EC ENGR 102 Week 8

Aidan Jan

November 19, 2024

Distortions

Causal Filters

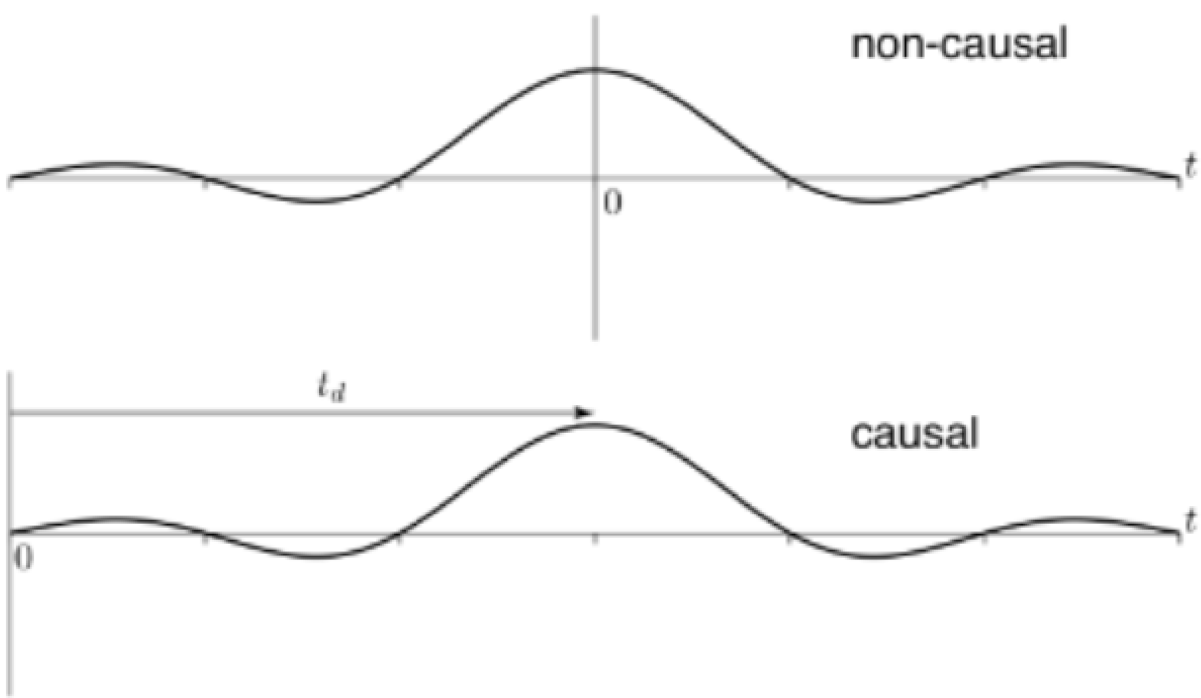
Ideal filters are not causal.

- The impulse responses of the ideal low pass, high pass, and band pass filters are all non-causal. This is obviously not practical in real-time scenarios, where the future is not known.
- To this end, it seems we have a problem. If we can't implement non-causal filters, then there has to be some approximations made (beyond introducing transition bands).
- What we ought to recognize is that we will never be able to take a signal at time t, given by $x(t)$, and immediately filter it. However, we could wait a bit of time, and then filter $x(t)$. This enables us to implement a causal filter, with the con being that the signal is delayed.

A practically implementable system that is causal must introduce a delay. For example, we could implement a practical and causal low pass filter by taking the impulse response of the ideal low pass filter, and:

- Truncate it (so that it is zero for some time $|t| > t_d$).
- Delay it by time t_d so that it is causal.

This is illustrated below:



Distortionless LTI Systems

When using filters, distortion could be introduced. In the frequency domain, depending on the filter being used, components of the signal at different frequencies may be amplified differently or may be delayed. We know that filtering causes:

- Amplitude scaling by $|H(j\omega)|$
- Phase shifting by $\angle H(j\omega)$

and depending on the frequencies of the signal, this could cause distortion to the system.

A system is without distortion if

$$y(t) = Kx(t - t_d)$$

where K is a scaling factor and t_d is a delay. This tells us that a distortionless signal is one that is identical to the input signal up to amplification and a constant delay.

What is $H(j\omega)$ for a distortionless LTI system?

