 3, 5, 7, \dots\}$

We have two standard ways of explicitly defining a set.

- First is simple (exhaustive) enumeration:

  $$A = \{\text{"Welcome"}, \text{"to"}, \text{"02-680"}\}$$

  In this case the set contains 3 elements, each of which is a string.

- The second way is to define a set by abstraction:

  $$B = \{x^2 | x = 2 \lor x = 3\}$$

  We could say that $4 \in B$, but $16 \notin B$.

  - The definition after the | is a statement written in **propositional logic**.

Set Membership

For a set S and an object x, the expression $x \in S$ is true when x is one of the objects in the set S.

- We read $x \in S$ as "x is an element of S", or "x is in S".
- There is also the notion of non-membership
  - if the set P is the set of all primes, then $4 \notin P$.
  - that is, 4 is not in P, or 4 is not prime.

Set Cardinality

For a set S, $|S|$ is the number of distinct elements in S.

Often we want to know how large a set is so we can compare sets.

- For the set of all algebraic operators described before $|O| = 4$
- For the set of all possible bit values B, $|B| = 2$.

but sometimes, we can't actually count how many there are. For example,

- $|\mathbb{Z}| = \infty$
- but, we still know that $|\mathbb{Z}| < |\mathbb{R}|$, since we know that $\mathbb{Z}$ is a subset of $\mathbb{R}$.

In the examples mentioned earlier:

$$A = \{\text{"Welcome"}, \text{"to"}, \text{"02-680"}\}$$

$$B = \{x^2 | x = 2 \vee x = 3\}$$

$|A| = 3$ and $|B| = 2$.

Note that if we define another set:

$$C = \{(2+2), (2-2), (2\cdot 2), (2 \div 2), (2^2)\}$$

the cardinality $|C| = 3$. It is not 5 since cardinality only includes unique elements.

Set Construction

As described earlier, there are two ways to construct a set. Exhaustive enumeration and set abstraction.

- Note that not all sets need to be assigned a name.
- Remember that sets are **unordered**, so the following sets are equivalent.
  - $\{2+2, 2\cdot 2, 2 \div 2, 2-2\}$
  - $\{0, 1, 4\}$
  - $\{4, 0, 1\}$

Set Abstraction

Let U be a set of possible elements called the **universe**. Let $P(x)$ be a condition, also called a **predicate**, that

- for every element $x \in U$
- $P(x)$ is **true** or **false**.

then we can write $\{x \in U \ : \ P(x)\}$

- which is all of the objects from $x \in U$
- for which $P(x)$ is **true**.

For example:

- set of even prime numbers: $\{x \in \mathbb{Z} \geq 1 \ : \ x \text{ is prime and } x \text{ is even}\}$
- set of primes between 10 and 20: $\{y \in \mathbb{Z} \ : \ y \text{ is prime and } 10 \leq y \leq 20\}$
- set of bits: $\{b \in \mathbb{Z} \ : \ b^2 = b\}$

For convenience, sometimes we don't use the universe if it's obvious

- this is particularly helpful when the predicate determines the universe
- In that case we could write something like $\{x \ : \ P(x)\}$

There is, of course, multiple ways to write these sets as well.

- Two digit perfect squares: $\{n \in \mathbb{Z} \ : \ \sqrt{n} \in \mathbb{Z} \text{ and } 10 \leq n \leq 99\}$
- $\{n^2 \ : \ n \in \mathbb{Z} \text{ and } 10 \leq n^2 \leq 99\}$
- $\{n^2 \ : \ n \in \{4, 5, 6, 7, 8, 9\}\}$ are the same set

Special Sets

There are some sets that we use that are so important we define them on their own, they also usually have **special symbols**.

The **Empty Set**, also called the **null set**, is a set that contains no elements.

$$\emptyset = \{\}$$

Since there are no elements in $\emptyset$, which we can write as $\forall x : x \notin \emptyset$, or $\nexists x : x \in \emptyset$, $|\emptyset| = 0$.

- Any set with **no elements** can be reduced to $\emptyset$
- Can also be defined as $\{x : \text{False}\}$ (similar to an if statement with "false" in the condition)

Additionally, we have common sets for groups of numbers.

- $\mathbb{N}$: natural numbers (positive whole numbers)
- $\mathbb{Z}$: integers (all whole numbers)
- $\mathbb{Q}$: rational numbers (numbers representable as fractions)
- $\mathbb{R}$: real numbers (decimal numbers)
- $\{0, 1\}$: booleans (no special symbol, represents two states)
- Σ: characters (sometimes, "alphabet")

Sometimes, we want to slightly restrict these sets, so sometimes you may see something like $\mathbb{Z}^+$, which is equivalent to $x \in \mathbb{Z} | x > 0$, or $\mathbb{Z}^{\geq 0}$, which is equivalent to $x \in \mathbb{Z} | x \geq 0$.

