EC ENGR 102 Week 9

Aidan Jan

November 26, 2024

Laplace Transform of the Unit Step

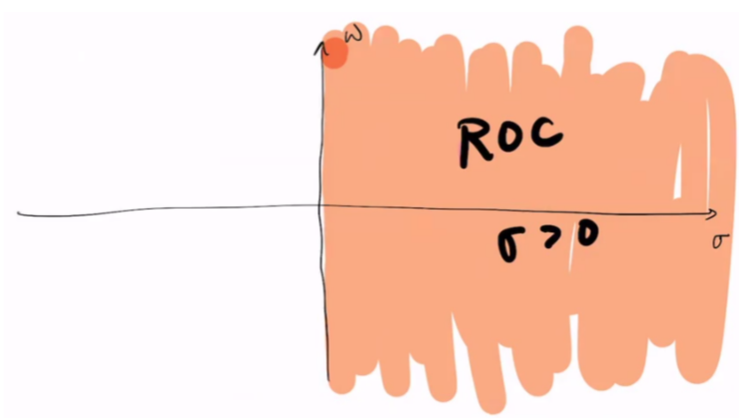
Let $f(t) = u(t)$.

$$\begin{aligned} F(s) &= \int_{0^-}^{\infty} u(t) \cdot e^{-st} \mathrm{d}t \\ &= \int_{0^-}^{\infty} e^{-st} \mathrm{d}t \\ &= -\frac{1}{s} e^{-st} \Big|_0^{\infty} \\ &= (0) - (-\frac{1}{s} e^{-s-0}) \\ &= \frac{1}{s} \end{aligned}$$

as long as $e^{-st} \to 0$ as $t \to \infty$

- In this case, the R.O.C. (region of convergence): $\boxed{\mathrm{Re}(s) = \sigma > 0}$

What is the ROC of the unit step Laplace transform?



- This R.O.C. does NOT contain the $j\omega$ axis ($\sigma = 0$).
- As a result, the Fourier transform is different from the Laplace transform for the unit step, since if $\sigma = 0$ in the Laplace transform, the integral does not converge.
- The Laplace and Fourier transforms are similar though.

– Fourier transform of unit step is:

$$\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

– This is similar to the Laplace transform with $s = j\omega$, but with the additional $\pi\delta(\omega)$ term.

Laplace Transform of Cosine

$$\begin{aligned} f(t) &= \cos(\omega t) \\ &= \frac{1}{2}\big[e^{j\omega t} + e^{-j\omega t}\big] \end{aligned}$$

Then,

$$\begin{aligned} F(s) &= \int_0^\infty \frac{1}{2}\big[e^{j\omega t} + e^{-j\omega t}\big]e^{-st}\mathrm{d}t \\ &= \frac{1}{2}\int_0^\infty e^{(-s+j\omega)t} + e^{(-s-j\omega)t}\mathrm{d}t \\ &= \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) \\ &= \frac{s}{s^2+\omega^2} \end{aligned}$$

- Note that in the final equation, the ω and the $j\omega$ within the s are different ω's. The ω present in the final equation is the ω of the cosine function. The one contained in the s is the Fourier transform variable.

The region of convergence is for when $e^{(-s\pm j\omega)t} \to 0$ as $t \to \infty$ and thus is $mathfrakR(s) > 0$. Like the unit step, the Laplace and Foruier transforms disagree, as the Laplace region of convergence does not include the $j\omega$ axis.

Laplace Transform of Powers of t

Let $f(t) = t^n$, for $n \geq 1$. Then,

$$F(s) = \int_0^\infty t^n e^{st}\mathrm{d}t$$

Integrate by parts: $u(t) = t^n$, $v'(t) = e^{-st}$, $u'(t) = nt^{n-1}$, and $v = -\frac{1}{s}e^{-st}$.

$$\begin{aligned} \mathcal{L}[t^n] &= -\left.\frac{t^n e^{-st}}{s}\right|_{t=0}^{t=\infty} + \int_0^\infty \frac{1}{s}e^{-st}\cdot n\cdot t^{n-1}\mathrm{d}t \\ &= \frac{-(\infty)^n e^{-s\cdot\infty}}{s} - \left(-\frac{0^n\cdot e^0}{s}\right) + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\mathrm{d}t \end{aligned}$$

