EC ENGR 102 Week 1

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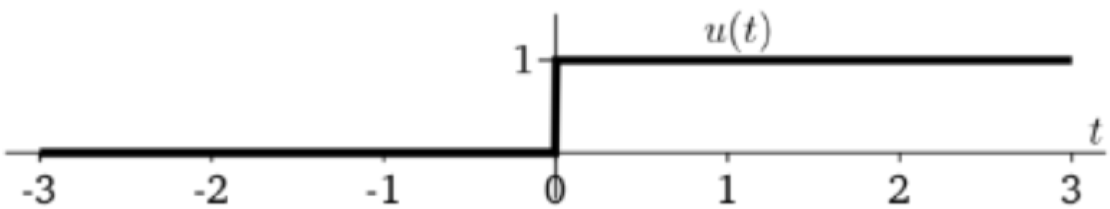
The Unit Step Function

The unit step function denoted by $u(t)$ in this class, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

It is also called the Heavyside step function. Drawn below:



Example:
Suppose I wanted to write

$$x(t) = \begin{cases} e^{-t} & t \geq 1 \\ 0 & t < 1 \end{cases}$$

We can do that in terms of the unit step function as

$$x(t) = e^{-t}u(t-1)$$

The Unit Rectangle

There are two definitions:

1. $$rect(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

   - This is a rectangle with height 1, from t = -0.5 to t = 0.5.
   - Notice the area under the curve is 1.

2. $$rect_\Delta(t) = \begin{cases} \frac{1}{\Delta} & |t| < \frac{\Delta}{2} \\ 0 & \text{else} \end{cases}$$

   - This is a general case of the function, where the Δ represents some number.
   - $rect_\Delta(t)$ where $\Delta = 2$ would make a rectangle going from $t = -1$ to $t = 1$, with height 0.5
   - Notice that any value of Δ will still have the area of the rectangle be 1

- Most of the time, we use the first definition of rectangle; we use this one for intuition.

Example:
How do we write the rectangle in terms of $u(t)$? There are also two ways:

$$rect(t) = u(t+0.5) - u(t-0.5)$$
$$rect(t) = u(t+0.5) \cdot u(-t-0.5)$$

We can use the step function and the rectangle function as building blocks for other functions.

Using building blocks

Consider the following example:

How do we write this using the step function as a building block? Each burst have a form of $A\cos(\omega t)$ and last for a width 0.5 around each integer.

Solution:
The burst around 0 can be written as

$$rect(2t) \cdot A\cos(\omega t)$$

The burst around 1 can be written as

$$rect(2(t-1)) \cdot A\cos(\omega t)$$

and et cetera. As a result, we can write the entire function as:

$$y(t) = \sum_{i=-\infty}^{\infty} rect(2(t-i)) \cdot A\cos(\omega t)$$

Unit Ramp

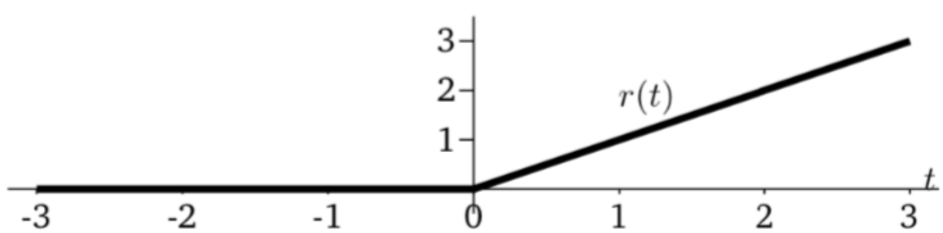
The unit ramp is defined as:

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Note that the unit ramp is the integral of the unit step, i.e.

$$r(t) = \int_{-\infty}^{t} u(r)\mathrm{d}r$$

The unit ramp is illustrated below:



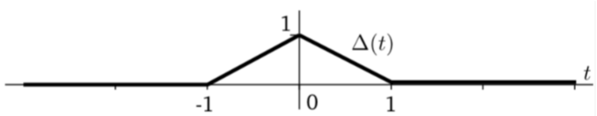
This is known in the AI world as ReLU().

