BIOMATH 208 Week 3

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Curves and Surfaces in the Brain

- The cortex and subcortical structures of the brain can be well represented by surfaces.
- The gyri and sulci of the cortex can be well represented by curves.

[FILL 5]

Discrete Curves

Curves are modeled as piecewise linear, parameterized by a list of vertices, e.g.:

[FILL 6] [FILL 7]

The first graph is actually in 3D! Notice how the z-coordinate of all the coordinates is zero.

Goal

- Build a vector space structure for discrete curves and surfaces.
- Design an inner product and associated norm that is small when two curves (resp. surfaces) and their tangents (normals) are close.
- With a norm and inner product, we will be able to use many standard machine learning algorithms.
- We follow the approach of "Large Deformation Diffeomorphic Metric Curve Mapping" (2008) or "Surface Matching via Currents" (2005).

Curves as Integral Operators

Definition: (Action of a curve on a smooth vector field)
Let our curve be parameterized by a function $\gamma : [0, 1] \to \mathbb{R}^3$. Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector valued function. The curve acts:

$$\int_0^1 v(\gamma(t)) \cdot \gamma'(t) \mathrm{d}t$$

where $\cdot$ is the "standard" dot product in $\mathbb{R}^3$. The term $\gamma'(t)$ is the derivative of γ at the point t, which can be thought of as a vector tangent to the curve whose magnitude is its speed.

Example: Action close to 0 because far away

[FILL 9]

Example: Action close to 0 because orthogonal

[FILL 10]

Example: A large action

[FILL 11]

Curves are covectors dual to smooth functions

This action is linear, and therefore curves are covectors.

- $+$: [FILL 12]
- $\cdot$: [FILL 12]

Parameterization Invariance

If $\gamma : [0,1] \to [0,1]$ is an increasing differentiable function with $\varphi(0) = 0$ and $\varphi(1) = 1$, then a different parameterization could be $\kappa(t) = \gamma(\varphi(t))$. Parameterization invariance means that

$$\int_0^1 v(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t = \int_0^1 v(\kappa(t)) \cdot \kappa'(t)\mathrm{d}t$$

Proof.

[FILL 13]

Changing Direction

If $\varphi(t) = 1 - t$ (i.e., we change direction), then a different parameterization could be $\kappa(t) = \gamma(\varphi(t))$. Then we have, $\kappa = -\gamma$:

$$\int_0^1 v(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t = -\int_0^1 v(\kappa(t)) \cdot \kappa'(t)\mathrm{d}t$$

Proof.

[FILL 14]

Distance between curves

- Two curves are close if their action on all smooth vector fields are similar. They are (weakly) equal if their action on all smooth vector fields are equal.
- How do we deal with the "all" part of this statement? Consider them all and take the worst one.

Definition: Operator norm for curves
We will use the operator norm

$$\|\gamma\|v^* = \sup_{v \in V, \|v\|_{v=1}} \gamma(v)$$

The Maximizer

The maximizer of the previous expression is given by a unit vector in the direction of $\gamma^\sharp$

$$v_{\max} = \frac{\gamma^\sharp}{|\gamma^\sharp|v}$$

Proof.

[FILL 16]

The Explicit Norm

Plugging in the maximizer gives

$$\|\gamma\|v^* = \|\gamma^\sharp\|v$$

This means we can define a norm on V^*, as long as we can define on V!

Proof.

[FILL 17]

Why smooth vector fields? Example: Importance of Smoothness

[FILL 18]

Enforcing Smoothness

We will use two approaches to make sure the above cannot occur:

1. Parameterize our vector fields as a superposition of smooth functions
2. Build an inner product so that non-smooth (or rough) functions have an infinite norm

Parameterization

We restrict ourselves to vector fields that are a superposition of Gaussians, $K(x) = \exp(-\frac{1}{2\sigma^2}|x|^2)$ where σ^2 controls smoothness.

$$V = \left\{ f \,:\, f(x) = \sum_{i=1}^{N} p_i K(x - x_i), \quad \forall x_i, p_i \in \mathbb{R}^3, \quad \forall N \in \mathbb{N} \right\}$$

Any vector field can be parameterized by a list of centers (x_i) and positions (p_i).