$t^n e^{-st} \to 0$ as $t \to \infty$ as long as $\mathrm{Re}\{s\} = \sigma$ is greater than zero. (R.O.C. $\sigma > 0$)

$$\begin{aligned} &= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\mathrm{d}t \\ &= \frac{n}{s}\cdot\mathcal{L}\big[t^{n-1}\big] \end{aligned}$$

Now, we have a recursive equation.

$$\mathcal{L}\big[t^n\big] = \frac{n}{s}\mathcal{L}\big[t^{n-1}\big]$$

By inspection, we get $\mathcal{L}[1]$ when $n = 1$, which is just the unit step. The first few n terms would be:

$$\begin{aligned}
\mathcal{L}\left[t^0\right] &= \frac{1}{s} \\
\mathcal{L}\left[t^1\right] &= \frac{1}{s} \cdot \mathcal{L}\left[t^0\right] = \frac{1}{s^2} \\
\mathcal{L}\left[t^2\right] &= \frac{2}{s} \cdot \mathcal{L}\left[t^1\right] = \frac{2}{s^3}
\end{aligned}$$

We can simplify this series to

$$\boxed{\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}}$$

The region of convergence for this Laplace transform is R.O.C. $\sigma > 0$.

- For this class, we assume that n is an integer. We will not go into what happens if n is not an integer; probably there will be a Γ function involved.

Laplace Transform of the Impulse

Let $f(t) = \delta(t)$. Then,

$$\begin{aligned}
F(s) &= \int_{0^-}^{\infty} \delta(t) e^{-st} \mathrm{d}t \\
&= e^{-s \cdot 0} \\
&= 1
\end{aligned}$$

Thus,

$$\boxed{\mathcal{L}[\delta(t)] = 1}$$

Pattern for Integration and Differentiation?

Notice the following trends:

$$\begin{aligned}
\delta(t) &\Longleftrightarrow 1 \\
u(t) &\Longleftrightarrow \frac{1}{s} \\
tu(t) &\Longleftrightarrow \frac{1}{s^2} \\
\frac{1}{2}t^2 u(t) &\Longleftrightarrow \frac{1}{s^3} \\
\frac{1}{6}t^3 u(t) &\Longleftrightarrow \frac{1}{s^4}
\end{aligned}$$

We see a clear pattern: differentiating a signal is equivalent to multiplying the Laplace transform by s while integrating is equivalent to multiplying the Laplace transform by $1/s$.

Review: Basic Laplace Transforms:

$$\mathcal{L}\big[e^{at}u(t)\big] = \frac{1}{a+s} \qquad \text{as long as } \sigma > -a$$

$$u(t) \Longleftrightarrow \frac{1}{s} \qquad \Re(s) > 0$$

$$\cos(\omega t) \Longleftrightarrow \frac{s}{s^2+\omega^2} \qquad \Re(s) > 0$$

$$\boxed{\mathcal{L}\big[t^n\big] = \frac{n!}{s^{n+1}}} \qquad \Re(s) > 0$$

$$\boxed{\mathcal{L}[\delta(t)] = 1} \qquad \Re(s) > 0$$

Laplace Transform Properties (copied from last week)

1. **Linearity:**

$$\boxed{\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)}$$

2. **Time scaling:**

$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)}$$

3. **Time shift:**

$$\boxed{\mathcal{L}[f(t-T)] = e^{-sT}F(s)}$$

4. **Frequency shift:**

$$\boxed{\mathcal{L}[f(t)e^{s_0 t} = F(s-s_0)]}$$

5. **Convolution:**

$$\boxed{\mathcal{L}[f_1(t) * f_2(t) = F_1(s)F_2(s)]}$$

6. **Integration:**

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)\mathrm{d}\tau\right] = \frac{1}{s}F(s)}$$