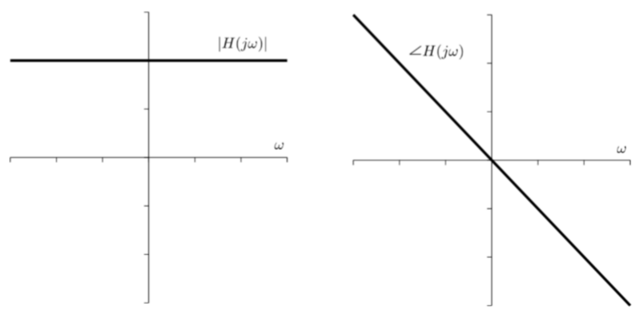
This is a system with

$$|H(j\omega)| = K$$

and

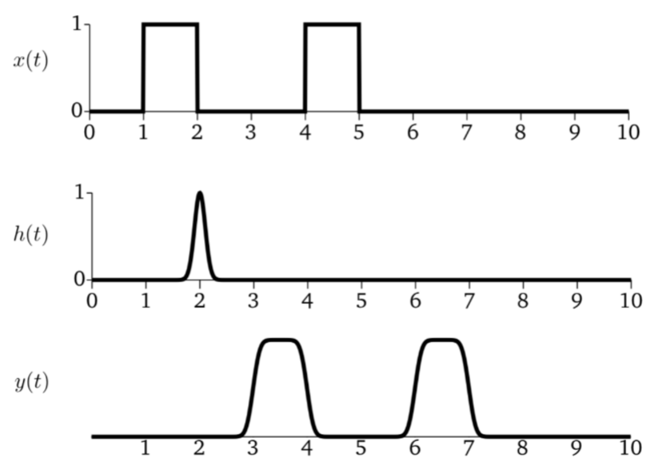
$$\angle H(j\omega) = -\omega t_d$$

It has the following amplitude and phase spectra:



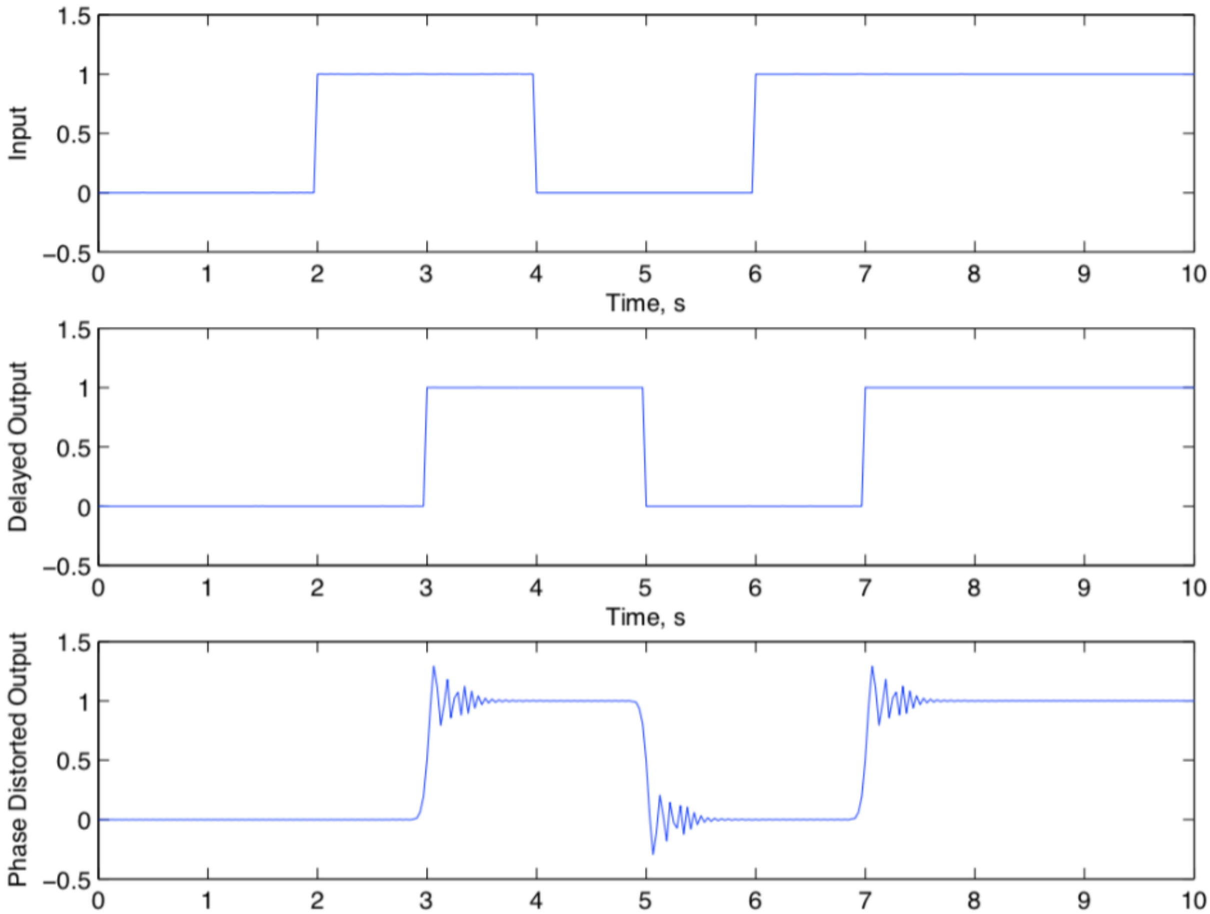
Amplitude Distortion

Amplitude distortion is relatively straightforward.



If we have an impulse which is not really an impulse (due to real life physics limitations), we get some distortion in the output.

Phase Distortion



- Phase distortion is caused when the different complex exponentials that compose a function are changed by slightly different frequencies, so when they sum together, something weird happens.
- Think: Suppose $\angle(Y(j\omega)) = \angle(x(j\omega)) + \angle(h(j\omega))$. If a distortion in phase is made to both x and h, does Y move by the same amount? Especially with a frequency change, they may not line up well.

Group Delay

We know that convolution with $\delta(t - t_d)$ only delays the signal, but introduces no distortion. Further, $\mathcal{F}[\delta(t - t_d)] = e^{-j\omega t_d}$, and so its phase is

$$\angle H(j\omega) = -\omega t_d$$

It turns out that if the phase is linear, the signal is only delayed in time without distortion.

Consider a simple example, a signal $x(t) = \cos(\omega t) + \cos(2\omega t)$. Delaying this signal by π/ω, i.e., $y(t) = x(t) * \delta(t - \pi - \omega)$ gives

$$\begin{aligned} y(t) &= \cos(\omega(t - \pi/\omega)) + \cos(2\omega(t - \pi/\omega)) \\ &= \cos(\omega t - \pi) + \cos(2\omega t - 2\pi) \end{aligned}$$

This makes intuitive sense, if the signal is higher frequency, we need a larger phase shift to gain the same temporal offset.

When the phase is not linear, different frequencies will experience different temporal delays. Therefore, the amount of delay will be frequency dependent, i.e., t_d will be a function of ω. Since $\angle H(j\omega) = -\omega t_d$, it is straightforward to derive what the frequency-dependent temporal delay is, i.e., it's

$$t_d(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\angle H(j\omega)$$

From this, we can see that if $\angle H(j\omega)$ is a line with slope $-k$, then its derivative is simply $-k$, leading to every frequency having the same delay $t_d = k$. This confirms our intuition from before.

This quantity, $t_d(\omega)$, is called *group delay.*

Distortions

Importance of amplitude and phase distortion depends on application.

For audio or speech:

- Amplitude distortion is very is important.
- Humans are relatively insensitive to phase distortion.

For images or video:

- Amplitude distortion is relatively unimportant, as long as it is slowly varying.
- Phase distortion is very important. Small amounts of non-linear phase result in very blurry looking images.

Sampling

Motivation

In reality, we could never store a continuous time signal. Instead, as we see in `MATLAB`, we store a signal's value at various times.

A key variable of interest is the sampling frequency, i.e., the time in between our samples, denoted T in the above diagram.

This is related to discrete signals, i.e., $x[n] = x(nT)$.

How to sample a continuous signal?

How do we sample a continuous signal? You may have several intuitions to do so already using the $\delta(t)$ signal and its property that $f(t)\delta(t) = f(0)\delta(t)$.

