Euclid's 5 axioms of geometry

objects : points, lines

1) Can ~~to~~ draw a straight line from any point to any point.

2) Can extend a finite straight lines continuously in a straight line.

3) ~~To describe~~ Can describe a circle with any centre & radius.

4) All right angles are equal to each other.

5) Given a straight line & a point not on it, there is exactly one line that can be drawn passing through the given point that doesn't meet the given line.

Propositional Logic

Atomic propositions $P = \{p, q, \text{----}\}$ countably infinite set
assumption for technical convenience

Set of logical connectives ~~operators~~
& operators

$C = \{\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg\}$, Parentheses: $T = \{(,)\}$

conjunction

p
$p \wedge q$ → well-formed formulas $(P \cup C \cup T)^*$

$p \wedge \wedge$ → not well-formed

Clear notion for checking "well-formed"-ness of a formula.

* Syntax

The set of WFF (well formed formulas) Φ is the smallest set satisfying the following:

i) $P \subseteq \Phi$

ii) if $\varphi \in \Phi$ then $\neg(\varphi) \in \Phi$.

iii) $\varphi_1, \varphi_2 \in \Phi$ then $(\varphi_1)\wedge(\varphi_2) \in \Phi$, $(\varphi_1)\vee(\varphi_2) \in \Phi$, $(\varphi_1) \Rightarrow (\varphi_2) \in \Phi$ & $(\varphi_1) \Leftrightarrow (\varphi_2) \in \Phi$

$$\varphi = p \mid \neg(\varphi) \mid (\varphi)\wedge(\varphi) \mid (\varphi)\vee(\varphi) \mid (\varphi) \Rightarrow (\varphi) \mid (\varphi) \Leftrightarrow (\varphi)$$

(WFFs ~~context~~ form context-free language.)

* Truth valuations: function $v : P \longrightarrow \{T, \perp\}$

Now, we want to extend truth valuation to the set of all WFFs (semantics)

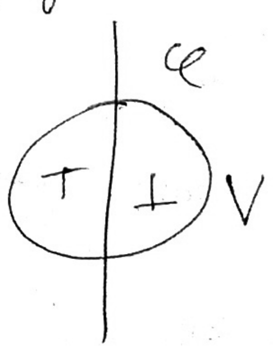
Let V = set of all truth valuations.

Assign meaning to the set of all WFFs.

$[\![\varphi]\!] \leftarrow$ symbol used to denote the meaning of the WFF φ.

$$[\![\varphi]\!] : V \longrightarrow \{T, \perp\}$$

$[\![\varphi]\!](v) = T$, $v \models \varphi$



$\varphi_1 = \neg(p \wedge q)$; $\varphi_2 = \neg p \vee \neg q$

How to say φ_1 & φ_2 ~~mean~~ have the same meaning?

One way is, for all valuations $v \in V$, $v \models \varphi_1$ iff $v \models \varphi_2$

(or)

$$[\![\varphi_1]\!] = [\![\varphi_2]\!]$$

* By induction on structure of φ.

& Suppose Π : some property of WFFs

To prove : Π holds $\forall$ WFFs.

idea: Principle of induction on structure

Suppose Π is true $\forall$ $p \in P$

If Π is true for φ, then it is also true for $\neg(\varphi)$.

If Π is true for $\varphi_1, \varphi_2 \in \Phi$, then it is true for $(\varphi_1) \vee (\varphi_2)$, $(\varphi_1) \wedge (\varphi_2)$, $(\varphi_1) \Rightarrow (\varphi_2)$ & $(\varphi_1) \Leftrightarrow (\varphi_2)$

The Π is true for all WFFs.

