EC ENGR 102 Week 5

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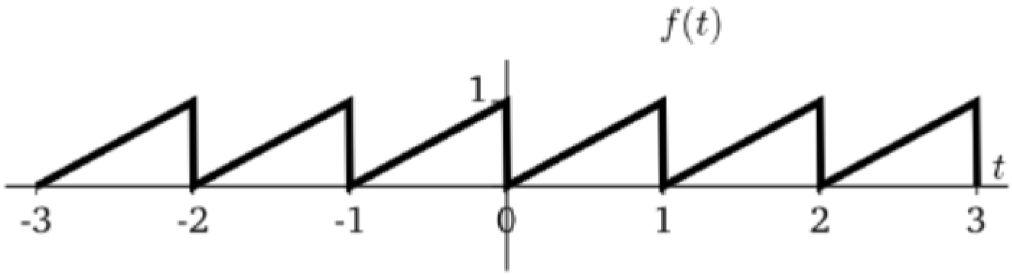
October 29, 2024

Sawtooth Signal

The sawtooth signal is given by $f(t) = t \mod 1$. It is plotted below:



This signal has a period of $T_0 = 1$. Now, when $k = 0$,

$$\begin{aligned} c_0 &= \int_0^1 te^0 \mathrm{d}t \\ &= \frac{t^2}{2}\bigg|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

This is also the Fourier series of the time-limited signal $f(t) = t$ on the interval $[0, 1)$. The time-limited signal can be made periodic via a periodic extension.

Fourier Series Properties

There are interesting symmetries and properties of the Fourier series that are worth expanding upon.

- c_0 **is the average of the signal**. Not that for $k = 0$, we have that

  $$c_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\mathrm{d}t$$

  Thus, c_0 is exactly the time-averaged mean of the signal and corresponds to a constant value (i.e., it has no sinusoidal component). For this reason, it is sometimes called the "DC component." DC stands for direct current in circuits, and refers to non-alternating (sinusoidal) currents. The DC component is the average value taken on by a signal.

Fourier Symmetry

We can apply Euler's formula to re-write the Fourier coefficients, and reveal some symmetries:

$$\begin{aligned} c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-j\frac{2\pi k t}{T_0}}\,\mathrm{d}t \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right]\mathrm{d}t \quad = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t - \frac{j}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t \end{aligned}$$

In the above equation, the left term is the real part, the right term is the imaginary part.

If $f(t)$ is real, then so are:

$$\begin{aligned} \Re(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\ \Im(c_k) &= -\frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t \end{aligned}$$

Therefore, for $f(t)$ real, and using the fact that $\cos(k)$ is even and $\sin(k)$ is odd, we have the following symmetries:

$$\begin{aligned} \Re(c_k) &= \Re(c_{-k}) && (1) \\ \Im(c_k) &= \Im(c_{-k}) && (2) \\ c_k^* &= c_{-k} && (3) \\ |c_k| &= |c_{-k}| && (4) \\ \angle c_k &= -\angle c_k^* && (5) \\ c_k &= c_{-k} \qquad \text{(only if } x(t) \text{ is even)} && (6) \\ c_k &= -c_{-k} \qquad \text{(only if } x(t) \text{ is odd)} && (7) \end{aligned}$$

Proof of (1)

$$\begin{aligned} \Re(c_{-k}) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cdot\cos\left(-\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cdot\cos\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\ &= \Re(c_{-k}) \end{aligned}$$

Proof of (2)

$$\begin{aligned} \Im(c_{-k}) &= -\frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cdot\sin\left(-\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cdot\sin\left(\frac{2\pi k}{T_0}t\right)\mathrm{d}t \\ &= \Im(c_{-k}) \end{aligned}$$

Proof of (3)

$$\begin{aligned} c_k &= \Re(c_k) + j\cdot\Im(c_k) \\ c_k^* &- \Re(c_k) - j\cdot\Im(c_k) \\ &= \Re(c_-k) + j\cdot\Im(c_-k) \\ &= c_{-k} \end{aligned}$$

Proof of (4)

$$
\begin{aligned}
|c_k| &= \sqrt{\Re^2(c_k) + \Im^2(c_k)} \\
|c_{-k}| &= \sqrt{\Re^2(c_k) + \Im^2(c_{-k})} \\
&= \sqrt{\Re^2(c_k) + (-\Im(c_k))^2} \\
&= |c_k|
\end{aligned}
$$

Proof of (5)

$$
\begin{aligned}
\angle c_k &= \arctan\left(\frac{\Re(c_k)}{\Im(c_k)}\right) \\
&= \arctan\left(\frac{\Re(c_{-k})}{\Im(c_{-k})}\right) \\
&= \arctan\left(\frac{\Re(c_k)}{-\Im(c_k)}\right) \\
&= \angle c_{-k}
\end{aligned}
$$

Proof of (6)

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\mathrm{d}t \\
c_{-k} &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{jk\omega_0 t}\mathrm{d}t
\end{aligned}
$$

Let $u = -t$.

$$
\begin{aligned}
&= -\frac{1}{T_0}\int_0^{-T_0} x(-u)\cdot e^{-jk\omega_0 u}\mathrm{d}u \\
&= \frac{1}{T_0}\int_{-T_0}^{0} x(u)\cdot e^{-jk\omega_0 u}\mathrm{d}u \\
&= c_k
\end{aligned}
$$