- We will arrive at sampling by first studying a related problem: the Fourier transform of periodic signals.
- The reason we approach this is that Fourier series are discrete coefficients, c_k, while the Fourier transform is typically some continuous signal. i.e., it seems like there may be a relationship whereby the Fourier series is like a sampled Fourier transform.
- So we ask: what is the relationship between the Fourier series and the Fourier transform?
- To see this, we can begin by identifying the relationship between the Fourier series and the Fourier transform.

Fourier transform of a periodic signal

We cannot directly take the Fourier transform of a periodic signal, since they do not have finite energy. However, we can use a few tricks (like in the Generalized Fourier Transform lecture) to calculate the FT of a periodic signal.

Let $f(t)$ have a Fourier series (with period $T_0 = \omega_0/2\pi$)

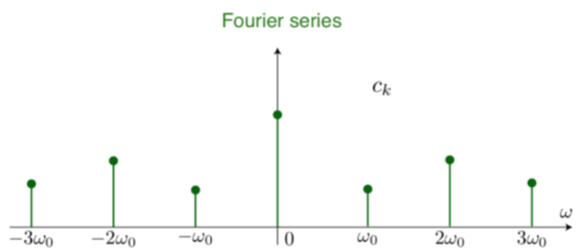
$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

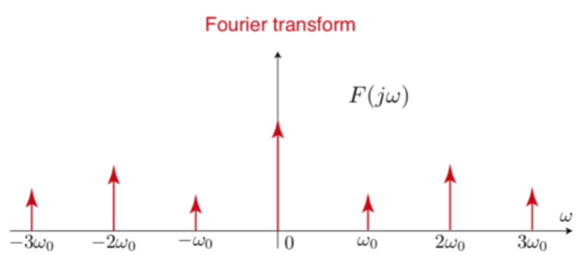
with

$$c_k = \frac{1}{T_0}\int_0^{T_0} f(t)e^{jk\omega_0 t}\mathrm{d}t$$

There's a close relationship between the two, as the Fourier series equation looks like the Fourier transform equation but with a $\sum$ instead of an $\int$.

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \iff \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0)$$

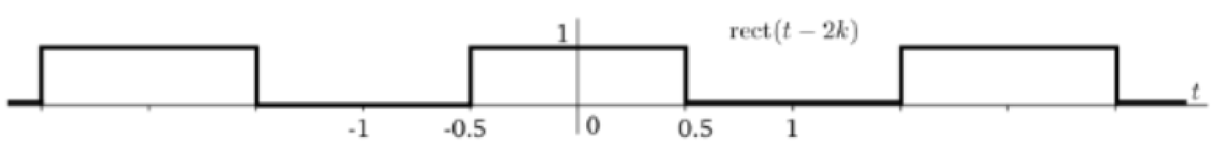




Example:
Consider the square wave below:

$$f(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}(t - 2k)$$

This is illustrated below:



In the Fourier series lecture (slides 8-32), we calculated that the Fourier series of this signal is:

$$c_k = \frac{1}{2}\mathrm{sinc}(k/2)$$

What is its Fourier transform?

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Longleftrightarrow \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0)$$

$$F(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{2}\text{sinc}\left(\frac{k}{2}\right) \cdot 2\pi \cdot \delta(\omega - k\pi)$$

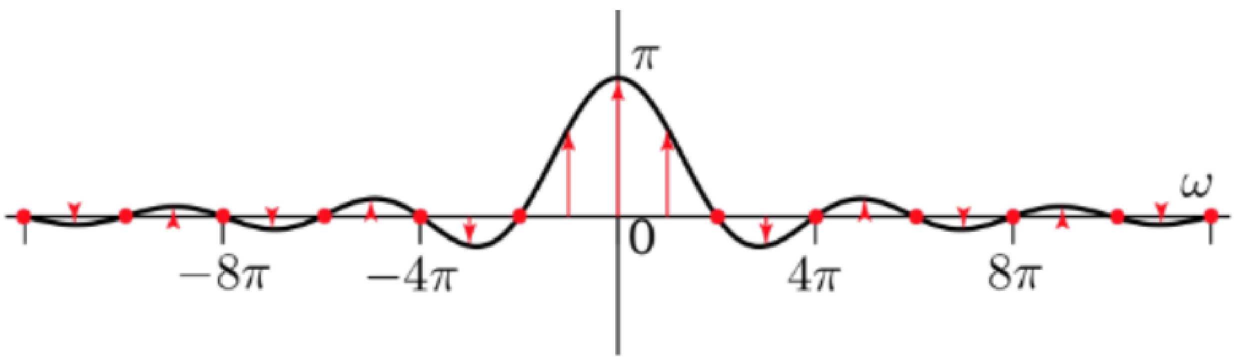
The $\delta(\omega - k\pi)$ is an impulse at $\omega = k\pi$. Therefore, $k = \frac{\omega}{\pi}$.

