Math 170E Week 6

Aidan Jan

February 19, 2023

Expected Value of Functions

$$\mathbb{E}_X = \sum_k p_X(k) \cdot k$$

Some properties we have covered:
1. $\mathbb{E}(aX) = a \cdot \mathbb{E}(X)$
For example, $\mathbb{E}(2X) = 2 \cdot \mathbb{E}(X)$
2. $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

In that case, X and Y can also be **functions**.
For example. $\mathbb{E}(x^2 + e^x) = \mathbb{E}(x^2) + \mathbb{E}(e^x)$
In the example above, x^2 is a function rather than just a random variable. This just means you are finding the expected value of the function. If $Y = x^4$, then $\mathbb{E}(x^4) = \mathbb{E}(Y)$, where Y is a random variable dependent on x.
In fact, the following is also true.

$$\mathbb{E}(x^4) = \mathbb{E}(Y) = \sum_k p_Y(k) \cdot k$$

However, this method is not necessarily easy to use, since calculating another probability mass function would be required.
Instead, for this example, it would be much more efficient to use:

$$\mathbb{E}(Y) = \mathbb{E}(x^4) = \sum_k p_x(k) \cdot k^4$$

In this formula, you would sum the same probabilities, but with the values of x^4 instead of just x, which gives the expected value of $\mathbb{E}(x^4)$

Example:
Let $f : \mathbb{R} \to \mathbb{R}$, $f(a) = \sin(a)$ X is a random variable, and thus $f(X)$ is also a random variable.

$$\mathbb{E}(f(X)) = \sum_k p_x(k) \cdot f(k)$$

The special cases are $f_0(a) = a$, and $f_0(x) = x$ It turns out this is the general form of the expected value formula.

$$mathbbE(X) = \sum_k p_X(k) \cdot k$$

$$\mathbb{E}(f(X)) = \sum_k p_x(k) \cdot f(k)$$

Example:
Let $X \sim \text{unif}[[1,6]]$.

$$\mathbb{E}(x^2) = \sum_k p_x(k) \cdot k^2$$

Example:
Let's prove $\mathbb{E}(f(x)+g(x)) = \mathbb{E}f(x) + \mathbb{E}g(x)$.

$$\begin{aligned}
\mathbb{E}(f(x)+g(x)) &= \sum_k p_x(k) \cdot (f(k)+g(k)) \\
&= \sum_k p_x(k) \cdot f(k) + \sum_k p_x(k) \cdot g(k) \\
&= \mathbb{E}f(x) + \mathbb{E}g(x)
\end{aligned}$$

Example:

$$\begin{aligned}
&X \sim B(20, 0.2) \\
&Y = \sqrt{X}
\end{aligned}$$

Find $p_Y(k)$.

$$p_Y(k) = \begin{cases} \binom{20}{k^2} \cdot 0.2^{k^2} \cdot 0.8^{20-k^2} & k^2 \in [[0,20]], k \geq 0 \\ 0 & otherwise \end{cases}$$

Now, what if $Y = (x-1)^2$ instead of just x^2? This causes a problem because when $Y = 1$, $x = 0, 2$. Thus, $P_Y(1) = \mathbb{P}(Y=1) = \mathbb{P}(\{X=0\} \cup \{X=2\})$. In this case, this equals

$$\mathbb{P}(X=0) + \mathbb{P}(X=2) = \binom{20}{0} \cdot 0.2^0 \cdot 0.8^{20} + \binom{20}{2} \cdot 0.2^2 \cdot 0.8^{18}$$

