Math 170E Week 10

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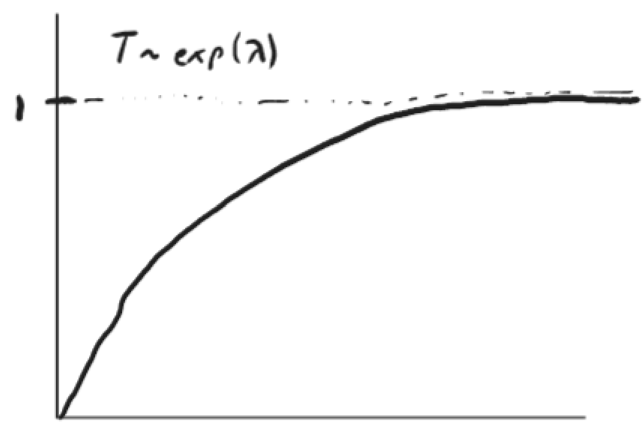
Exponential Random Variables

$$X \sim \exp(\lambda)$$

$$\mathbb{P}(X \leq a) = F_X(a) = 1 - e^{-\lambda a}, \quad \mathbb{P}(X > a) = e^{-\lambda a}$$

$$f_X(a) = \lambda e^{-\lambda a}, \quad a \geq 0$$

This type of random variable follows a curve that looks something like this:



An example of an exponential random variable follows:

Example:

Consider an earthquake. Every few hundred years, the Earth will experience a major earthquake. However, what is the probability it will happen at a given time? If t represents the time since the last major earthquake, T starts small. However, it starts grows relatively rapidly to 1, or $\lim_{T \to 100}$ towards the start, then levels off. Other examples of exponential random variables include time periods before receiving your next email, time period before next relationship/breakup, etc.

These quantities can be approximated by exponential random variables because

$$\mathbb{P}(T > 2) = \mathbb{P}(T > 2 | T > 1) \cdot \mathbb{P}(T > 1)$$

and it turns out that

$$\mathbb{P}(T > 2 | T > 1) \approx \mathbb{P}(T > 1)$$

This makes intuitive sense because the fact they you did not get an email in the first hour should not predict, or affect, whether or not you will get an email in the second hour.
Since this is true, the equation can be simplified to:

$$\mathbb{P}(T > 2) \approx [\mathbb{P}(T > 1)]^2$$

Similarly,

$$\mathbb{P}(T > 4) \approx [\mathbb{P}(T > 2)]^2$$

and so on...

$$\mathbb{P}(T > 5) \approx [\mathbb{P}(T > \frac{5}{2})]^2$$

$$\mathbb{P}(T > 1) \approx [\mathbb{P}(T > \frac{1}{2})]^2$$

It turns out the only type of function that has this property is the exponential function, since $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$, a property that most other mathematical functions do not have.
The general form of the exponential random variable is:

$$g(a) = e^{-\lambda a} = \left(e^{-\lambda}\right)^k = (k)^a$$

Expected Value of Exponential Random Variables

$$\mathbb{E}T = \frac{1}{\lambda}$$

Example:
If T is an exponential variable, and $\mathbb{E}T = 100$, what is the distribution for the variable?
The solution would be $T \sim \exp(\frac{1}{100})$, since if the expected value is $\mathbb{E}T = 100$, then $\lambda = \frac{1}{100}$. The distribution would follow the equation

$$f_T(a) = \frac{1}{100}e^{-\frac{a}{100}}, \qquad a > 0$$

If instead of estimating the number of times something happens in a given amount of time, rather than the next time something would happen, for example estimating the number of major earthquakes in the next 500 years instead of when the next earthquake will happen, that distribution would be a Poisson distribution. If $N = \{\text{number of earthquakes in the next 500 years}\}$, then $N \sim \text{Poi}(**)$.

