Math 170E Week 3

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Independence

Definition: If two events, A and B are independent, then $A \perp B$, and $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. If either A or B have a probability of 0 or 1, then the two events are automatically independent.

If $A \perp B$, then $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. Dividing both sides by $\mathbb{P}(B)$, assuming that $\mathbb{P}(B) \neq 0$, this yields $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A) \implies \mathbb{P}(A|B)$

$$\frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(B^c)} = \frac{\mathbb{P}(A) - \mathbb{P}(A \cap B)}{1 - \mathbb{P}(B)} = \frac{\mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}(B)}{1 - \mathbb{P}(B)} \implies \mathbb{P}(A)$$

Therefore, this also implies if $A \perp B$, then $\mathbb{P}(A|B^c)$. This shows that if two events are independent, then the other event cannot be predicted.

Example:
Suppose $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B) = 0.4$, and $\mathbb{P}(A) \perp \mathbb{P}(B)$. Find $\mathbb{P}(A \cap B^c)$.

$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$

Independence of 3 events

Let A, B, and C be independent events. Even if $A \perp B$, $B \perp C$, and $C \perp A$, it does not imply $A \perp B \perp C$. However, $A \perp B \perp C$ does imply $A \perp B$, $B \perp C$, and $C \perp A$.
If A, B, and C are independent, then $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$. This implies that $\mathbb{P}(A|B \cap C) = \mathbb{P}(A)$

Independence of k events

If A_1, A_2, ..., A_k are independent, then this implies that $\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_k) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot ... \cdot \mathbb{P}(A_k)$. The implication also goes the opposite direction.

Example: Suppose you toss two coins. There are four possible combinations in the way they land: TT, TH, HT, HH. Each are independent because the chance of the first coin landing on heads does not change the chance of the second coin landing on heads. The chance of landing each side for a given coin is 0.5. Thus, each combination has a 0.25 chance of occurring.

Example: Suppose A, B, C, and D are independent. What is the meaning of the following?

$$(A \cap B^c) \perp (C^c \cup D)^c$$

. To answer this, let $F = (A \cap B^c)$ and $G = (C^c \cup D)^c$. F and G are events composed of A, B, C, and D. $F \perp G$. If an event is composed of independent events, then the new event is also independent to the other events it is not composed of.

Example:
Let $A = \{$"Student supports Tom"$\}$. We want to know $\mathbb{P}(A)$.
Let $B = \{$Roll a die, get 1, 2, 3, or 4$\}$.
Let $C = \{$Student writes down "Yes"$\}$
The rule is that the student has an opinion on whether they support Tom or not. On the paper, they write whether they support (Yes or no) if they roll 1, 2, 3, or 4 on the dice, and the opposite (no or yes) if they roll 5 or 6.
$\mathbb{P}(B) = \frac{2}{3}$ since there is a $\frac{2}{3}$ chance of rolling 1, 2, 3, or 4 on the dice.
The probability of C is dependent on events A and B, such that

$$\begin{aligned}
C &= (A \cap B) \cup (A^c \cap B^c) \\
\mathbb{P}(C) &= \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B^c) \\
&= \mathbb{P}(A) \cdot \frac{2}{3} + (1 - \mathbb{P}(A)) \cdot \frac{1}{3} \\
&= \frac{1}{3} + \frac{1}{3} \cdot \mathbb{P}(A)
\end{aligned}$$

Example:
Suppose you have three bags. The first has 50 green balls and 50 purple balls, the second has 1 green ball and 99 purple balls, and the third bag has 0 green balls and 100 purple balls. Suppose you randomly pick one bag and one ball inside the bag. Given the ball picked is green, what is the probability that the first bag was picked?
First, let $A = \{$Bag #1 is picked$\}$ and let $B = \{$Green ball is picked$\}$. Thus, the problem can be rephrased as find $\mathbb{P}(A|B)$.
To solve this, $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(B|A) \cdot \mathbb{P}(A) \\
&= \frac{1}{2} \cdot \frac{1}{3} \\
&= \frac{1}{6}
\end{aligned}$$

To find $\mathbb{P}(B)$, use the equation

$$\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(C \cap B)$$

This simplifies to:

$$\begin{aligned}
&= \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|C) \cdot \mathbb{P}(C) \\
&= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{10} \cdot \frac{1}{3} \\
&= \frac{1}{5}
\end{aligned}$$

Therefore, $\mathbb{P}(A|B) = \frac{\frac{1}{6}}{\frac{1}{5}} = \boxed{\frac{5}{6}}$.

Relation between P(A—B) and P(B—A)

The two conditional probabilities can be written as the following:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

Dividing them would yield:

$$\frac{\mathbb{P}(A|B)}{\mathbb{P}(B|A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}$$

Thus,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \cdot \mathbb{P}(B|A)$$

Choosing the correct conditional probability

Suppose you are hiring employees, and the only difference between them is whether or not they are good at math. Let $\mathbb{P}(T)$ represent the probability the employee will eventually become a top employee, and let $\mathbb{P}(M)$ represent the probability the employee is good at math. Out of the top 100 employees, 30 are good at math, 70 are not. Out of all the other employees (not a top employee), 100 are good at math, 300 are not. The question is: is it worth being good at math if you want to become a top employee?

