BIOMATH 208 Week 8

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1 Review

[FILL]

Image Registration

- Minimize the sum of square error:
$$\mathrm{SSE}(T \cdot I, J)$$
- T is a translator or affine transformation
- For T translator:
$$\mathrm{d}f(T) = \int (I(x-T) - J(x)) \cdot \mathrm{d}I(x-T)\mathrm{d}x$$
- For affine:
$$\mathrm{d}f(T) = \int (I(T^{-1}x) - J(x)) \cdot \mathrm{d}[I(T^{-1}x)]^T \cdot [T^{-1}x]^T \mathrm{d}x$$

Metric Manifolds (Riemannian Manifolds)

[FILL manifold drawing]

$$L(\gamma) = \int_0^1 g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\mathrm{d}t$$
$$A(\gamma) = \int_0^1 \text{no square root}\mathrm{d}t$$

- Any minimizer of A is also a minimizer of L
- Any minimizer of A is constant speed
- Optimizing over A makes things easier (no square root) and gives a unique parameterization: constant speed geodesics

The Constant Speed Geodesic Equation

In a given coordinate chart, with components of a metric tensor field g written as g_{ij}, and its inverse written g^{ij}, constant speed geodesics are determined by

$$\ddot{q}^i + \frac{1}{2} g^{ij}(-\partial_j g_{kl} + \partial_k g_{jl} + \partial_l g_{kj})\dot{q}^k \dot{q}^l = 0$$

Proof

- We will consider a path $q(t)$ and a perturbation $q \mapsto q + \epsilon\delta q$.
- This denotes a corresponding perturbation in the components of velocity $\dot{q} \mapsto \dot{q} + \epsilon\delta\dot{q}$ (with $\delta\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\delta q$).
- We seek to find stationary solutions via

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} A(q + \epsilon\delta q)\bigg|_{\epsilon=0}$$

  - e.g., find the directional derivative in the direction of δq.

First, we plug into our equation

$$= \frac{\mathrm{d}}{\mathrm{d}\epsilon} A(q + \epsilon\delta q)\bigg|_{\epsilon=0}$$

We'll apply chain rule and get derivatives of g

$$= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_0^1 g_{ij}(q(t) + \epsilon\delta q(t))(\dot{q}^i(t) + \epsilon\delta\dot{q}^i(t))(q^{ij}(t) + \epsilon\delta\dot{q}^j(t))$$

Notice that the last two terms are quadratic in velocity.

Not trivial: We will apply integration by parts to write things as gradient dot direction for direction $=$ δq. e.g., $\delta q(0) = \delta q(1) = 0$ (fixed endpoints). The boundary terms are zero in integration by parts.

$$= \int \partial_k g_{ij}(q(t))\delta g^k(t)\dot{q}^i(t)\dot{q}^j(t) + g_{ij}(q(t))\delta\dot{q}^i(t)\dot{q}^j(t) + g_{ij}(g(t))\dot{q}^i(t)\delta\dot{q}^j(t)\mathrm{d}t$$

The first term is something acting linearly on the direction. The next two terms are not, since $\delta\dot{q}^i(t)$ is the time derivative of the direction.

$$= \int_0^1 \partial_k g_{ij}\dot{q}i(t)\dot{q}^j(t)\delta q^k(t) - \frac{\mathrm{d}}{\mathrm{d}t}(g_{ij}(q)\dot{q}^j)\delta q^i - \frac{\mathrm{d}}{\mathrm{d}t}(g_{ij}(q)\dot{q}^i)\delta q^j \mathrm{d}t$$

Note that δq has 3 different indices, but they are just dummy variables

$$\begin{aligned}
&= \int_0^1 \partial_k g_{ij}\dot{q}^i\dot{q}^j\delta q^k - \frac{\mathrm{d}}{\mathrm{d}t}(g_{kj}(q)\dot{q}^j)\delta q^k - \frac{\mathrm{d}}{\mathrm{d}t}(g_{ik}(q)\dot{q}^i)\delta q^k \mathrm{d}t \\
&= \int partial_k g_{ij}(q)\dot{q}^i\dot{q}^j\delta q^k - \partial_i g_{kj}(q)\dot{q}^i\dot{q}^j\delta q^k - g_{kj}(q)\ddot{q} + \delta q^k \\
&\quad - \partial_j g_{ik}(q)\dot{q}^j\dot{q}^i\delta q^k - g_{ik}(q)\dot{q}^i\delta\dot{q}^k \mathrm{d}t
\end{aligned}$$

Now we simplify the equation. We want to simplify to the form

$$\int_0^1 \text{something}(t)^k \cdot q^k(t)\mathrm{d}t = 0$$

Since every term has a q^k, we can factor that out.

$$0 = \partial_k g_{ij}(q)\dot{q}^i\dot{q}^j - \partial_i g_{kj}(q)\dot{q}^i\ddot{q}^j - g_{kj}(q)\ddot{q}^j - \partial_j g_{ik}(g)\dot{q}^j\dot{q}^k - g_{ik}(q)\ddot{q}^i$$