Unit Triangle

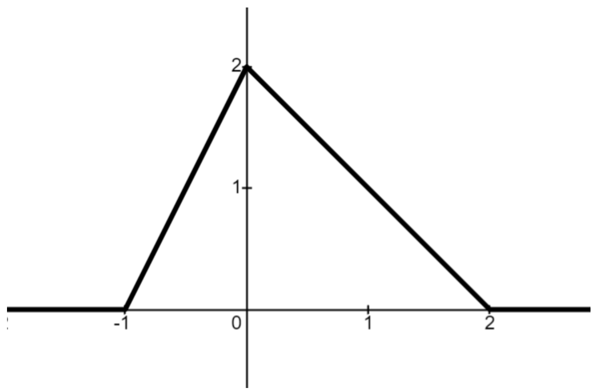
The unit triangle is defined as:

$$\triangle(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{else} \end{cases}$$

The unit triangle is illustrated below:



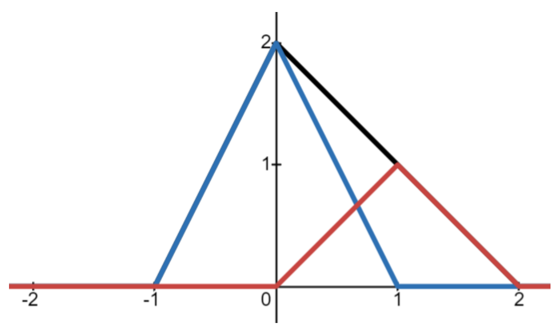
Example:
Lets say we want to make a skewed triangle.



Solution:

$$x(t) = 2\triangle(t) + \triangle(t-1)$$

The intuition is the following:



The $2\triangle(t)$ term represents the blue line and the $\triangle(t-1)$ term represents the red line. We can add the two lines between $0 \geq t \geq 1$ to get the negative 1 slope we need.

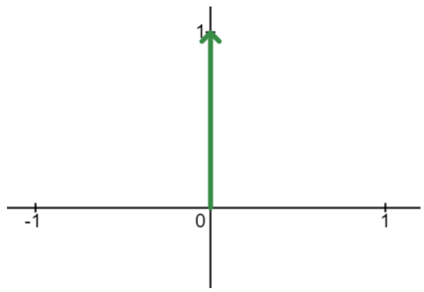
Impulse Function

This is an extremely important signal.

- This is defined as $\delta(t)$, or "impulse", "delta", or "Dirac" function. It is **not** a rigorous mathematical function.
- Features of the impulse function:
  1. It is very large (i.e., approaching infinity), at $t = 0$

2. It's zero everywhere else, $t \neq 0$.
3. Area $= 1$

- Shown on the graph as an arrow pointing up at $t = 0$



- The intuition of the impulse function is a $rect_\Delta(t)$ where Δ approaches 0. (e.g., the width of the rectangle goes to 0 and the height goes to infinity.)
- $\delta(t) \cdot x(t) = x(0) \cdot \delta(t)$, where $x(t)$ is any function.
  - The impulse is still affected by other functions because by intuition it is a very, very thin rectangle. The height is an infinitely large number, but not infinity.
  - The area of the impulse is also scaled by $x(0)$.
  - The shape of the impulse does not change with scaling, but its area does.
  - This is called the **impulse dampening property**.
- The impulse can be moved by an amount T:

$$x(t) \cdot \delta(t - T) = x(T) \cdot \delta(t - T)$$

  - This is called the **impulse sampling property**.
- What happens when we take the integral?

$$\begin{aligned} &\int_{-\infty}^{\infty} x(t) \cdot \delta(t)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x(0) \cdot \delta(t)\mathrm{d}t \\ &= x(0) \cdot \int_{-\infty}^{\infty} \delta(t)\mathrm{d}t \\ &= x(0) \end{aligned}$$

- What if it is shifted?

$$\begin{aligned} &\int_{-\infty}^{\infty} x(t) \cdot \delta(t - T)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x(T) \cdot \delta(t - T)\mathrm{d}t \\ &= x(T) \cdot \int_{-\infty}^{\infty} \delta(t - T)\mathrm{d}t \\ &= x(T) \end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} x(t)\delta(t - T)\mathrm{d}t = x(T)}$$

  - This is called the **impulse sifting property**.

