

Computing with Real Numbers

I. The LFT Approach to Real Number Computation

II. A Domain Framework for Computational Geometry

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Abstract. We introduce, in Part I, a number representation suitable for exact real number computation, consisting of an exponent and a mantissa, which is an infinite stream of signed digits, based on the interval $[-1, 1]$. Numerical operations are implemented in terms of *linear fractional transformations* (LFT's). We derive lower and upper bounds for the number of argument digits that are needed to obtain a desired number of result digits of a computation, which imply that the complexity of LFT application is that of multiplying n -bit integers. In Part II, we present an accessible account of a domain-theoretic approach to computational geometry and solid modelling which provides a data-type for designing robust geometric algorithms, illustrated here by the convex hull algorithm.

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