EC ENGR 102 Week 3

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Memory

- A system has *memory* if its output depends on past or future values of the input. If the output depends only on present values of the input, the system is called *memoryless*.
- e.g., $x(t) = A\cos(\omega t)$ is memoryless.
- $x(t) = \int_{-\infty}^{t} e^{-\tau^2} \mathrm{d}\tau$ is not.

Invertibility

- A system is called *invertible* if an input can always be exactly recovered from the output. That is, a system S is invertible if there exists an S^{inv} such that

$$x = S^{\text{inv}}(S(x))$$

- e.g., $y(t) = [x(t)]^2$ and $y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ are not invertible
- $y(t) = ax(t)$ for $a \neq 0$ is invertible. Its inverse is $x(t) = \frac{1}{a} \cdot y(t)$.

System Impulse Response

- This lecture introduces time-domain analysis of systems, including the impulse response. It also discusses linear time-invariant sysstems. Topics include:
  - Impulse response definition
  - Impulse response of LTI systems
  - The impulse response as a sufficient characterization of an LTI system
  - Impulse response and the convolution integral
- We've built a foundation on signal operations, signal models, and systems. Today will be the first lecture where we present a new idea key to signals and systems: the impulse response.
- This impulse response will start us on the path to new concepts including convolution, Fourier series, and Fourier Transform

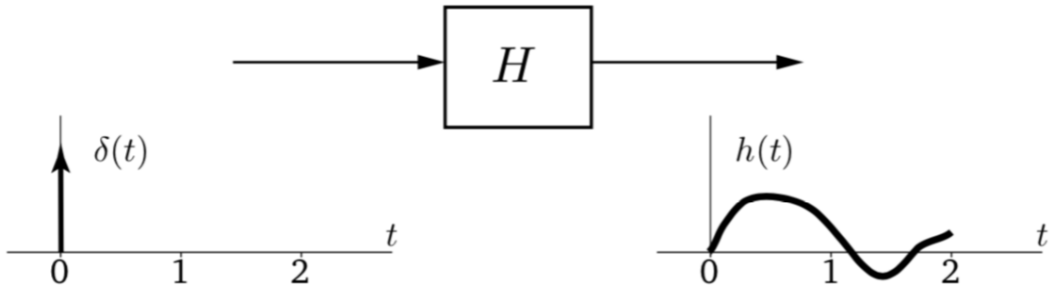
Impulse Response

Why do we need the impulse response?

- In real life, we often do not have the luxury of knowing exactly what S is, or perhaps we only know it imperfectly. And even if we did know it, it could take on a very complicated form.
- The *impulse response* is a characterization of the system that, for linear time-invariant systems, *enables to calculate the output for* ***any*** *input.* In this manner, it is a full time-domain description of the system.

Impulse Response Computation

$$h(t) = H(\delta(t))$$



- In this case, our input signal is $x(t) = \delta(t)$, and our output is $y(t) = h(t)$.

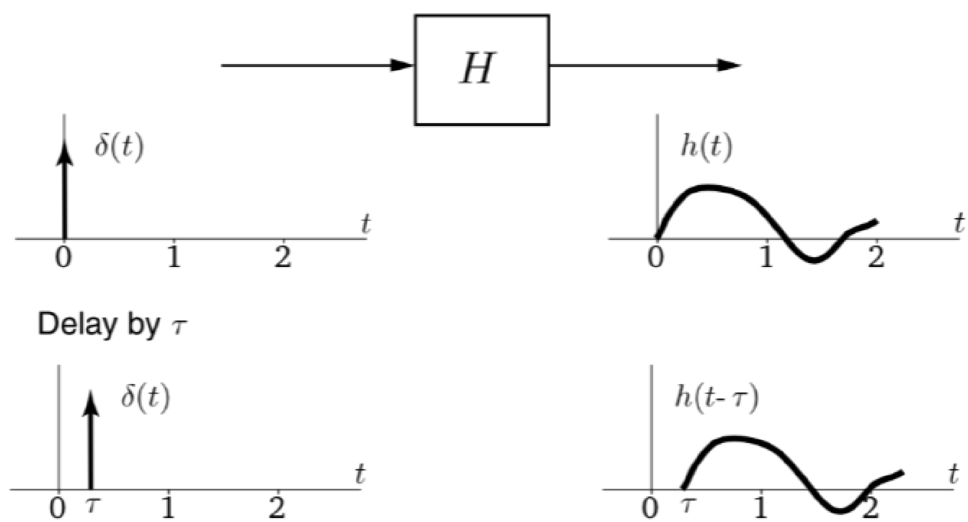
Impulse response formal vs time invariant notation

<u>Formal:</u>

$$y(t) = H(x(t))$$
$$h(t) = H(\delta(t))$$
$$h(t, \tau) = H(\delta(t - \tau))$$

<u>Time-invariance:</u>

$$h(t) = H(\delta(t))$$
$$[FILL]$$



A word on "t" in this formula

$$h(t) = H(\delta(t))$$

In the above equation, the two t's are NOT the same.

