EC ENGR 102 Week 5

Aidan Jan

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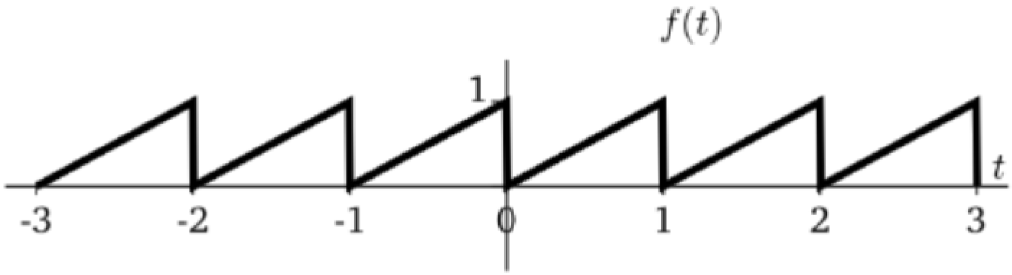
Sawtooth Signal

The sawtooth signal is given by $f(t) = t \mod 1$. It is plotted below:



This signal has a period of $T_0 = 1$. Now, when $k = 0$,

$$\begin{aligned} c_0 &= \int_0^1 t e^0 \mathrm{d}t \\ &= \left.\frac{t^2}{2}\right|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

This is also the Fourier series of the time-limited signal $f(t) = t$ on the interval $[0, 1)$. The time-limited signal can be made periodic via a periodic extension.

Fourier Series Properties

There are interesting symmetries and properties of the Fourier series that are worth expanding upon.

- c_0 **is the average of the signal**. Not that for $k = 0$, we have that

$$c_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) \mathrm{d}t$$

  Thus, c_0 is exactly the time-averaged mean of the signal and corresponds to a constant value (i.e., it has no sinusoidal component). For this reason, it is sometimes called the "DC component." DC stands for direct current in circuits, and refers to non-alternating (sinusoidal) currents. The DC component is the average value taken on by a signal.

Fourier Symmetry

We can apply Euler's formula to re-write the Fourier coefficients, and reveal some symmetries:

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-j\frac{2\pi k t}{T_0}}\,\mathrm{d}t \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right]\mathrm{d}t \quad = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t - \frac{j}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t
\end{aligned}
$$

In the above equation, the left term is the real part, the right term is the imaginary part.

If $f(t)$ is real, then so are:

$$
\begin{aligned}
\Re(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\
\Im(c_k) &= -\frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t
\end{aligned}
$$

Therefore, for $f(t)$ real, and using the fact that $\cos(k)$ is even and $\sin(k)$ is odd, we have the following symmetries:

$$
\begin{aligned}
\Re(c_k) &= \Re(c_{-k}) && (1)\\
\Im(c_k) &= \Im(c_{-k}) && (2)\\
c_k^* &= c_{-k} && (3)\\
|c_k| &= |c_{-k}| && (4)\\
\angle c_k &= -\angle c_k^* && (5)\\
c_k &= c_{-k} \qquad \text{(only if } x(t) \text{ is even)} && (6)\\
c_k &= -c_{-k} \qquad \text{(only if } x(t) \text{ is odd)} && (7)
\end{aligned}
$$

Proof of (1)

$$
\begin{aligned}
\Re(c_{-k}) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cdot\cos\left(-\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cdot\cos\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\
&= \Re(c_{-k})
\end{aligned}
$$

Proof of (2)

$$
\begin{aligned}
\Im(c_{-k}) &= -\frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cdot\sin\left(-\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cdot\sin\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\
&= \Im(c_{-k})
\end{aligned}
$$

Proof of (3)

$$
\begin{aligned}
c_k &= \Re(c_k) + j\cdot\Im(c_k) \\
c_k^* &- \Re(c_k) - j\cdot\Im(c_k) \\
&= \Re(c_-k) + j\cdot\Im(c_-k) \\
&= c_{-k}
\end{aligned}
$$

Proof of (4)

$$\begin{aligned}
|c_k| &= \sqrt{\Re^2(c_k) + \Im^2(c_k)} \\
|c_{-k}| &= \sqrt{\Re^2(c_k) + \Im^2(c_{-k})} \\
&= \sqrt{\Re^2(c_k) + (-\Im(c_k))^2} \\
&= |c_k|
\end{aligned}$$