If generalized from this example,

$$P_Y(k) = \mathbb{P}(Y=k) = \sum_{a\,:\,f(a)=k} \mathbb{P}(X=a)$$

General formula:

$$P_{f(X)}(k) = \sum_{a\,:\,f(a)=k} p_X(a)$$

Random Variables and Independence

Let X be a random variable. Let B be an event. The question: is $X \perp B$? If $X \perp B$: then,

$$\mathbb{P}(X = 1|B) = \mathbb{P}(X = 1)$$

$$\mathbb{P}(x = k|B) = \mathbb{P}(X = k)\ \forall k \in \mathbb{R}$$

$$\Updownarrow$$

$$X \perp B$$

Example:
Consider two dice rolls. $S = \{(x_1, x_2)\ :\ x_1, x_2 \in [[1, 6]]\}$. Let $X = x_1$, $B = \{x_1 = x_2\}$. Let's check if they are independent.

$$\mathbb{P}(x = k|B) = \mathbb{P}(X = k)\ \forall k \in \mathbb{R}$$

$\mathbb{P}(X = 1) = \frac{1}{6}$, and $B = \{(1, 1), (2, 2), ..., (6, 6)\}$

$$\mathbb{P}(X = 1|B) = \frac{\mathbb{P}(X = 1 \cap B)}{\mathbb{P}(B)} = \frac{|X = 1 \cap B|}{|B|} = \frac{1}{6}$$

This is similar for $k = 2, 3, 4, 5, 6$. Since the equation is true, $X \perp B$.

Side: Notation That equation can be written as $P_X = P_{X|B}$. $P_{X|B}$ is the probability mass function of X given B. It tells us the distribution of X under the event B.

$$P_{X|B} R \to \mathbb{R}$$

$$P_{X|B}(k) = P(X = k|B)$$

Similarly, if a summation is present, then

$$\sum_k P(X = k|B) = \mathbb{P}(\bigcup_k \{X = k\}|B) = 1$$

Also,

$$\mathbb{P}(A_1 \cup A_2|B) = \sum_{k=1}^{2} \mathbb{P}(A_k|B)$$

$$\sum_k \mathbb{P}_{X|B}(k) = 1 \ \ \text{(Always holds)}$$

$$\sum_k P_X(k) = 1$$

Expectation value can also be written in a similar fashion.

$$\mathbb{E}X = \sum_k \mathbb{P}_X(k) \cdot k$$

$$\mathbb{E}[X|B] := \sum_k \mathbb{P}_{X|B}(k) \cdot k$$

Example:

Valentines problem:

Suppose you are trying to find a date. How do you find "the one?" If you are dating, "the one" you like may not like you back if you keep dating all the other girls. When do you stop? The solution happens to be $\frac{n}{e}$, where n represents the total pool of girls you can choose from, and e is the number 2.71...

Total Probability Theorem

Consider two events A and B.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Now consider X, a random variable.

$$P_{X|B}(k) = \mathbb{P}(X = k|B)$$

The **Total Probability Theorem** states that if $\{B_k\}$ is a partition, then

$$\mathbb{P}(A) = \sum_k \mathbb{P}(A|B_k) \cdot \mathbb{P}(B_k)$$

Total Expected Value Theorem

Similar to the total probability theorem, if $\{B_k\}$ is a partition, then

$$\mathbb{E}X = \sum_k \mathbb{E}(X|B_k) \cdot \mathbb{P}(B_k)$$
$$\mathbb{E}(X|A) = \sum_a a \cdot P_{X|A}(a)$$

The derivation of this comes from the total probability theorem. Substitute $A = \{X = m\}$. If this was plugged into the total probability theorem, you get

$$\mathbb{P}(X = m) = \sum_k \mathbb{P}(X = m|B_k) \cdot \mathbb{P}(B_k)$$
$$\mathbb{P}_X(m) = \sum_k \mathbb{P}_{X|B_k}(m) \cdot \mathbb{P}(B_k)$$

Note: Let $X = 1_A$ in the total expected value theorem, then we obtain the total probability theorem. The two equations are very closely related.

$$1_A = \begin{cases} 1 & A \text{ is a success} \\ 0 & A \text{ is a failure} \end{cases}$$

Example:

Suppose you randomly pick N in $[[1, 10]]$. Add N red balls into a bag with 20 blue balls, and randomly pick a ball. Let P represent the probability that the ball is red.

