Linear Algebra Take Home Exam 1

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Question 1

```
import numpy as np
import sympy as sp
from sympy import init_printing, Matrix
init_printing()

# Question 1

A = Matrix([
    [1,1,0,1,-1,-1,1],
    [0,1,-1,1,-1,0,0],
    [-1,0,1,1,-1,1,-1],
    [0,-1,1,1,-1,-1,-1],
    [1,1,0,0,0,-1,1],
    [1,1,-1,0,-1,-1,-1]])
```

Below is the RREF of Matrix A

```
A.rref()[0]
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Question 2

Pivot Columns: 1, 2, 3, 4, 5, 6

Question 3

Assuming that A is a coefficient matrix, this means that A has 7 vectors with 6 entries per vector. According to Theorem 8, If a set contains more vectors than there are entries in a vector, this means that if A is consistent, it is linearly dependent

Question 4

```
fifth_column = A[:,4]
other_columns = Matrix(np.delete(A, 4, axis=1))

newA = Matrix(np.append(other_columns, fifth_column, axis = 1))
newArref= newA.rref()[0]
x = newArref[:,-1]
```

The solution set is below

```
x # Solution Set
```

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

I can show that x is the correct solution set with the code below: In the form $A\vec{x} = \vec{b}$, we know $A =$ A (without the fifth column), $\vec{x}$ is our unique solution set, and $\vec{b}$ is our fifth column of A. Thus $A\vec{x}$ (as shown in the code below) is equal to the fifth column of A.

```
print(other_columns*x)
```

```
Matrix([[-1], [-1], [-1], [-1], [0], [-1]])
```

Question 5

The columns of A span $\mathbb{R}^6$. According to theorem 4, The columns of A span $\mathbb{R}^6$ if and only if A has a pivot position in each row. Since the RREF of A (as seen in Question 1) has a pivot position in every row, We know that A spans $\mathbb{R}^6$.

Question 6

```
zeroes = Matrix(np.zeros((6,1)))
A_augmented = Matrix(np.append(A, zeroes, axis = 1))
A_augmented.rref()[0]
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

This shows the following

$$\begin{cases} x_1 = -2x_7 \\ x_2 = x_2 \\ x_3 = x_3 \\ x_4 = -2x_7 \\ x_5 = -2x_7 \\ x_6 = -x_7 \\ x_7 = x_7 \end{cases} \text{... thus: } \vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -2 \\ 0 \\ 0 \\ -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

An example of a solution is $\begin{bmatrix} -2 \\ 0 \\ 0 \\ -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$.

Question 7

```
b = Matrix([1,0,0,0,0,0])
A_augmented2 = Matrix(np.append(A, b, axis = 1))

A_aug2_RREF = A_augmented2.rref()[0]

A_aug2_RREF
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -2 \\ 0 \\ 0 \\ -2 \\ -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

An example of a solution is 'c' below where x_3 and x_7 are 0.

```
c = Matrix(np.append(A_aug2_RREF[:,-1],0))
print(c)
```

```
Matrix([[3], [-1], [0], [1], [0], [2], [0]])
```

This means $3\vec{a}_1 - 1\vec{a}_2 + 0\vec{a}_3 + \vec{a}_4 + 0\vec{a}_5 + 2\vec{a}_6 + 0\vec{a}_7 = \vec{b}$

```
add_q7 = Matrix(np.zeros(6))

for i in range(7):
    add_q7 += c[i]*A[:,i]

print(add_q7==b) #Proof
```

```
True
```

Question 8

Matrix A is a 6×7 matrix.

Part (a)

The domain of T is $\mathbb{R}^6$

Part (b)

The codomain of T is $\mathbb{R}^6$

Part (c)

The range of T is $\mathbb{R}^6$ Columns 1,2,3,4,6 and 7 of A have 6 entries in each and are linearly independent meaning that they span $\mathbb{R}^6$ (Theorem 4). While adding the fifth column may make Matrix A linearly dependent, this doesn't change the span.

Part (d)

If Part (c) is correct, there is no vector in $\mathbb{R}^6$ that is not in the range T.