Proof of (5)

$$\begin{aligned}
\angle c_k &= \arctan\left(\frac{\Re(c_k)}{\Im(c_k)}\right) \\
&= \arctan\left(\frac{\Re(c_{-k})}{\Im(c_{-k})}\right) \\
&= \arctan\left(\frac{\Re(c_k)}{-\Im(c_k)}\right) \\
&= \angle c_{-k}
\end{aligned}$$

Proof of (6)

$$\begin{aligned}
c_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\mathrm{d}t \\
c_{-k} &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{jk\omega_0 t}\mathrm{d}t
\end{aligned}$$

Let $u = -t$.

$$\begin{aligned}
&= -\frac{1}{T_0}\int_0^{-T_0} x(-u)\cdot e^{-jk\omega_0 u}\mathrm{d}u \\
&= \frac{1}{T_0}\int_{-T_0}^{0} x(u)\cdot e^{-jk\omega_0 u}\mathrm{d}u \\
&= c_k
\end{aligned}$$

Fourier Series Properties

- If $x(t)$ is even, then $x(t) = x(-t)$, and therefore, $c_k = c_{-k}$. You can see this by realizing that kt only appears in the complex exponential, and therefore negating t has the same effect as negating k.

$$x(t) \text{ even} \Longrightarrow c_k = c_{-k}$$

- If $x(t)$ is odd, then $x(t) = -x(-t)$, and therefore, $c_k = -c_{-k}$. This holds for the same reason as for the even case.

$$x(t) \text{ odd} \Longrightarrow c_k = -c_{-k}$$

- Combining facts, we have that if $x(t)$ is even and real, then $c_k = c_{-k}$ and $c_{-k} = c_k^*$, and so $c_k = c_k^*$. This means that c_k must be real.

$$x(t) \text{ even and real} \Longrightarrow c_k \text{ real}$$

- If $x(t)$ is odd and real, then $c_k = -c_{-k}$, and because $c_{-k} = c_k^*$, then $c_k = -c_k^*$. This means that c_k must be imaginary.

$$x(t) \text{ odd and real} \Longrightarrow c_k \text{ imaginary}$$

Perseval's Theorem

Suppose we want to find the power of a complex signal:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 \, \mathrm{d}t$$

Since $x(t)$ is complex, we split the square to $x(t) \cdot x(t)^*$. Therefore,

$$\begin{aligned}
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)x(t)^* \mathrm{d}t \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right]\left[\sum_{n=-\infty}^{\infty} c_n^* e^{-jn\omega_0 t}\right] \mathrm{d}t
\end{aligned}$$

We can then switch the order of the summation and integral.

$$= \frac{1}{T_0}\sum_{k=-\infty}^{\infty} c_k \sum_{k=-\infty}^{\infty} c_n^* \cdot \int_{t_0}^{t_0+T_0} e^{j(k-n)\omega_0 t} \mathrm{d}t$$

Notice that the integral returns 0 when $k \neq n$, and T_0 when $k = n$. This is because if you expand the exponential using Euler's formula, then you are integrating a cosine and sin over one period, the periods of which will cancel out. Therefore,

$$\begin{aligned}
&= \frac{1}{T_0}\sum_{-\infty}^{\infty} c_k \cdot c_k^* \cdot T_0 \\
&= \sum_{k=-\infty}^{\infty} |c_k|^2
\end{aligned}$$

Everything before this point is fair game on Midterm 1.

Aperiodic Signals

- The Fourier series can model (almost) any **periodic** or **time-limited** function as a sum of complex exponentials. However, most signals we encounter are not necessarily periodic or time-limited.
- The **Fourier transform** allows us to calculate the spectrum of aperiodic signals.

Intuition of going from Fourier series to Fourier transform

Extending Fourier series to the Fourier transform is fairly intuitive.

The idea is the following:

- We can calculate the Fourier series of a periodic or time-limited signal, over some interval of length T_0.
- A signal that is not periodic can be viewed as a periodic signal, where T_0 is infinite. As T_0 is infinite, it never repeats.
- But the point is that we can replace our Fourier series calculation as, instead of being over a finite period, T_0, being over all time, from $t = -\infty$ to ∞.