Now, we can (1) combine the pairs of items that look the same, and (2) act with the matrix inverse of g (sharp map.)

$$\begin{aligned}
0 &= g_{ik}(q)\ddot{g}^i + \frac{1}{2}(-\partial_k g_{ij}(q) + \partial_i g_{kj}(q) + \partial_j g_{ik}(q))\dot{q}^i\dot{q}^j \\
0 &= \ddot{q}^l + \frac{1}{2}(g^{-1})^{kl}(-\partial_k g_{ij}(q) + \partial_i g_{kj}(q) + \partial_j g_{ik}(q))\dot{q}^i\dot{q}^j
\end{aligned}$$

[FILL]

Christoffel Symbols

We simplify by introducing the notation $\ddot{q}^k + \Gamma^k_{ij}\dot{q}^i\dot{q}^j = 0$. **Definition: Christoffel Symbol of the first kind**

$$\Delta_{kij} = \frac{1}{2}(-\partial_k g_{ij} + \partial_i g_{kj} + \partial_j g_{ik})$$

Definition: Christoffel symbol of the second kind

$$\begin{aligned}\Gamma^l_{ij} = g^{lk}\Gamma_{kij} \\ &= \frac{1}{2}g^{lk}(-\partial_k g_{ij} + \partial_i g_{kj} + \partial_j g_{ik})\end{aligned}$$

Note they are symmetric in the last two indices.

Geodesics in Euclidean Space

If g is constant everywhere in some chart, then the resulting geodesics are linear equations in this chart

$$q^i(t) = a^i t + b^i$$

Proof:

If g is constant then $\partial_k g_{ij} = 0$ for all i, j, k. The equation becomes

$$\ddot{q}^i = 0$$

Integrating once gives

$$\dot{q}^i = a^i$$

for some a. Integrating again gives the solution for arbitrary b.

Geodesics in polar coordinates

Earlier we showed that the Euclidean dot product in polar coordinates is

$$g_{ij}(r,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}_{ij}$$

Geodesics in this coordinate system are given by

$$\ddot{r} - r\dot{\theta}^2 = 0$$
$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0$$

Proof

[FILL]

Now, start calculating partial derivatives. Note they are only nonzero when we take the derivative of the $\theta - \theta$ component ($i = 1, j = 1$), with respect to r.

$$[FILL]$$

Now Christoffel symbols of the first kind.

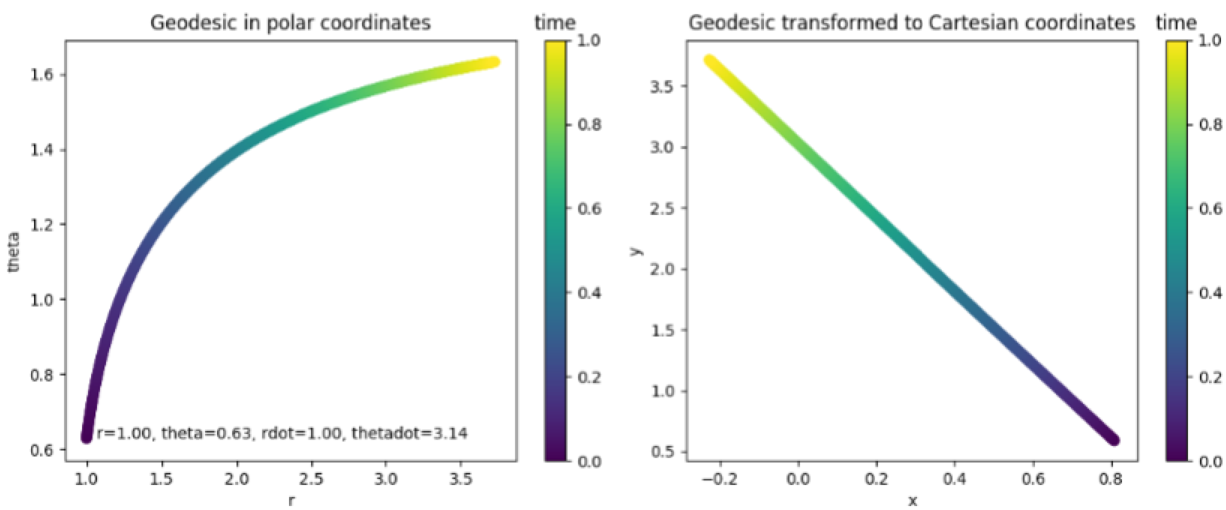
$$\begin{aligned}&\Gamma_{kij} = -\frac{1}{2}(-\partial_k g_{ij} + \partial_i g_{kj} + \partial_j g_{ik}) \\ &\Gamma_{000} = 0 \\ &\Gamma_{010} = 0 \\ &\Gamma_{100} = 0 \\ &\Gamma_{110} = \Gamma_{101} = r\Gamma_{001} &= [FILL]\Gamma_{011} = \Gamma_{101} &= \Gamma_{111} =\end{aligned}$$