- You cannot say $h(2) = H(\delta(2))$.
- The t on the right is an independent variable that defines the input.

- The t on the left is an independent variable that defines the output.

We can show this by making $h(t)$ the unit step function. $h(2)$ would return 1, while $\delta(2)$ would return 0.

Extended Linearity

Recall that a system, H, is linear if for $y_n = H(x_n)$ where n is a subscript denoting different signals, and a_n are constants, we have that:

$$\sum_n a_n y_n = H\left(\sum_n a_n x_n\right)$$

i.e., it has both homogeneity and superposition. Thus, summation and the system operator can be interchanged.

In particular, this holds over integration (which is summation over infinitesimal intervals). That is, if $y = H(x)$, then:

$$\int_{-\infty}^{\infty} a(\tau)y(t-\tau)\ \mathrm{d}\tau = H\left(\int_{-\infty}^{\infty} a(\tau)x(t-\tau)\ \mathrm{d}\tau\right)$$

Important fact about the impulse response

FACT: If H is an LTI (linear time-invariant system) with impulse response

$$h(t) = H(\delta(t))$$

then we can calculate $H(x(t))$ for ANY $x(t)$ **IF** we know $h(t)$.

- Said differently, it is completely characterized by $h(t)$.
- I can calculate $y(t)$ for any $x(t)$ as long as I know $h(t)$.

Derivation of this fact

Approach: write $x(t)$ in terms of $\delta(\tau)$'s.
Suppose $t = 0$, $x(0)$.

$$x(\tau) \cdot \delta(\tau) = x(0) \cdot \delta(\tau)$$

If we integrate this using the sampling property, we get $x(0)$.

If we now want to know what the value is at $x = 3$, then we do $t = 3$, $x(3)$.

$$x(\tau) \cdot \delta(\tau - 3) = x(3) \cdot \delta(\tau - 3)$$

By integrating this, we get the result of $x(3)$.

Now let's consider the general case: $x(t)$.

$$x(\tau) \cdot \delta(\tau - t) = x(t) \cdot \delta(\tau - t)$$

If we integrate this equation, we get $x(t)$. Therefore, by sending an impulse through the circuit at particular values of τ, we can find the result of the circuit.

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau) \cdot \delta(\tau - t)\mathrm{d}\tau &= \int_{-\infty}^{\infty} x(t) \cdot \delta(\tau - t)\mathrm{d}\tau \\ &= x(t) \cdot \int_{-\infty}^{\infty} \delta(\tau - t)\mathrm{d}\tau \\ &= x(t)\end{aligned}$$

We get the following two cases:

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} x(\tau)\delta(\tau - t)\mathrm{d}\tau \\
x(t) &= \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t - \tau)\mathrm{d}\tau
\end{aligned}
$$

From these two cases, we can assert that $\tau = t$.

The Convolution Integral

$$
\begin{aligned}
y(t) &= H(x(t)) \\
&= H\left(\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\mathrm{d}\tau\right) \\
&= \int_{-\infty}^{\infty} x(\tau)H(\delta(t - \tau))\mathrm{d}t && \text{this is possible since } H \text{ is linear} \\
&= \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau)\mathrm{d}\tau && \text{H is time invariant.}
\end{aligned}
$$

If we now set the output to this function,

$$
y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\mathrm{d}\tau
$$

This is called the "Convolution", or "Convolution integral".

Examples of computing the impulse response

To find the impulse response,

1. Set $x(t)$ to $\delta(t)$.
2. We compute the system's output, $h(t) = H(\delta(t))$

Example 1: What is the impulse response of $y(t) = \int_{-\infty}^{t} x(\tau)\mathrm{d}\tau$?

Solution:
It is the unit step function!

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{t} \delta(t)\mathrm{d}\tau \\
&= u(t)
\end{aligned}
$$

Note that $y(t) = \int_{-\infty}^{\infty} x(\tau)u(t - \tau)\mathrm{d}\tau$, the convolution integral, is equal to the original $y(t)$ equation!

Example 2:

$$
y(t) = x(t - \alpha)
$$

Solution:
To find the impulse response, we apply the substitutions as described above:

$$
h(t) = \delta(t - \alpha)
$$

Looking at the convolution integral,

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-T-\alpha)\mathrm{d}\tau \\
&= x(t-\alpha)
\end{aligned}
$$