2 Lecture 10.09.21

We presented a solution to Enderton, exercise 1.2.10.

1.2.10(a) Show that if $\Sigma \subset \mathcal{S}$ is finite, then there is a $\Delta \subseteq \Sigma$ such that Σ is equivalent to Δ and Δ is independent. We proceeded by induction on the size of Σ and noted the importance of considering the case where Σ is empty, since any set of tautologies is equivalent to the $\emptyset$. We reduced the induction step to showing that for every $\Sigma \subseteq \mathcal{S}$ and $\alpha \in \mathcal{S}$, if $(\Sigma - \{\alpha\}) \models \alpha$, then $(\Sigma - \{\alpha\})$ is equivalent to Σ.

It follows at once from the lemma below that for every $\Sigma \subseteq \mathcal{S}$ and $\alpha \in \mathcal{S}$, if $(\Sigma - \{\alpha\}) \models \alpha$, then $(\Sigma - \{\alpha\})$ is equivalent to Σ. First, for any $\Sigma \subseteq \mathcal{S}$ we define $\mathsf{Mod}(\Sigma) = \{h \in \mathcal{H} \mid h \models \Sigma\}$.

Lemma 1 *For every* $\Sigma, \Gamma \subseteq \mathcal{S}$, *if* $\mathsf{Mod}(\Sigma) = \mathsf{Mod}(\Gamma)$, *then* $\mathrm{Cn}(\Sigma) = \mathrm{Cn}(\Gamma)$.

We suggested proving the converse of the foregoing lemma as an exercise (hint: apply the Compactness Theorem).

Exercise 1 *For every* $\Sigma, \Gamma \subseteq \mathcal{S}$, *if* $\mathrm{Cn}(\Sigma) = \mathrm{Cn}(\Gamma)$, *then* $\mathsf{Mod}(\Sigma) = \mathsf{Mod}(\Gamma)$.

1.2.10(b) Let $\varphi_n = (A_0 \wedge \ldots \wedge A_n)$. Let $\Sigma = \{\varphi_n \mid n \in N\}$. Note that if $\Gamma \subseteq \Sigma$ and Γ is independent, then Γ contains at most one sentence, whereas if Γ is equivalent to Σ then Γ is infinite. It follows that no subset of Σ is both independent and equivalent to Σ.

1.2.10(c) Let $\Sigma = \{\sigma_0, \sigma_1, \ldots\}$. We may suppose without loss of generality that no finite subset of Σ is equivalent to Σ. We define a sequence of sentences $\delta_i \in \Sigma$ by induction as follows. Let $\delta_0 = \sigma_i$ where i is the least j such that σ_j is not a tautology. Some such j exists, for otherwise Σ would be equivalent to $\emptyset$. Let $\delta_{n+1} = \sigma_i$ where i is the least j such that $\{\delta_0, \ldots, \delta_n\} \not\models \sigma_j$. Such a j exists, for otherwise Σ is equivalent to $\{\delta_0, \ldots, \delta_n\}$. Now, let $\gamma_0 = \delta_0$ and $\gamma_{n+1} = (\delta_0 \wedge \ldots \wedge \delta_n) \rightarrow \delta_{n+1}$. Let $\Gamma = \{\gamma_n \mid n \in N\}$. Next time, we will verify that Γ is equivalent to Σ and that Γ is independent.

We began to study the expressive power of sentential logic. We first addressed the finite case.

1. $\mathcal{O}_n = \{A_i \mid i \leq n\}$;
2. $\mathcal{S}_n =$ the set of sentences generated from $\mathcal{O}_n$ using the sentential connectives;
3. $\mathcal{H}_n = \{h \mid h : \mathcal{O}_n \longrightarrow \{\top, \bot\}\}$;
4. for $\varphi \in \mathcal{S}_n, \mathrm{Mod}_n(\varphi) = \{h \in \mathcal{H}_n \mid h \models \varphi\}$.

Theorem 1 (Expressive Completeness Theorem for Sentential Logic) *For every n and for every $X \subseteq \mathcal{H}_n$, there is a $\varphi \in \mathcal{S}_n$, such that $Mod_n(\varphi) = X$.*

A proof of this theorem, essentially the same as we presented in class (with slightly different terminology), may be found in Enderton, section 1.5.

We then began to consider the infinite case. We showed that

Theorem 2 (Cantor's Diagonal Theorem) *$\mathcal{H}$ is not countable.*

Proof: Let $\{h_1, h_2, \ldots\} \subseteq \mathcal{H}$. We show that there is an $h \in \mathcal{H}$ such that for every i, $h \neq h_i$. Let **change**$(\top) = \bot$ and **change**$(\bot) = \top$. For every i, let $h(A_i) = \textbf{change}(h_i(A_i))$. ■

Exercise 2 *Show that there is a $P \subseteq \mathcal{H}$ such that for all $\Sigma \subseteq \mathcal{S}$, $\mathsf{Mod}(\Sigma) \neq P$. Can you give an example of such a P which is countable?*

Exercise 3 *Show that for every finite $P \subseteq \mathcal{H}$ there is a $\Sigma \subseteq \mathcal{S}$ such that $\mathsf{Mod}(\Sigma) = P$.*

Next time, we will go over solutions to these exercises. We will then proceed to prove the compactness theorem for sentential logic (see Enderton, section 1.7). As part of the proof, we will present solutions to Enderton, exercises 1.7.1 & 2.