02-680 Module 15

Essentials of Mathematics and Statistics

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Multiple Random Variables

Sometimes, we need to consider interactions between more than one random variable. These situations arise in many real-world and statistical applications. Consider tossing a (fair) coin 3 times, and define two random variables:

- X - the number of heads in the first toss
- Y - the number of heads in all 3 tosses

We want to know the **joint probability** (that is, the probability $p(X = x, Y = y)$)

		Y			
		0	1	2	3
X	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0
	1	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

Marginal Probabilities

What if we're given a **joint distribution** and want to compute the probabilities of individual variables? This leads us to **marginal probabilities**:

For example, unfair coins. Let

- X = outcome of coin 1
- Y = outcome of coin 2

These are two unfair coins modeled as **random variables.** Suppose we're given the following joint probabilities:

		Y		
		`Heads`	`Tails`	
X	`Heads`	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$
	`Tails`	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{7}{10}$
		$\frac{4}{10}$	$\frac{6}{10}$	

If we want to determine, say $P(X = \text{Tails})$, it turns out we can sum over all the possibilities of Y:

$$P(X = \texttt{Tails}) = \sum_{y \in \{\texttt{Heads, Tails}\}} p(X = \texttt{Tails}, Y = y) = \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$$

Doing all the math, both coins are biased toward Tails, the first coin (X) more-so.

What are Marginal Probabilities?

They are individual probabilities of one variable, found by summing over the other:

$$
\begin{aligned}
P(X = \texttt{Heads}) &= \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \\
P(X = \texttt{Tails}) &= \frac{3}{10} + \frac{4}{10} = \frac{7}{10} \\
P(Y = \texttt{Heads}) &= \frac{1}{10} + \frac{3}{10} = \frac{4}{10} \\
P(Y = \texttt{Tails}) &= \frac{2}{10} + \frac{4}{10} = \frac{6}{10}
\end{aligned}
$$

Continuous Random Variables

For continuous variables, we move from summation to integration:

$$p(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x, y)\, \mathrm{d}y\, \mathrm{d}x$$

And to get a **marginal distribution function**

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, \mathrm{d}y$$

Let's say we're drawing points uniformly over the unit square, in this case:

$$f(x, y) = \begin{cases} 1 & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if we want to find

$$p\left(X \le \frac{1}{2}, Y \le \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} 1\, \mathrm{d}y\, \mathrm{d}x = \frac{1}{4}$$

Independence of Random Variables

Similar to independence for event probabilities, two random variables X and Y are **independent** if and only if

$$\forall x, y \;:\; p(X = x, Y = y) = p(X = x) \cdot p(Y = y)$$

Conditional Distributions

Conditional distribution can be derived from the joint distribution $P(X, Y)$

$$p(X|Y) = \frac{p(X, Y)}{p(Y)}$$