02-680 Module 3

Essentials of Mathematics and Statistics

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Ordered Collections

Recall that a set is a well-defined collection of distinct objects.

- No duplicates
- No order
- Suitable for representing "which" elements are in a group.

However, in many contexts, order matters. For example, a set like {latitude, longitude} cannot distinguish between $(31.2, 121.4)$ and $(121.4, 31.2)$. These are different locations!

Sequences

A sequence, also known as a list or tuple, is an **ordered collection** of objects. (typically called components or entries)

- When the number of objects in the collection is 2, 3, 4, or n
- the sequence is called an (ordered pair), triple, quadruple, or n-tuple.
- In some conventions they may be written using **angle brackets** $\langle\rangle$, but parenthesis also work.

Tuples may be used for representing positions, color, etc.

Tuples

When these have small cardinality (length) we can use terms like

- Ordered pair
- Triple
- Quadruple
- or more generically an "n-tuple"

Cartesian Product

A very useful way to **construct** set of tuples is using the **cartesian product** operator, which in essence creates all possible pairs of elements from two sets.

$$S \times T = \{\langle x, y\rangle | x \in S \wedge y \in T\}$$

As an example, let's remember the first two sets from our examples before:

$$A = \{\text{"Welcome"}, \text{"to"}, \text{"02-680"}\}$$
$$B = \{x^2 | x = 2 \vee x = 3\} = \{4, 9\}$$

In this case, the **cartesian product** is

$$A \times B = \{\langle\text{"Welcome"}, 4\rangle, \langle\text{"to"}, 4\rangle, \langle\text{"02-680"}, 4\rangle, \langle\text{"Welcome"}, 9\rangle, \langle\text{"to"}, 9\rangle, \langle\text{"02-680"}, 9\rangle\}$$

It doesn't have to be different sets in the cartesian product though, we can have the product with a set and itself. In fact, this is performed so often it has its own notation:

$$B \times B = B^2 = \{\langle 4, 4\rangle, \langle 4, 9\rangle, \langle 9, 4\rangle, \langle 9, 9\rangle\}$$

This notation also generalizes, so $S^3 = S \times S \times S$, etc.

- Note that S^2 will contain all the tuples of length 2, S^3 will contain all the tuples of length 3, etc.
- If a tuple is raised to the zeroth power, it results in an empty tuple.

Sets of Tuples with Varying Lengths

Sometimes we want to have a set of tuples of different lengths. Remember sets don't need to be over objects of the same type.

Let $B = \{4, 9\}$. In this case, B^2 represents all the 2-tuples over B, and B^3 represents all the 3-tuples over B.

- Therefore, the union $B^2 \cup B^3$ will contain both, even though the tuples are not the same length.

Kleene Star

If we wanted to enumerate all binary numbers up to 8 digits (while omitting leading 0's), how do we express this set?

- We can use a cartesian product and a union to define the set:

$$\{1\} \times \bigcup_{i=1}^{7} \{0, 1\}^i$$

The $\{1\}$ ensures the first digit is 1. The union represents the other digits. However, sometimes we want the set of all tuples of **any length**, then we use the **Klenne Star**, denoted by $*$ to generate all finite-length tuples formed from elements of a set.

- For some set S, we define $S^* = \bigcup_{i=0}^{\infty} S^i$.
- $S^0 = \langle\rangle$ (the empty tuple) for any S, and in the case of Σ^0, we often call it the **empty string**.

Strings as Tuples Over an Alphabet

In certain contexts, sequences of elements from the same set are called **strings**. For a set Σ called an alphabet, a string over Σ is an element of Σ^n, for some nonnegative integer n.

In other words, a string is any element of $\bigcup_{n \in Z \geq 0} \Sigma^n$. The length of a string $x \in \Sigma^n$ is n.

- For example, the set of all 5-letter English words is a subset of:

$$\{A, B, \ldots, Z\}^5$$

Notation Conventions for Strings

When writing strings, we usually omit angle brackets and commas (unlike regular tuples.)

Empty String

Strings are just sets of characters in an alphabet. They can have any non-negative length, including zero. For an alphabet Σ:

$$\Sigma^0\{\epsilon\}$$

Here, ϵ is the empty string:

- It contains zero elements
- It is unique (only one such string exists)

String	**Belongs to**	**Explanation**
ABRACADABRA	$\{A, B, \ldots, Z\}^{11}$	A string of 11 capital letters
11010011	$\{0, 1\}^8$	A binary string of length 8