Operators

We have four standard operators used to compare sets, referred to as **set operators**.

- **Union** ($\cup$): The union of two sets S and T, denoted $S \cup T$, is the set of all elements in either S or T (or both).
$$S \cup T = \{x : x \in S \text{ or } x \in T\}$$
- **Intersection** ($\cap$): The intersection of two sets S and T, denoted $S \cap T$, is the set of all elements in both S and T.
$$S \cap T = \{x : x \in S \text{ and } x \in T\}$$
- **Set Difference** ($\backslash$): Similar to numerical subtraction, given two sets S and T, $S \backslash T$ is the set of elements in S but not T.
$$S \backslash T = \{x | x \in S \land x \notin T\}$$
  - For set difference, order matters. $S \backslash T$ and $T \backslash S$ are different sets.
  - Sometimes, set difference is represented using the minus ($-$) symbol instead of backslash ($\backslash$).
- **Complement** ($\bar{\ }$): While we can write something like $\tilde{\ }S$, it is normally written as $\bar{S}$. This represents the set of elements that are not in S.
$$\bar{S} = \tilde{\ }S := \{x \in U | x \notin S\}$$

Properties of Union and Intersection

Instead of writing long unions and intersections over multiple sets, we can use indexed notation, just like we do with summation and products.

$$\bigcup_{i=1}^{n} S_i = S_1 \cup S_2 \cup \cdots \cup S_n$$
$$\bigcap_{i=1}^{n} S_i = S_1 \cap S_2 \cap \cdots \cap S_n$$

Since a union combines two sets, the following is true:

$$\max\{|S|, |T|\} \leq |S \cup T| \leq |S| + |T|$$

Since an intersection only contains elements present in both sets, the following is true:

$$0 \leq |S \cap T| \leq \min\{|S|, |T|\}$$

Arithmetics and Sets

We previously saw the use of sums and products like

$$\sum_{i=3}^{5} 2^i = 8 + 16 + 32 = 56$$

but we can do the same using a set. Define $S = \{3, 4, 5\}$, then we can say:

$$\sum_{x \in S} 2^x = 8 + 16 + 36 = 56$$

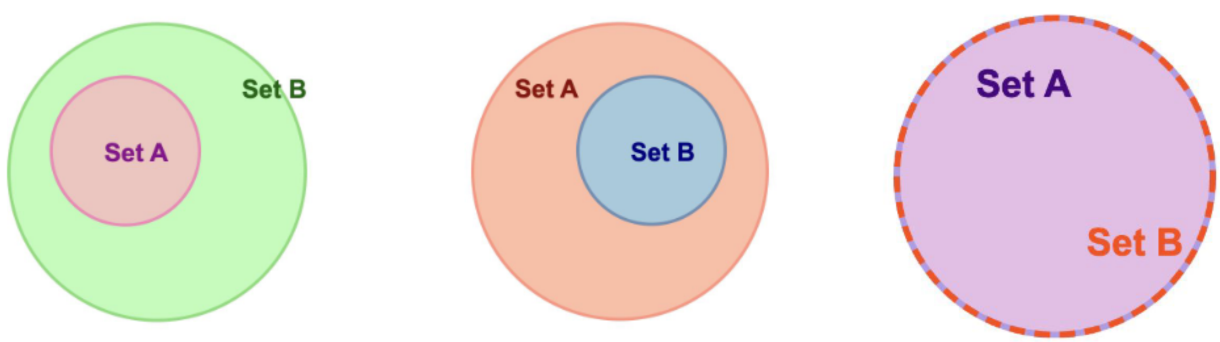
We can do the same with product, max, and min:

$$\prod_{x \in S} x \qquad \max_{x \in S} x \qquad \min_{x \in S} x$$

Comparing Sets

We often need to know if one set is contained within another. (notice while similar this is different from saying one is larger than another.)

- Subset $A \subseteq B$: A is fully contained within B.
- Superset $A \supseteq B$: B is fully contained within A.
- Equality $A = B$: A contains all elements of, but no more than, B.



$A \subseteq B$ $\qquad$ $A \supseteq B$ $\qquad$ $A = B$

Subset ($\subseteq$)

We say one set is a **subset** of another if all of its elements are in both sets.

$$S \subseteq T \Leftrightarrow \forall x \in S \,:\, x \in T$$

A set S is a subset of a set T, written $S \subseteq T$, if every $x \in S$ is also an element of T.

