Math 170E Week 5

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Poisson Distribution Continued

$$X \sim \text{Poi}(\lambda)$$

$$P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \text{ ,where } n >> 1$$

For n events $A_1, ..., A_n$, if each have the same probability of occurring and all are independent, then

$$X = \sum_k 1_{A_n}$$

$$X \sim \text{Poi}(\sum_k \mathbb{P}(A_k))$$

Reviewing from last time, some examples of Poisson distributions are muons hitting a plate in a set amount of time, number of car accidents in a set amount of time, number of babies born in a set amount of time, etc.

Example:

The hat problem:

Suppose you have a room full of people all wearing hats. They all put their hats on a table, and one at a time, pick a random hat and leave with it. What is the probability that none of them leave with their own hat? This can be solved using a Poisson distribution; the answer is e^{-1}. This problem is not covered in this class, but is also an example of a Poisson distribution.

Revisiting Binomial Distribution

Binomial distribution considers events $A_1, ..., A_n$, and checks whether $\mathbb{P}(A_k) = p$.

$$X = \sum_k 1_{A_k}$$

This happens to be similar to the poisson distribution, only binomial distribution does not necessarily need a large number of different events. In fact, if an infinite number of events are given, the poisson distribution would equal the binomial distribution.

Lemma:

$$\text{As } n \to \infty \text{ and } p \to 0, B(n,p) \to \text{Poi}(np)$$

This is because $\lim_{x\to\infty} \frac{1}{x} = 0$.

Geometric Distribution

$$X \sim \text{Geo}(p), 0 \leq p \leq 1$$
$$P_X(k) = (1-p)^{k-1} \cdot p$$

Consider an example. You have 6 trials, and each have two outcomes, success or fail. Let $A_k = \{$The first successful trial is the
Thus,

$$\{X = k\} = A_1^c \cap A_2^c \cap ... \cap A_{k-1}^c \cap A_k$$

$$\begin{aligned}\mathbb{P}(X = k) &= \mathbb{P}(A_1^c) \cdot \mathbb{P}(A_2^c) \cdot \dots \cdot \mathbb{P}(A_{k-1}^c) \cdot \mathbb{P}(A_k) \\ &= (1-p)^{k-1} \cdot p\end{aligned}$$

Expectation Value

Denoted as $\mathbb{E}$, it represents the value related to the value of the outcome.

$$\mathbb{E}_X = \sum_k \mathbb{P}_X(k) \cdot k$$

It is simply the product of the probability of success, times the value of the success. For example, the expectation value of a dice toss is $\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + ... + \frac{1}{6} \cdot 6 = 3.5$ Also known as expected value, and very close to average or mean. In this course, only expected value matters. If

$$X \sim \text{Bern(p)}, \mathbb{E}_X = p$$
$$X \sim \text{B}(n,p), \mathbb{E}_X = np$$
$$X \sim \text{Poi}(\lambda), \mathbb{E}_X = \lambda$$
$$X \sim \text{Geo}(p), \mathbb{E}_X = \frac{1}{p}$$

To derive these:
Bernoulli: $X \sim \text{Bern}(p)$

$$\begin{aligned}\mathbb{E}_X &= \sum_k P_x(k) \cdot k \\ &= P_X(1) \cdot 1 + P_X(0) \cdot 0 \\ &= \boxed{p}\end{aligned}$$

Binomial: $X \sim \mathrm{B}(n, p)$

$$\begin{aligned}
\mathbb{E}_X &= \sum_k k \cdot P_X(k) \\
&= \sum_k k \cdot \binom{n}{p} \cdot p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} k \cdot \frac{n!}{(n-k)!k!} \cdot p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \cdot p^k (1-p)^{n-k} \\
&= np \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= 1
\end{aligned}$$

For the last step, the summation converges to 1. This is true because following the binomial distribution formula:

$$(a+b)^n = a^n + na^{n-1}b + \cdots = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

if $a = b = 1$, then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Now, assume $a = (1-p)$, and $b = p$.

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

Poisson: $X \sim \mathrm{Poisson}(X)$

$$\sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

To solve this, we must consider the harmonic series for e^x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

Going back to the Poisson distribution,

$$\begin{aligned}
\mathbb{E}_X &= \sum_k k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\
&= \lambda \cdot \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda \cdot \sum_{m=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^m}{m!} \\
&= \lambda
\end{aligned}$$

Thus, consider $X \sim \mathrm{Poisson}(X), \mathbb{E}_X = 2$. Therefore, $X \sim \mathrm{Poisson}(2)$
Geometric: $X \sim \mathrm{Geo}(p)$

$$
\begin{aligned}
\mathbb{E}_X &= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} p \\
&= p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}
\end{aligned}
$$

From here, consider the property that $1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}$. Therefore, $1 + 2a + 3a^2 + \cdots = \frac{1}{(1-a)^2}$.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a}(a + a^2 + a^3 + \dots) \\
=&\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{1}{1-a} - 1\right) \\
=&\frac{1}{(1-a)^2}
\end{aligned}
$$

Therefore,

$$
p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = \frac{1}{p^2} \cdot p = \frac{1}{p}
$$

Pascal's Story

There goes a story:
In the 1650's, two pirates, call them A and B, were playing a gambling game. Whoever wins takes the pot of 500 gold coins. They both agreed to play five rounds, winner of at least three takes all. However, after their third round, something happened, and they had to stop the game. After the event, both A and B were killed. Their families decided to split the money instead. At the time the games ended, A had won two rounds and B had won one.

The question Pascal asked was what was the best way to split the money? Some proposed that since A had won more games, family A should get all the money. Some said that A should receive $\frac{2}{3}$ of the money, since A had won $\frac{2}{3}$ of the games. However, Pascal argued that neither was correct. He said that the probabilities should be considered. The last two games had the sample space of $\{(A, A), (A, B), (B, A), (B, B)\}$. Three of these four possibilities would result in A winning the money. Thus, the expected value for A would be $\frac{1}{4} \cdot 500 \cdot 3$, while the expected value for B would be $\frac{1}{4} \cdot 500 \cdot 1$. Thus, A should receive 375 coins, while B should receive 125 coins.