- Mathematically, we can calculate the Fourier series of $f(t)$ over the interval $[-T/2.T/2)$ via:

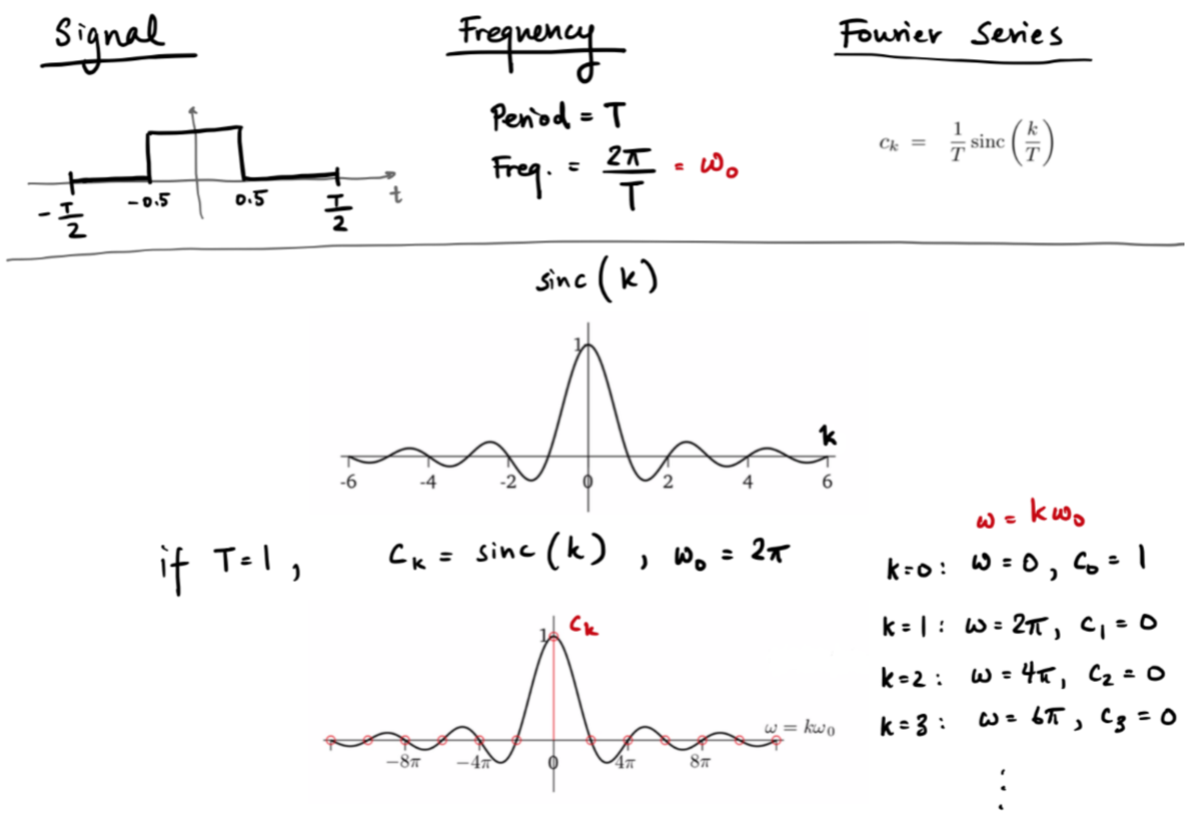
$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega-0}\mathrm{d}t$$

where $\omega_0 = 2\pi/T$. In the Fourier transform, we're now going to let $T \to \infty$.

Example:
This is the rect() function.