Example:
What if we want to find $\mathbb{P}(\text{No earthquake in 500 years})$? In this case, since $\lambda = \frac{1}{100}$, and $a = 500$, then the probability follows the expression e^{-5}. If we find this using Poisson, we will get the same answer. Let N represent the number of earthquakes in 500 years.

$$N \sim \text{Poi}(b)$$

$$= e^{-b} \cdot \frac{b^k}{k!}$$

$$\mathbb{P}(N = 0) = e^{-b} \Rightarrow b = 5$$

Expected Value of Continuous Random Variables

Suppose $X \sim f_X$, where f_X is a piecewise continuous function on X. Then,

$$\mathbb{E}X = \int_{-\infty}^{+\infty} f_X(a) \cdot a \,\mathrm{d}a$$

The letter s is also commonly used in this case:

$$\mathbb{E}X = \int_{-\infty}^{+\infty} f_X(s) \cdot s \,\mathrm{d}s$$

This is similar to the expected value formula for the discrete variables:

$$\boxed{\mathbb{E}X = \int f_X(a) \cdot a \,\mathrm{d}a}$$

versus

$$\mathbb{E}X = \sum_a p_X(a) \cdot a$$

For the two cases: discrete variables uses the probability distribution function while continuous variables use the cumulative distribution function, both use the value times the probability ($f(x) \cdot x$), and discrete variables use $\sum$ while continuous variables use $\int$.

Aside: Riemann Sum

Assume $f_X(a) = 0$, if $a \notin [0, 1]$.

$$Y = \frac{k}{100} \Leftrightarrow X \in \left[\frac{k}{100}, \frac{k+1}{100}\right], k = Z$$

e.g. $X = 0.172 \Rightarrow Y = 0.17$, X is a continuous random variable and Y is a discrete random variable, $X \approx Y$. In this case, $\mathbb{E}X \approx \mathbb{E}Y$.

$$\begin{aligned}
\mathbb{E}Y &= \sum_a p_Y(a) \cdot a \\
&= \sum_{k=0}^{100} p_Y(\frac{k}{100}) \cdot \frac{k}{100}
\end{aligned}$$

$$Y = 0.05 \Leftrightarrow X \in [0.05, 0.06]$$

$$\begin{aligned}
& p_Y(\frac{k}{100}) \\
& = \mathbb{P}(X \in \left[\frac{k}{100}, \frac{k+1}{100}\right)) \\
& \approx f_X\left(\frac{k}{100}\right) \cdot \left|\frac{k+1}{100} - \frac{k}{100}\right|
\end{aligned}$$

Therefore,

$$\mathbb{E}Y \approx \sum_k f_X\left(\frac{k}{100}\right) \cdot \frac{k}{100} \cdot \left|\frac{k+1}{100} - \frac{k}{100}\right|$$

This is close to finding the area below the curve. Instead of using 100 as the denominator, if ∞ were used, it would be equal to the integral. It can be said that this is similar to $\int_0^1 f_X(a) \cdot a \, \mathrm{d}a$. In this case, Y is a discrete random variable, but it is close to the continuous random variable, X.

Example: Unit Disk, Random Point

You have a disk with radius 1. Choose a random point inside and a is the distance between the origin and the point. What is $\mathbb{E}X$? In this case, the solution can be thought of as

$$\mathbb{E}X = \int_0^1 2a \cdot a \, \mathrm{d}a$$

This is because the area of the small circle with radius a divided by the total area represents the probability of $\mathbb{P}(X \le a)$, and this simplifies to $\frac{\pi \cdot a^2}{\pi} = a^2$. Treating a^2 as the cumulative distribution function would give the probability density function of $2a$, since $F'(a) = f(a)$. Since $\mathbb{E}X = \int f_X(a) \cdot a \, \mathrm{d}a$, then $\mathbb{E}X = \int_0^1 2a \cdot a \, \mathrm{d}a = \frac{2}{3}$.

What if the value is also a function?

$$\begin{aligned} \mathbb{E}g(X) &= \sum_k g(k) \cdot p_X(k) \\ &= \sum_k p_X(a) \cdot g(a) \\ &= \int f_X(a) \cdot g(a) \, \mathrm{d}a \end{aligned}$$

Formulas

Many formulas used for discrete random variables can be extended to continuous random variables.