Integral of an impulse

$$\int_{\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$
$$\int_{\infty}^{0+} \delta(t)\mathrm{d}t = 1$$
$$\int_{\infty}^{0-} \delta(t)\mathrm{d}t = 0$$

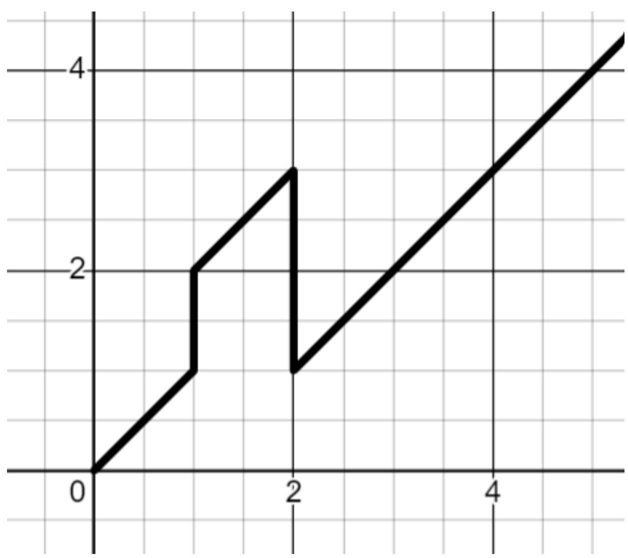
The $0+$ represents approaching zero from the right (e.g., infinitely close to zero on the positive side), and $0-$ represents approaching zero on the left (e.g., infinitely close to zero on the negative side). **Example:** Given the signal $x(t)$:

$$x(t) = 1 + \delta(t-1) - 2\delta(t-2)$$

What is $y(t) = \int_{o}^{t} x(\tau)\mathrm{d}\tau$?

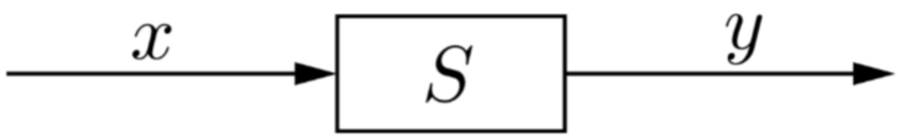
- Based on the equation, the graph is always one, with an impulse at $t = 1$, and a negative, double impulse at $t = 2$.
- If we take the integral (which is continuous), we get a ramp from $0 < t < 1$, then a step of 1 (since area of impulse $= 1$), then ramp from $1 < t < 2$, then two steps down (since the negative double impulse has area -2), then a ramp continuing to infinity.
- The ramp is present because the constant $(+1)$ creates a line with slope 1 on the integral.

Graph of the integral:



Systems

What is a system?
A system transforms an *input signal*, $x(t)$, into an output signal, $y(t)$.



- Systems, like signals, are also *functions.* However, their inputs and outputs are signals.
- Systems can have either single or multiple inputs (SI or MI, respectively), and single or multiple outputs (SO and MO). In this class, we focus on *single input, single output* systems (SISO).

Example Systems

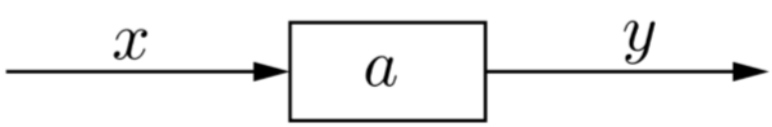
Scaling System

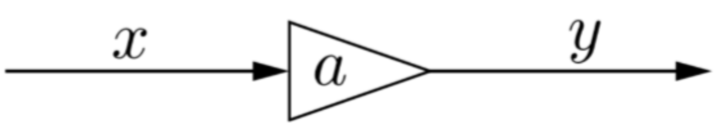
Consider an input $x(t)$ and an output $y(t)$. The scaling system is:

$$y(t) = ax(t)$$

with the following properties:

- If $|a| > 1$, the system is called an *amplifier*.
- If $|a| < 1$, the system is called an *attenuator*.
- If $a < 0$, the system is called *interventing.*
- It is common that a block diagram denotes this with a triangle.



