The nim game and the mathematical language

R. Ramanujam

Azim Premji University, Bengaluru
email: jam@imsc.res.in

Nature of Math Course, Week 5

February 1, 2022

Nim

Nim, from the German word nehmen, analysed in 1904 by C. L. Bouton of Harvard University.‘

- Two players I and II, move alternately.
- The game is played with m piles of counters.
- When a player moves, she picks a pile and removes some non-zero many counters from that pile.
- When a player cannot move, he loses (and the other wins).

Ingredients

Every game has three main ingredients:

- The set of players, often $\{I, II\}$. In general, $[n] = \{1, 2, \ldots, n\}$.
- The rules of the game, that specify, at any game position, whose turn it is to move, what moves are applicable, and the resulting new game position after any move.
- Outcomes or winning conditions, that specify at which positions the game is over, and perhaps depending on the course of play, the outcome at those positions.

Backward induction

Zermelo 1913: In every finite extensive form game of perfect information, we can compute whether player i can win (or not).

- Theorem: Backward induction shows who wins, gives a winning strategy in the case of win / lose games, and an NE for general games.
- Note that the game arena for any Nim heap is acyclic and hence the unfolding is a finite tree, so BI applies.

End of theory?

Backward induction completely solves finite extensive form games of perfect information, so we might as well go home.

- So the Nim game is solved, isn't it ?

End of theory?

Backward induction completely solves finite extensive form games of perfect information, so we might as well go home.

- So the Nim game is solved, isn't it ?
- If we are only interested in existence of winning strategies, this suffices. If we also wish to look at the structure of strategies, this leaves us quite unsatisfied.
- Indeed, in the case of Nim, combinatorial analysis offers more.

Simple cases

Consider the Nim game $(1, 1, 2)$, and you are to move. How would you play ?

Simple cases

Consider the Nim game $(1, 1, 2)$, and you are to move. How would you play ?

- It is easy to see that removing one from the heap with 2 counters is not a good idea.

Simple cases

Consider the Nim game $(1, 1, 2)$, and you are to move. How would you play ?

- It is easy to see that removing one from the heap with 2 counters is not a good idea.
- But removing all counters from that heap works.
- We can call $(1, 1, 2)$ a winning position (for whoever plays) and $(1, 1)$ a losing position (for whoever plays).

Simple cases

Consider the Nim game $(1,1,2)$, and you are to move. How would you play ?

- It is easy to see that removing one from the heap with 2 counters is not a good idea.
- But removing all counters from that heap works.
- We can call $(1,1,2)$ a winning position (for whoever plays) and $(1,1)$ a losing position (for whoever plays).
- We can see that $(1,1,1)$ is a winning position.

Simple cases

Consider the Nim game $(1,1,2)$, and you are to move. How would you play ?

- It is easy to see that removing one from the heap with 2 counters is not a good idea.
- But removing all counters from that heap works.
- We can call $(1,1,2)$ a winning position (for whoever plays) and $(1,1)$ a losing position (for whoever plays).
- We can see that $(1,1,1)$ is a winning position.
- Can we see that a k-tuple $(1,1,\ldots,1)$ is a winning position iff k is odd ?

The copy strategy

Consider the case of two heaps (m, n).

- Suppose $m = n = 4$, say. Now, whatever move I plays on one heap, II can copy that move on the other heap, thus making the heaps equal again. So this is a losing position for player I.

The copy strategy

Consider the case of two heaps (m, n).

- Suppose $m = n = 4$, say. Now, whatever move I plays on one heap, II can copy that move on the other heap, thus making the heaps equal again. So this is a losing position for player I.
- On the other hand, given heaps of unequal size, player I can equalize them and present II with equal heaps (which is losing for II).

The copy strategy

Consider the case of two heaps (m, n).

- Suppose $m = n = 4$, say. Now, whatever move I plays on one heap, II can copy that move on the other heap, thus making the heaps equal again. So this is a losing position for player I.
- On the other hand, given heaps of unequal size, player I can equalize them and present II with equal heaps (which is losing for II).
- Lemma: For all $m, n \geq 0$. (m, n) is winning iff $m \neq n$.

Subgames

With three or more heaps, analysis becomes difficult: what can we say about $(2, 3, 6)$, for instance ?