* Definition of $v \models \varphi$ by induction on structure of φ

$v \models p$ iff $v(p) = \top$

$v \models \neg(\varphi)$ iff $v \not\models \varphi$

$v \models (\varphi_1) \wedge (\varphi_2)$ iff $v \models (\varphi_1)$ and $v \models \varphi_2$

$v \models (\varphi_1) \vee (\varphi_2)$ iff $v \models \varphi_1$ or $v \models \varphi_2$

$v \models (\varphi_1) \Rightarrow (\varphi_2)$ iff $v \not\models \varphi_1$ or $v \models \varphi_2$

$v \models (\varphi_1) \Leftrightarrow (\varphi_2)$ iff $(v \models \varphi_1$ and $v \models \varphi_2)$
or $(v \not\models \varphi_1$ and $v \not\models \varphi_2)$

Defⁿ: A formula is satisfiable if there is a truth ~~reduction~~ valuation v s.t. $v \models \varphi$.

Defⁿ: A formula φ is valid if for every truth valuation v, $v \models \varphi$.

Example instance

objects : slots, courses, students, halls

Problem : Schedulability.

Axioms of this system :

- Every course is associated with 2 slots.
- Students can take courses.
- Every course should be associated with a hall
- If a student is taking 2 courses then these 2 courses can't be associated with the same slot.
- If a slot is associated with two courses then these two courses can't be associated with the same hall.

Object sets :

$$\text{Students} = \{S_1, \ldots, S_m\}$$

$$\text{Courses} = \{C_1, \ldots, C_n\}$$

$$\text{Slots} = \{T_1, \ldots, T_r\}$$

$$\text{Halls} = \{H_1, \ldots, H_n\}$$

$$s\_c : \text{Students} \times \text{Courses} \to \{\top, \bot\}$$

$$\{a_{ct} \mid c \in \text{Courses}, t \in \text{slots}\}$$

$$\{a_{ch} \mid c \in \text{courses}, h \in \text{Halls}\}$$

$$\bigwedge_{c \in \text{Courses}} \bigvee_{\substack{(t_1,t_2) \\ \in \text{slots} \times \text{slots} \\ t_1 \neq t_2}} \left((a_{ct_1} \wedge a_{ct_2}) \wedge \bigwedge_{\substack{t \neq t_1 \\ t \neq t_2}} \neg a_{ct} \right)$$

$$\bigwedge_{c \in \text{Courses}} \bigvee_{h \in \text{Halls}} \left(a_{ch} \wedge \bigwedge_{h' \in \text{Halls} \setminus \{h\}} \neg a_{ch'} \right)$$

$$\bigwedge_{s}$$

Suppose each edge of an undirected graph is colored using one of the three colors - red, blue or green. Consider the following property of such graphs: if any vertex is the end point of a red colored edge, then it is either an end point of a blue colored edge or not an end point of any green colored edge. If a graph doesn't satisfy this property, which of the following statements about G are true?

(a) There is a red colored edge

(b). Any vertex that is the end point of a red colored edge is also the end point of a green colored edge.

(c) There is a vertex that is not an end point of any blue colored edge but is an end point of a green colored & a red colored edge.

(d) (a) & (c).

r : ~~v~~ v is the end point of a red colored edge

b : v is an end pt. of a blue colored edge

g : v is an end pt. of a green colored edge.

$\forall v \; (r(v) \Rightarrow b(v) \vee \neg g(v))$

$\exists$ v ✓

$\exists v \; r(v) \wedge (\neg b(v) \wedge g(v))$ true ✓

~~$\forall v \; \neg r(v)$~~ $\forall v \; r(v)$ ~~$\forall v \; \neg r(v)$~~

$\exists v \; r(v)$. ~~$\forall$~~ $\forall v \; r(v) \Rightarrow$

$\neg b(v) \wedge g(v)$

---

v_r : vertex v is the end point of a red colored edge

v_b : blue

v_g : green

$\neg (v_r \Rightarrow (v_b \vee \neg v_g))$

$\Rightarrow v_r \wedge (\neg v_b \wedge v_g)$

Q) Abs - Sh

not - ap - Ati - bay

φ_1

φ_2

φ_3

$\varphi_1 \wedge \varphi_2 \quad \varphi_3$

Any truth assignment that satisfies $\varphi_1 \wedge \varphi_2$ also satisfies φ_3

$\varphi \wedge \varphi_2$ logically entails φ_3 , $\varphi_1 \wedge \varphi_2 \models \varphi_3$

Q) p : "x is a prime no"

q : x is odd

1) x being prime is a sufficient condition for x being odd.