Hence, the Fourier transform of the square wave is the Fourier transform of a rect multiplied by evenly spaced δ's, i.e.,

$$\begin{aligned} F(j\omega) &= \pi \sum_{k=-\infty}^{\infty} \text{sinc}(\omega/2\pi)\delta(\omega - k\pi) \\ &= \pi \cdot \text{sinc}(\omega/2\pi) \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi) \end{aligned}$$

Impulse train notation: (this is discussed below.)

$$= \pi \cdot \text{sinc}\left(\frac{\omega}{2\pi}\right) \cdot \delta_\pi(\omega)$$

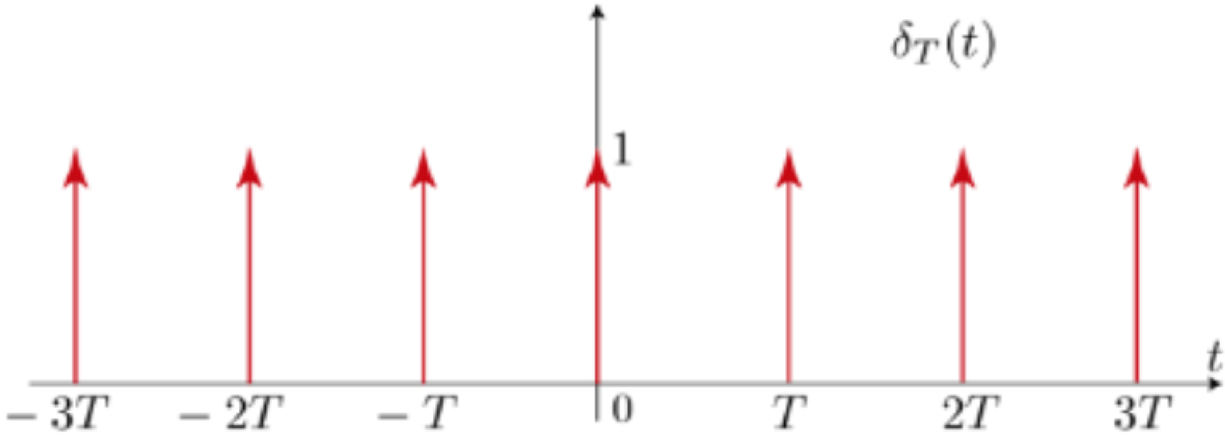


Impulse Trains

To simplify the notation here, we can define an *impulse train* which ends up being our sampling function. We let $\delta_T(t)$ be a sequence of unit δ functions spaced by T.

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

This is illustrated below:



Fourier Series of Impulse Train

By intuition:

- We know that the Fourier transform of the square wave is a sinc multiplied by a $\delta_\pi(\omega)$.
- From the convolution theorem, this means that the inverse Fourier transform (i.e., the square wave) is the inverse Fourier transform of a sinc convolved with the inverse Fourier transform of a impulse train.
- We know that a square is simply a rect repeated over and over again, i.e., convoluted with a impulse train.
- So intuitively, the Fourier transform of a impulse train should be a impulse train.
- Note: we will sometimes use the term 'delta train' to describe an impulse train.

Proof:

$$
\begin{aligned}
c_k &= \frac{1}{T} \int_{1 \text{ period}} f(t) e^{-jk\omega_0 t} \mathrm{d}t \\
&= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta_T(t) \cdot e^{-jk\omega_0 t} \mathrm{d}t
\end{aligned}
$$

Note that $\omega_0 = 2\pi/T$

$$
\begin{aligned}
&= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{0} \mathrm{d}t \\
&\boxed{c_k = \frac{1}{T}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{F}[\delta_T(t)] &= \mathcal{F}\left[\frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}\right] \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} \mathcal{F}\left[e^{-jk\omega_0 t}\right]
\end{aligned}
$$

Recall that $e^{jk\omega_0 t} \Longleftrightarrow 2\pi \cdot \delta(\omega - k\omega_0)$.

$$
\begin{aligned}
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} 2\pi \cdot \delta(\omega - k\omega_0) \\
&= \omega_0 \cdot \delta_{\omega_0}(\omega)
\end{aligned}
$$

By this, the Fourier transform of an impulse train is an impulse train.

