CS 174C Week 2

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Vectors

- Vectors are n-tuples of scalar elements.
  - $\vec{v} = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{R}$
  - Magnitude: $|v| = \sqrt{x_1^2 + \cdots + x_n^2}$
  - Unit Vectors: $v \,:\, |v| = 1$
  - Normalizing a vector: $\hat{v} = \frac{v}{|v|}$
- **Addition**
$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
- **Multiplication with scalar (scaling)**
$$ax = (ax_1, \ldots, ax_n), a \in \mathbb{R}$$
- **Properties**
$$u + v = v + u$$
$$(u + v) + w = u + (v + w)$$
$$a(u + v) = au + av, a \in \mathbb{R}$$
$$u - u = 0$$

Linear Combination of Vectors

A linear combination of the m vectors $v_1, \ldots, v_m$ is a vector of the form:

$$w = a_1 v_1 + \cdots + a_m v_m, a_1, \ldots a_m \in \mathbb{R}$$

Special Cases:

- Linear Combination
$$w = a_1 v_1 + \cdots + a_m v_m, a_1, \ldots a_m \in \mathbb{R}$$
- Affine Combination

  - A linear combination for which $a_1 + \cdots + a_m = 1$

- Convex Combination
  - An affine combination for which $a_i \geq 0 \forall i = 1, \ldots, m$

Linear Independence

For vectors $v_1, \ldots, v_m$, if $a_1 v_1 + \cdots + a_m v_m = 0$ if and only if $a_1 = a_2 = \cdots = a_m = 0$, then the vectors are linearly independent.

Generators and Base Vectors

How many vectors are needed to generate a vector space?

- Any set of vectors that generate a vector space is called a generator set
- Given a vector space $\mathbb{R}^n$ we can prove that we need a minimum of n vectors to generate all vectors v in $\mathbb{R}^n$
- A generator set of minimum size is called a basis for the given vector space

Standard Unit Vectors

$$v = (x_1, \ldots x_n), x_i \in \Re$$

$$\begin{aligned}(x_1, x_2, \ldots, x_n) &= x_1(1, 0, 0, \ldots, 0, 0) \\ &+ x_2(0, 1, 0, \ldots, 0, 0) \\ &\ldots \\ &+ x_n(0, 0, 0, \ldots, 0, 1)\end{aligned}$$

For any vector space $\Re^n$:

$$\begin{aligned}i_1 &= (1, 0, 0, \ldots, 0, 0) \\ i_2 &= (0, 1, 0, \ldots, 0, 0) \\ \ldots \\ i_n &= (0, 0, 0, \ldots, 0, 1)\end{aligned}$$

Standard Unit Vectors

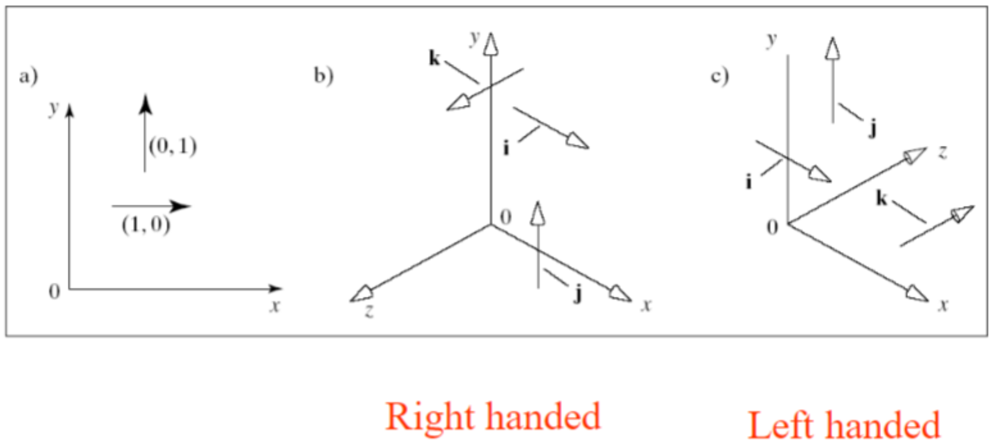
In 2D, the standard vectors are:

- $i = (1, 0)$
- $j = (0, 1)$

In 3D, the standard vectors are:

