

GPU Teaching Kit

Accelerated Computing



Module 12 – Floating-Point Considerations

Lecture 12.2 - Numerical Stability

Objective

- Understand numerical stability in linear system solver algorithms
 - Cause of numerical instability
 - Pivoting for increased stability



Numerical Stability

- Linear system solvers may require different ordering of floating-point operations for different input values in order to find a solution
- An algorithm that can always find an appropriate operation order and thus a solution to the problem is a numerically stable algorithm
 - An algorithm that falls short is numerically unstable

Gaussian Elimination Example

Original

$$3X + 5Y + 2Z = 19$$
 $2X + 3Y + Z = 11$
 $X + 5/3Y + 2/3Z = 19/3$
 $X + 3/2Y + 1/2Z = 11/2$
 $X + 2Y + 2Z = 11$

Step 1: divide equation 1 by 3, equation 2 by 2

$$X + \frac{5}{3}Y + \frac{2}{3}Z = \frac{19}{3}$$

$$- \frac{1}{6}Y - \frac{1}{6}Z = -\frac{5}{6}$$

$$\frac{1}{3}Y + \frac{4}{3}Z = \frac{14}{3}$$

Step 2: subtract equation 1 from equation 2 and equation 3

Gaussian Elimination Example (Cont.)

$$X + 5/3Y + 2/3Z = 19/3$$

 $- 1/6Y - 1/6Z = -5/6$
 $1/3Y + 4/3Z = 14/3$
 $X + 5/3Y + 2/3Z = 19/3$
 $Y + Z = 5$
 $Y + 4Z = 14$

Step 3: divide equation 2 by -1/6 and equation 3 by 1/3

$$X + \frac{5}{3}Y + \frac{2}{3}Z = \frac{19}{3}$$

$$Y + Z = 5$$

$$+ 3Z = 9$$

Step 4: subtract equation 2 from equation 3

Gaussian Elimination Example (Cont.)

$$X + 5/3Y + 2/3Z = 19/3$$

 $Y + Z = 5$
 $+ 3Z = 9$
 $X + 5/3Y + 2/3Z = 19/3$
 $Y + Z = 5$
 $Z = 3$

Step 5: divide equation 3 by 3 We have solution for Z!

$$X + 5/3Y + 2/3Z = 19/3$$

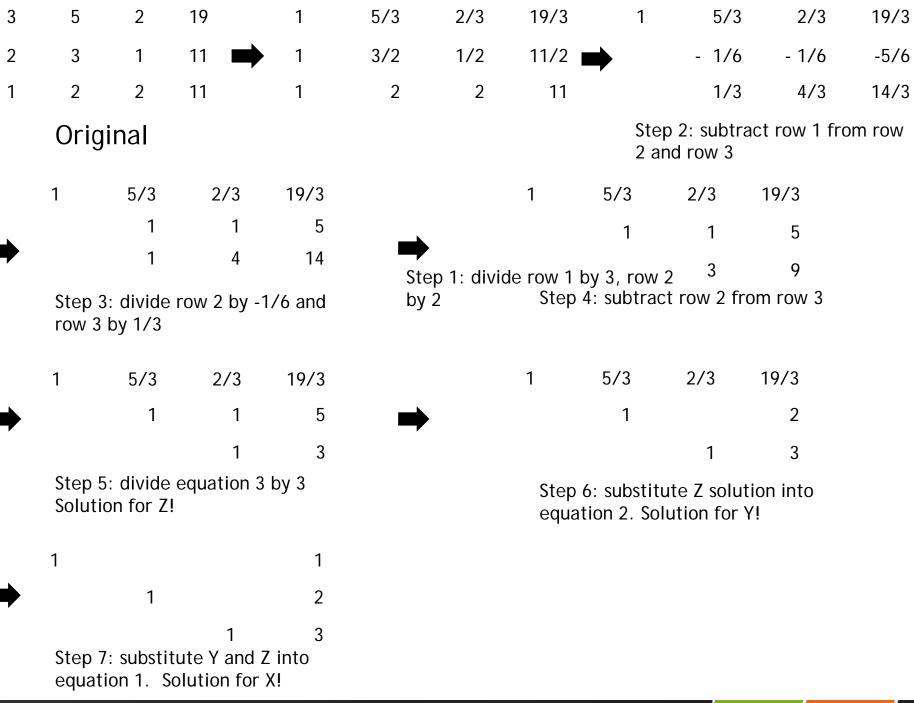
 $Y = 2$

Step 6: substitute Z solution into equation 2. Solution for Y!

