02-680 Module 9

Essentials of Mathematics and Statistics

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Linear Independence

First, we need to first define linear dependence.

- A set of vectors is **linearly dependent** if at least one of the vector lies in the span of the others, meaning it adds no new dimension
- In other words, by summing or scalar multiplying the other vectors, it is possible to produce the given vector.

Linear Combinations of Vectors

for a list of vectors $v_1, \dots, v_m$, a linear combination is written in the form:

- $a_1 v_1 + \cdots + a_m v_m$
- where $a_1, \dots, a_m$ are scalars.

Building a Vector Space

Vectors are elements of a **vector space**, and we can:

- Add them
- Scale them with scalars
- Use them to build other vectors through combinations

Vector spaces follow the **closure property**:

- Addition and scalar multiplication are closed
- The result always stays within the *same* vector space.

Why Linear Independence Matters

To build the whole space efficiently from a vector set, we want:

- No redundancy in our set.
- No vector in the set can be written as a combination of the others.
- Each vector adds a new direction.

This leads to the concept of **Linear Independence** - a foundational idea before we define a basis.

Definition of Linear Independence

Let $S = \{v_1, v_2, \ldots, v_n\} \subseteq V$ be a set of vectors in a vector space V. We say that S is linearly independent if:

$$a_1v_1 + a_2v_2 + \cdots a_nv_n = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0$$

A linear combination of vectors in S equals the zero vector only **when all the coefficients are zero**.

- This is because linear independence is defined this way to ensure that each vector contributes a new, non-redundant direction to the span. It guarantees a minimal set of generators, uniqueness in representation, and full dimensionality of the space spanned by the vectors.
- In other words, the equation says that for any vector $v_i \in S$ there is **no linear combination** of $S \setminus \{v_i\}$ that is equal to v_i. Here, $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ with $\forall i\ :\ a_i \in \mathbb{R}$ is a linear combination of the vectors in S.
- This is usually simplified to:

$$\sum_{i=1}^{n} \alpha_i v_i$$

Note that linear dependence and linear independence are notions that apply to a **collection** of vectors. It does not make sense to say things like "this vector is linearly dependent on these other vectors," or "this matrix is linearly independent."

Example

Is the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$$

linearly independent? Equivalently, we are asking ifthe homogeneous vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can solve this by forming a matrix and row reducing.

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This says $x = -2z$ and $y = -z$. So there exist nontrivial solutions: for instance, taking $z = 1$ gives this equation of **linear dependence**:

$$-2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2v_1 - v_2 + v_3 = 0 \Rightarrow v_3 = 2v_1 + v_2$$

This shows that the three vectors are linearly dependent.

- We can test whether any set of vectors are independent using this method. If the only solution to $a_1v_1 + a_2v_2 + a_3v_3 = 0$ is the trivial solution, $a_1 = a_2 = a_3 = 0$, then the vectors are linearly independent.
- If a non-zero solution exists (meaning that one vector can be written in terms of the other two), then it is not.

Facts about Linear Independence

1. Two vectors are linearly dependent if and only if they are colinear, i.e., one is a scalar multiple of the other.

2. Any set containing the zero vector is linearly dependent.

3. If a subset of $\{v_1, v_2, \dots, v_k\}$ is linearly dependent, then $\{v_1, v_2, \dots, v_k\}$ is linearly dependent as well.

Span

A **span** is the set of all possible linear combinations of a list of vectors $v_1, \dots, v_m$:

$$\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + \dots a_m v_m \; : \; a_1, \dots, a_m \in \mathbb{R}\}$$

The span is the smallest subspace that contains all the elements of the list.

- Drawing a picture of $\text{span}\{v_1, v_2, \dots, v_k\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \dots, v_k$.

Finding Span

For a set of vectors, we say that **span** is another set of vectors that consists of all linear combinations. So in the case above:

$$\langle 8, 7\rangle \in \text{span}(\{\langle 1, 2\rangle, \langle 2, 1\rangle\}) = \{a_1\langle 1, 2\rangle + a_2\langle 2, 1\rangle | a_1, a_2 \in \mathbb{R}\}$$

We want to know **whether** $\langle 8, 7\rangle$ can be written as such a linear combination.

To do this, we assume:

$$\langle 8, 7\rangle = a_1\langle 1, 2\rangle + a_2\langle 2, 1\rangle$$

First, we distribute the variables:

$$\langle a_1 + 2a_2, 2a_1 + a_2\rangle = \langle 8, 8\rangle$$

Now, we can convert it into a linear system:

$$\begin{aligned} a_1 + 2a_2 &= 8 \\ 2a_1 + a_2 &= 7 \end{aligned}$$

We can solve this using substitution, elimination, or converting it to a matrix and reducing. In this case, $a_1 = 2$ and $a_2 = 3$.

- Since the solution exists, yes, $\langle 8, 7\rangle$ is in the span of $\{\langle 1, 2\rangle, \langle 2, 1\rangle\}$

- Formally, for a set of vectors D with cardinality $|D| = n$,

$$\text{span}(D) \; := \left\{ \sum_{i=1}^{n} \alpha_i D_i \middle| \forall i \in [n] \; : \; \alpha_i \in \mathbb{R} \right\}$$

- If $D \subseteq V$ for a vector space V, then D is always a valid subspace of V.

- So in the example above since $\{\langle 1, 2\rangle, \langle 2, 1\rangle\} \subseteq \mathbb{R}^2$, the vector space defined by the set is also a subspace of $\mathbb{R}^2$.