$p \Rightarrow q$

2) $\underbrace{x \text{ being odd}}_{c_1}$ is a necessary condition for $\underbrace{x \text{ being prime}}_{c_2}$ $\quad c_2 \Rightarrow c_1$

$p \Rightarrow q$

3)

$\cancel{(I, i, j)}$

$\cancel{(I, i, j) = T \quad \forall i = \{1, 2, \dots, 9\} \quad \forall j}$

$$\bigwedge_{(k,r)\in I\times I} \; \bigvee_{n\in I} (m, n, k)$$

$$\bigwedge_{(k,r)\in I\times I} \; \bigvee_{m\in I} (m, n, k)$$

$I_1 = \{1, 2, 3\}$

$\forall k \in I \quad \exists \; m, n \in \{1, 2, 3\} \; (m, n, k) = T$

$$\bigwedge_{k\in I} \; \bigvee_{m,n\in I_1\times I_1} (m, n, k)$$

Limits of logical reasoning

Lemma: A formula φ is valid iff $\neg\varphi$ is not satisfiable.

Pf: ($\Rightarrow$): Suppose φ is valid

$\Rightarrow$ Every truth valuation v satisfies φ

i.e., $\forall v, v \models \varphi$

~~Hence, $v \not\models \neg\varphi$~~

Hence for every v, $v \not\models \neg\varphi$. Hence $\neg\varphi$ is not satisfiable
(By semantics of logic)

($\Leftarrow$): Suppose $\neg\varphi$ is not satisfiable

$\Rightarrow$ For every v, $v \not\models \neg\varphi$ (By defn of satisfiability)

$\Rightarrow$ For every v, $v \models \varphi$ (By semantics of prop. logic)

$\square$.

Hence φ is valid.

$\mathbb{P}$ = set of atomic propositions

$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \Rightarrow \varphi \mid \varphi \Leftrightarrow \varphi$

$[\![\varphi]\!] : V \longrightarrow \{\top, \bot\}$ (Semantics (assigning meaning))

Def^{ns}:

Vocabulary of a formula φ: set of atomic propositions occurring in φ.

$Voc(\varphi) \subseteq \mathbb{P}$

$Voc(p) = \{p\}$, $Voc(\neg\varphi) = Voc(\varphi)$, ~~Voc~~

$Voc(\varphi_1 \wedge \varphi_2) = Voc(\varphi_1 \vee \varphi_2) = voc(\varphi_1 \Rightarrow \varphi_2) = voc(\varphi_1 \Leftrightarrow \varphi_2)$

$= voc(\varphi_1) \cup voc(\varphi_2)$

* Lemma: Let φ be a formula & v_1, v_2 be 2 truth assignments s.t. $v_1(p) = v_2(p)\ \forall p \in voc(\varphi)$. Then $v_1 \models \varphi$ iff $v_2 \models \varphi$.

Pf: By induction on the structure of φ.

Base case: $\varphi = p$

$v_1 \models \varphi$ iff $v_1(p) = \top \Leftrightarrow v_2(p) = \top \Leftrightarrow v_2 \models \varphi$

Induction step: (cases)

$\varphi = \neg\psi$:

$v_1 \models \varphi$ iff $v_1 \not\models \psi$ iff $v_2 \not\models \psi$ iff $v_2 \models \varphi$

$\underline{\varphi = \varphi_1 \wedge \varphi_2:}$

$v_1 \models \varphi_1 \wedge \varphi_2$ iff $(v_1 \models \varphi_1 \ \& \ v_1 \models \varphi_2)$ iff $(v_2 \models \varphi_1 \ \& \ v_2 \models \varphi_2)$
iff $v_2 \models \varphi_1 \wedge \varphi_2$

Similar argument follows for the cases:-
$\varphi = \varphi_1 \Rightarrow \varphi_2$, $\varphi = \varphi_1 \Leftrightarrow \varphi_2$

<u>Def</u>^{n}: A formula φ logically entails ψ $(\varphi \models \psi)$ if every truth assignment that satisfies φ also satisfies ψ.