Arriving at the Fourier transform

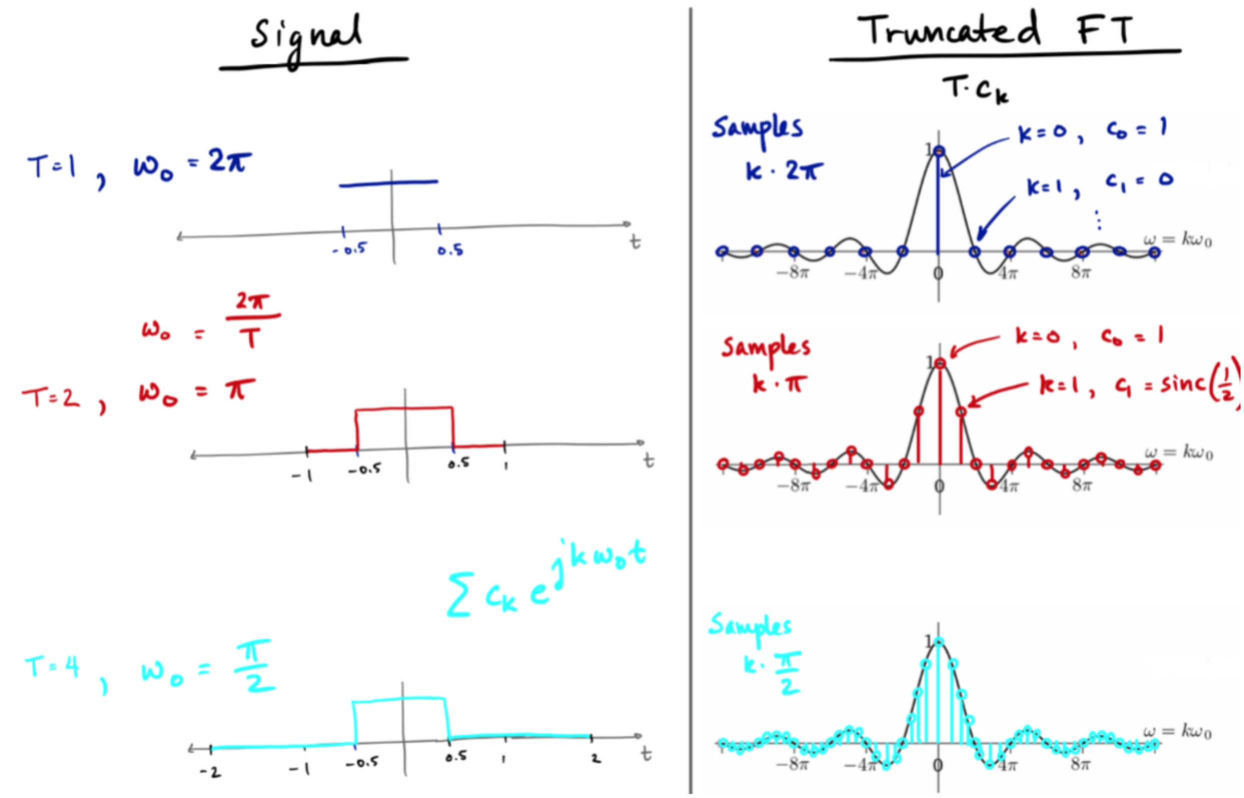
When $T \to \infty$, $c_k \to 0$.

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{jk\omega_0 t}\mathrm{d}t$$

To prevent this, we introduce the truncated Fourier transform.

$$\begin{aligned} & F_T(j\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\omega t}\mathrm{d}t \\ \Rightarrow \quad & F_T(jk\omega_0) = T \cdot c_k \end{aligned}$$

Remember that we replace $k\omega_0$ with ω.



The intuition: as $T \to \infty$, we more finely sample the truncated fourier transform. $k\omega_0 \to \omega$, since $\omega_0 \to 0$ as $T \to \infty$.

Now, let's set $T \to \infty$. If we do this, then $\omega_0 = 2\pi/T$ will approach 0. So suppose instead that we define a continuous variable,

$$\omega = \frac{2\pi k}{T}$$

which means that k increases with T, so that $\omega = k\omega_0$ is fixed.

The Fourier transform is the limit of the truncated Fourier transform.

$$\begin{aligned} F(k\omega) &= \lim_{T\to\infty} F_T(j\omega) \\ &= \lim_{T\to\infty} \int_{-T/2}^{T/2} f(t) = e^{-j\omega t} \mathrm{d}t \\ &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \mathrm{d}t \end{aligned}$$

This is the Fourier transform, which takes you from the time domain, $f(t)$, to the frequency domain, $F(j\omega)$.