Now, Christoffel symbols of the second kind

$$\Gamma^l_{ij} = g^{lk}\Gamma_{kij}$$

$$\begin{aligned}
\Gamma^0_{00} &= g^{00}\Gamma_{000} + g^{01}\Gamma_{100} = 0 \\
\Gamma^0_{01} &= g^{00}\Gamma_{001} + g^{01}\Gamma_{101} = 0 \\
\Gamma^0_{10} &= g^{00}\Gamma_{001} + g^{01}\Gamma_{101} = 0 \\
\Gamma^0_{10} &= g^{00}\Gamma_{010} + g^{01}\Gamma_{110} = -r \\
\Gamma^1_{00} &= g^{10}\Gamma_{000} + g^{11}\Gamma_{100} = 0 \\
\Gamma^1_{01} &= g^{10}\Gamma_{001} + g^{11}\Gamma_{101} = y_r \\
\Gamma^1_{10} &= g^{10}\Gamma_{001} + g^{11}\Gamma_{101} = y_r \\
\Gamma^1_{10} &= g^{10}\Gamma_{010} + g^{11}\Gamma_{110} = 0
\end{aligned}$$

An example of geodesic curves are shown below:



Christoffel Symbols are not Tensors

Why? Because a tensor that is 0 in one chart induced basis will be 0 in all others.

- Change of basis involved multiplying by Jacobians, these are always invertible matrices (smoothly compatible atlas).

Size Data

Size data is very common in medical imaging, where it is often called volumetry or morphometry. It gives a means to quantify data in structural images.

Geodesics on size Consider $\mathcal{M}$ the space of sizes (or the scale group), with the "natural" coordinate chart (i.e., a number in $\mathbb{R}^+$). Let $g(q) = \frac{1}{q^2}$, and so $g^{-1}(q) = q^2$. Note this metric is left invariant. To see this, just push forward vectors with q^{-1}.

$$g(q)(u, v) = \frac{1}{q^2}uv = g(I)(\frac{1}{q}u, \frac{1}{q}v)$$

- e.g., we push forward, back to identity, with q^{-1}.

For $a, b \in \mathbb{R}$, geodesics are given by

$$q(t) = \exp(at + b)$$

Proof

First calculate g^{-1}

$$\left(\frac{1}{q^2}^{-1}\right) = q^2 \qquad \rightarrow \text{ sharp map}$$

Now, calculate the derivative of g.

$$\partial_0 g_{00} = \frac{\mathrm{d}}{\mathrm{d}q}\frac{1}{q^2} = -2q^{-3}$$

Now calculate the Christoffel symbol of the first kind.

$$\Gamma_{000} = \frac{1}{2}(-2q^{-3})$$

Note, in 1D, two of the terms cancel out because of a minus sign.

Now,calculate the Christoffel symbol of the second kind. [FILL]

Write the geodesic equation:

$$\ddot{q} - \frac{1}{q}\dot{q}^2 = 0$$

Notice that the coefficients are the Christoffel symbols.

Show exponentials are a solution:

$$\begin{aligned} q(t) &= \exp(at+b) \\ \dot{q}(t) &= a\exp(at+b) \\ \ddot{q}(t) &= a^2\exp(at+b) \end{aligned}$$

Plugging in,

$$a^2\exp(at+b) - \frac{1}{\exp(at+b)}(a\exp(at+b))^2 = 0$$

Distances on Size Data

The Riemannian distance between two points $0 < a < b$ is given by the log of their ration

$$d(a,b) = \log(b/a) = \log(b) - \log(a)$$

Note the distance between sa and sb is the same as that between a and b (left invariance).

Proof

Let $q(t)$ be any increasing function with $q(0) = a$ and $q(1) = b$. Then,

$$d(a,b) = \int_0^1 \sqrt{\frac{\dot{q}^2(t)}{q^2(t)}}\mathrm{d}t$$
$$[FILL]$$

Normal coordinates on sizes

We chose a base of point 1, and use a chart where

$$x(p) = d(1,p)\mathrm{sign}(p-1) = \log(p)$$

This chart has the property that the Euclidean distance from 1 to p in the chart, is the Riemnanian distance on the manifold.

This property defines "normal coordinates" on a manifold.

Probability data

Probability data shows up in image segmentation/classification tasks, where an image or a pixel is assigned to a given category with some probability (e.g., foreground vs. background)

- Consider $\mathcal{M}$ the space of probabilities, with the "natural" coordinate chart (i.e., a number in (0, 1)). Let's choose the metric tensor $g(q) = \frac{1}{q^2(1-q)^2}$ which says that the parameter changes near the end-points are larger than those near the middle. Note, g is invariant under $q \mapsto 1 - q$.

- For $a, b \in \mathcal{R}$, geodesics are given by the logistic function (sigmoid)

$$q(t) = \frac{1}{1 + \exp(at + b)}$$