EC ENGR 102 Week 1

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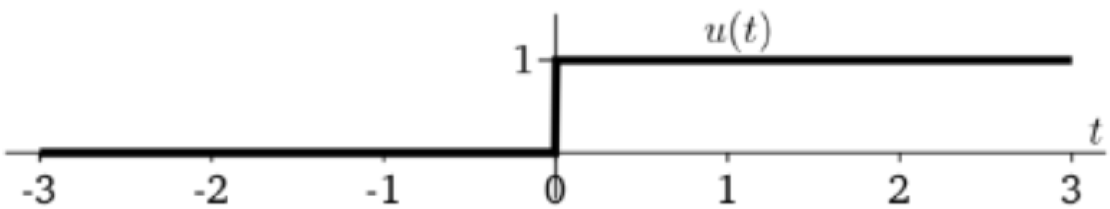
The Unit Step Function

The unit step function denoted by $u(t)$ in this class, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

It is also called the Heavyside step function. Drawn below:



Example:
Suppose I wanted to write

$$x(t) = \begin{cases} e^{-t} & t \geq 1 \\ 0 & t < 1 \end{cases}$$

We can do that in terms of the unit step function as

$$x(t) = e^{-t}u(t-1)$$

The Unit Rectangle

There are two definitions:

1. $$rect(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

   - This is a rectangle with height 1, from t = -0.5 to t = 0.5.
   - Notice the area under the curve is 1.

2. $$rect_{\Delta}(t) = \begin{cases} \frac{1}{\Delta} & |t| < \frac{\Delta}{2} \\ 0 & \text{else} \end{cases}$$

   - This is a general case of the function, where the Δ represents some number.
   - $rect_{\Delta}(t)$ where $\Delta = 2$ would make a rectangle going from $t = -1$ to $t = 1$, with height 0.5
   - Notice that any value of Δ will still have the area of the rectangle be 1

- Most of the time, we use the first definition of rectangle; we use this one for intuition.

Example:
How do we write the rectangle in terms of $u(t)$? There are also two ways:

$$rect(t) = u(t+0.5) - u(t-0.5)$$
$$rect(t) = u(t+0.5) \cdot u(-t-0.5)$$

We can use the step function and the rectangle function as building blocks for other functions.

Using building blocks

Consider the following example:

How do we write this using the step function as a building block? Each burst have a form of $A\cos(\omega t)$ and last for a width 0.5 around each integer.

Solution:
The burst around 0 can be written as

$$rect(2t) \cdot A\cos(\omega t)$$

The burst around 1 can be written as

$$rect(2(t-1)) \cdot A\cos(\omega t)$$

and et cetera. As a result, we can write the entire function as:

$$y(t) = \sum_{i=-\infty}^{\infty} rect(2(t-i)) \cdot A\cos(\omega t)$$

Unit Ramp

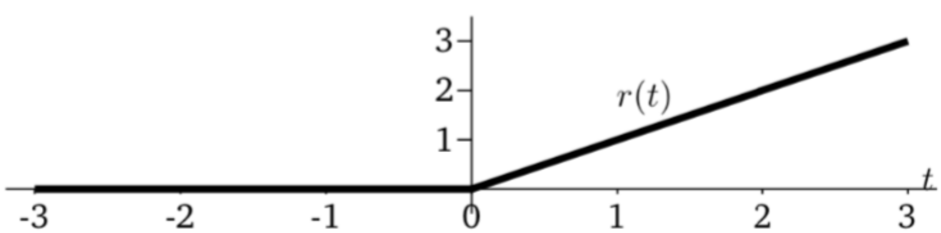
The unit ramp is defined as:

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Note that the unit ramp is the integral of the unit step, i.e.

$$r(t) = \int_{-\infty}^{t} u(r) \mathrm{d}r$$

The unit ramp is illustrated below:



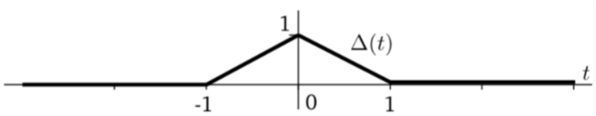
This is known in the AI world as ReLU().