- $i = (1, 0, 0)$

- $j = (0, 1, 0)$
- $k = (0, 0, 1)$



Representation of Vectors Through Basis Vectors

Given a vector space R^n, a set B of basis vectors $\{b_i \in R^n, i = 1, \ldots, n\}$, and a vector v in R^n we can always find scalar coefficients such that:

$$v = a_1 b_1 + \cdots a_n b_n$$

So, vector v expressed with respect to B is:

$$v_B = (a_1, \ldots, a_n)$$

That is, the elements of a vector v in R^n are the scalar coefficients of the linear combination of the base vectors that equals v

Dot Product

Definition:

$$\vec{w}, \vec{v} \in \Re^n$$

$$\vec{w} \cdot \vec{v} = \sum_{i=1}^{n} w_i v_i = w_0 \cdot v_0 + w_1 \cdot v_1 + \ldots w_n \cdot v_n$$

Properties:

1. Symmetry: $a \cdot b = b \cdot a$
2. Linearity: $(a + b) \cdot c = a \cdot c + b \cdot c$
3. Homogeneity: $(sa) \cdot b = s(a \cdot b)$
4. $|b|^2 = b \cdot b$
5. $a \cdot b = |a| \cdot |b| \cos(\theta)$

- Two vectors are **perpendicular** if their dot product equals 0.
  - Acute if their dot products are greater than 0
  - Obtuse if their dot products are less than 0
- A vector $\vec{v}$ dot product'ed with itself would produce the same vector, but with a magnitude of $\|v\|^2$

Orthogonal Projection:

$$u_v = \frac{(u \cdot v) \cdot v}{(v \cdot v)}$$

Perpendicular Vectors

Vectors a and b are perpendicular if and only if $a \cdot b = 0$.

- Also called normal or orthogonal vectors
- The standard unit vectors form an orthogonal basis:
  - $i \cdot j = 0$
  - $j \cdot k = 0$
  - $i \cdot k = 0$

Cross Product

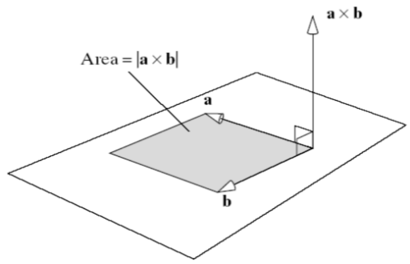
Defined only for 3D vectors and with respect to the standard unit vectors

$$a \times b = (a_y b_z - a_z b_y)i + (a_z b_x - a_x b_z)j + (a_x b_y - a_y b_x)k$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Properties of the Cross Product

1. $i \times j = k$, $i \times k = -j$, $j \times k = i$
2. Antisymmetry: $a \times b = -b \times a$
3. Linearity: $a \times (b + c) = a \times b + a \times c$
4. Homogeneity: $(sa) \times b = s(a \times b)$
5. The cross product is normal to both vectors: $(a \times b) \cdot a = (a \times b) \cdot b = 0$
6. $|a \times b| = |a||b| \sin(\theta)$



Recap of Vectors

- Vector Spaces
  - Operations with vectors
- Representing vectors through a basis

$$v = a_1 b_1 + \cdots + a_n b_n; v_b = (a_1, \ldots, a_n)$$

- Standard unit vectors
- Dot product
  - Perpendicularity
- Cross product
  - Normal to both vectors of the product

Matrices

Definition: Rectangular arrangement of scalar elements

$$A_{3\times 3} = \begin{pmatrix} -1 & 2.0 & 0.5 \\ 0.2 & -4.0 & 2.1 \\ 3 & 0.4 & 8.2 \end{pmatrix}$$

$$A = (A_{ij})$$

Special Square Matrices

- Zero: $A_{ij} = 0 \forall i, j$
- Identity: $I_n = \begin{cases} I_{ii} = 1 \forall i & I_{ij} = 0 \forall i \neq j \end{cases}$
- Symmetric: $(A_{ij})_{n\times n} = (A_{ji})_{n\times n}$ or $A = A^T$

Operations with Matrices

Addition

$$A_{m\times n} + B_{m\times n} = (a_{ij} + b_{ij})$$

Properties:

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $f(A + B) = fA + fB$
4. Transpose: $A^T = (a + ij)^T = (a_{ji})$

Multiplication

$$C_{m\times r} = A_{m\times n} B_{n\times r}$$

$$(C_{ij} = (\sum_{k=1}^{n} a_{ik} b_{kj}))$$

Properties:

1. $AB \neq BA$
2. $A(BC) = (AB)C$
3. $f(AB) = (fA)B$
4. $A(B + C) = AB + AC$, $(B + C)A = BA + CA$
5. $(AB)^T = B^T A^T$

Inverse of a Square Matrix

$$MM^{-1} = M^{-1}M = I$$

Important property:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Dot Product as a Matrix Multiplication

A vector is a column matrix

$$\begin{aligned} a \cdot b &= a^T b \\ &= (a_1, a_2, a_3) \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

Vectors vs Points

- Vectors have size and direction but no location
- Points have location but no size or direction
- Problem: We represent both as triplets!

Relationship Between Points and Vectors

- A difference between two points is a vector
- A point plus an offset vector is a point

This leads to the convention of representing points and vectors as column matrices:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{pmatrix} \qquad P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix}$$

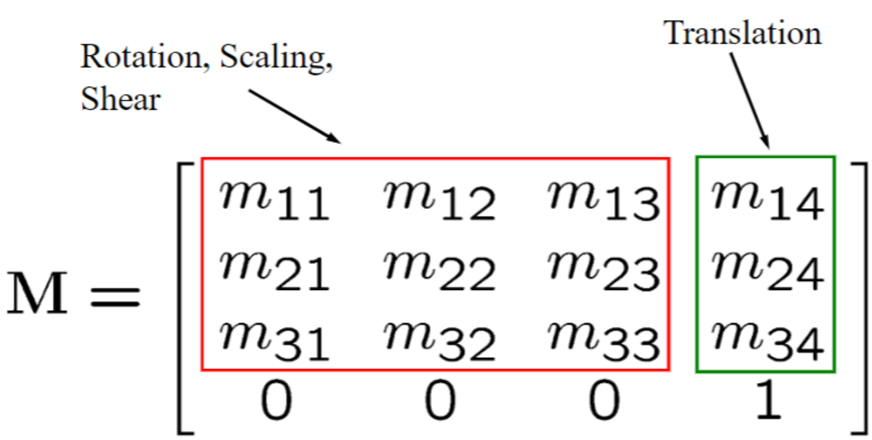
Coordinate Systems

Defined by: a, b, c, O

$$\begin{aligned} v &= v_1 a + v_2 b + v_3 c \\ P - O &= p_1 a + p_2 b + p_3 c \\ P &= O + p_1 a + p_2 b + p_3 c \end{aligned}$$

Affine Transformations in 3D

General Form:



Translations

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Scale Around the Origin

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Shear Around the Origin

Along x-axis:

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Rotation Around the Origin

There are three axes to rotate around.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rigid Body Transformations

- Includes **translations and rotations**
- Preserves angles and distances

Inversion of Transformations

- Translation: $T^{-1}(t_x, t_y, t_z) = T(-t_x, -t_y, -t_z)$
- Rotation: $R_{axis}^{-1}(\theta) = R_{axis}(-\theta)$
- Scaling: $S^{-1}(s_x, s_y, s_z) = S(\frac{1}{s_x}, \frac{1}{s_y}, \frac{1}{s_z})$
- Shearing: $Sh^{-1}(a) = Sh(-a)$

Inverse of Rotations

Pure rotation only, no scaling or shear

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Then,

$$M^{-1} = M^T$$

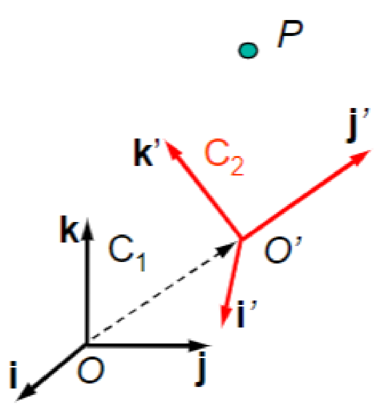
Since the rotation matrix M is an orthonormal matrix