<u>Lemma</u>: Let φ & ψ be formulas. Then $\varphi \models \psi$ iff $\varphi \Rightarrow \psi$ is valid.

Pf- $(\Rightarrow:)$ Suppose $\varphi \models \psi$. T.P: $\varphi \Rightarrow \psi$ is valid.
Let v be any truth assignment.
Case: if $v \not\models \varphi$ then $v \models \varphi \Rightarrow \psi$ by semantics
Case otherwise: $v \models \varphi \Rightarrow v \models \psi$ ($\because \varphi \models \psi$) So $v \models \varphi \Rightarrow \psi$)

$(\Leftarrow:)$ Suppose $\varphi \Rightarrow \psi$ is valid
T.P: $\varphi \models \psi$, ~~Let v~~
Let $v \models \varphi$
$\because \varphi \Rightarrow \psi$ is valid, $v \models \varphi \Rightarrow \psi$, so by semantics
$v \models \psi$. $\square$

We write $v \models X$ (X is a set of formulas) when $v \models \varphi$ for all $\varphi \in X$. $\boxed{X \Rightarrow \varphi}$ $v \models \varphi$.

$X \models \varphi$ iff for every truth assignment v, if $v \models X$ then $v \models \varphi$.

$\varphi = \varphi_1 \wedge \varphi_2$

$v \models \varphi$ $\tilde{\varphi} = \varphi_1 \vee \varphi_2$.

$X \models \varphi$ iff $X \Rightarrow \varphi$ is valid $v \models \{\varphi_1, \varphi_2\}$.

$\leftrightarrow v \models \varphi$

* We need a set X s.t. X is not satisfiable but every ~~propery~~ proper subset is satisfiable.

$$X_4 = \{p \Rightarrow q,\ q \Rightarrow r,\ r \Rightarrow \neg p,\ p\}$$

* Finite satisfiability lemma:

Let X be a set of formulas s.t. every finite subset of X is satisfiable, then X is satisfiable.

Pf. (we prove the contrapositive of the statement)

Suppose X is not satisfiable.

This means any truth valuation will falsify at least one formula in X.

$$V \models X^{-} \longleftrightarrow V \VDash X$$

Compactness: If $X \models \varphi$, then there is a finite subset $X_f \subseteq X$ s.t. $X_f \models \varphi$.

Pf: If $X \models \varphi$, $X \cup \{\neg\varphi\}$ is not satisfiable.

So there is a finite subset $X_f \subseteq X \cup \{\neg\varphi\}$ that is not satisfiable.

Case I: X_f doesn't contain $\neg\varphi$. So $X_f \subseteq X$ & X_f is not satisfiable, so $X_f \models \varphi$.

Case 2: X_f contains $\neg\varphi$

$\Rightarrow X_f = X_f' \cup \{\neg\varphi\}$. So $X_f' \models \varphi$ (claim)

Suppose $X_f' \not\models \varphi$

$\Rightarrow \exists$ a valuation v s.t. $v \models X_f'$ & $v \not\models \varphi$

$\Rightarrow v \models \neg\varphi$

$\Rightarrow v \models X_f$ (as $X_f = X_f' \cup \{\neg\varphi\}$)

$\Rightarrow\Leftarrow$ (as X_f is not satisfiable).
(By assumption)

× **Proof Systems** (Systems meant to write formal proofs)

Axioms : Formulas taken to be true for granted.

Derivation rules : ways to derive new formulas from those that have already been derived.

Eg.

$$\frac{\varphi \quad \varphi \Rightarrow \psi}{\psi} \text{ Modus Ponens}$$

~~Derivation~~

Points:

i) Give a sound basis for a system of inference.

ii) Mechanical verifiability of the correctness of proofs. → Reliable (Here correctness matters more than comprehensibility)

Syntactic : $\varphi = p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi$ (we don't care about the meaning assigned / semantics)

(No need to assign any meaning)

Derivation : A sequence (finite) of formulas $\varphi_1, \ldots \varphi_n$ s.t. each φ_i is either an instance of an axiom or is derived from formulas that appear earlier in the sequence using one or more of the derivation rules.

