BIOMATH 208 Week 4

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Review

Curves and Surfaces

On a computer:

- list of points (vertices)
- list of connectivity elements (line segments, triangles)
  - Integers (2 for segment, 3 for traingle), which are indices for the array of vertices
  - orientation matters (we often want consistent tangent or normal vectors)

In a vector space:

- consider them as integral operators, compute path or flux integrals when acting on a smooth vector field.
- Therefore, these are covectors in the space dual to smooth vector fields
- We looked at the operator norm
$$|\delta|_{v^*} = \sup_{v \in V, |v|_v = 1} \gamma(v) = \|\gamma^\sharp\|_v$$
- We can model smooth functions instead of weird curves

We chose to use the reproducing kernel inner product for our space of smooth functions, and parameterize them as $v(x) = \sum_{i=1}^{N} p_i k(x - c_i)$. ($p_i$ is the direction, not normalized, and c_i is the center of the vector.), for any $N \in \mathbb{N}$.

- k as a gaussian with a fixed width

The reproducing kernel inner product is:
$$\langle ak(\cdot - a), bk(\cdot - y)\rangle_v = a \cdot bk(x - y)$$

- The $\cdot$ on the right hand side is a dot product in $\mathbb{R}^3$.
- This shows that non-smooth functions will have an infinite norm.

The Flat Map

While there are many objects included in the dual space, we will focus on the ones that result from the flat map.

The flat map is given by
$$\flat(aK(\cdot - x)) = a\delta_x$$

- K is a gaussian blob
- a is a vector
- x is a center

Definition (Linear evaluation functional)

δ_x acts linearly on a function, and returns its value at a point.

$$\delta_x(v) = v(x)$$

We define the action of $a\delta_x$ as

$$a\delta_x(v) = a \cdot v(x)$$

- The $\cdot$ is a dot product in $\mathbb{R}^3$.

The flat map for smooth vector fields

We can expand our definition using linearity

$$\flat\left(\sum_i a_i K(\cdot - x_i)\right) = \sum_i \flat(a_i K(\cdot - x_i)) = \sum_i a_i \delta_{x_i}$$

Proof

This flat map is the one defined by our inner product

$$\flat(aK(\cdot - x)) = a\delta_x$$

By the definition of inner product,

$$\begin{aligned}\langle ak(\cdot - x), bk(\cdot - y)\rangle_v &= a \cdot bk(x-y)\\ &= b(ak(\cdot - x))(bk(\cdot - y))\\ &= a\delta_x(bk(\cdot - y))\end{aligned}$$

The Sharp Map

By definition, the sharp map is the inverse of the flat map:

$$\sharp(a\delta_x) = aK(\cdot - x)$$

It is extended to all ("nice") linear evaluation functionals by linearity.

Discrete Line Integrals

Approximate our curve γ with a sequence of points $x_1, \ldots, x_N$. The center of the ith edge is $c_i \frac{x_i + x_{i+1}}{2}$ for $i \in \{1, \ldots, N-1\}$. The tangent to the ith edge is $\tau_i = x_{i+1} - x_i$. Then:

$$\gamma(v) = \int v(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t \simeq \sum_{i=1}^{N-1} v(c_i) \cdot \tau_i$$

- Centers: average of two consecutive points
- Tangent: difference between two consecutive points
- This is like a riemann sum.
- We can evaluate the right side of the equation (the summation) with evaluation functionals.

Integrals as evaluation functionals

We can rewrite this as:

$$\sum_{i=1}^{N-1} v(c_i) \cdot \tau_i = \left(\sum_{i=1}^{N-1} \tau_i \delta_{c_i} \right)(v)$$

The right side element $\left(\sum_{i=1}^{N-1} \tau_i \delta_{c_i}\right) \in V^*$, and is a discrete curve that can be written as a "nice" covector, a weighted sum of evaluation functions.

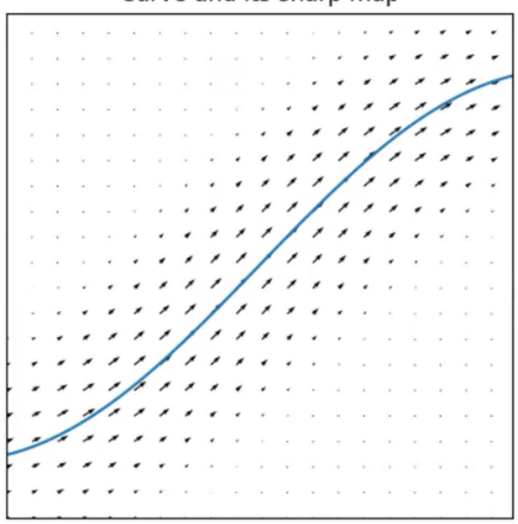
Sharp map for discrete curves

If γ is a discrete curve, then its sharp map is given by

$$\delta^\sharp(x) = \sum_{i=1}^{N} \tau_i K(x - c_i)$$

- We just replaced the δ_x with a K.
- In the below image, the vector field (black) is the sharp map applied to our curve.