Step 7: substitute Y and Z into equation 1. Solution for X!

3

3



Basic Gaussian Elimination is Easy to Parallelize

- Have each thread to perform all calculations for a row
 - All divisions in a division step can be done in parallel
 - All subtractions in a subtraction step can be done in parallel
 - Will need barrier synchronization after each step
- However, there is a problem with numerical stability



Pivoting



Pivoting: Swap row 1 (Equation 1) with row 2 (Equation 2)

—	1	3/2	1/2	11/2
—		5	2	16
	1	2	2	11

Step 1: divide row 1 by 3, no need to divide row 2 or row 3

Pivoting (Cont.)



Step 2: subtract row 1 from row 3 (column 1 of row 2 is already 0)

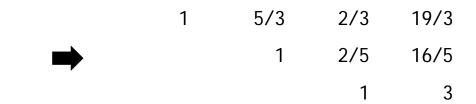
 1	3/2	1/2	11/2
	1	2/5	16/5
	1	3	11

Step 3: divide row 2 by 5 and row 3 by 1/2

Pivoting (Cont.)

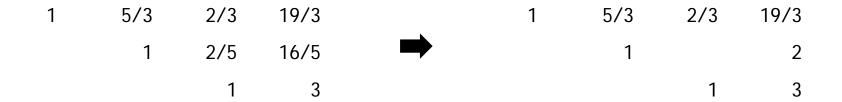


Step 4: subtract row 2 from row 3



Step 5: divide row 3 by 13/5 Solution for Z!

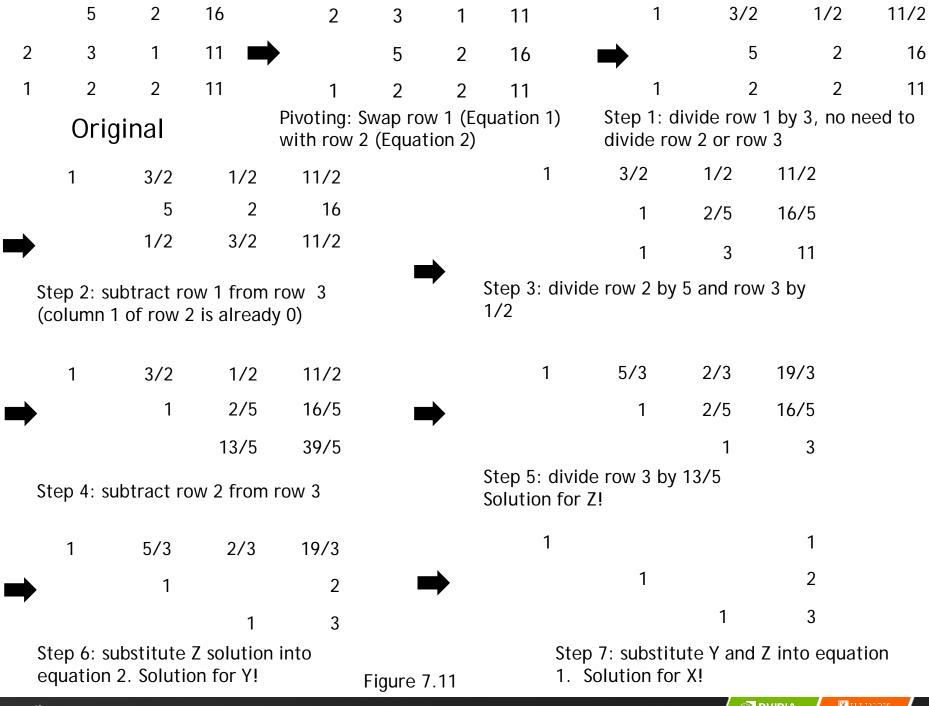
Pivoting (Cont.)



Step 6: substitute Z solution into equation 2. Solution for Y!



Step 7: substitute Y and Z into equation 1. Solution for X!



Why is Pivoting Hard to Parallelize?

- Need to scan through all rows (in fact columns in general) to find the best pivoting candidate
 - A major disruption to the parallel computation steps
 - Most parallel algorithms avoid full pivoting
 - Thus most parallel algorithms have some level of numerical instability





GPU Teaching Kit

Accelerated Computing





The GPU Teaching Kit is licensed by NVIDIA and the University of Illinois under the Creative Commons Attribution-NonCommercial 4.0 International License.