* **Hibert's Proof system for propositional logic:**

Purpose of this system (main) : To be able to prove meta-theorems

↳ about the system itself (not inside the system)

↓

Compromises on human comprehension.

Syntax : $\varphi = p \mid \neg\varphi \mid \varphi \Rightarrow \varphi$

Axioms:

$A_1 : \varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_1)$

$A_2 : \varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_3) \Rightarrow ((\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\varphi_1 \Rightarrow \varphi_3))$

$A_3 : (\neg\varphi_1 \Rightarrow \neg\varphi_2) \Rightarrow ((\neg\varphi_1 \Rightarrow \varphi_2) \Rightarrow \varphi_1)$ → Reductio ad Asurdum

$$\frac{\varphi_1 \quad \varphi_1 \Rightarrow \varphi_2}{\varphi_2}$$ Modus Ponens (Only 1 inference rule)

Derive : $\varphi \Rightarrow \varphi$

1) A_2 : $\varphi_1 = \varphi$, $\varphi_2 = \varphi \Rightarrow \varphi$, $\varphi_3 = \varphi$ (An instance of A_2)

$\underbrace{(\varphi \Rightarrow ((\varphi \Rightarrow \varphi) \Rightarrow \varphi))}_{\varphi_1} \Rightarrow ((\varphi \Rightarrow (\varphi \Rightarrow \varphi)) \Rightarrow (\varphi \Rightarrow \varphi))$

2) $\underbrace{(\varphi \Rightarrow ((\varphi \Rightarrow \varphi) \Rightarrow \varphi))}_{\varphi_1}$ [Instance of A_1]

3) $(\varphi \Rightarrow (\varphi \Rightarrow \varphi)) \Rightarrow (\varphi \Rightarrow \varphi)$ [MP 1,2]

1. $\varphi \Rightarrow (\varphi \Rightarrow \varphi)$ [A_1]

5. $\varphi \Rightarrow \varphi$ [MP 3, 4]

~~Defn~~: ~~Derive~~

φ is <u>derivable</u> if there is a derivation where the last formula is φ.

* In Hibert proof system,

φ is derivable $\overset{\text{soundness}}{\Longrightarrow}$ φ is valid

" " " $\underset{\text{completeness}}{\Longleftarrow}$ " " "

Both are true in Hibert's system

$\Gamma \vdash \varphi$ → means φ is derivable from the system Γ ~~(set of axioms & inference rules)~~ (set of props)

$\varphi_0, \varphi_1, \ldots, \varphi_n$

~~$\vdash$~~ To prove:

$$\{\varphi_1 \Rightarrow \varphi_2, \varphi_2 \Rightarrow \varphi_3\} \vdash \varphi_1 \Rightarrow \varphi_3$$

1) $\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_3) \Rightarrow ((\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\varphi_1 \Rightarrow \varphi_3))$ [A_2]

2) $\varphi_2 \Rightarrow \varphi_3$ [Γ]

3) $(\varphi_2 \Rightarrow \varphi_3) \Rightarrow (\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_3))$

4) $\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_3)$ [MP 2,3]

5) $(\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\varphi_1 \Rightarrow \varphi_3)$ [MP 1,4]

6) $\varphi_1 \Rightarrow \varphi_2$ (Γ)

7) $\varphi_1 \Rightarrow \varphi_3$ (MP 5,6)

Intuition says : $\models \psi_1 \wedge \psi_2 \Rightarrow \psi_3$

<u>Deduction</u> <u>theorem</u> : $\Gamma \cup \{\varphi\} \vdash \psi$ iff $\Gamma \vdash \varphi \Rightarrow \psi$

<u>Proof</u> : ($\Leftarrow$): Suppose $\Gamma \vdash \varphi \Rightarrow \psi$

claim : $\Gamma \cup \{\varphi\} \vdash \psi$

$$\Gamma \cup \{\varphi\} \vdash \varphi \Rightarrow \psi$$

Also, $\Gamma \cup \{\varphi\} \vdash \varphi$ (By the extended def^n of derivabilty)