Subgames

With three or more heaps, analysis becomes difficult: what can we say about $(2, 3, 6)$, for instance ?

- Observe that every finite extensive form game is of the form 0 or
$$g_1 + g_2 + \ldots + g_m$$
.
- 0 can be thought of as the empty game (in which no player can make any move).
- $g_1, g_2, \ldots, g_m$ are subgames.

Sum of games

Choosing between subgames has an interesting algebraic structure.

- Suppose $g = g_1 + g_2 + \ldots + g_m$.
- Also suppose $h = h_1 + h_2 + \ldots + h_n$.
- Then

$$g + h = (g_1 + h) + \ldots + (g_m + h) + (g + h_1) + \ldots + (g + h_n)$$

.

Sum of games

Choosing between subgames has an interesting algebraic structure.

- Suppose $g = g_1 + g_2 + \ldots + g_m$.
- Also suppose $h = h_1 + h_2 + \ldots + h_n$.
- Then

  $$g + h = (g_1 + h) + \ldots + (g_m + h) + (g + h_1) + \ldots + (g + h_n)$$

  .
- This suggests the notation $1 + 3 + 6$ for the nim game $(1, 3, 6)$.
- When g is a subgame of h, we write $g \leq h$.

Equivalence of games

Once we have game expressions, a natural question arises:
when do we consider two games g and g' to be equivalent ?

Equivalence of games

Once we have game expressions, a natural question arises: when do we consider two games g and g' to be equivalent ? (This is only for the analysis of Nim, we will return to these considerations again later.)

- Definition: $g_1 \equiv g_2$ if, for all h, $g_1 + h$ is winning (losing) iff $g_2 + h$ is winning (losing).

Equivalence of games

Once we have game expressions, a natural question arises: when do we consider two games g and g' to be equivalent ? (This is only for the analysis of Nim, we will return to these considerations again later.)

- Definition: $g_1 \equiv g_2$ if, for all h, $g_1 + h$ is winning (losing) iff $g_2 + h$ is winning (losing).
- Sanity check: $\equiv$ is indeed an equivalence relation.

Equivalence of games

Once we have game expressions, a natural question arises: when do we consider two games g and g' to be equivalent ? (This is only for the analysis of Nim, we will return to these considerations again later.)

- Definition: $g_1 \equiv g_2$ if, for all h, $g_1 + h$ is winning (losing) iff $g_2 + h$ is winning (losing).
- Sanity check: $\equiv$ is indeed an equivalence relation.
- Sanity check: $g + 0 = g$.

Equivalence of games

Once we have game expressions, a natural question arises: when do we consider two games g and g' to be equivalent ? (This is only for the analysis of Nim, we will return to these considerations again later.)

- Definition: $g_1 \equiv g_2$ if, for all h, $g_1 + h$ is winning (losing) iff $g_2 + h$ is winning (losing).
- Sanity check: $\equiv$ is indeed an equivalence relation.
- Sanity check: $g + 0 = g$.
- Therefore, if $g_1 \equiv g_2$ then g_1 is winning iff g_2 is winning. But the converse is not true.

Equivalence of games

Once we have game expressions, a natural question arises: when do we consider two games g and g' to be equivalent ? (This is only for the analysis of Nim, we will return to these considerations again later.)

- Definition: $g_1 \equiv g_2$ if, for all h, $g_1 + h$ is winning (losing) iff $g_2 + h$ is winning (losing).
- Sanity check: $\equiv$ is indeed an equivalence relation.
- Sanity check: $g + 0 = g$.
- Therefore, if $g_1 \equiv g_2$ then g_1 is winning iff g_2 is winning. But the converse is not true.
- However, all losing games are equivalent, to 0.

The loser's lemma

Lemma: If g is losing then $g \equiv 0$.

The loser's lemma

Lemma: If g is losing then $g \equiv 0$.

- Fix a losing game g.

The loser's lemma

Lemma: If g is losing then $g \equiv 0$.

- Fix a losing game g.
- To prove: for all h, $g + h$ is losing iff h is losing.

The loser's lemma

Lemma: If g is losing then $g \equiv 0$.

- Fix a losing game g.
- To prove: for all h, $g + h$ is losing iff h is losing.
- We prove by induction on h that whenever h is losing, so is $g + h$.

The loser's lemma

Lemma: If g is losing then $g \equiv 0$.