Fourier Transform Formula

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \mathrm{d}t$$

Inverting the Fourier Transform

How do we go from $F(j\omega)$ back to $f(t)$?. Let's begin by writing the Fourier series.

$$\begin{aligned} f(t) &= \lim_{T\to\infty} f_T(t) \\ &= \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(jk\omega_0) e^{jk\omega_0 t} \end{aligned}$$

Now as $T \to \infty$, what we see is that this approaches an integral. This is an infinite sum, where the integration "widths" are the infinitesimal $1/T$ and the "heights" are $F_T(jk\omega_0)e^{jk\omega_0 t}$. To make this more clear, we denote $\Delta\omega = 2\pi/T$, and note that $\omega = k\Delta\omega$. Then, this sum becomes

$$\begin{aligned} f(t) &= \lim_{\Delta\omega\to 0} \sum_{k=-\infty}^{\infty} F_T(jk\Delta\omega) e^{jk\Delta\omega t} \frac{\Delta\omega}{2\pi} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} \mathrm{d}\omega \end{aligned}$$

This is the inverse Fourier transform, which takes you from the frequency domain, $F(j\omega)$ to the time domain, $f(t)$.

Fourier Transform Summary

The fourier transform is:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty}} f(t) e^{-j\omega t} \mathrm{d}t$$

The inverse fourier transform is:

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} \mathrm{d}\omega}$$

A few notes:

- Like in Fourier series, the inversion formula for $f(t)$ is accurate when $f(t)$ is continuous, but produces the midpoint when $f(t)$ has jumps.
- These two are almost identical in form, except for the sign of the complex exponential and the factor of $1/2\pi$.
- Check your intuition when you look at these formulas: to go from the time domain to frequency domain (Fourier transform) you should integrate away time (giving a function of frequency). Likewise, to go from the frequency domain to the time domain (inverse Fourier transform) you should integrate away frequency (giving a function of time).

A sufficient condition for the existence of the Fourier transform

- From $F(j\omega)$, we can determine $f(t)$ and vice versa (if it's well-behaved; e.g., at discontinuities, the Fourier transform will return the midpoint).
- Not every function has a Fourier transform. For example, a sufficient condition for a Fourier transform

is that it should have finite energy. Note,

$$
\begin{aligned}
|F(j\omega)| &= \left| \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \mathrm{d}t \right| \\
&\leq \int_{-\infty}^{\infty} \left| f(t) e^{-j\omega t} \right| \mathrm{d}t \\
&= \int_{-\infty}^{\infty} |f(t)| \mathrm{d}t \\
< \infty
\end{aligned}
$$

- The above is a sufficient (but not necessary) requirement for the existence of the Fourier transform.

Example: Fourier Transform of rect()

$$
\begin{aligned}
F(j\omega) &= \int_{-\infty}^{\infty} \mathrm{rect}(t/T) e^{-j\omega t} \mathrm{d}t \\
&= \int_{-T/2}^{T/2} e^{-j\omega t} \mathrm{d}t \\
&= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2} \\
&= \frac{1}{-j\omega} \left(e^{-j\omega T/2} - e^{jwT/2} \right) \\
&= \frac{1}{-j\omega} (-2j \sin(\omega T/2)) \\
&= \frac{2 \sin(\omega T/2)}{\omega} \\
&= \frac{T \sin(\pi(\omega T/2\pi))}{\pi(\omega T/2\pi)} \\
&= T \mathrm{sinc}(\omega T/2\pi)
\end{aligned}
$$

Note that here, we went through some extra algebra to get things into the sinc$(\cdot)$ form. This is out of convenience. Thus, we have that