Unit Triangle

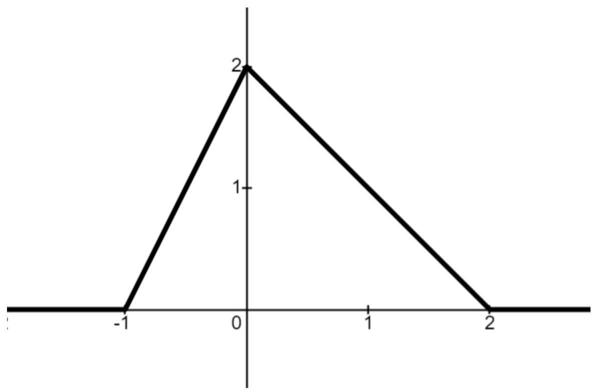
The unit triangle is defined as:

$$\triangle(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{else} \end{cases}$$

The unit triangle is illustrated below:



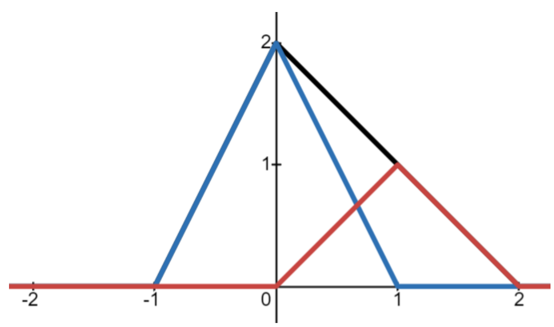
Example:
Lets say we want to make a skewed triangle.



Solution:

$$x(t) = 2\triangle(t) + \triangle(t-1)$$

The intuition is the following:



The $2\triangle(t)$ term represents the blue line and the $\triangle(t-1)$ term represents the red line. We can add the two lines between $0 \geq t \geq 1$ to get the negative 1 slope we need.

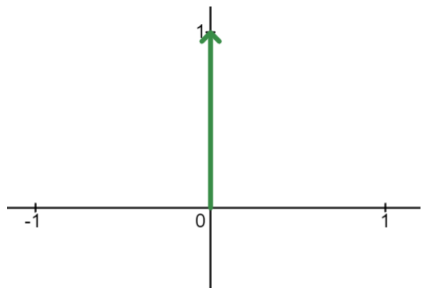
Impulse Function

This is an extremely important signal.

- This is defined as $\delta(t)$, or "impulse", "delta", or "Dirac" function. It is **not** a rigorous mathematical function.
- Features of the impulse function:
  1. It is very large (i.e., approaching infinity), at $t = 0$

2. It's zero everywhere else, $t \neq 0$.
3. Area $= 1$

- Shown on the graph as an arrow pointing up at $t = 0$



- The intuition of the impulse function is a $rect_{\Delta}(t)$ where Δ approaches 0. (e.g., the width of the rectangle goes to 0 and the height goes to infinity.)
- $\delta(t) \cdot x(t) = x(0) \cdot \delta(t)$, where $x(t)$ is any function.
  - The impulse is still affected by other functions because by intuition it is a very, very thin rectangle. The height is an infinitely large number, but not infinity.
  - The area of the impulse is also scaled by $x(0)$.
  - The shape of the impulse does not change with scaling, but its area does.
  - This is called the **impulse dampening property**.
- The impulse can be moved by an amount T:

$$x(t) \cdot \delta(t-T) = x(T) \cdot \delta(t-T)$$

  - This is called the **impulse sampling property**.
- What happens when we take the integral?

$$\begin{aligned} & \int_{-\infty}^{\infty} x(t) \cdot \delta(t) \mathrm{d}t \\ = & \int_{-\infty}^{\infty} x(0) \cdot \delta(t) \mathrm{d}t \\ = & \, x(0) \cdot \int_{-\infty}^{\infty} \delta(t) \mathrm{d}t \\ = & \, x(0) \end{aligned}$$

- What if it is shifted?

$$\begin{aligned} & \int_{-\infty}^{\infty} x(t) \cdot \delta(t-T) \mathrm{d}t \\ = & \int_{-\infty}^{\infty} x(T) \cdot \delta(t-T) \mathrm{d}t \\ = & \, x(T) \cdot \int_{-\infty}^{\infty} \delta(t-T) \mathrm{d}t \\ = & \, x(T) \end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} x(t)\delta(t-T)\mathrm{d}t = x(T)}$$

  - This is called the **impulse sifting property**.

Integral of an impulse

$$\int_{\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$
$$\int_{\infty}^{0+} \delta(t)\mathrm{d}t = 1$$
$$\int_{\infty}^{0-} \delta(t)\mathrm{d}t = 0$$

The $0+$ represents approaching zero from the right (e.g., infinitely close to zero on the positive side), and $0-$ represents approaching zero on the left (e.g., infinitely close to zero on the negative side).