- In other words, $S \subseteq T$ is equivalent to $S - T = \{\}$.

Superset ($\supseteq$)

If a set contains another set, we can say the original is a **superset** of the other.

$$S \supseteq T \Leftrightarrow \forall x \in T \,:\, x \in S$$

Equality ($=$)

If two sets contain the same elements, they are equal.

$$S = T \Leftrightarrow (\forall x \in S \,:\, x \in T) \wedge (\forall y \in T \,:\, y \in S)$$

S is equal to T if and only if every element of S is also in T, and every element of T is also in S.

$$S = T \Leftrightarrow (S \subseteq T) \wedge (S \supseteq T)$$

Proper Subset and Proper Supersets

We also have the proper subset ($\subset$) and proper superset ($\supset$) operators, which represent subset and superset (but not equality).

- We define a proper subset as

$$S \subset T \Leftrightarrow (\forall x \in S \,:\, x \in T) \wedge (\exists y \in T \,:\notin S)$$

  - A set S is a proper subset of a set T, written $S \subset T$, if $S \subseteq T$ and $S \neq T$. In other words, $S \subset T$ whenever $S \subseteq T$, but $T \nsubseteq S$.

- We define a proper superset as

$$S \supset T \Leftrightarrow (\forall x \in T \,:\, x \in S) \wedge (\exists y \in S \,:\notin T)$$

  - S is a proper superset of T if and only if every element of T is in S, and there exists at least one element in S that is not in T.

Power Sets

The **power set** represents the collection of all subsets of a given set. It is a set of sets.

$$P(S) := \{T | T \subseteq S\}$$

The **power set** of S is denoted by (S).

For example, the power set of $0, 1, 2$:

$$(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

Also, a power set of a power set is a set of set of sets...

Cardinality of a Power Set

An interesting fact is that the size of the power set of a set is

$$|P(S)| = 2^{|S|}$$

Think of each element of $P(S)$ as a binary number of length $|S|$. If the i-th bit (or position) is 1, then the i-th element from S is in that subset. Then, each binary number represents a unique subset of S.

Power Set of the Empty Set

The cardinality of the power set of the empty set is 1. ($2^0 = 1$). This is because even though the empty set is empty, it still has one subset, which is itself.

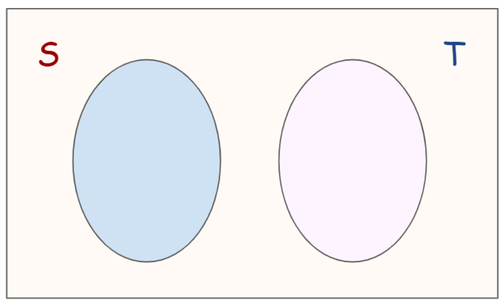
$$P(\emptyset) = \{\emptyset\}$$

Note that $\{\emptyset\} \neq \emptyset$. This is a set containing the empty set, not the empty set itself. In fact,

$$P(\emptyset) = \{\emptyset, \{\emptyset\}\}$$

Disjoint Sets

Two sets S and T are disjoint if there is no $x \in S$ where $x \in T$. In other words, if $S \cap T =$.



Two disjoint sets S and T.

For example, the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are disjoint, since their intersection is $\{\}$. However, $\{2, 3, 5, 7\}$ and $\{2, 4, 6, 8\}$ are not disjoint, because 2 is an element of both.

Partitions

The first interesting use of a set of sets is to form a partition of S into a set of disjoint subsets whose union is precisely S.

- A partition of a set S is a set $\{A_1, A_2, \ldots, A_k\}$ of nonempty sets $A_1, A_2, \ldots, A_k$, for some $k \geq 1$, such that
  - $A_1 \cup A_2 \cup \cdots \cup A_k = S$
  - for any i, j, where $j \neq i$, the sets A_i and A_j are disjoint.