$$
\mathrm{rect}(t/T) \Longleftrightarrow T \mathrm{sinc}(\omega T/2\pi)
$$

Example: Fourier Transform of a Causal Exponential

Let's find the Fourier transform of

$$
f(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

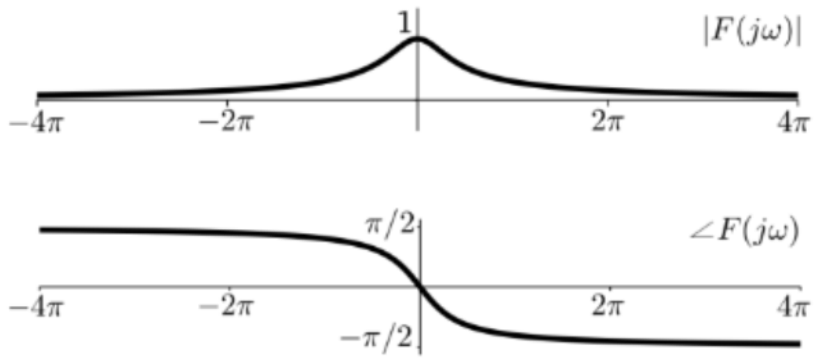
for $a > 0$.

Its Fourier transform is

$$\begin{aligned}
F(j\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\mathrm{d}t \\
&= \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t}\mathrm{d}t \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t}\mathrm{d}t \\
&= -\frac{1}{a+j\omega} e^{-(a=j\omega)\cdot t}\Big|_{t=0}^{t=\infty} \\
&= \frac{1}{a+j\omega}
\end{aligned}$$

$$e^{at}u(t) \Longleftrightarrow \frac{1}{a+j\omega}$$

Below is the spectrum of the causal exponential for $a = 1$.



Fourier Transforms we know

$$\mathrm{rect}(t/T) \Longleftrightarrow T\mathrm{sinc}(\omega T/2\pi)$$
$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a+j\omega}$$

Fourier Symmetries

Derivations analogous to Fourier series

- For any $f(t)$, whether it be real, imaginary, or complex:
  - $f(t)$ even $\rightarrow F(j\omega)$ even.
  - $f(t)$ odd $\rightarrow F(j\omega)$ odd.
- A *real* signal has a Hermitian Fourier transform:

$$F(-j\omega) = F^*(j\omega)$$

- An *imaginary* signal has an anti-Hermitian Fourier transform:

$$F(-j\omega) = -F^*(j\omega)$$

- Furthermore,

- For $f(t)$ real and even, $F(j\omega)$ is real and even.
- For $f(t)$ real and odd, $F(j\omega)$ is imaginary and odd.
- For $f(t)$ imaginary and odd, $F(j\omega)$ is real and odd.
- For $f(t)$ imaginary and even, $F(j\omega)$ is imaginary and even.

The Fourier Transform Operator

To denote the operation of taking the Fourier transform, we use $\mathcal{F}(\cdot)$ or $\mathcal{F}[\cdot]$. That is, if

$$f(t) \Longleftrightarrow F(j\omega)$$

we may alternately write this as

$$F(j\omega) = \mathcal{F}[f(t)]$$

Likewise, the operator $\mathcal{F}^{-1}$ refers to the inverse Fourier transform. Therefore,

$$\mathcal{F}^{-1}[F(j\omega)] = f(t)$$

This also means that

$$\mathcal{F}^{-1}[\mathcal{F}[f(t)]] = f(t)$$

at all points of continuity in $f(t)$.

Summary of all properties of Fourier Transform (can be used without proof)

1. **Linearity:**

$$\boxed{\mathcal{F}[af_1(t) + bf_2(t)] = a\mathcal{F}[]f_1(t) + b\mathcal{F}[f_2(t)]}$$

2. **Time scaling:**

$$\boxed{\mathcal{F}[f(at)] = \frac{1}{|a|}F\left(j\frac{\omega}{a}\right)}$$

3. **Time reversal:**

$$\boxed{\mathcal{F}[f(-t)] = F(-j\omega)}$$

4. **Complex conjugate:**

$$\boxed{f^*(t) \Longleftrightarrow F^*(-j\omega)}$$

5. **Duality:**

$$\boxed{F(t) \Longleftrightarrow 2\pi f(-j\omega)}$$

6. **Time-shifting:**

$$\boxed{\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}F(j\omega)}$$

7. **Derivative:**

$$\boxed{\mathcal{F}[f'(t)] = j\omega F(j\omega)}$$

8. **Convolution:**

$$\boxed{\mathcal{F}[(f_1 * f_2)(t)] = F_1(j\omega)F_2(j\omega)}$$

9. **Perseval's Theorem:**

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 \mathrm{d}\omega}$$

10. **Multiplication:**

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}(F_1 * F_2)(j\omega)}$$

11. **Modulation:**

$$\boxed{\mathcal{F}[f(t)e^{j\omega_0 t}] = F(j(\omega - \omega_0))}$$

Proof of Linearity

For two signals, $f_1(t)$ and $f_2(t)$ and two complex numbers a and b,

$$\boxed{\mathcal{F}[af_1(t) + bf_2(t)] = a\mathcal{F}[]f_1(t) + b\mathcal{F}[f_2(t)]}$$