Fourier Series Properties

- If $x(t)$ is even, then $x(t) = x(-t)$, and therefore, $c_k = c_{-k}$. You can see this by realizing that kt only appears in the complex exponential, and therefore negating t has the same effect as negating k.

$$x(t) \text{ even} \implies c_k = c_{-k}$$

- If $x(t)$ is odd, then $x(t) = -x(-t)$, and therefore, $c_k = -c_{-k}$. This holds for the same reason as for the even case.

$$x(t) \text{ odd} \implies c_k = -c_{-k}$$

- Combining facts, we have that if $x(t)$ is even and real, then $c_k = c_{-k}$ and $c_{-k} = c_k^*$, and so $c_k = c_k^*$. This means that c_k must be real.

$$x(t) \text{ even and real} \implies c_k \text{ real}$$

- If $x(t)$ is odd and real, then $c_k = -c_{-k}$, and because $c_{-k} = c_k^*$, then $c_k = -c_k^*$. This means that c_k must be imaginary.

$$x(t) \text{ odd and real} \implies c_k \text{ imaginary}$$

Perseval's Theorem

Suppose we want to find the power of a complex signal:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 \, \mathrm{d}t$$

Since $x(t)$ is complex, we split the square to $x(t) \cdot x(t)^*$. Therefore,

$$\begin{aligned}
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)x(t)^* \mathrm{d}t \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right]\left[\sum_{n=-\infty}^{\infty} c_n^* e^{-jn\omega_0 t}\right] \mathrm{d}t
\end{aligned}$$

We can then switch the order of the summation and integral.

$$= \frac{1}{T_0}\sum_{k=-\infty}^{\infty} c_k \sum_{k=-\infty}^{\infty} c_n^* \cdot \int_{t_0}^{t_0+T_0} e^{j(k-n)\omega_0 t}\mathrm{d}t$$

Notice that the integral returns 0 when $k \neq n$, and T_0 when $k = n$. This is because if you expand the exponential using Euler's formula, then you are integrating a cosine and sin over one period, the periods of which will cancel out. Therefore,

$$\begin{aligned}
&= \frac{1}{T_0}\sum_{-\infty}^{\infty} c_k \cdot c_k^* \cdot T_0 \\
&= \sum_{k=-\infty}^{\infty} |c_k|^2
\end{aligned}$$

Everything before this point is fair game on Midterm 1.

Aperiodic Signals

- The Fourier series can model (almost) any **periodic** or **time-limited** function as a sum of complex exponentials. However, most signals we encounter are not necessarily periodic or time-limited.
- The **Fourier transform** allows us to calculate the spectrum of aperiodic signals.

Intuition of going from Fourier series to Fourier transform

Extending Fourier series to the Fourier transform is fairly intuitive.

The idea is the following:

- We can calculate the Fourier series of a periodic or time-limited signal, over some interval of length T_0.
- A signal that is not periodic can be viewed as a periodic signal, where T_0 is infinite. As T_0 is infinite, it never repeats.
- But the point is that we can replace our Fourier series calculation as, instead of being over a finite period, T_0, being over all time, from $t = -\infty$ to ∞.

- Mathematically, we can calculate the Fourier series of $f(t)$ over the interval $[-T/2.T/2)$ via:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

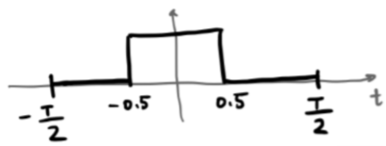
with

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega-0}\mathrm{d}t$$

where $\omega_0 = 2\pi/T$. In the Fourier transform, we're now going to let $T \to \infty$.

Example:

Signal



Frequency

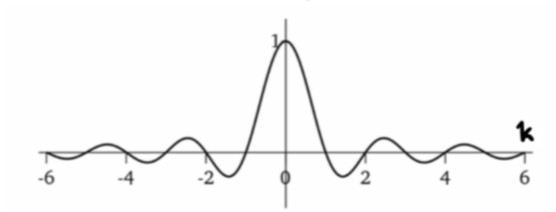
Period = T

Freq. = $\frac{2\pi}{T} = \omega_0$

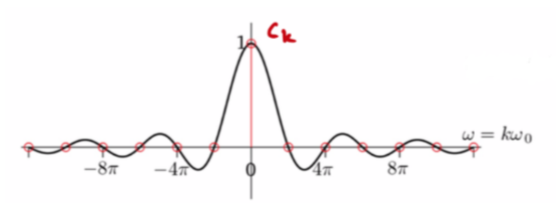
Fourier Series

$$c_k = \frac{1}{T}\,\mathrm{sinc}\left(\frac{k}{T}\right)$$

sinc(k)



if T=1, $c_k = \mathrm{sinc}(k)$, $\omega_0 = 2\pi$



$\omega = k\omega_0$

$k=0$: $\omega = 0$, $c_0 = 1$

$k=1$: $\omega = 2\pi$, $c_1 = 0$

$k=2$: $\omega = 4\pi$, $c_2 = 0$

$k=3$: $\omega = 6\pi$, $c_3 = 0$

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