Math 170E Week 2

Aidan Jan

January 23, 2023

Equiprobable Outcomes

Formula (from last lecture):
$S = \{\omega_1, \omega_2, ...\}$, $|S| < \infty$

$$\forall A \subset S, \mathbb{P}(A) = \frac{|A|}{|S|}$$

Ex.
$S = \{1, 2, 3, 4, 5, 6\}$ Therefore,

$$\mathbb{P}(\{2,4\}) = \frac{|\{2,4\}|}{|\{1,...,6\}|} = \frac{1}{3}$$

Ex. Consider a bowl with 4 blue balls and one red ball. Find $\mathbb{P}$(picked the red ball).

The equiprobable outcome formula does not work for this case because the chances are not equiprobable. However, if the balls were numbered 1 - 5, with the red one being "5", this question can be made equiprobable. It would be analyzing the chance the ball labeled "5" is picked, rather than the red ball is picked.

Ex. (extended)

Now, instead of picking one ball, pick two balls. What is the chance that at least one of them is red if you pick two balls, one after the other, without replacement?

$$S = \{(x_1, x_2) \,:\, x_1, x_2 \in [[1,5]], x_1 \neq x_2\}$$

$$A = \{(x_1, x_2) \in S \,:\, x_1 = 5 \text{ or } x_2 = 5\}$$

Using this way of viewing it, this is a permutation problem. Using the permutation formula, there are $5 \times 4 = 20$ ways to pick two balls. For A, there are 8 outcomes. (They are listed here:)

$$(1,5), (2,5), (3,5), (4,5), (5,1), (5,2), (5,3), (5,4)$$

Therefore, the probability is $\frac{8}{20} = \frac{2}{5}$.

Another way this problem can be viewed is using the following:

$$S = \{\{x_1, x_2\} \,:\, x_1, x_2 \in [[1,5]], x_1 \neq x_2\}$$

$$A = \{\{x_1, x_2\} \in S \,:\, 5 \in \{x_1, x_2\}\}$$

Thus,

$$S = \{(1,5),(2,5),(3,5),(4,5),(1,4),(2,4),(3,4),(1,3),(2,3),(1,2)\}$$

$$A = \{(1,5),(2,5),(3,5),(4,5)\}$$

Using the equiprobable outcome formula, we get the same answer of $\frac{4}{10} = \frac{2}{5}$.

Limits of Probabilities

Consider a coin flip. $X_n =$ Heads, $Y_n =$ Tails The number of heads can be written as $\frac{X_n}{X_n+Y_n}$

Now, suppose we want to test whether a coin is fair, and we flip it an infinite number of times.

If

$$\lim_{n\to\infty} \frac{X_n}{X_n + Y_n} = 0.5$$

then the coin is fair.

$$\lim_{n\to\infty} \frac{X_n}{X_n + Y_n} = Z, \text{ where } Z \sim \text{unif}[0,1]$$

is the formula.

Ex.

You have 80 balls and 40 boxes, and you randomly put balls into the boxes. What is $\mathbb{P}$(12th box is empty)?

$$S = \{(x_1, x_2, ..., x_{80}\} \,:\, x_k \in [[1,40]].1 \leq k \leq 80\}$$

$$A = \{\vec{x} \in S \,:\, 12 \notin \{x_1, ..., x_{80}\}\}$$

(can also be written as)

$$S = \{\vec{x} \in N^{80} \,:\, x_k \in [[1,40]].1 \leq k \leq 80\}$$

$$A = \{\vec{x} \in S \,:\, \forall k, x_k \neq 12, 1 \leq k \leq 80\}$$

Thus,

$$|A| = 39 \times 39 \times ... = 39^{80}$$

$$\mathbb{P}(A) = \frac{|A|}{|S|} = \left(\frac{39}{40}\right)^{80}$$

Conditional Probability

Let $A, B \subset S$. **Conditional Probability** is the probability of an event given another event occurs. For example, a question would read as "What is the probability of B, given A occurs. It is written as $\mathbb{P}(A|B)$

Conditional Probability Formula

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B}{\mathbb{P}(B)}$$

Ex. Professor Monz has 4 kids. Out of the four kids, at least two of them are girls. What is the probability that the oldest child is a son?

Solution:
This can be written as $\mathbb{P}$(The oldest one is a boy | " ≥ 2 girls")

Let B_k represent the case where Monz has k boys.

$$S = \{(x_1, x_2, x_3, x_4), x_k \in \{\text{G}, \text{B}\}, k = 1, 2, 3, 4\}$$

$$B = B_2 \cup B_3 \cup B_4$$

$$\mathbb{B} = \mathbb{P}(B_2) + \mathbb{P}(B_3) + \mathbb{P}(B_4)$$

First, we need to find the possibility of having k boys.

$$\begin{aligned}
\mathbb{P}(B_2) &= \frac{|B_2|}{|S|} = \binom{4}{2} \times \frac{1}{|S|} = \frac{6}{16} \\
\mathbb{P}(B_3) &= \frac{|B_3|}{|S|} = \binom{4}{3} \times \frac{1}{|S|} = \frac{4}{16} \\
\mathbb{P}(B_4) &= \frac{|B_4|}{|S|} = \frac{1}{16} \\
\therefore \mathbb{P}(B) &= \frac{11}{16}
\end{aligned}$$

The chance of any one child being the oldest is $\frac{1}{4}$ since there are four children. Therefore, if A represents the oldest being a boy, and B represents the given that there are at least two girls,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A|B)}{\mathbb{P}(B)} = \frac{\frac{1}{4}}{\frac{11}{16}} = \frac{4}{11}$$

Thus, $\frac{4}{11}$ is the answer.

Conditional Probability Rules

Suppose A and B are independent events.
1. $\mathbb{P}(A_1 \cup A_2|B) = \mathbb{P}(A_1|B) + \mathbb{P}(A_2|B)$
2. $\mathbb{P}(A|B) \neq \mathbb{P}(B|A)$
3. $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$
4. $\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \cdot \mathbb{P}(B)$

Example:
Suppose you have a bowl with 10 balls in it. 4 are red, 6 are blue. Let A = first ball picked is red. Let B = second ball picked is red. $\mathbb{P}(A \cap B) = \mathbb{P}(B|A) \cdot \mathbb{A} = \frac{3}{9} \cdot \frac{4}{10} = \frac{2}{15}$

Example:
Now, what if the second one is not red? The same formula can be applied... yielding the solution $\frac{4}{15}$
If we add the two solutions, then we find just the probability that the first ball is red, since we covered all the options for the second choice. $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$

Using the same expansion as before, this can be written as:

$$\mathbb{P}(A) = \mathbb{P}(A|B) \cdot \mathbb{P}(B) + \mathbb{P}(A|B^c) \cdot \mathbb{P}(B^c)$$

Plugging in the numbers from before of $\mathbb{P}(A|B)$, $\mathbb{P}(A|B^c)$, $\mathbb{P}(B)$, and $\mathbb{P}(B^c)$,

$$\mathbb{P}(A) = \frac{3}{9} \cdot \frac{4}{10} + \frac{4}{9} \cdot \frac{6}{10} = \frac{2}{15} + \frac{4}{15} = \frac{2}{5}$$

This yields the original probability of picking a red ball first, $\frac{4}{10}$.