Another way to write this is

$$af_1(t) + bf_2(t) \Longleftrightarrow aF_1(j\omega) + bF_2(j\omega)$$

where $F_1(j\omega) = \mathcal{F}[f_1(t)]$ and $F_2(j\omega) = \mathcal{F}[f_2(t)]$.

To show this, note:

$$\begin{aligned}\mathcal{F}(af_1(t) + bf_2(t)) &= \int_{-\infty}^{\infty} (af_1(t) + bf_2(t))e^{-j\omega t}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} af_1(t)e^{-j\omega t}\mathrm{d}t + \int_{-\infty}^{\infty} bf_2(t)e^{-j\omega t}\mathrm{d}t \\ &= a\mathcal{F}[f_1(t)] + b\mathcal{F}[f_2(t)]\end{aligned}$$

This extends to finite combinations, i.e.,

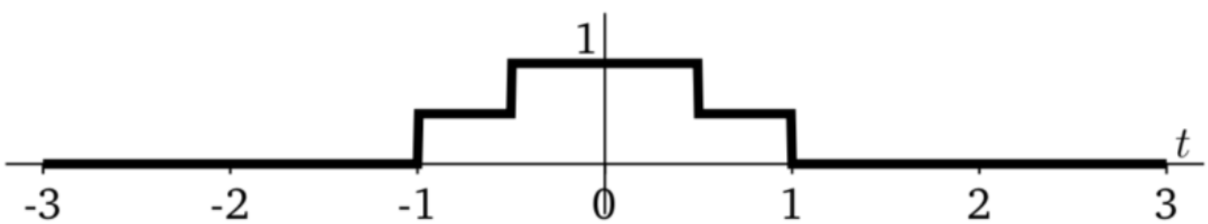
$$\mathcal{F}\left[\sum_{k=1}^{K} a_k f_k(t)\right] = \sum_{k=1}^{K} a_k \mathcal{F}[f_k(t)]$$

Linearity example

Consider the signal:

$$f(t) = \begin{cases} \frac{1}{2} & \frac{1}{2} \le |t| \le 1 \\ 1 & |t| \ge \frac{1}{2} \end{cases}$$

This signal steps up and then steps down, as shown below.



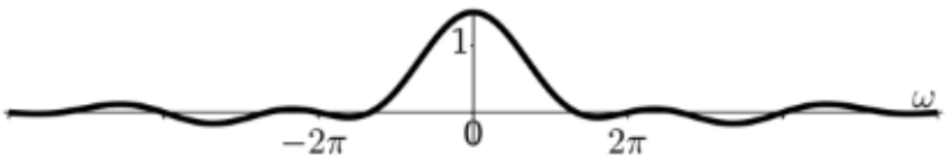
$$f(t) = \frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right) + \frac{1}{2}\mathrm{rect}(t)$$

$$\mathrm{rect}\left(\frac{t}{T}\right) \Longleftrightarrow T\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$$

and therefore,

$$\begin{aligned}F(j\omega) &= \frac{1}{2}2\mathrm{sinc}(2\omega/2\pi) + \frac{1}{2}\mathrm{sinc}(\omega/2\pi) \\ &= \mathrm{sinc}(\omega/\pi) + \frac{1}{2}\mathrm{sinc}(\omega/2\pi)\end{aligned}$$

This is shown below:



Proof of Time-scaling Property

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(at)] = \frac{1}{|a|}F\left(j\frac{\omega}{a}\right)}$$

Note, for real a:

- If $a > 1$, $f(t)$ contracts, but its Fourier transform expands.
- If $0 < a < 1$, then $f(t)$ expands, but its Fourier transform contracts.
- Thus, stretching a signal in time compresses its Fourier transform, and compacting the signal expands its Fourier transform.

