Processes in *doing* mathematics

R. Ramanujam

Azim Premji University, Bengaluru
email: jam@imsc.res.in

Nature of Math Course, Week 3

January 24, 2022

Transitions

The movement from ‘real world’ problems to formalization to mathematical discovery includes many transitions. (V.I. Arnold).

- Acknowledging and identifying these transitions is essential for understanding the nature of mathematics, both at the school and at the college level.

Transitions

The movement from ‘real world’ problems to formalization to mathematical discovery includes many transitions. (V.I. Arnold).

- Acknowledging and identifying these transitions is essential for understanding the nature of mathematics, both at the school and at the college level.
- Perhaps not surprisingly, these transitions are co-located with what are considered *difficult* topics for learning mathematics.

Transitions

The movement from ‘real world’ problems to formalization to mathematical discovery includes many transitions. (V.I. Arnold).

- Acknowledging and identifying these transitions is essential for understanding the nature of mathematics, both at the school and at the college level.
- Perhaps not surprisingly, these transitions are co-located with what are considered *difficult* topics for learning mathematics.
- They sharply contrast school mathematics and the discipline of mathematics.

Processes

The content areas of mathematics are clear: arithmetic, algebra, geometry, trigonometry, data analysis, probability, calculus.

Processes

The content areas of mathematics are clear: arithmetic, algebra, geometry, trigonometry, data analysis, probability, calculus. What about mathematical processes?

- formal problem solving,
- use of heuristics,
- estimation and approximation,
- optimisation,
- use of patterns, visualisation,
- abstraction and representation,
- reasoning and argumentation,
- making connections,
- mathematical communication.

Problem solving

Mathematics remains the main arena for students to experience and learn formal problem solving.

- General tactics: abstraction, quantification, analogy, case analysis, reduction to simpler situations, guess-and-verify, ...

Problem solving

Mathematics remains the main arena for students to experience and learn formal problem solving.

- General tactics: abstraction, quantification, analogy, case analysis, reduction to simpler situations, guess-and-verify, ...
- The use of heuristics.

Problem solving

Mathematics remains the main arena for students to experience and learn formal problem solving.

- General tactics: abstraction, quantification, analogy, case analysis, reduction to simpler situations, guess-and-verify, ...
- The use of heuristics.
- Problem posing is as important as problem solving.

Role of problem solving

Problem solving plays a major role in learning / doing mathematics.

- Problem solving, in the sense of Polya! The answer, if there is one, is not the end of the process. A shift has to be made from getting answers to gaining insight or constructing arguments.

Role of problem solving

Problem solving plays a major role in learning / doing mathematics.

- Problem solving, in the sense of Polya! The answer, if there is one, is not the end of the process. A shift has to be made from getting answers to gaining insight or constructing arguments.

The last is a transition that has nothing to do with mathematics itself, and is the big difference between school mathematics and the discipline.

Problem solving

In most mathematics classrooms, problem solving only means end of chapter exercises. Actually there is plenty:

- Problem solving for learning mathematical content.

Problem solving

In most mathematics classrooms, problem solving only means end of chapter exercises. Actually there is plenty:

- Problem solving for learning mathematical content.
- Learning problem solving.

Problem solving

In most mathematics classrooms, problem solving only means end of chapter exercises. Actually there is plenty:

- Problem solving for learning mathematical content.
- Learning problem solving.
- Learning mathematics through problem solving.

Problem solving

In most mathematics classrooms, problem solving only means end of chapter exercises. Actually there is plenty:

- Problem solving for learning mathematical content.
- Learning problem solving.
- Learning mathematics through problem solving.
- Open ended, exploratory problems are rare in mathematics classrooms.

What are good problems?

- Big and small problems: whose solutions require different durations.
  We must dispel the myth that any problem that cannot be solved in half an hour is unsolvable.

What are good problems?

- Big and small problems: whose solutions require different durations.
  We must dispel the myth that any problem that cannot be solved in half an hour is unsolvable.
- Open problems: The fact that there are problems which no adult has been able to solve for centuries, and yet may be solved by someone in the future, tells the student that mathematics is alive.

Abstract problems

It is not necessary that the problem context is a “real life” situation. Abstract problems are quite alright as well.

Abstract problems

It is not necessary that the problem context is a "real life" situation. Abstract problems are quite alright as well.