- Fix a losing game g.
- To prove: for all h, $g + h$ is losing iff h is losing.
- We prove by induction on h that whenever h is losing, so is $g + h$.
- Assuming this, the other direction is easy:

The loser's lemma

Lemma: If g is losing then $g \equiv 0$.

- Fix a losing game g.
- To prove: for all h, $g + h$ is losing iff h is losing.
- We prove by induction on h that whenever h is losing, so is $g + h$.
- Assuming this, the other direction is easy: Suppose h is winning. Then there is a move to h' that's losing.

The loser's lemma

Lemma: If g is losing then $g \equiv 0$.

- Fix a losing game g.
- To prove: for all h, $g + h$ is losing iff h is losing.
- We prove by induction on h that whenever h is losing, so is $g + h$.
- Assuming this, the other direction is easy: Suppose h is winning. Then there is a move to h' that's losing. By claim above, $g + h'$ is losing.

The loser's lemma

Lemma: If g is losing then $g \equiv 0$.

- Fix a losing game g.
- To prove: for all h, $g + h$ is losing iff h is losing.
- We prove by induction on h that whenever h is losing, so is $g + h$.
- Assuming this, the other direction is easy: Suppose h is winning. Then there is a move to h' that's losing. By claim above, $g + h'$ is losing. Hence there is a winning move in $g + h$, and hence $g + h$ is winning.

The loser's lemma

We prove by induction on h that whenever h is losing, so is $g + h$.

The loser's lemma

We prove by induction on h that whenever h is losing, so is $g + h$.

- IH1: for all $g' \leq g$, if h is losing, then so is $g' + h$.

The loser's lemma

We prove by induction on h that whenever h is losing, so is $g + h$.

- IH1: for all $g' \leq g$, if h is losing, then so is $g' + h$.
- IH2: for all $h' \leq h$, if h' is losing, then so is $g + h'$.

The loser's lemma

We prove by induction on h that whenever h is losing, so is $g + h$.

- IH1: for all $g' \leq g$, if h is losing, then so is $g' + h$.
- IH2: for all $h' \leq h$, if h' is losing, then so is $g + h'$.
- Every initial move in $g + h$ is either in g or in h. First consider the latter.

The loser's lemma

We prove by induction on h that whenever h is losing, so is $g + h$.

- IH1: for all $g' \leq g$, if h is losing, then so is $g' + h$.
- IH2: for all $h' \leq h$, if h' is losing, then so is $g + h'$.
- Every initial move in $g + h$ is either in g or in h. First consider the latter.
- h is losing, so every move in h to h' is winning, hence there exists a move in h' to h'' which is losing. By IH2, $g + h''$ is losing.

The loser's lemma

We prove by induction on h that whenever h is losing, so is $g + h$.

- IH1: for all $g' \leq g$, if h is losing, then so is $g' + h$.
- IH2: for all $h' \leq h$, if h' is losing, then so is $g + h'$.
- Every initial move in $g + h$ is either in g or in h. First consider the latter.
- h is losing, so every move in h to h' is winning, hence there exists a move in h' to h'' which is losing. By IH2, $g + h''$ is losing.
- g is losing, so every move in g to g' is winning, hence there exists a move in g' to g'' which is losing. By IH2, $g'' + h$ is losing.

The loser's lemma

We prove by induction on h that whenever h is losing, so is $g + h$.

- IH1: for all $g' \leq g$, if h is losing, then so is $g' + h$.
- IH2: for all $h' \leq h$, if h' is losing, then so is $g + h'$.
- Every initial move in $g + h$ is either in g or in h. First consider the latter.
- h is losing, so every move in h to h' is winning, hence there exists a move in h' to h'' which is losing. By IH2, $g + h''$ is losing.
- g is losing, so every move in g to g' is winning, hence there exists a move in g' to g'' which is losing. By IH2, $g'' + h$ is losing.
- Thus every move in $g + h$ is losing, and we are done.

Applications

We now have: if h is losing then for all g, $g + h = g$.

Applications

We now have: if h is losing then for all g, $g + h = g$.

- This means that every losing subgame can be ignored !

Applications

We now have: if h is losing then for all g, $g + h = g$.

- This means that every losing subgame can be ignored !
- Let us use this knowledge to analyse $1 + 2 + 3 + 4 + 5$.