Transformations as a Change of Basis

$$P_{C_1} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \mathrm{M}P_{C_2}$$

Note:

- $O = O_{C_1} = [0, 0, 0]^T$
- $\mathbf{M}O = O' = [O'_x, O'_y, O'_z]^T$
- $i = i_{C1} = [1, 0, 0]^T$
- $\mathbf{M}i = i' = [i'_x, i'_y, i'_z]^T$
- $j = j_{C1} = [0, 1, 0]^T$
- $\mathbf{M}j = j' = [j'_x, j'_y, j'_z]^T$
- $k = k_{C1} = [0, 0, 1]^T$
- $\mathbf{M}k = k' = [k'_x, k'_y, k'_z]^T$



Composition of 3D Affine Transformations

The composition of affine transformations is an affine transformation. Any 3D affine transformation can be performed as a series of elementary affine transformations

Rotation Representation Revisited

There are several possible representations

- Rotation matrix
- Fixed angle
- Euler angle
- Axis-angle
- Quaternion
- Exponential map

Composition and interpolation are desirable properties.

Rotation Matrix Representation

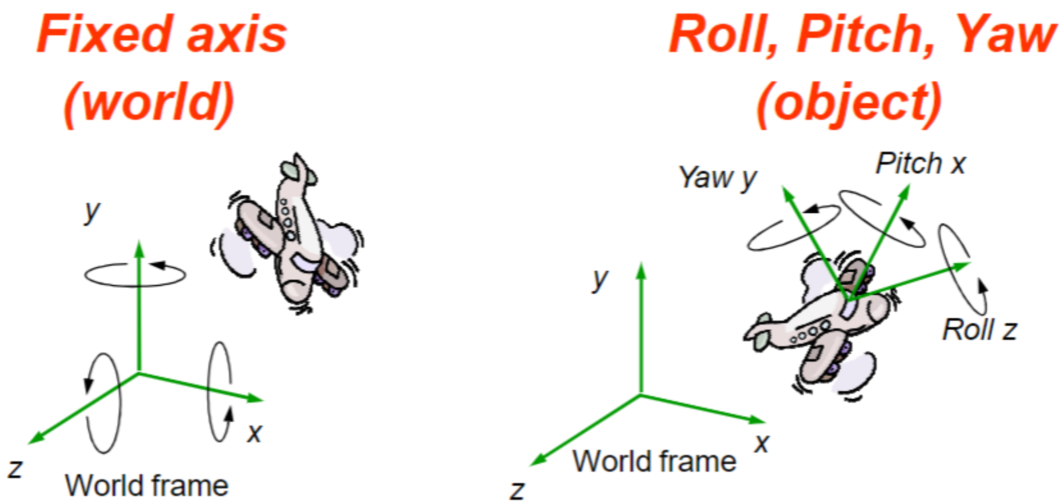
- Extracting pure rotational component
- 3x3 matrix - 9 elements
- 3 orthogonality constraints
- 3 normalization constraints

$$R = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$\begin{aligned} a \cdot b = 0, |a| = 1, \\ b \cdot c = 0, |b| = 1, \\ c \cdot a = 0, |c| = 1 \end{aligned}$$

$$R^{-1} = R^T, \det(R) = +1$$

Fixed Angle vs Euler Angle Representations



- Many possible choices: x-y-z, y-x-z, z-x-y, etc.
- $R = R_z(\theta_3)R_y(\theta_2)R_x(\theta_1)$

Any Euler angle choice is equivalent to a reverse fixed angle formulation.

- Example:
  - Euler angles: z-x-y = Fixed angles: y-x-z

Serious Problems with Euler Angles

- Gimbal Lock (loss of a rotational degree of freedom when interpolating using Euler angles)
  - Can create weird paths (swinging out of plane)
  - We would like minimum length path

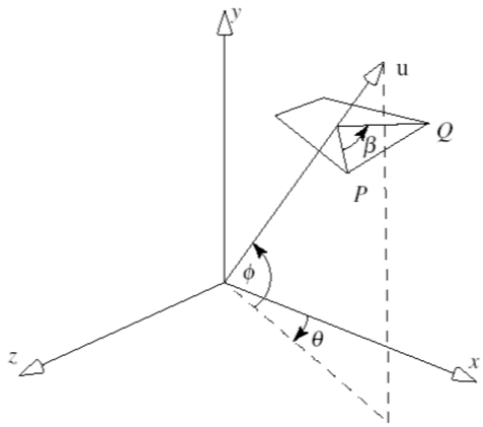
Axis-Angle Representation

Vector(axis): u
Rotation angle: β
Method:

1. Two rotations to align **u** with the x-axis: $R_z(-\phi)R_y(\theta)$
2. Do x-roll by β: $\mathbf{R}_x(\beta)$
3. Undo the alignment: $R_y(-\theta)R_z(\phi)$

All together:

$$R_u(\beta) = R_y(-\theta)R_z(\phi)R_x(\beta)R_z(-\phi)R_y(\theta)$$



Complex Numbers and Rotation

Complex numbers can represent 2D rotations

$$z = a + ib = |z|(\cos(\theta) + i\sin(\theta)) = |z|e^{i\theta}$$

Multiplication is equivalent to rotation around the origin

$$zw = |z||w|e^{i(\theta+\phi)}$$

Quaternions

Extension of complex numbers using three imaginary quantities i, j, k.

$$q = a + bi + cj + dk, a, b, c, d \in \Re$$

Where:

- $i^2 = j^2 = k^2 = -1$
- $ij = -ji = k$
- $jk = -kj = i$
- $ki = -ik = j$

Properties and Definitions

- $q = [s, x, y, z] = [s, v]$
- $[s_1, v_1] + [s_2, v_2] = [s_1 + s_2, v_1 + v_2]$
- $[s_1, v_1][s_2, v_2] = [s_1 s_2 - v_1 v_2, s_1 v_2 + s_2 v_1 + v_1 \times v_2]$
- $(q_1 q_2) q_3 = q_1 (q_2 q_3)$

- $q_1 q_2 \neq q_2 q_1$
- $|q| = \sqrt{s^2 + x^2 + y^2 + z^2}$

Other Properties and Definitions

- Identity: $q[1, 0, 0, 0] = q$
- Inverse: $q^{-1} = (\frac{1}{|q|})^2 (s, -v)$ and $q^{-1} q = q q^{-1} = (1, 0, 0, 0)$
- Conjugate: $\bar{q} = (s, -v)$
- $(pq)^{-1} = q^{-1} p^{-1}$

Unit Quaternions

- Unit quaternions have unit norms
- Isomorphic to orientations
- General form:

$$q = (\cos(\theta), \sin(\theta) v), v \in \mathbb{R}^3, |v| = 1$$

  - Equivalent to rotation by angle 2θ around the axis defined by **v**
  - q and -q are equivalent when interpreted as orientation

Rotations with Quaternions

Definition:

- Quaternion $q = (s, x, y, z) = (s, v)$
- Point(vector) $u = (x, y, z) \rightarrow \hat{u} = (0, x, y, z)$
- $u' = \mathrm{Rot}(u) = q \hat{u} q^{-1}$
- For unit quaternions the inverse is equivalent to the conjugate

Successive Rotations

Rotate first by p, and then by q.

$$\begin{aligned}
\mathrm{Rot}_q(\mathrm{Rot}_p(\hat{u})) &= q(p \hat{u} p^{-1}) q^{-1} \\
&= (qp) \hat{u} (p^{-1} q^{-1}) \\
&= (qp) \hat{u} (qp)^{-1} \\
&= \mathrm{Rot}_{qp}(\hat{u})
\end{aligned}$$

What Rotation Does $-q$ Represent?

That is, what angle and what axis?

$$\text{Rot}(\theta, v) \rightarrow q = [\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})v]$$

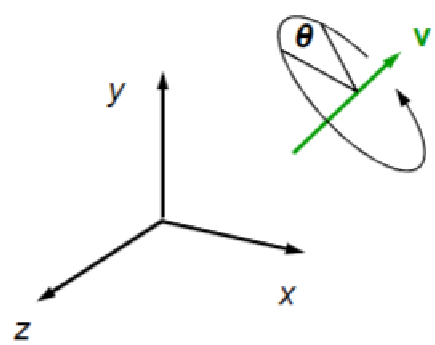
Now, for $-\theta$ around $-v$, or $2\pi - \theta$ around $-v$

$$\begin{aligned} q' &= [\cos(\frac{2\pi - \theta}{2}), \sin(\frac{2\pi - \theta}{2})(-v)] \\ &= [\cos(\pi - \frac{\theta}{2}), -\sin(\pi - \frac{\theta}{2})v] \\ &= [-\cos(\frac{\theta}{2}), -\sin(\frac{\theta}{2})v] \\ &= -q \end{aligned}$$

