$\tau_N^f = \hat{f}_N - f_N$; $(f_N)_i = 2\partial_t h(x_i, n\Delta t) + U\partial_x h(x_i, n\Delta t)$.

$(\hat{f}_N)_i = 2\,\frac{h_i^n - h_i^{n-1}}{\Delta t} + U\,\frac{h_i^n - h_{i-1}^n}{\Delta x}$.

$h_i^n = h_i^{n-1} + \left(\frac{\partial h}{\partial t}\right)_{i,n}\Delta t + \frac{\Delta t^2}{2}\frac{\partial^2 h}{\partial t^2}(x_i, \xi_i)$ $\quad \xi_i \in](n-1)\Delta t, n\Delta t[$.

$h_i^n = h_{i-1}^n + \left(\frac{\partial h}{\partial x}\right)_{i,n}\Delta x + \frac{\Delta x^2}{2}\frac{\partial^2 h}{\partial x^2}(\tilde{x}_i, t_n)$ $\quad \tilde{x}_i \in]x_{i-1}, x_i[$

$\Rightarrow \left(\frac{\partial h}{\partial t}\right)_{i,n} = \frac{h_i^n - h_i^{n-1}}{\Delta t} + O(\Delta t)$ If we assume ~~the~~ $\left|\frac{\partial^2 h}{\partial t^2}\right| \le C$ $\forall x, \forall t$

$\left(\frac{\partial h}{\partial x}\right)_{i,n} = \frac{h_i^n - h_{i-1}^n}{\Delta x} + O(\Delta x)$ " $\left|\frac{\partial^2 h}{\partial x^2}\right| \le C$ $\forall x, \forall t$.

Let $\phi = h^3 \partial_x p$.

x_{i-1} $\quad x_{i-\frac12}$ $\quad x_i$ $\quad x_{i+\frac12}$ $\quad x_{i+1}$

$\phi_{i+\frac12,n} = \phi_{i,n} + \frac{\Delta x}{2}\left(\frac{\partial\phi}{\partial x}\right)_{i,n} + \frac{(\Delta x)^2}{4}\left(\frac{\partial^2\phi}{\partial x^2}\right)_{i,n} + \frac{(\Delta x)^3}{48}\frac{\partial^3\phi}{\partial x^3}(\tilde{x}_i, t_n)$ $\quad \tilde{x}_i \in]x_i, x_{i+\frac12}[$

$\phi_{i-\frac12,n} = \phi_{i,n} - \frac{\Delta x}{2}\left(\frac{\partial\phi}{\partial x}\right)_{i,n} + \frac{(\Delta x)^2}{4}\left(\frac{\partial^2\phi}{\partial x^2}\right)_{i,n} - \frac{(\Delta x)^3}{48}\frac{\partial^3\phi}{\partial x^3}(\hat{x}_i, t_n)$ $\quad \hat{x}_i \in]x_{i-\frac12}, x_i[$

$\Rightarrow \left(\frac{\partial\phi}{\partial x}\right)_{i,n} = \frac{\phi_{i+\frac12,n} - \phi_{i-\frac12,n}}{\Delta x} + O(\Delta x^2)$ If we assume $\left|\frac{\partial^3}{\partial x^3}\left(h^3\frac{\partial p}{\partial x}\right)\right| \le C$ $\forall x, \forall t$

$p_{i+1,n} = p_{i+\frac12,n} + \frac{\Delta x}{2}\left(\frac{\partial p}{\partial x}\right)_{i+\frac12,n} + \frac{(\Delta x)^2}{8}\left(\frac{\partial^2 p}{\partial x^2}\right)_{i+\frac12,n} + \frac{(\Delta x)^3}{48}\left(\frac{\partial^3 p}{\partial x^3}\right)_{i+\frac12,n} + \frac{(\Delta x)^4}{4!\,2^4}\left(\frac{\partial^4 p}{\partial x^4}\right)_{i+\frac12,n} + (\Delta x)^5 K\frac{\partial^5 p}{\partial x^5}$

$p_{i,n} = p_{i+\frac12,n} - \;" \;+\; " \;-\; " \;+\; " \;- \Delta x^5 K\frac{\partial^5 p}{\partial x^5}$

$\Rightarrow \frac{p_{i+1,n} - p_{i,n}}{\Delta x} = \left(\frac{\partial p}{\partial x}\right)_{i+\frac12,n} + \frac{(\Delta x)^2}{24}\left(\frac{\partial^3 p}{\partial x^3}\right)_{i+\frac12,n} + O(\Delta x^4)$

$\frac{p_{i,n} - p_{i-1,n}}{\Delta x} = \left(\frac{\partial p}{\partial x}\right)_{i-\frac12,n} + \frac{(\Delta x)^2}{24}\left(\frac{\partial^3 p}{\partial x^3}\right)_{i-\frac12,n} + O(\Delta x^4)$.

$\Rightarrow \left(\frac{\partial\phi}{\partial x}\right)_{i,n} = \frac{1}{\Delta x^2}\left[h^3_{i+\frac12}p_{i+1,n} - (h^3_{i+\frac12} + h^3_{i-\frac12})p_{i,n} + h^3_{i-\frac12}p_{i-1,n}\right] - \frac{\Delta x}{2}\left(h^3_{i+\frac12,n}\left(\frac{\partial^3 p}{\partial x^3}\right)_{i+\frac12,n} - h^3_{i-\frac12,n}\left(\frac{\partial^3 p}{\partial x^3}\right)_{i-\frac12}\right) + O(\Delta^3)$.

If we ~~suppose~~ suppose that $x,t \mapsto h^3(x,t)\frac{\partial^3 p}{\partial x^3}$ is of class C^1 $\Rightarrow$

$\left(\frac{\partial\phi}{\partial x}\right)_{i,n} = \frac{1}{\Delta x^2}[\quad] + O(\Delta x^2)$.

Finally we obtain: $\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) - 2\frac{\partial h}{\partial t} - U\frac{\partial h}{\partial x} - \frac{(h^3_{i+\frac12}p_{i+1} \cdots)}{\Delta x^2} - 2\frac{h_i^n - h_i^{n-1}}{\Delta t} - U\frac{h_i^n - h_{i-1}^n}{\Delta x} = O(\Delta t) + O(\Delta x)$

Then the scheme is consistant.

This is the definition of the ~~stability~~ ~~and~~ consistance and the proof.

We say that it is of order 1 in time and 1 in x