Sampling with an impulse train

As we saw earlier, one of the things we will use the impulse train for is to sample signals.

Given a signal $f(t)$,

$$
\begin{aligned}
f(t)\delta_T(t) &= f(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT) \\
&= \sum_{k=-\infty}^{\infty} f(t) \cdot \delta(t - kT) \\
&\boxed{\hat{f}(t) = \sum_{k=-\infty}^{\infty} f(kT) \cdot \delta(t - kT)}
\end{aligned}
$$

Square Wave Example pt. 2

Let's revisit our square wave example, where

$$
f(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}(t - 2k)
$$

Another way to represent this square wave is as follows:

$$
f(t) = \mathrm{rect}(t) * \delta_2(t)
$$

Hence, we can calculate its Fourier transform by using the convolution theorem. Recall that, for $\omega_0 = 2\pi/T$.

$$
\mathrm{rect}(t) \Longleftrightarrow \mathrm{sinc}(\omega/2\pi)
$$

and

$$
\delta_T(t) \Longleftrightarrow \omega_0 \delta_{\omega_0}(\omega)
$$

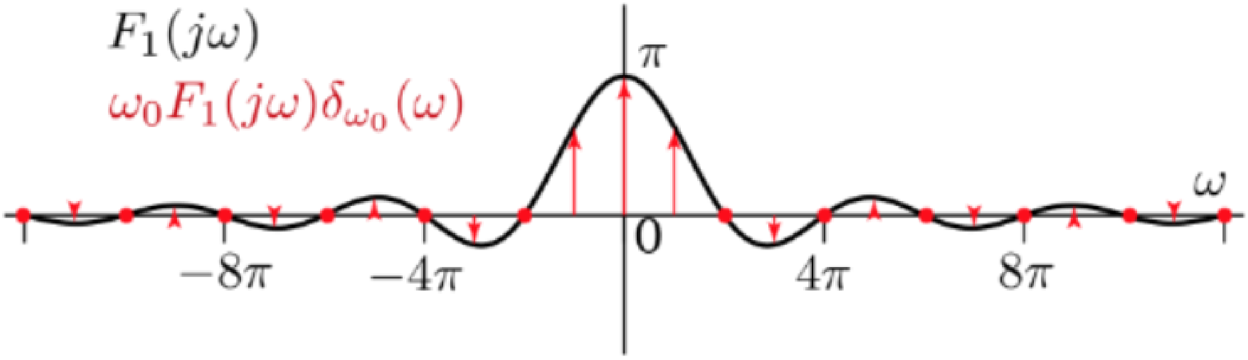
Note that when $T = 2$, then $\omega_0 = \pi$. Then, we have that,

$$
\begin{aligned}
\mathcal{F}[f(t)] &= \mathcal{F}[\mathrm{rect}(t) * \delta_2(t)] \\
&= \mathcal{F}[\mathrm{rect}(t)]\mathcal{F}[\delta_2(t)] \\
&= \mathrm{sinc}(\omega/2\pi)\pi\delta_\pi(\omega)
\end{aligned}
$$

This is exactly the same Fourier transform we calculated earlier using the Fourier series of the square wave.

Sampling and Periodicity

Another intuition to remember here is that the Fourier transform of a periodic signal is the Fourier transform of one period of the signal (which we can denote f_1), sampled by an impulse train at multiples of ω_0.



Discrete - periodic duality

We can determine the Fourier transform of a signal sampled in the time domain. Consider

$$\tilde{f}(t) = f(t)\delta_T(t)$$

Its Fourier transform is

$$\tilde{F}(j\omega) = \mathcal{F}[f(t)\delta_T(t)]$$

These are merely samples of $F(j\omega)$ repeated every ω_0, since

$$\tilde{F}(j\omega) = \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega)$$