7. **Derivative:**

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$$

8. **Multiplication by t:**

$$\boxed{\mathcal{L}[tf(t)] = -F'(s)}$$

Time Scaling Property

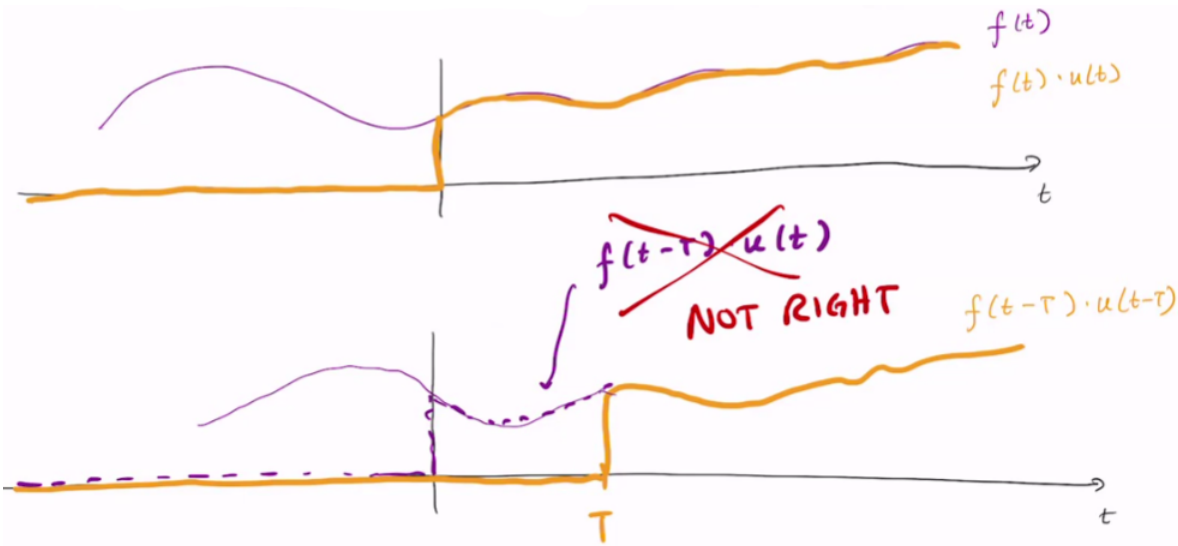
$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)}$$

- Note, we only consider $a > 0$, since if $a < 0$ and $f(t)$ is a causal signal, then $f(at)$ would be anticausal. The unilateral Laplace transform is for causal signals only.
- This is the reason why the a does not have an absolute value like the Fourier transform equation.

Time Shift Property

Let $f(t) \Longleftrightarrow F(s)$. If we delay a signal by T, i.e., $f(t-T)$, then we proceed with the understanding that:

- $T > 0$, since if $T < 0$, the signal would be noncausal.
- For delays $T > 0$, the signal $f(t-T)$ is zero in the interval from 0 to T.



Integration Property

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)\mathrm{d}\tau\right] = \frac{1}{s}F(s)}$$

To derive this, we use the convolution theorem.

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(\tau)\mathrm{d}\tau\right] &= \mathcal{L}[f(t) * u(t)] \\ &= \mathcal{L}[f(t)]\mathcal{L}[u(t)] \\ &= \frac{1}{s}F(s)\end{aligned}$$

Differentiation Property

Let $f'(t)$ be the derivative of $f(t)$ with respect to time. Then,

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}\mathrm{d}t$$

Integrating by parts, we set $u = e^{-st}$ and $v' = f'(t)$. Then, $u' = -se^{-st}$ and $v = f(t)$. Hence,

$$\begin{aligned}\mathcal{L}[f'(t)] &= \int_0^\infty f'(t)e^{-st}\mathrm{d}t \\ &= f(t)e^{-st}|_0^\infty + \int_0^\infty se^{-st}f(t)\mathrm{d}t \\ &= -f(0) + sF(s)\end{aligned}$$

if $e^{-st} \to 0$ (R.O.C. $\sigma > 0$) as $t \to \infty$. Hence,

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$$

Multiplication by t

If $f(t) \Longleftrightarrow F(s)$, then we can differentiate both sides to see that:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\mathrm{d}t \\ F'(s) &= \int_0^\infty (-t)e^{-st} f(t)\mathrm{d}t \\ &= \mathcal{L}[-tf(t)] \end{aligned}$$