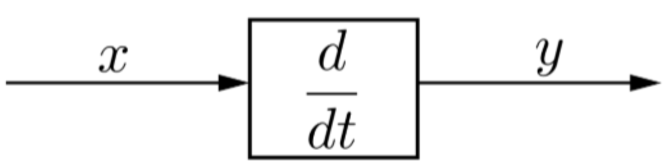


Differentiator

The differentiator is denoted:

$$\begin{aligned} y(t) &= x'(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} x(t) \end{aligned}$$

Block diagram below:

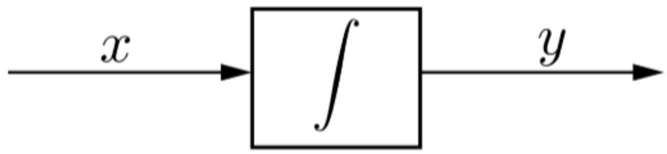


Integrator

The integrator is denoted:

$$y(t) = \int_a^t x(\tau)\mathrm{d}\tau$$

where a is often 0 or $-\infty$.
Block diagram below:



Time Shift System

The time shift system shifts a signal by T, i.e.,

$$y(t) = x(t - T)$$

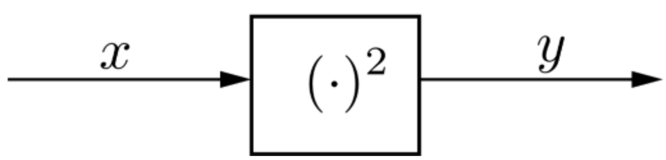
- If $T > 0$, then it is a *delay* system.
- If $T < 0$, then it is a *predictor* system.

Squarer

The squarer system squares a signal, i.e.,

$$y(t) = x^2(t)$$

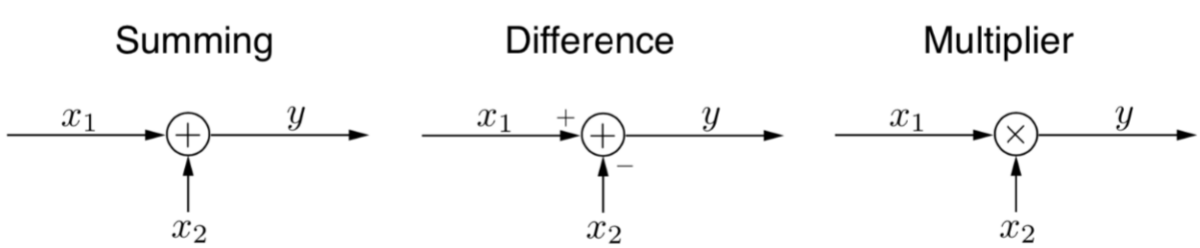
Block diagram below:



Systems with multiple inputs

The AM system (radios) is an example of a system that takes two input signals, $x(t)$ and $\cos(\omega_c t)$, and outputs one signal, $y(t)$. This is a multiple input, single output (MISO) system. Here are a few other examples of multiple input systems.

- Summing system: $y(t) = x_1(t) + x_2(t)$
- Difference system: $y(t) = x_1(t) - x_2(t)$
- Multiplier system: $y(t) = x_1(t)x_2(t)$

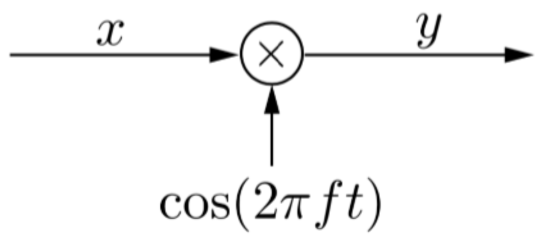


Amplitude modulation (AM radio)

Amplitude modulation takes an input "message", $x(t)$ and outputs a "transmitted signal", $y(t)$. This is denoted via:

$$y(t) = x(t)\cos(2\pi f_c t)$$

Here, f_c is called the carrier frequency. When you turn to AM radio at e.g., 880 kHz, that means the carrier frequency is $f_c = 880kHz$. The AM block diagram is shown below.



System Stability

A system is *bounded-input, bounded-output* (BIBO) stable if every bounded input leads to a bounded output.