For example, consider the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Here are some different ways to partition S:

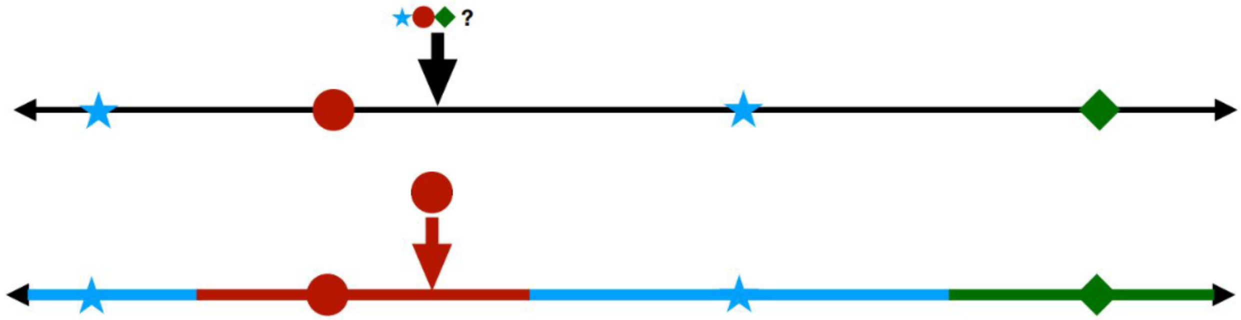
- $\{\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8, 10\}\}$
- $\{\{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \{10\}\}$
- $\{\{1, 4, 7, 10\}, \{2, 5, 8\}, \{3, 6, 9\}\}$
- $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}$
- $\{\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\}$

Use of Sets

In machine learning we often label data points. One way we can do that is using nearest neighbor:

- Given a set of labelled data and a new unlabeled point
- Use the "closest" point to label the new one.

This method creates a *partition* of the space into those that are closest to some label.



Jaccard Coefficient

Jaccard measures the **correlation** of two sets as a ratio of:

- how many elements are in both sets, over
- how many elements are in either set.

$$J(A,B) = \frac{|A \cap B|}{|A \cup B|} = \frac{|A \cap B|}{|A| + |B| - |A \cap B|}$$

Functions

We will say that a function provides a **mapping** from one set onto another. Formally we say that a function f that maps from a set S to a set T is written as

$$f\ :\ S \mapsto T$$

Formal Definition

Let S and T be sets. A function f from S to T, written $f\ :\ S \mapsto T$, assigns to each input value $a \in A$ a **unique** output value $b \in T$; the unique value b assigned to a is denoted by $f(a)$.

We sometimes say that f maps a to $f(a)$. Note that S and T are allowed to be in the same set.

For example, a function might have inputs and outputs that are both elements of $\mathbb{Z}$

Domain, Codomain, and Range

When writing $f\ :\ S \mapsto T$, we call S the **domain** set, and T the **codomain**. Note that the **codomain** is slightly different from the range of a function; the **range** is the subset of T that is reachable from an input in S. Formally the range is

$$y \in T | \exists x \in S\ :\ f(x) = y$$

The set of all y in T such that there exists an x in S with $f(x) = y$.

- Note that not all possible outputs in the codomain are actually achieved. There may be an element $b \in T$ for which there's no $a \in S$ with $f(a) = b$.

Range/Image

The range (or image) of a function $f : S \mapsto T$ is the set of all $b \in T$ such that $f(a) = b$ for some $a \in S$. The range of f is the set

$$\{y \in T \ : \ \text{there exists at least one } x \in S \text{ such that } f(x) = y\}$$

Domain and Codomain

The **domain** and **codomain** of a fuction are its sets of possible inputs and outputs: for a function $f : S \mapsto T$, the set S is called the **domain**, and the set T is called the **codomain** of the function f. Some examples:

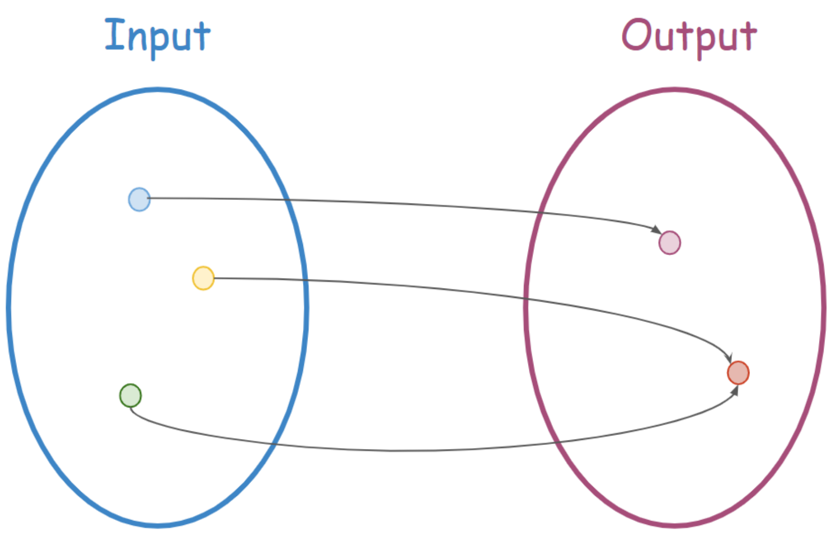
- Not has domain {True, False}, and codomain {True, False}. In this case, range = codomain.
- The square function maps $\mathbb{R} \to \mathbb{R}$. However, in this case, Range $\subset$ Codomain, since the range does not span negative numbers.