Curve and its sharp map



Inner product for discrete curves

Let $\mu = \sum_{i=1}^{n^\mu - 1} \tau_i^\mu \delta_{c_i^\mu}$ and $\nu = \sum_{i=1}^{n^\nu - 1} \tau_i^\nu \delta_{c_i^\nu}$. The inner product is:

$$g_{V^*}(\mu, \nu) = \sum_{i=1}^{n^\mu - 1} \sum_{j=1}^{n^\nu - 1} K(c_i^\mu - c_j^\nu)(\tau_i^\mu \cdot \tau_j^\nu)$$

This equation works due to bilinearity, since

$$\langle \tau_1 \delta_{c_1}, \tau_2 \delta_{c_2} \rangle = \langle \tau_1 k(\cdot - c_1), \tau_2 k(\cdot - c_2) \rangle_v = \tau_1 \cdot \tau_2 k(c_1 - c_2)$$

The double sum we take in this formula represents the sum of all possible pairs of i and j. Think: nested for loops.

Distance between discrete curves

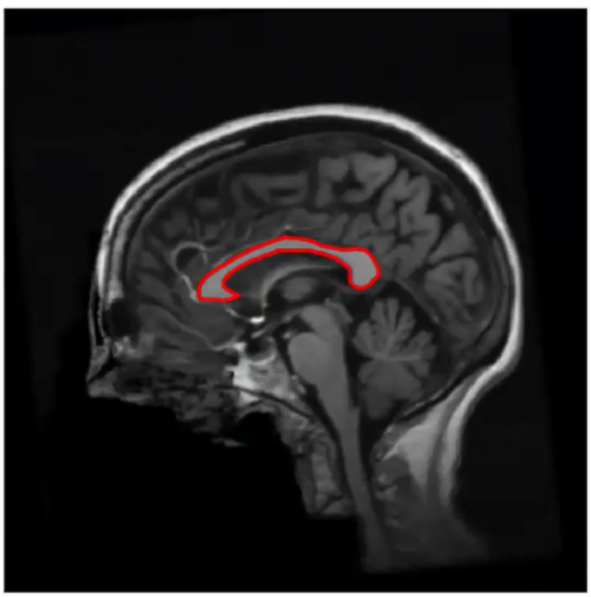
The distance between two curves is the norm of their difference.

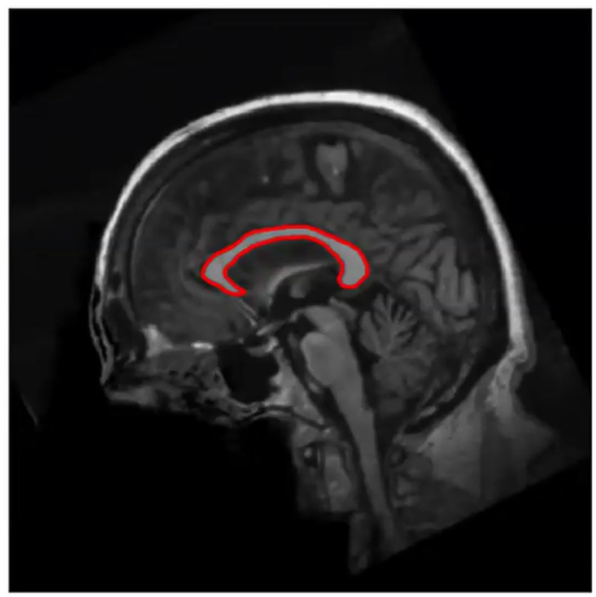
$$
\begin{aligned}
&\|\mu - \nu\|_{V^*}^2 \\
&= g_{V^*}(\mu, \mu) - 2g_{V^*}(\mu, \nu) + g_{V^*}(\nu, \nu) \\
&= \sum_{i,i'=1}^{n^\mu - 1} K(c_i^\nu - c_{i'}^\nu)(\tau_i^\nu \cdot \tau_{i'}^\nu) \\
&\quad - 2\sum_{i=1}^{n^\mu - 1}\sum_{j=1}^{n^\nu - 1} K(c_i^\mu - c_j^\nu)(\tau_i^\mu \cdot \tau_j^\nu) \\
&\quad + \sum_{j,j'=1}^{n^\nu - 1} K(c_j^\nu - c_{j'}^\nu)(\tau_j^\nu \cdot \tau_{j'}^\nu)
\end{aligned}
$$

We are essentially summing

- All the pairs in the first curve, $g_{V^*}(\mu, \mu)$,
- all the pairs between the two curves, $2g_{V^*}(\mu, \nu)$,
- and all the pairs in the second curve, $g_{V^*}(\nu, \nu)$

The Corpus Callosum





- One of the most visible and well-studied parts for the brain is the Corpus Callosum, because it contains such a large amount of white matter
- In the past, a common treatment for epilepsy was to sever the Corpus Callosum, as it prevents positive feedback loops between the two sides.
- Its shape changes depending on different diseases, phenotypes, etc.