In this case, $B_k = \{N = k\}$, where $B_1, B_2, ..., B_k$ is a partition. $A = \{\text{color is red}\}$. Thus,

$$\begin{aligned}
\mathbb{P}(A) &= \sum_k \mathbb{P}(A|B_k) \cdot \mathbb{P}(B_k) \\
&= \sum_{k=1}^{10} \frac{1}{10} \cdot \mathbb{P}(A|B_k)^{\frac{k}{20+k}} \\
&= \sum_{k=1}^{10} \frac{1}{10} \cdot \frac{k}{20+k}
\end{aligned}$$

The above problem is relatively simple. Now, let's change the question. Instead, keep picking random balls (with replacement) that the Xth-ball is the first red ball.

$$X|B_k \sim \text{Geo}\left(\frac{k}{20+k}\right)$$

$$\begin{aligned}
\mathbb{E}X &= \sum_k \mathbb{P}(B_k) \cdot \mathbb{E}(X|B_k) \\
&= \sum_{k=1}^{10} \cdot \frac{1}{10} \cdot \mathbb{E}(X|B_k)
\end{aligned}$$

If we define $\mathbb{P}(X = m|B_k) = \mathbb{P}(Y = m) \forall m$ and $Y \sim \text{Geo}\left(\frac{k}{20+k}\right)$, then

$$\mathbb{P}(Y = m) = \left(1 - \frac{k}{20+k}\right)^{k-1} \cdot \left(\frac{k}{20+k}\right)$$

Geometric Distribution Proof

Let's prove that $\mathbb{E}X = \frac{1}{p}$ by using the expected value theorem.

$$A = \{\text{the first trial is successful}\}$$

$$\mathbb{P}(A) = p$$

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(X|A) \cdot \mathbb{P}(A) + E(X|A^c) \cdot \mathbb{P}(A^c) \\
&= p + \mathbb{E}(x|A^c) \cdot (1-p)
\end{aligned}$$

However, $\mathbb{E}(X|A^c) = \mathbb{E}X + 1$, since if given that there is a fail, the expected value of X would increase, since X represents the probability of the first success being on event m. Thus,

$$\begin{aligned}
\mathbb{E}X &= p + (1 + \mathbb{E}(X)) \cdot (1-p) \\
\mathbb{E}X &= 1 + (1-p) \cdot \mathbb{E}X \\
\mathbb{E}X &= \frac{1}{p}
\end{aligned}$$

Example:

Suppose you are walking on a number line, from 0 to 10. You start at 5. Every step, you walk in a random direction to a neighboring number. Let T^5 represent the number of steps to arrive at either 0 or 10, starting at 5.

We are looking for $\mathbb{E}(T^5)$.

Let $A = \{\text{first step is moving to the left}\}$ Then,

$$\begin{aligned}
\mathbb{E}T^5 &= \mathbb{E}(T^5)|A) \cdot \mathbb{P}(A) + \mathbb{E}(T^5|A^c) \cdot \mathbb{P}(A^c) \\
&= \frac{1}{2}(\mathbb{E}(T^5|A) + \mathbb{E}(T^5|A^c)) \\
&= 1 + \frac{1}{2} \cdot (\mathbb{E}(T^4) + \mathbb{E}(T^6))
\end{aligned}$$

This simplification holds true because $\mathbb{E}T^4$ and $\mathbb{E}T^6$ are 1 closer to the edges than $\mathbb{E}T^5$, and to get from 5 to 4 or 6, you must take one step.

If $a_k = \mathbb{E}(T^k)$, then for all k,

$$a_k = \begin{cases} 1 + \frac{a_{k-1}+a_{k+1}}{2} & k \in [[1,9]] \\ 0 & k = 0, 10 \end{cases}$$

$$\implies a_k = k \cdot (10 - k)$$