Extra Details:

- Note that the specification of the function includes both the **domain** and the **codomain**.
- But if those three elements are not compatible, it's not a function.
- For example, if a function has domain $\mathbb{R}$ and codomain $\mathbb{Z}$, then this implies that any real number passed into the function must return an integer. For a function like $y = 3x^2$, this does not work, since if $x = 1.5$, then y is not an integer; the codomain is incorrect. Thus, the specification $f(x) = 3x^2, f : \mathbb{R} \mapsto \mathbb{Z}$ is not a function.

A Visual Representation of Functions

This is a many-to-one function, where multiple inputs can map to the same output, but each input has exactly one output.



Function Composition ($\circ$)

We may find ourselves needing to apply one function to the output of another. This is known as **function composition**, defined formally as

$$g \circ f : S \mapsto U \text{ assuming } f : S \mapsto T \wedge g : T \mapsto U$$

For two functions $f : S \mapsto T$ and $g : T \mapsto C$, the function $g \circ f : S \mapsto C$ maps an element $a \in S$ to $g(f(a)) \in C$. The function $g \circ f$ is called the composition of f and g.

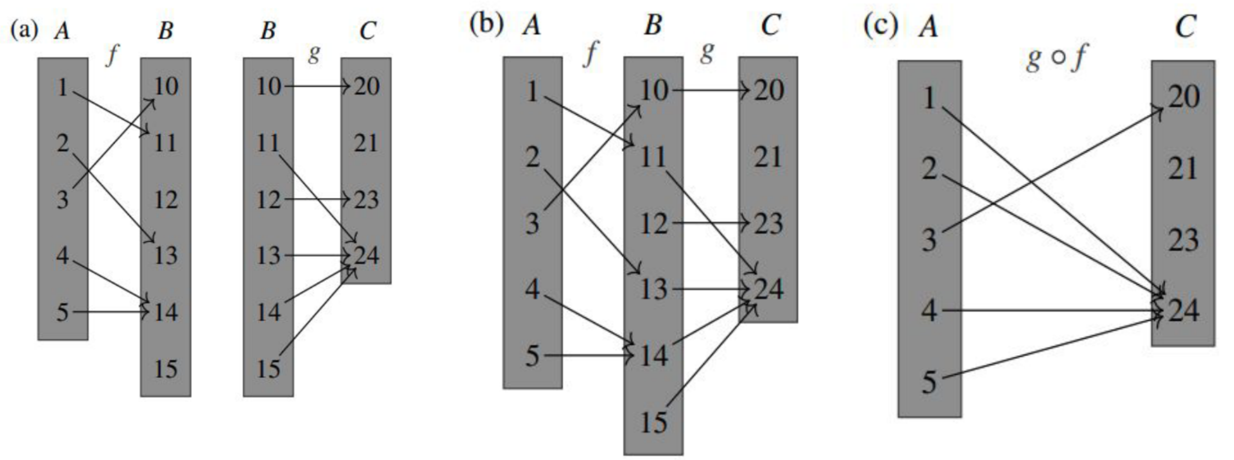
- Notice a slight oddity of the notation: $g \circ f$ applies the function f first and the function g second, even though g is written first.
- In use it may look like $g(f(x))$, so x would need to be from S since that's the input to f and since the input to g needs to be from T.
- That also needs to be the codomain of f then the codomain of g ends up as the codomain of the composite function.

Example: Function Composition

Let $f : \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2$. Then:

- $g \circ f = g(f(x)) = g(2x + 1) = (2x + 1)2 = 4x^2 + 4x + 1$
- $f \circ g = f(g(x)) = f(x^2) = 2x^2 + 1$
- $g \circ g = g(g(x)) = g(x^2) = \left(x^2\right)^2 = x^4$
- $f \circ f = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$

Visual Representation of Functions



For example, $(g \circ f)(1) = g(f(1))$, which is $g(11)$ because $f(1) = 11$. And $g(11) = 24$ because of g's arrow from 11 to 24.

https://www.cs.carleton.edu/faculty/dln/book/ch02_basic-data-types_2021_October_05.pdf