In this case, we want to compare $\mathbb{P}(T|M)$ with $\mathbb{P}(T|M^c)$, rather than $\mathbb{P}(M|T)$ and $\mathbb{P}(M^c|T)$. We want to compare whether the probability of becoming a top employee is higher given the employee is good at math, or the employee is not good at math. In other words, the proportion of people good at math and people not good at math that become top employees.

$$\mathbb{P}(T|M) = \frac{\mathbb{P}(T \cap M)}{\mathbb{P}(M)} = \frac{\frac{30}{500}}{\frac{130}{500}} = \frac{3}{13}$$

$$\mathbb{P}(T|M^c) = \frac{\mathbb{P}(T \cap M^c)}{\mathbb{P}(M^c)} = \frac{70}{370} = \frac{7}{37}$$

Since $\mathbb{P}(T|M^c) < \mathbb{P}(T|M)$, it is better to be good at math if you want to become a top employee.

Partitions:

Let B_1, B_2, B_3, ... B_n be partitions of S. By the definition of partition, the following two statements hold true:

$$B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n = S$$

$$B_1 \cap B_2 \cap B_3 \cap \cdots \cap B_n = \emptyset$$

Example:
A and A^c always form a partition.

Lemma: If B_1, ..., B_n is a partition of A, then

$$\mathbb{P}(A) = \sum_{k=1}^{n} \mathbb{P}(A|B_k) \cdot \mathbb{P}(B_k)$$

This can be thought of as an extension of the formula:

$$\mathbb{P}(A) = \mathbb{P}(A|B) \cdot \mathbb{P}(B) + \mathbb{P}(A|B^c) \cdot \mathbb{P}(B^c)$$

Since all the pieces B_k of A are disjointed, the following also holds true:

$$A = \bigcup_{k=1}^{n} (A \cap B_k)$$

$$\begin{aligned}\mathbb{P}(A) &= \sum_{k} \mathbb{P}(A \cap B_k) \\ &= \sum_{k} \mathbb{P}(A|B_k) \cdot \mathbb{P}(B_k)\end{aligned}$$

Example:
Suppose you are playing a game where your goal is to toss a ball into a target. First, you choose a ball from random; there are 10 red balls, 20 blue balls, and 70 purple balls. Then, based on color of the ball picked, you have to try tossing the ball from varying distances. The red ball is tossed from the red line, which is the closest to the target. You have a 60% chance of making it. The blue ball is tossed from the blue line, which you have a 40% chance of making it, and the purple ball is tossed from the purple line, which you have a 20% chance of making it.

Define the events as the following: Let A represent hitting the target. Let $B_1 = \{R\}$, $B_2 = \{B\}$, $B_3 = \{P\}$. Thus, $\mathbb{P}(B_1) = 0.1$, $\mathbb{P}(B_2) = 0.2$, $\mathbb{P}(B_3) = 0.7$, $\mathbb{P}(A|B_1) = 0.6$, $\mathbb{P}(A|B_2) = 0.4$, $\mathbb{P}(A|B_3) = 0.2$.

Your friend decides to play the game. Given that they hit the target, what is the chance they picked up the red ball?

Given they hit the target, picking the red ball is described by the probability $\mathbb{P}(B_1|A)$

To do this, use the formula:

$$\mathbb{P}(B_1|A) = \frac{\mathbb{P}(A|B_1) \cdot \mathbb{P}(B_1)}{\mathbb{P}(A)}$$

All the numbers here are already calculated.

$$\frac{0.6 \cdot 0.1}{0.28} = \boxed{\frac{3}{14}}$$

Note that this probability is greater than the original chance of picking a red ball.

As a side note, we can modify the formula to generalize the form:

$$\frac{\mathbb{P}(A|B_1) \cdot \mathbb{P}(B_1)}{\sum_k \mathbb{P}(A|B_k) \cdot \mathbb{P}(B_k)}$$

Example:

Suppose you have two bags, one has 50 red and 50 blue balls, and the other has 70 red and 30 blue balls. You are not allowed to look at more than one of the contents of each bag at a time. You have to decide which bag has more red balls.

Name the 50/50 bag the "left bag" and the 70/30 bag the "right bag". (suppose the 50/50 bag is left of the 70/30 bag).

$$\mathbb{P}(A|B) = \mathbb{P}(\text{Left bag is good bag}|\text{Pick one ball from left bag it's red}$$

Suppose in the first ball pick from a bag, you picked the red ball.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A^c) \cdot \mathbb{P}(A^c)} = \frac{\frac{1}{2} \cdot \mathbb{P}(B|A)}{\frac{1}{2}(\mathbb{P}(B|A) + \mathbb{P}(B|A^c))} = \frac{\frac{1}{2} \cdot 0.7}{\frac{1}{2}(0.7 + 0.5)} = \frac{7}{12}$$

From the new information, the left bag now has a greater probability of being the good bag. Next, you pick a ball from the right bag, and it is red. Doing the calculation, the probability of each bag being the good bag becomes 0.5 again. (You know this because the two bags are now balanced, but this can be shown with math too.)

Using this method, you can adjust the probability the left or right bag is the good bag using the new information.