To show this, let's consider $a > 0$. (The proof is essentially the same for $a0$) We will use a variable change, $\tau = at$, which means that $\mathrm{d}\tau = a\mathrm{d}t$.

$$\mathcal{F}(f(at)) = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}\mathrm{d}t$$

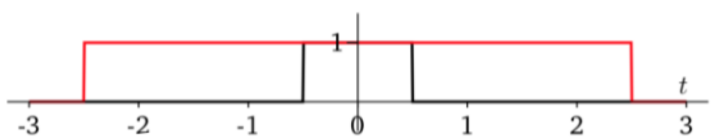
Example:
Knowing that

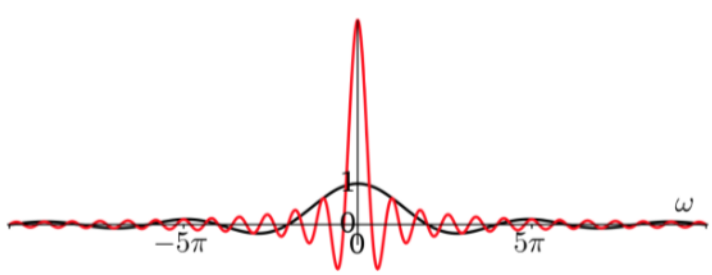
$$\mathrm{rect}(t/T) \Longleftrightarrow T\mathrm{sinc}(\omega T/2\pi)$$

we can then determine that the Fourier transform of $\mathrm{rect}(t)$ is $\mathrm{sinc}(\omega/2\pi)$

Time-Scaling Example

Bandwith: consider two rect pulses, $\mathrm{rect}(t)$ and $\mathrm{rect}(t/5)$. These are their graphs and Fourier transforms.





The fatter rect has a narrower spectrum. The width of the spectrum is called bandwidth. So a shorter pulse has a larger bandwidth.

Time-reversal

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(-t)] = F(-j\omega)}$$

To show this, apply the time-scaling result with $a = -1$.
Find the Fourier transform on $f(t) = e^{-a|t|}$ (for $a > 0$) without doing integration.

We know that

$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a + j\omega}$$

Suppose $f(t) = e^{-at}u(t) + e^{at}u(-t)$.
Then,

$$\begin{aligned} F(j\omega) &= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{a-j\omega}{a^2+\omega^2} + \frac{a+j\omega}{a^2+\omega^2} \\ &= \frac{2a}{a^2+\omega^2} \end{aligned}$$

Time-shift

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}F(j\omega)$$

$$\mathcal{F}[f(t-\tau)] = \int_{-\infty}^{\infty} f(t-\tau)e^{j\omega t}\mathrm{d}t$$

Let $\alpha = t - \tau$, $\mathrm{d}\alpha = \mathrm{d}t$, $t = \alpha = \tau$.

$$\begin{aligned} &= \int_{-\infty}^{\infty} f(\alpha)e^{-j\omega(\alpha+\tau)}\mathrm{d}\alpha \\ &= \int_{-\infty}^{\infty} f(\alpha)e^{-j\omega\alpha}e^{-j\omega\tau}\mathrm{d}\alpha \\ &= e^{-j\omega\tau} \cdot \int_{-\infty}^{\infty} f(\alpha)e^{-j\omega\alpha}\mathrm{d}\alpha \\ &= e^{-j\omega\tau}F(j\omega) \end{aligned}$$

Convolution Theorem (IMPORTANT)

If $f_1(t)$ and $f_2(t)$ are two signals with Fourier transforms $F_1(j\omega)$ and $F_2(j\omega)$, respectively, then

$$\boxed{\mathcal{F}[(f_1 * f_2)(t)] = F_1(j\omega)F_2(j\omega)}$$

Stated simply: **convolution in the time domain is multiplication in the frequency domain.** (and multiplication is easy.)

Time:	LTI	$y(t) = h(t) * x(t)$	$\Leftarrow$"Impulse response"
Spectrum:	FT	$Y(j\omega) = H(j\omega)X(j\omega)$	$\Leftarrow$"Frequency response"