- A seemingly ill-structured problem like this one: Consider six towns in flatland. The least distance between any two is m, the maximum distance between any two is M. Show that $M/m \geq \sqrt{3}$.

Abstract problems

It is not necessary that the problem context is a “real life” situation. Abstract problems are quite alright as well.

- A seemingly ill-structured problem like this one: Consider six towns in flatland. The least distance between any two is m, the maximum distance between any two is M. Show that $M/m \geq \sqrt{3}$.
- Or a “classical” but definitely exploratory problem such as: Show that $5555^{2222} + 2222^{5555}$ is divisible by 7.

Abstract problems

It is not necessary that the problem context is a "real life" situation. Abstract problems are quite alright as well.

- A seemingly ill-structured problem like this one: Consider six towns in flatland. The least distance between any two is m, the maximum distance between any two is M. Show that $M/m \geq \sqrt{3}$.
- Or a "classical" but definitely exploratory problem such as: Show that $5555^{2222} + 2222^{5555}$ is divisible by 7.
- On a geoboard (or equivalently on graph paper) draw as many rhombi as possible whose sides are 5 units long.

The practice of mathematics

Doing mathematics often means:

- Selecting between representations or devising new ones,
- Looking for invariances,
- Observing extreme cases and typical ones to come up with conjectures,
- Looking actively for counterexamples,
- Simplifying or generalising problems to make them easier to address,
- Building on answers to generate new questions for exploration,

and so on. These are mostly missing in our mathematics learning.

Reasoning

Mathematics remains the principal domain for formal reasoning.

- George Polya talks of Demonstration, Deduction and Derivation.

Reasoning

Mathematics remains the principal domain for formal reasoning.

- George Polya talks of Demonstration, Deduction and Derivation.
- A proof provides both justification and explanation.

Reasoning

Mathematics remains the principal domain for formal reasoning.

- George Polya talks of Demonstration, Deduction and Derivation.
- A proof provides both justification and explanation.
- A derivation in a formal system provides only justification.

Reasoning

Mathematics remains the principal domain for formal reasoning.

- George Polya talks of Demonstration, Deduction and Derivation.
- A proof provides both justification and explanation.
- A derivation in a formal system provides only justification.
- A demonstration is a deduction whose premises are known to be true; according to Aristotle, a demonstration produces knowledge.

Communication

A distinct characteristic of mathematics is the use of a highly stylised formal symbolic language.

- Mathematical communication

Communication

A distinct characteristic of mathematics is the use of a highly stylised formal symbolic language.

- Mathematical communication
- Rigour in formulation

Communication

A distinct characteristic of mathematics is the use of a highly stylised formal symbolic language.

- Mathematical communication
- Rigour in formulation
- Choice of notation

Communication

A distinct characteristic of mathematics is the use of a highly stylised formal symbolic language.

- Mathematical communication
- Rigour in formulation
- Choice of notation
- Conventions to help make connections across areas

Multiplicity

The tyranny of the one right answer, to be obtained by the one algorithm that has been taught in class.

- How do people subtract in shops ?

Multiplicity

The tyranny of the one right answer, to be obtained by the one algorithm that has been taught in class.

- How do people subtract in shops ?
- How do farmers estimate yields of crops ?

Multiplicity

The tyranny of the one right answer, to be obtained by the one algorithm that has been taught in class.

- How do people subtract in shops ?
- How do farmers estimate yields of crops ?
- How do sculptors decide proportions of statues ?

Multiplicity

The tyranny of the one right answer, to be obtained by the one algorithm that has been taught in class.

- How do people subtract in shops ?
- How do farmers estimate yields of crops ?
- How do sculptors decide proportions of statues ?
- How do analysts estimate convergence of series ?

Multiplicity

The tyranny of the one right answer, to be obtained by the one algorithm that has been taught in class.

- How do people subtract in shops ?
- How do farmers estimate yields of crops ?
- How do sculptors decide proportions of statues ?
- How do analysts estimate convergence of series ?
- How do geometers construct surfaces ?

A quote

Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now. ...
The pupil himself should reinvent mathematics. During this process, the learner is engaged in an activity where experience is described, organised and interpreted by mathematical means. This activity is mathematising.

Hans Freudenthal, Revisiting Mathematics Education, 1991.