02-680 Module 6

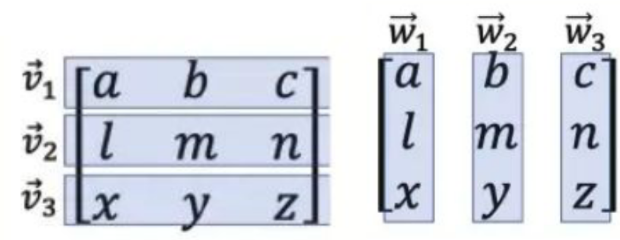
Essentials of Mathematics and Statistics

Aidan Jan

September 25, 2025

Matrices

You can almost think of a **matrix** as a 2-dimensional vector. We say that an "n-by-m" matrix $M \in \mathbb{R}^{n\times m}$ has n rows and m columns, and we usually write it as:

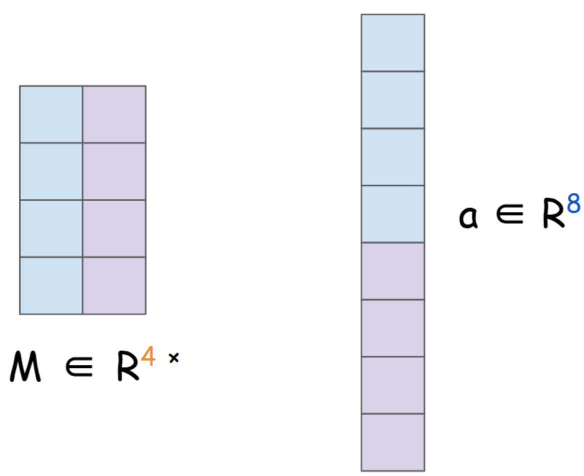


$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,m} \\ M_{2,1} & M_{2,2} & \dots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & M_{n,2} & \dots & M_{n,m} \end{bmatrix}$$

Matrix to Vector Transformation

$\mathbb{R}^{n\times m}$ is the set of all real-valued (n, m)-matrices. $M \in \mathbb{R}^{n\times m}$ can be equivalently represented as $M \in \mathbb{R}^{n\times m}$ by stacking all n columns of the matrix into a long vector.

- By stacking its columns, a matrix M can be represented as a long vector a.



Matrix Element

Let A be an $m \times n$ matrix. We will generally write $a_{i,j}$ for the entry in the ith row and the jth column. It is called the i, j entry of the matrix.

Matrix Transpose

For a given matrix $M \in \mathbb{R}^{n \times m}$, the transpose $M^T \in \mathbb{R}^{m \times n}$ is defined such that:

$$\forall i \in [1, n], j \in [1, m] \; : \; M^T_{j,i} = M_{i,j}$$

This operation works for both matrices and vectors (which are really just $n \times 1$ matrices.)

$$\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3} \right)^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}_{3\times 3}$$

Matrix Operations

Matrix Addition

Like with vectors, the addition of two matrices as well as scalar multiplication are element-wise operations, so for matrices $M, N \in \mathbb{R}^{n \times m}$:

$$O = M + N \rightarrow O_{i,j} = M_{i,j} + N_{i,j} \quad \forall 1 \leq i \leq n, 1 \leq j \leq m$$

Note that to be able to add, both matrices must be the same size.

Scalar Multiplication

Let M be an $n \times m$ matrix and scalar $a \in \mathbb{R}$. Denote the columns of M by $v_1, v_2, \ldots, v_p \; : \; O = aM \rightarrow O_{i,j} = aM_{i,j} \quad \forall 1 \leq i \leq n, 1 \leq j \leq m$.

$$M = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{bmatrix} \qquad aM = \begin{bmatrix} | & | & & | \\ av_1 & av_2 & \cdots & av_p \\ | & | & & | \end{bmatrix}$$

Matrix Multiplication

In general, to multiply a $m \times n$ matrix by an $n \times p$ matrix, the n must be the same, and the result is an $m \times p$ matrix.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 8 & 1 \cdot 6 + 2 \cdot 9 & 1 \cdot 7 + 2 \cdot 10 \\ 3 \cdot 5 + 4 \cdot 8 & 3 \cdot 6 + 4 \cdot 9 & 3 \cdot 7 + 4 \cdot 10 \end{bmatrix} = \begin{bmatrix} 21 & 24 & 27 \\ 47 & 54 & 61 \end{bmatrix}$$

This also works for vectors, where a vector is essentially a matrix with one of the dimensions being 1. Two vectors multiplied together (one must be a row vector, one must be a column vector) is just equal to the dot product between the two vectors.