By MP, $\Gamma \cup \{\varphi\} \vdash \psi$ //

($\Rightarrow$): Suppose $\Gamma \cup \{\varphi\} \vdash \psi$

To prove : $\Gamma \vdash \varphi \Rightarrow \psi$

let $\Gamma \cup \{\varphi\} \vdash \psi$ be $\psi_1, \psi_2, \ldots, \psi_n$, where $\psi_n = \psi$

we will prove that $\Gamma \vdash \varphi \Rightarrow \psi_i \ \forall \ i \leq n$, by induction on i.

<u>Base case</u> : $i = 1$:

<u>Case 1</u> : ψ_1 is an instance of an axiom

1. $\vdash \psi_1$
2. $\vdash \psi_1 \Rightarrow (\varphi \Rightarrow \psi_1)$ [A_1]
3. $\varphi \Rightarrow \psi_1$ (MP 1,2)
4. $\Gamma \vdash \varphi \Rightarrow \psi_1$ (Without any hypothesis, we can derive it so throwing in any hypothesis won't change the outcome)

Case 2: $\Psi_1 \in \Gamma$

1. $\Gamma \vdash \Psi_1$

2. $\Gamma \vdash \Psi_1 \Rightarrow (\varphi \Rightarrow \Psi_1)$ {can be derived without any hypothesis (axiom A1) so, no problem by putting in a hypo}

3. $\Gamma \vdash \varphi \Rightarrow \Psi_1$ (MP 1, 2)

Case 3: $\Psi_1 = \varphi$

T.P: $\Gamma \vdash \varphi \Rightarrow \varphi$

Induction hypo: $\forall i < k, \Gamma \vdash \varphi \Rightarrow \Psi_i$.

Inductive step: T.P: $\Gamma \vdash \varphi \Rightarrow \Psi_k$

Case: Ψ_k is an instance of an axiom or $\Psi_k \in \Gamma \cup \{\varphi\}$ ✓ we are done

Case: Ψ_k is obtained by applying MP to $\Psi_j, \Psi_j \Rightarrow \Psi_k$

By I.H, $\Gamma \vdash \varphi \Rightarrow \Psi_j$ and $\Gamma \vdash \varphi \Rightarrow (\Psi_j \Rightarrow \Psi_k)$

$\Gamma \vdash \varphi \Rightarrow (\Psi_j \Rightarrow \Psi_k) \Rightarrow ((\varphi \Rightarrow \Psi_j) \Rightarrow (\varphi \Rightarrow \Psi_k))$ [A_2]

By MP, $\Gamma \vdash (\varphi \Rightarrow \Psi_j) \Rightarrow (\varphi \Rightarrow \Psi_k)$.

By MP, $\Gamma \vdash \varphi \Rightarrow \Psi_k$. $\square$

* <u>Soundness</u> : If $\vdash \varphi$, then $\models \varphi$

Pf- Let $\vdash \varphi$ be $\psi_1, \psi_2, \psi_3, \ldots, \psi_n$ s.t. $\psi_n = \varphi$.
We will prove that $\models \psi_i, \forall i \le n$, by induction on i

<u>Base Case</u>: <u>$i=1$</u> : ~~Only one~~

$\because$ ψ_1 is derivable & its the first hypothesis.

ψ_1 must be an instance of an axiom.

$\models \psi_1$, by exhaustive check (draw the truth table & bruteforce to show its valid)

<u>Induction step</u>: For all $i < k$, $\models \psi_i$

T.P : $\models \psi_k$

If ψ_k is an instance of an axiom, $\models \psi_k$ by exhaustive check.

otherwise, ψ_k is obtained by MP on $\psi_j, \psi_j \Rightarrow \psi_k$.
By IH, $\models \psi_j$ & $\models \psi_j \Rightarrow \psi_k$. Any truth assignment that satisfies ψ_j also satisfies ψ_k.
Every truth assignment satisfies ψ_j, so every truth assignment satisfies ψ_k, i.e., $\models \psi_k$. $\square$