Hence,

$$\boxed{\mathcal{L}[tf(t)] = -F'(s)}$$

General Laplace Transforms

$g(t)$	$G(s)$
$\int_0^t f(\tau)\mathrm{d}\tau$	$\frac{1}{s}F(s)$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

More examples: Laplace Transform of Sine

We know that

$$\cos(\omega t) \Longleftrightarrow \frac{s}{s^2 + \omega^2}$$

We can write the sine function in terms of cosine:

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos(\omega t) = -\sin(\omega t)\cdot\omega \Longrightarrow \sin(\omega t) = -\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\cos(\omega t)$$

Therefore,

$$\begin{aligned} \mathcal{L}[\sin(\omega t)] &= -\frac{1}{\omega}\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\cos(\omega t)\right] \\ &= -\frac{1}{\omega}\left[s\cdot\mathcal{L}[\cos(\omega t)] - \cos(\omega\cdot 0)\right] \\ &= -\frac{1}{\omega}\left[\frac{s^2}{s^2+\omega^2} - 1\right] \\ &= -\frac{1}{\omega}\left[\frac{-\omega^2}{s^2+\omega^2}\right] \\ &= \frac{\omega}{s^2+\omega^2} \end{aligned}$$

Laplace Transform to Solve Differential Equations

Suppose we want to solve the differential equation

$$y'(t) + y(t) = u(t) \qquad \text{where } y(0) = 0$$

First, we can take the Laplace transform of both sides:

$$sY(s) - y(0) + Y(s) = \frac{1}{s}$$

Simplifying,

$$
\begin{aligned}
sY(s) - y(0) + Y(s) &= \frac{1}{s} \\
Y(s)(s+1) &= \frac{1}{s} \\
Y(s) &= \frac{1}{s(s+1)} \\
&\boxed{Y(s) = \frac{1}{s} - \frac{1}{s+1}}
\end{aligned}
$$

Therefore, (refer to Laplace pairs; $u(t) \iff \frac{1}{s}$)

$$\mathcal{L}^{-1}[Y(s)] = u(t) - e^{-t}u(t)$$

So,

$$y(t) = 1 - e^{-t} \qquad \text{for } t \geq 0$$

Key Take-home Point:

With Laplace Transform, *differential equations* are turned into *algebraic equations*.

$$y(t) = (h * x)(t) \iff Y(s) = H(s)X(s) \implies H(s) = \frac{Y(s)}{X(s)}$$

$H(s)$ is known as the "transfer function".

General Method to Solve Any-Order Differential Equations

Suppose $y^{(k)}(t) = \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}$, and all initial conditions are zero. Then,

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t) = b_m x^{(m)}(t) + b_m x^{(m-1)}(t) + \cdots + b_m x^(1)(t) + b_0 x(t)$$

$$\Big\Updownarrow \mathcal{L}$$

$$a_n \cdot s^n Y(s) + a_{n-1} \cdot s^{n-1} Y(s) + \cdots + a_1 \cdot sY(s) + a_0 Y(s) = b_m s^m X(s) + b_{m-1} s^{m-1} X(s) + \cdots + b_1 s \cdot X(s) + b_0 X(1)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Inverse Laplace Transforms We Should Know

$$
\begin{aligned}
\mathcal{L}\big[e^{-at}u(t)\big] &= \frac{1}{a+s} \\
\mathcal{L}[\cos(\omega t)] &= \frac{s}{s^2 + \omega^2} \\
\mathcal{L}[\sin(\omega t)] &= \frac{\omega}{s^2 + \omega^2} \\
\mathcal{L}\big[e^{-at}\cos(\omega t)\big] &= \frac{(s+a)}{(s+a)^2 + \omega^2} \\
\mathcal{L}\big[e^{-at}\sin(\omega t)\big] &= \frac{\omega}{(s+a)^2 + \omega^2} \\
\mathcal{L}^{-1}\left[\frac{r}{(s-\lambda)^k}\right] &= \frac{r}{(k-1)!} t^{k-1} e^{\lambda t}
\end{aligned}
$$