- Bounded input: $x(t)$ is a bounded input if there exists a constant M_x, such that $|x(t)| < M_x < \infty$ for all t.
- Boudned output: $y(t)$ is a bounded output if there exists a constant M_y, such that $|y(t)| < M_y < \infty$ for all t.
- Essentially, a system is BIBO stable if both the input and output is always finite and between some threshold value.
  - In real life, a non-BIBO stable system can cause electrical fires if the output voltage is too high.

Example:
AM Radio: $y(t) = x(t) \cdot \cos(\omega_c t)$ Assume that $|x(t)| < M_x < \infty$. Then,

$$\begin{aligned} |y(t)| &= |x(t) \cdot \cos(\omega_c t)| \\ &= |x(t)| \cdot |\cos(\omega_c t)| \\ &\leq |x(t)| \cdot 1 \\ &\leq M_x \end{aligned}$$

Therefore, AM radio is BIBO stable.
Example:
Hyperbola: $y(t) = \frac{1}{x(t)}$ This function is not BIBO stable - if $x(t) = 0$, $x(t)$ is a bounded input, but $y(t)$ is not a bounded output.

What to look for:

- $x(t) < |x(t)|$
- Triangle inequality: $|\sum_i x_i| \leq \sum_i |x_i|$
- $\left|\int_{t_1}^{t_2} x(\tau)\mathrm{d}\tau\right| \leq \int_{t_1}^{t_2} x(\tau)\mathrm{d}\tau$

Causal Systems

- A system is causal if its output only depends on past and present values of the input.
- All real world systems are causal (i.e., we can't use information from the future). An example of a non-causal system is the system $x(-t) = S(x(t))$.

Time Invariance

A system is *time invariant* if a time shift in the input only produces an identical time shift of the output. i.e., S is time invariant if

$$y(t) = S(x(t)) \Rightarrow y(t-\alpha) = S(x(t)-\alpha)$$

Example:
Squarer: $y(t) = [x(t)]^2$

- Let's say we shift the input: $x(t) \to x(t-\alpha)$
- Then, if the function is time invariant, the output becomes $[x(t-\alpha)]^2$.
- To check, we shift output. $y(t) \to y(t-\alpha) = [x(t-\alpha)]^2$.
- They match, therefore, the function is time invariant.

Example:
AM Radio: $y(t) = x(t) \cdot \cos(\omega_c t)$

- We shift the input: $x(t-\alpha)$
- If the function is time invariant, the output becomes $x(t-\alpha) \cdot \cos(\omega_c t)$.
- Shift output: $y(t-\alpha) = x(t-\alpha) \cdot \cos(\omega_c (t-\alpha))$
- Since they don't match, this function is not time invariant.

Linearity

A system is *linear* if the following two properties hold:

1. **Homogeneity:** for any signal, x, and any scalar a,

$$S(ax) = aS(x)$$

2. **Superposition:** for any two signals, x and $\widetilde{x}$,

$$S(x+\widetilde{x}) = S(x) + S(\widetilde{x})$$

3. To check if a system is linear, we can combine:

$$S(ax+b\widetilde{x}) = a \cdot S(x) + b \cdot S(\widetilde{x})$$

Example:
AM radio: $y(t) = x(t) \cdot \cos(\omega_c t) = AM(x(t))$.
Show that: $AM(ax(t) + b\widetilde{x}(t)) = a \cdot AM(x(t)) + b \cdot AM(\widetilde{x}(t))$.
Left hand side:

$$\begin{aligned} \text{LHS} &= (ax(t) + b\widetilde{x}(t)) \cdot \cos(\omega_c t) \\ &= ax(t) \cdot \cos(\omega_c t) + b\widetilde{x}(t) \cdot \cos(\omega_c t) \\ &= a \cdot AM(x(t)) + b \cdot AM(\widetilde{x}(t)) \\ &= \text{RHS} \end{aligned}$$

Therefore, AM radio is linear.

Example:
<u>Squarer</u>: $y(t) = [x(t)]^2$

$$\text{LHS} = (ax(t) + b\widetilde{x}(t))^2 = a^2x^2(t) + 2abx(t)\widetilde{x}(t) + b^2\widetilde{x}^2(t)$$
$$\text{RHS} = a \cdot x^2(t) + b \cdot \widetilde{x}^2(t)$$

Since they are not equal, the squarer system is nonlinear.