Thus, $\text{Rot}_{-q} = \text{Rot}_q$

Quaternions vs Axis-Angle Representation

Rotate by θ around v



Equivalent quaternion

$$\text{Rot}(\theta, v) \rightarrow q = [\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})v]$$

Exponential Map Representation

Three parameters: (v_1, v_2, v_3)

- Vector direction: axis of rotation
- Vector magnitude: amount of rotation

$$\begin{aligned} v = [0,0,0] &\rightarrow e^{[0,0,0]^T} = [1,0,0,0] \\ v \neq 0 &\rightarrow e^v = \sum_{m=0}^{\infty} \left(\frac{1}{2}\hat{v}\right)^m = (\cos(\frac{1}{2}\theta), \sin(\frac{1}{2}\theta)\bar{v}) \end{aligned}$$

where $|v| = \theta$ and $\bar{v} = \frac{v}{|v|}$

- Singularities for $2n\pi$

- Numerically unstable when $|v|$ is close to zero

Which Representation Should We Use?

More than one, and there is no panacea!

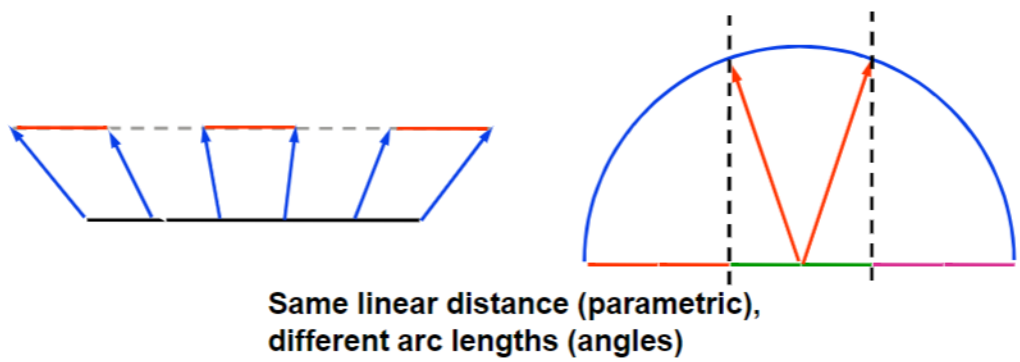
- Interface based on Euler angles
- Internal representation using quaternions
- Drawing using matrices

Depends on the application.

Interpolating Quaternions

Linear interpolation: non-linear change in orientation

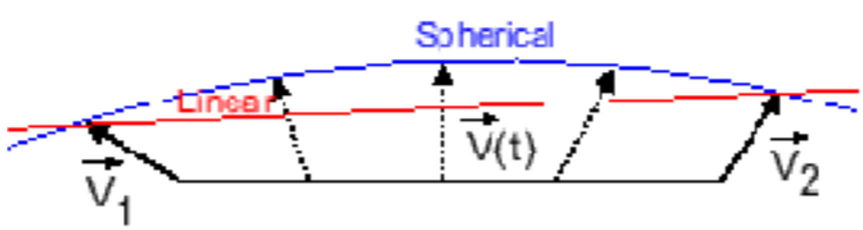
$$q = lerp(q_1, q_2, t) mt \in [0, 1]$$



Spherical linear interpolation: Interpolate along a sphere (angles instead of trigonometric values)

$$\theta = q_1 \cdot q_2$$

$$slerp(q_1, q_2, t) = \frac{\sin((1-t)\theta)}{\sin(\theta)} q_1 + \frac{\sin(t\theta)}{\sin(\theta)} q_2$$



Issues With Slerp

- Not necessarily unit result, needs renormalization

- First order discontinuity at keyframes
  - Need polynomial interpolation for smooth results
  - Polynomials on a sphere