Partial Fraction Expansion

Let

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots a_n s^n}$$

From the fundamental theorem of algegra,

- $b(s)$ has m roots (called "zeros" of $F(s)$, since if $b(s) = 0 \Rightarrow F(s) = 0$)
- $a(s)$ has n roots (called "poles" of $F(s)$, since if $a(s) = 0 \Rightarrow F(s) \to \infty$)

For partial fraction expansion, let's first assume that no poles are repeated and that $m < n$ (i.e., more poles than zeros = "proper" rational function). Then, $F(s)$ can be written in its *partial fraction* expansion:

$$F(s) = \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}$$

where

- $\lambda_1, \ldots, \lambda_n$ are the poles of F.
- The numbers $r_1, \ldots, r_n$ are called *residues*
- It turns out when $\lambda_k = \lambda_l^*$, then $r_k = r_l^*$.
- Note the poles can be complex numbers

Inversion of a Partial Fraction

In partial fraction form, inverting the Laplace transform is easy because

$$\begin{aligned}
\mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left[\frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}\right] \\
&= r_1 \cdot \mathcal{L}^{-1}\left[\frac{1}{s - \lambda_1}\right] + \cdots + r_n \cdot \mathcal{L}^{-1}\left[\frac{1}{s - \lambda_n}\right] \\
&= r_1 \cdot e^{-\lambda_1 t} + r_2 \cdot e^{-\lambda_2 t} + \cdots + r_n \cdot e^{-\lambda_n t} \qquad \text{for } t \geq 0
\end{aligned}$$

How to Find the Partial Fraction Expansion

To find the partial fraction expansion, we

- Find the polds $\lambda_1, \ldots, \lambda_n$, which means we find the zeros of $a(s)$.
- Find the residues of $r_1, \ldots, r_n$

There are several methods to calculate partial fraction expansions.

Method 1: Partial Fractions via Solving Linear Equations

- No one actually uses this method, because Method 2 is just better. However, it is here for completion.
- In this method, we factor $a(s)$ to find the poles, then solve linear equations to find the residues. Say $m = 2$ and $n = 3$. Then,

$$\frac{b_0 + b_1 s + b_2 s^2}{(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)} = \frac{r_1}{s - \lambda_1} + \frac{r_2}{s - \lambda_2} + \frac{r_3}{s - \lambda_3}$$

First, we clear the denominators by multiplying both sides by

$$(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)$$

This gives:

$$b_0 + b_1 s + b_2 s^2 = r_1(s-\lambda_2)(s-\lambda_3) + r_2(s-\lambda_1)(s-\lambda_3) + r_3(s-\lambda_1)(s-\lambda_2)$$

At this point, we equate the coefficients of each power of s.

$$\begin{aligned} b_0 + b_1 s + b_2 s^2 &= r_1(s-\lambda_2)(s-\lambda_3) + r_2(s-\lambda_1)(s-\lambda_3) + r_3(s-\lambda_1)(s-\lambda_2) \\ &= r_1\lambda_2\lambda_3 + r_2\lambda_1\lambda_3 + r_3\lambda_1\lambda_2 + \ldots \\ &+ s[r_1(-\lambda_3-\lambda_2) + r_2(-\lambda_3-\lambda_1) + r_3(-\lambda_2-\lambda_1)] \\ &+ s^2[r_1 + r_2 + r_3] \end{aligned}$$

Thus,

$$\begin{aligned} b_2 &= r_1 + r_2 + r_3 \\ b_1 &= r_1(-\lambda_3-\lambda_2) + r_2(-\lambda_3-\lambda_1) + r_3(-\lambda_2-\lambda_1) \\ b_0 &= r_1\lambda_2\lambda_3 + r_2\lambda_1\lambda_3 + r_3\lambda_1\lambda_2 \end{aligned}$$

Now we have n poles and n equations for n unknowns.

