02-613 Week 3

Algorithms and Advanced Data Structures

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Graph Traversal Algorithms

Depth First Search (DFS)

- Start from the root node and go down as deep as possible into a branch, and record nodes along the way.
- If there are no more child nodes, backtrack the path taken, and go down a different branch.
- Since each node in the graph is visited exactly once, this is a $O(n)$ operation.
- Typically implemented using recursion or a stack.

Theorem

If $(x, y) \in E$, either x is an ancestor of y or x is a descendant of y. Proof:

- Without loss of generality, x is an ancestor of y.
- All nodes between initially seeing x or leaving x are descendents of x.
- y must be explored before leaving x

Breadth First Search (BFS)

- Very similar to DFS but rather than going down all one branch all the way, it visits every child in order of level.
- Start from the root and record it plus every child. Repeat for each child in order.
- Also $O(n)$, since each node in the graph is visited exactly once.
- Typically implemented using a queue.

Theorem

If $(x, y) \in E$, then $|\text{layer}(x) - \text{layer}(y)| \leq 1$. Proof:

- Without loss of generality, assume that layer(x) < layer(y) - 1.
- All neighbors of x are added in or before layer(x) + 1

$$\text{layer}(y) \leq \text{layer}(x) + 1$$
$$\text{layer}(y) > \text{layer}(x) + 1$$

- The above is a contradiction.

- Basically, every time a node is visited, all its neighbors are added to the list to be iterated in the next level iteration.

The following is an example implementation of BFS.

```
TreeGrowing(graph G, node V, function findNext):
    T = ({v}, {})
    S = set of nodes incident to V
    while S != {}
        e = nextEdge(G, S)
        T += e
        S = updateFrontier(G, S, e)
```

Aside: Stacks and Queues

A queue is an array that views items in a First-in, First-out (FIFO) order. Basically elements added into the queue first will also be visited first.

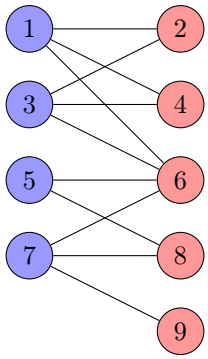
- `Dequeue()`: removes the first element of the queue, and moves the head pointer to the next element.
- `Enqueue(e)`: adds the element e to the end of the list.

A stack is an array that views items in a Last-in, First-out (LIFO) order. Basically elements added into the queue last will be visited first.

- `Pop(e)`: removes the last element of the list and decrements the tail pointer to the previous element.
- `Push(e)`: adds an element e to the end of the list.

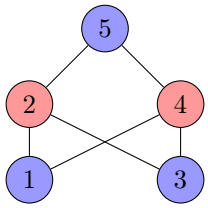
Bipartite Graphs

A **bibartite graph** is one with nodes in two sets such that there are no edges within a partition.



- $G = (U \cup V, E)$ such that $U \cap V = \emptyset$
- For a graph to be bipartite, it must have a two-coloring.
- It must also have no odd-length cycles.

Another example:



It is easy to tell that this graph is bipartite just by looking at it and coloring it in. However, what if the graph was bigger?

Breadth-First-Search Strategy

1. First, pick a random node, and BFS through the graph. Notice that every tree is bipartite since there are no cycles.
2. Now, insert in all the non-tree-edges. If there are no edges within the same level, then it is bipartite. (No "monolayer" edegs.)

```
def determineIfBipartite(G):
    T = BFS(G)
    for each layer:
        for each node pair u, v in layer:
            if (u, v) in G.E:
                return False
    return True
```

Proof of Correctness

For the purpose of contradiction, assume that there exists a $(u, v) \in E$ that is monolayer. Let Z be the common ancestor of u and v and let the path length from Z to u and v be l_i. In this case, edge (u, v) will create a cycle with length $2l_i+1$. This length is always odd for any $l_i \in \mathbb{Z}$, and results in a contradiction since bipartite graphs cannot have odd-length cycles. This algorithm will have a time complexity of $O(|V| + |E|)$ since it is limited by the BFS step.

Topological Sort

Graph Decomposition Method

Given a DAG (directed, acyclic graph), find a bijective mapping f from v to $\{1, \ldots, |V|\}$ such that $\forall(u, v) f(u) < f(v)$.

Theorem

Every DAG has at least one node with no incoming edges.

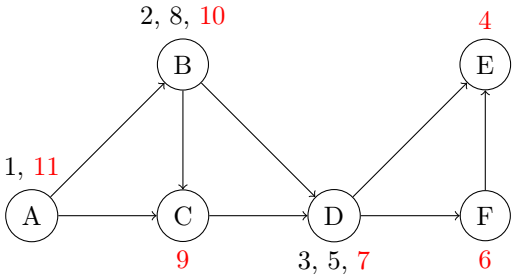
Because of this fact, we can sort the nodes of the DAG, such that any node would have no parents that come after it.

1. First, for every vertex, store the number of incoming edges in an array. $O(|V|)$.
2. While not every vertex is sorted,
    - Delete every node in the graph (and its outgoing edges) if it has 0 incoming edges
    - Store those nodes.

In this case, every vertex is visited once for the preprocessing, and afterwards, every vertex and every edge is visited. Therefore, the total runtime is $O(|V| + |E|)$.

DFS Method

Consider a DFS run on a graph. Every node can be assigned entering and leaving numbers, which represent the order in which the node was first visited and last visited. We start from A since it is the only node with no incoming edges.



Notice how a valid topological sorting is the leaving numbers of every node on the graph, in reverse order.

Shortest Path Algorithm

Suppose we have a directed, weighted graph (which may contain cycles) and we want to find the shortest path from node A to every other node.

Dijkstra's Algorithm:

- $\forall u d[u] = \infty$, $p[u] = \emptyset$, $F = V$
  - Let $d[]$ represent the distance from a starting node, let s be the starting node, let $p[u]$ represent the parent of u (e.g., which parent is part of the shortest path to u)
- $d[s] = 0$
- while $F \neq 0$:
  - u = vertex from F with minimum $d[u]$
  - remove u from F
  - $\forall$ neighbors v of u:
    - if $d[u]$ length$(u, v) < d[v]$
      - update$(d[v], p[v])$

Proof of Dijkstra's

First, a corrolary:

Corr. 1: let T be the set of nodes explored at some point in the algorithm. $\forall u \in T$, the path found by Dijkstra's is shortest. We can prove this by induction.

- Base case: $|T| = 1$, $T = (\{s\}, \{\})$, $d[s] = 0$. This is obviously correct, since the shortest distance between a starting node and itself must be 0.
- Hypothesis: Assume Corr. 1 at $|T| = k$. Let v be the $(k+1)$-th node. Let P_v be the path by Dijkstra. Let p' be any other $s \to v$ path. If $s \to u \to v < p'$, by design or Dijkstra since otherwise we would have added y.