Review - Curve Fitting and Interpolation

Consider a curve fitting problem: You have a lot of data points. The goal is to find a "nice" $f(x)$ that passes through my data.

What does "nice" mean? It is a curve with no cusps or discontinuities.

To do this, we first find the minimizer of f.

$$\text{argmin}_{f \in V} \langle f, f \rangle_v \text{ such that } f(x_i) = y_i \qquad \forall i \in \{1, \dots, N\}$$

This is a costumed plurization, use Lagrange multipliers p_i.

$$\begin{aligned} L &= \sum_{i=1}^{N} p_i \cdot (y_i - f_i) + \langle f, f \rangle_v \\ &= \sum_{i=1}^{N} p_i f(x_i) + \frac{1}{2} \langle f, f \rangle_v + \sum_{i=1}^{N} p_i y_i \end{aligned}$$

for a fixed p, find the best f. (Notice the last term in the above equation is not dependent on f.)

$$\begin{aligned} &= \sum_{i=1}^{N} P_i \delta_{x_i}(f) + \langle f, f \rangle_v + \sum_{i=1}^{N} p_i y_i \\ &= \langle \sum_{i=1}^{N} p_i \cdot K(\cdot - x_i), f \rangle_v + \langle f, f \rangle_v + \cdots \\ &= \langle g, f \rangle_v + \langle f, f \rangle_v + \cdots \end{aligned}$$

We can solve this by completing the square. The result would look something like

$$\langle f - g, f - g \rangle_v + \cdots$$

The constants of this equation are independent of f. This equation is minimized when $f = g$.

Therefore, the optimal f is $f(x) = \sum_{i=1}^{N} p_i K(x - x_i)$.

- We now need to solve for p.
- In these types of problems, p is a lagrange multiplier, so we would have to refer to the constraints.

$y_j = f(x_j) = \sum_{i=1}^{N} p_i K(x_j - x_i)$

- p_i is a N by 1 array
- $k(x_j - x_i)$ is an N by N array.
- Therefore, the right side can be thought of as matrix multiplication.

solve for p by solving linear equations!

Smooth Manifolds

Motivation: Many useful data types in medical imaging are not elements of a vector space. (Not closed under $+$ and $\cdot$.)

- Rotation matrices

- Diffusion tensors
- Probabilities

We can still analyze them quantitatively by modeling them as elements of a manifold.

We will discuss two main types of data

1. Pixels that are manifold valued objects
2. Manifold valued objects that act on imaging data

Intuition for Smooth Manifolds

A manifold is a set (possibly curved), such that if you zoom in close it looks like a (flat) vector space (i.e., $\mathbb{R}^d$ for some d).

- A classic example is a sphere like the earth. When we walk around in a small area it looks flat.

Example - Not Manifolds

1. A line segment:
   - Imagine you are standing at the end of a line segment. You see a cliff! The segment ends, but $\mathbb{R}$ doesn't.
2. An "x":
   - Consider the intersection between two lines. If you zoom in on the intersection, it always looks the same! It does not smooth out.

Definition of a Smooth Manifold

A smooth manifold is a triple

1. A set $\mathcal{M}$
2. A topology $\mathcal{O}$
3. A collection of smoothly compatible charts called an atlas $\mathcal{A}$, where every point is in at least one chart.

Topologies

- We will not cover topologies in detail in lecture. Please see the notes if you are interested.

Working definition of topologies: We can think of a topology as a collection of open sets (including $\mathcal{M}$ and $\emptyset$), that allow us to define continuous functions:

- A function f is continuous if the inverse image of any open set is also open set.

Charts

Charts will make precise what "looks like $\mathbb{R}^d$ means. Definition: A chart is a pair (U, x) in $\mathcal{A}$, where U is an open subset of $\mathcal{M}$ and $x : \mathcal{M} \to \mathbb{R}^d$ is a continuous and invertible map, with continuous inverse (a homeomorphism), called the coordinate map, for some $d \in \mathbb{N}$. Definitiion: d is called the dimension of the manifold.