- Note that matrix multiplication is **not** commutative! If we swap the order of the matrices, the result will not be the same.

Square Matrices

A square matrix is one that has an equal number of rows and columns. (e.g., $m = n$). These matrices have a few special properties.

- **Main diagonal** is the entries where the horizontal and vertical component are equal.
  - A **diagonal matrix** is one where all the numbers on the main diagonal are nonzero, and all the other numbers are zero.

- **Symmetry:** a square matrix is **symmetric** if $A = A^T$
- **Anti-symmetry:** a matrix's anti-symmetric is $A = -A^T$. Below: Left - symmetric. Right - anti-symmetric.

$$\begin{bmatrix} x & a & b \\ a & y & c \\ b & c & z \end{bmatrix} \quad \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

- **Trace:** the trace of a matrix $\mathrm{tr}(A)$ is the sum of the diagonal elements.
- **Identity:** an identity matrix of size n is an $n \times n$ matrix where the main diagonal values are 1 and all other values are 0. Symbolized by $I_{n\times n}$.
  - If I is multiplied with any other matrix, the other matrix does not change. $AI = IA = A$.

Determinant of a Square Matrix

The determinant of a square matrix A is a real number $\det(A)$. It is defined via its behavior with respect to row operations; this means we can use row reduction to compute it.

- Written with the $\det(A)$ function, or $|A|$.

2x2 Determinant Example

Let us compute

$$\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}\right)$$

We obtain

$$\det = 2 \cdot 6 - 1 \cdot 1 = 12 - 1 = 11$$

Basically, sum the products of diagonals going towards the right, and subtract products of diagonals going to the left.

Recursive Determinant Example

Compute

$$\det\left(\begin{bmatrix} 2 & 6 & 1 \\ 3 & 2 & 5 \\ 2 & 3 & 6 \end{bmatrix}\right)$$

$$\begin{aligned} \det(A) &= A_{11} \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 3 * 6 & \end{bmatrix}\right) - A_{12} \cdot \det\left(\begin{bmatrix} 3 & 5 \\ 2 & 6 \end{bmatrix}\right) + A_{13} \cdot \det\left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\right) \\ &= 2(-3)06(8) + 1(5) \\ &= -6 - 48 + 5 \\ &= -49 \end{aligned}$$

Using the notation of sets of column/row indices ($A = A_{[n],[n]}$) can then use set math to manipulate those rows/columns (mainly using \\):

$$A_{[n]\setminus i,[n]\setminus j}$$

Which is A with all but row i and all but column j/

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad A_{[3]\setminus 2,[3]} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

To make this easier we will shorten this to:

$$A_{[n]\setminus i,[n]\setminus j} \Leftrightarrow A_{\setminus i,\setminus j}$$

We need that notation to more easily define the determinate for any chosen j:

$$|A| := \sum_{i=1}^{n}(-1)^{(i+j)}A_{ij}|A_{\setminus i,\setminus j}$$

Determinant Transpose Property

$$\det(A) = \det(A^T)$$

Adjoint / adjugate matrix

Define $\tilde{A}$ to be:

$$\tilde{A} = \begin{bmatrix} (-1)^{1+1}|A_{\setminus 1,\setminus 1}| & (-1)^{1+2}|A_{\setminus 1,\setminus 2}| & \cdots & (-1)^{1+n}|A_{\setminus 1,\setminus n}| \\ (-1)^{2+1}|A_{\setminus 2,\setminus 1}| & (-1)^{2+2}|A_{\setminus 2,\setminus 2}| & \cdots & (-1)^{2+n}|A_{\setminus 2,\setminus n}| \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}|A_{\setminus n,\setminus 1}| & (-1)^{n+2}|A_{\setminus n,\setminus 2}| & \cdots & (-1)^{n+n}|A_{\setminus n,\setminus n}| \end{bmatrix}$$

to be the matrix of coefficients. The transpose $\tilde{A}$ (or $\tilde{A}^T$) is called the adjoint of A, denoted simply $\text{adj}(A)$.

- If A is nonsingular, then:

$$A^{-1} = \frac{1}{\det(A)}(\text{adj}(A))$$