EC ENGR 102 Week 6

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Proof of the Convolution Theorem

$$\begin{aligned}
\mathcal{F}[(f_1 * f_2)(t)] &= \int_{-\infty}^{\infty} \left(\int --\infty^{\infty} f_1(\tau) f_2(t-\tau) \mathrm{d}\tau \right) e^{-j\omega t} \mathrm{d}t \\
&= \int_{-\infty}^{\infty} f_1(t) \int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t} \mathrm{d}t \, \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} f-1(t) \left[e^{-j\omega\tau} F_2(j\omega) \right] \mathrm{d}\tau \\
&= F_2(j\omega) \cdot \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} \mathrm{d}\tau \\
&= F_2(j\omega) \cdot F_1(j\omega)
\end{aligned}$$

Example

What is the Fourier transform of the unit triangle,

$$\triangle(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Duality of the Fourier Transform

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{F(t) \Longleftrightarrow 2\pi f(-j\omega)}$$

This expression may be opaque at first. What this is saying is that if I take a Fourier transform pair, I can find the dual pair by replacing all the ω's with t's in $F(j\omega)$ and all the t's with $-\omega$'s in $f(t)$. After scaling by 2π, this results in another Fourier transform pair.

Essentially, every Fourier transform pair we derive really gives us two Fourier transform pairs.

Duality Proof

To show this, recognize that as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} \mathrm{d}\omega$$

then

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega) e^{-j\omega t} \mathrm{d}\omega$$

Now, the right hand side of this equation is the Fourier transform of $F(j\omega)$ with the roles of ω and t reversed. Hence, $2\pi f(-t)$ is the Fourier transform of $F(j\omega)$ and after we swap the ω and the t's, we arrive at the duality result.

Duality Examples

- Since $\text{rect}(t) \Longleftrightarrow \text{sinc}(\omega/2\pi)$, then

$$\begin{aligned}\text{sinc}(t/2\pi) &\Longleftrightarrow 2\pi\text{rect}(-\omega)\\ &= 2\pi\text{rect}(\omega)\end{aligned}$$

  Thus, we have that $\text{sinc}(t/2\pi) \Longleftrightarrow 2\pi\text{rect}(\omega)$.

- Since

$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a+j\omega}$$

  then

$$\frac{1}{a+jt} \Longleftrightarrow 2\pi e^{a\omega}u(-\omega)$$

Frequency Domain Convolution

The frequency domain convolution theorem is that for $f_1(t) \Longleftrightarrow F_1(j\omega)$ and $f_2(t) \Longleftrightarrow F_2(j\omega)$, then

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(jv)F_2(j(\omega - v))\mathrm{d}v}$$

We typically write this as:

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}(F_1 * F_2)(j\omega)}$$

but note that the convolution is with respect to ω, not $j\omega$. (Remember that j is constant!)

This means that multiplication in the time domain is convolution in the frequency domain. This proof is very similar to the time domain proof.

Modulation: duality of time-shifting

Dual intuition: Time shift in the time domain is multiplication by a complex exponential in frequency domain. Thus, multiplication by a complex exponential in the time domain ought be a shift in the frequency domain.

Recall that:

$$\boxed{\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}F(j\omega)}$$

Another FT pair (derived later) is

$$\mathcal{F}[f(t)e^{j\omega_0 t} = F(j(\omega - \omega_0))]$$

Using linearity, we also see that:

$$\begin{aligned}\mathcal{F}[f(t)\cos(\omega_0 t)] &= \frac{1}{2}(F(j(\omega - \omega_0)) + F(j(\omega + \omega_0)))\\ \mathcal{F}[f(t)\sin(\omega_0 t)] &= \frac{1}{2j}(F(j(\omega - \omega_0)) - F(j(\omega + \omega_0)))\end{aligned}$$

To prove the modulation result, note that if $\mathcal{F}[f(t)] = F(j\omega)$ then

$$\begin{aligned}\mathcal{F}[f(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}\mathrm{d}t\\ &= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}\mathrm{d}t\\ &= F(j(\omega - \omega_0))\end{aligned}$$

To get the cosine and sine results, we note that e.g., for cosine,

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

From here, we can use linearity to compute the Fourier transform.