Applications

We now have: if h is losing then for all g, $g + h = g$.

- This means that every losing subgame can be ignored !
- Let us use this knowledge to analyse $1 + 2 + 3 + 4 + 5$.
- Consider $1 + 2 + 3$. Note that removing an entire heap leads to a winning position. Reducing any heap leads to two equal heaps, which is losing and can be ignored. Thus $1 + 2 + 3$ is losing.

Applications

We now have: if h is losing then for all g, $g + h = g$.

- This means that every losing subgame can be ignored !
- Let us use this knowledge to analyse $1 + 2 + 3 + 4 + 5$.
- Consider $1 + 2 + 3$. Note that removing an entire heap leads to a winning position. Reducing any heap leads to two equal heaps, which is losing and can be ignored. Thus $1 + 2 + 3$ is losing.
- We know that $4 + 5$ is winning, so $1 + 2 + 3 + 4 + 5$ is winning.

A principle

Can we get some more general mileage than analysing simple Nim heaps ?

- Question: How do you ensure that you do not lose in a Chess game against a Grandmaster ?

The copycat principle

We can now enunciate an important principle in bipartisan games.

The copycat principle

We can now enunciate an important principle in bipartisan games.

- Lemma $g + g \equiv 0$.

The copycat principle

We can now enunciate an important principle in bipartisan games.

- Lemma $g + g \equiv 0$.
- Given g, assume that the claim holds for $g' \leq g$.

The copycat principle

We can now enunciate an important principle in bipartisan games.

- Lemma $g + g \equiv 0$.
- Given g, assume that the claim holds for $g' \leq g$.
- A move in $g + g$ is of the form $g' + g$.

The copycat principle

We can now enunciate an important principle in bipartisan games.

- Lemma $g + g \equiv 0$.
- Given g, assume that the claim holds for $g' \leq g$.
- A move in $g + g$ is of the form $g' + g$.
  This is winning by the move to the subgame $g' + g'$ which is losing by IH.

The copycat principle

We can now enunciate an important principle in bipartisan games.

- Lemma $g + g \equiv 0$.
- Given g, assume that the claim holds for $g' \leq g$.
- A move in $g + g$ is of the form $g' + g$.
  This is winning by the move to the subgame $g' + g'$ which is losing by IH.
  So $g + g$ is losing and by loser's lemma, equivalent to 0.

The algebraic structure

The class of games with addition forms an abelian group.

The algebraic structure

The class of games with addition forms an abelian group.

- Commutativity and associativity are easy.

The algebraic structure

The class of games with addition forms an abelian group.

- Commutativity and associativity are easy.
- 0 is the neutral element.

The algebraic structure

The class of games with addition forms an abelian group.

- Commutativity and associativity are easy.
- 0 is the neutral element.
- Every element has an inverse, and as we just saw, has itself as inverse.

The algebraic structure

The class of games with addition forms an abelian group.

- Commutativity and associativity are easy.
- 0 is the neutral element.
- Every element has an inverse, and as we just saw, has itself as inverse. Reminiscent of some very familiar algebraic structures?

The algebraic structure

The class of games with addition forms an abelian group.

- Commutativity and associativity are easy.
- 0 is the neutral element.
- Every element has an inverse, and as we just saw, has itself as inverse. Reminiscent of some very familiar algebraic structures?

What we have seen is a glimpse of the Sprague-Grundy theory of impartial games.

Games and numbers

John Conway took this much farther.

- There is a distinguished sub-group of games called numbers which can also be multiplied and which form a field.
- This field contains both the real numbers and the ordinal numbers.
- In fact, Conway's definition generalizes both Dedekind cuts and von Neumann ordinals.
- A beautiful microcosmos of numbers and games which are infinitesimally close to zero, and ones which are infinitely large.

Games and numbers

John Conway took this much farther.

- There is a distinguished sub-group of games called numbers which can also be multiplied and which form a field.
- This field contains both the real numbers and the ordinal numbers.
- In fact, Conway's definition generalizes both Dedekind cuts and von Neumann ordinals.
- A beautiful microcosmos of numbers and games which are infinitesimally close to zero, and ones which are infinitely large.
- Donald Knuth's novel: Surreal numbers: every real number is surrounded by a whole lot of new numbers that lie closer to it than any other 'real' value does.