Independence

All the independence formulas still work: If $X \perp Y$,

$$\begin{aligned} \mathbb{E}(2X) &= 2 \cdot \mathbb{E}X \\ \mathbb{E}(X+Y) &= \mathbb{E}X + \mathbb{E}Y \\ \mathbb{E}(XY) &= \mathbb{E}X \cdot \mathbb{E}Y \end{aligned}$$

Distributions

The $p_X(a)$ in discrete random variables is similar to $f_X(a)$ for continuous random variables. Also,

$$\sum_a p_X(a) \cdots \to \int f_X(a) \ldots \mathrm{d}a$$

Multiple Variables

In discrete random variables,

$$p_{X,Y}(k,l) = \sum_{k,l} p_{X,Y}(k,l) \cdot g(k,l)$$

In continuous random variables,

$$\int f_{X,Y}(a,b) \cdot g(a,b) \, \mathrm{d}a \, \mathrm{d}b$$

Additionally,

$$f_{X,Y} = \partial_a \partial_b \, F_{X,Y}$$

Conditionals

Suppose that X is a continuous random variable and B is an event. The following is still true:

$$p_{X|B} = \mathbb{P}(X = k|B)$$

There is also a version of the total probability theorem. The one for discrete variables is

$$p_X(k) = \sum_n p_{X|B_n}(k) \cdot \mathbb{P}(B_n)$$

The continuous variable version is

$$f_X(a) = \sum_n f_{X|B}(a) \cdot \mathbb{P}(B_n)$$

The expected value formula is also similar: In discrete variables,

$$\mathbb{E}(X|B) = \sum_k p_{X|B}(k) \cdot k$$

In continuous variables,

$$\mathbb{E}(X|B) = \int f_{X|B}(a) \cdot a \, \mathrm{d}a$$

$$\mathbb{E}(g(x)|B) = \int f_{X|B}(a) \cdot g(a) \, \mathrm{d}a$$

Also,

$$f_{X|B}(a) = F'_{X|B}(a) = \frac{\mathrm{d}}{\mathrm{d}a} \mathbb{P}(X \leq a|B)$$

$$f_X(a) = F'_X(a) = \frac{\mathrm{d}}{\mathrm{d}a} \mathbb{P}(X \leq a)$$

Examples

Example:
Suppose you have a coin: e.g.

$$H: \qquad X \sim \text{unif}[0,1]$$

$$T: \qquad X \sim \text{unif}[0,2]$$

Then,

$$f_{X|B_H}(a) = \begin{cases} 1 & a \in [0,1] \\ 0 & \text{others} \end{cases}$$

$$f_{X|B_T}(a) = \begin{cases} 1/2 & a \in [0,2] \\ 0 & \text{others} \end{cases}$$

Thus, using the formula given above

$$f_X(a) \ \sum_n f_{X|B_a}(a) \cdot \mathbb{P}(B_n)$$

we can find the total probably density function.

$$f_X(a) = \begin{cases} 3/4 & a \in [0,1] \\ 1/4 & a \in [1,2] \\ 0 & \text{others} \end{cases}$$

Example:
Suppose that

$$F_{X,Y}(a,b) = 1 - e^{-ab}$$

Since $f_X = F'_X$, we can differentiate.

$$\partial a(1 - e^{-ab}) = be^{-ab}$$

$$\partial b(1 - e^{-ab}) = ae^{-ab}$$

The product of the partials would give $f_{X,Y}$.

Example:
Suppose that $X \perp Y$ and $X, Y \sim \mathrm{N}(0,1)$. Find $\mathbb{P}(X + Y \leq 3)$. Thus,

$$f_X(a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}$$

$$f_Y(b) = \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2}}$$

Therefore,

$$f_{X,Y}(a,b) = \frac{1}{2\pi} e^{-\frac{a^2+b^2}{2}}$$

Thus,

$$\mathbb{P}(a + b \leq 3) = \int\int_{a+b\leq 3} \frac{1}{2\pi} \cdot e^{-\frac{a^2+b^2}{2}} \, \mathrm{d}a \, \mathrm{d}b$$

Example:
Suppose you are rolling three fair die. What is the expected number of rolls required to roll both 3 and 18?

$$\mathbb{P}(X = 3) = \frac{1}{6^3} = \frac{1}{216}$$

Thus, $\mathbb{P}(X = 3 or 18) = \frac{1}{108}$

$$\mathbb{E}(\ldots 3\text{or}18) = 108$$

$$\mathbb{E}(3\text{and}18) = 108 + 216 = 324$$