Method 2: Partial Fractions via the "Cover-Up" Procedure

Here, we solve for each residual individually in the following way. E.g., to get r_1, we first multiply both sides by $(s-\lambda_1)$.

$$\frac{b_0 + b_1 s + b_2 s^2}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \frac{r_3}{s-\lambda_3}$$

becomes

$$\frac{(s-\lambda_1)(b_0 + b_1 s + b_2 s^2)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = r_1 + \frac{r_2(s-\lambda_1)}{s-\lambda_2} + \frac{r_3(s-\lambda_1)}{s-\lambda_3}$$

Now, we set $s = \lambda_1$ to get:

$$r_1 = \frac{b_0 + b_1 \cdot \lambda_1 + b_2 \cdot \lambda_1^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$$

In general,

$$r_k = (s-\lambda_k)F(s)\big|_{s=\lambda k}$$

Example: Let's find the partial fraction expansion of:

$$\frac{s^2-2}{s(s+1)(s+2)} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

$$\begin{aligned} r_1 &= \left.\frac{s^2-2}{(s+1)(s+2)}\right|_{s=0} = \frac{(0)^2-2}{(0+1)(0+2)} = -1 \\ r_2 &= \left.\frac{s^2-2}{(s)(s+2)}\right|_{s=-1} = \frac{1-2}{(-1)(1)} = 1 \\ r_3 &= \left.\frac{s^2-2}{(s)(s+1)}\right|_{s=-2} = \frac{4-2}{-2(-1)} = 1 \end{aligned}$$

Therefore,

$$F(s) = \frac{s^2-2}{s(s+1)(s+2)} = -\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}$$

Method 3: Partial Fractions via L'Hopital's Rule

Another way to find the kth residual is to calculate:

$$r_k = \frac{b(\lambda_k)}{a'(\lambda_k)}$$

The idea behind this approach is to still use the cover-up method (i.e., multiply the partial fraction expansion by $(s - \lambda_k)$) and set s to λ_k. This technique finds another formula for the residual.

$$r_k = \lim_{s \to \lambda_k} \frac{(s - \lambda_k) b(s)}{a(s)}$$

Expanding,

$$\begin{aligned}
&= \lim_{s \to \lambda_k} \frac{s \cdot b(s) - \lambda_k \cdot b(s)}{a(s)} \\
&= \lim_{s \to \lambda_k} \frac{s \cdot b'(s) + 1 \cdot b(s) - \lambda_k \cdot b'(s)}{a'(s)} \\
&= \lim_{s \to \lambda_k} \frac{b'(s)(s - \lambda_k) + b(s)}{a'(s)} \\
&= \frac{b(\lambda_k)}{a'(\lambda_k)}
\end{aligned}$$

Example: Let's do the same example as in Method 2:

$$\frac{s^2 - 2}{s(s+1)(s+2)} = \frac{s^2 - 2}{s^3 + 3s^2 + 2s}$$

Differentiating the denominator, we get:

$$a'(s) = 3s^2 + 6s + 2$$

Therefore,

$$\begin{aligned}
r_1 &= \left. \frac{s^2 - 2}{3s^2 + 6s + 2} \right|_{s=0} = \frac{-2}{2} = -1 \\
r_2 &= \left. \frac{s^2 - 2}{3s^2 + 6s + 2} \right|_{s=-1} = \frac{1 - 2}{3 - 6 + 2} = 1 \\
r_3 &= \left. \frac{s^2 - 2}{3s^2 + 6s + 2} \right|_{s=-2} = \frac{4 - 2}{12 - 12 + 2} = 1
\end{aligned}$$

ODE Example

Let's solve the following ODE:

$$v'''(t) - v(t) = 0$$

where

- $v(0) = 1$
- $v'(0) = 0$
- $v''(0) = 0$

Following the General Laplace Transform list, we get

$$\begin{aligned} f'''(t) &\Longleftrightarrow s \cdot (s^2F(s) - sf(0) - f'(0)) - f''(0) \\ &= s^3F(s) - s^2f(0) - sf'(0) = f''(0) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}[v'''(t)] &= s^3V(s) - -s^2v(0) - sv'(0) - v''(0) \\ &= s^3V(s) - s^2 \end{aligned}$$

From our equation, we get:

$$\begin{aligned} s^3V(s) - s^2 - V(s) = 0 \\ V(s)(s^3 - 1) = s^2 \\ V(s) = \frac{s^2}{s^3 - 1} \end{aligned}$$

Now, we do partial fraction decomposition.