Handout: Applications of the Compactness Theorem

Recall the compactness theorem:

Compactness Theorem. *For a set Σ of wffs and a wff φ, we have the following.*

1. *If $\Sigma \models \varphi$ then there is a finite set $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \varphi$.*
2. *If Σ is finitely satisfiable then Σ is satisfiable.*

This section is devoted to applications of the theorem.

1 Applications of the Sentential Compactness Theorem

Theorem 1 (Four Color Theorem, Appel and Haken 1976)**.** *Every finite planar map can be colored with four colors.*[1]

While this theorem is very difficult to prove. The following is quite easy to show using the the Sentential Compactness Theorem (that is the compactness theorem for sentential logic, which we proved it earlier in the semester).

Theorem 2. *If every* finite *planar map can be colored with four colors, then every* countably infinite *planar map can be colored with four colors.*

Proof. Assume that every finite planar map can be colored with four colors. Now, consider an arbitrary infinite planar map M with countries $C_0, C_1, C_2, \ldots$. We will use the following sentence symbols:

- For all $n \in \mathbb{N}$ we have sentence symbols R_n, B_n, G_n, Y_n (for "red", "blue", "green", "yellow").
- For all $n, m \in \mathbb{N}$ we have the sentence symbol $N_{m,n}$ (for "neighboring")

We let Σ be the following set of sentential formulas of the form:

1. $N_{n,m}$ whenever C_n and C_m are neighboring countries
2. $\neg N_{n,m}$ whenever C_n and C_m are neighboring countries
3. $(R_n \vee B_n \vee G_n \vee Y_n)$
4. $\neg(R_n \wedge B_n) \wedge \neg(R_n \wedge G_n) \wedge \neg(R_n \wedge Y_n) \wedge \neg(B_n \wedge G_n) \wedge \neg(B_n \wedge Y_n) \wedge \neg(G_n \wedge Y_n)$

[1] There are some basic assumptions. For example as each region must be connected—so Alaska and the Continental United States may get two different colors. Also touching at one point does not count, so Colorado and Arizona could be colored the same.

5. $N_{m,n} \rightarrow \neg((R_m \wedge R_n) \vee (B_m \wedge B_n) \vee (G_m \wedge G_n) \vee (Y_m \wedge Y_n))$

Conditions (1) and (2) "hard code" which countries are neighboring. Conditions (3) and (4) expresses that each country has one and only one color. Condition (5) expresses that no two neighboring countries can have the same colors.

Notice the map M is four colorable iff Σ is satisfiable. For, any assignment of the sentence symbols satisfying Σ tells us how to color the map M.

Now, to show that Σ is satisfiable, by the Sentential Compactness Theorem we only have to show Σ is finitely satisfiable. Let $\Sigma' \subseteq \Sigma$ be finite. Since Σ' is finite, it contains only finitely many symbols. Let n be the largest index used in Σ'. (For example, R_n can occur in Σ', but not R_{n+1}.) Now, consider the finite map M_n which only uses the countries $C_1, \ldots, C_n$. This map is finite, and therefore four-colorable by assumption. So there is an assignment of the sentence symbols $R_0, \ldots, R_n$, $B_0, \ldots, B_n$, $G_0, \ldots, G_n$, $Y_0, \ldots, Y_n$, and the sentence symbols $N_{1,1}, \ldots, N_{i,j}, \ldots, N_{n,n}$ which satisfies Σ'.

Therefore, Σ is finitely satisfiable. By the Sentential Compactness Theorem Σ is satisfiable. Therefore, M is four colorable. $\square$

Exercise 3. Use the Sentential Compactness Theorem to prove that every partial order can be extended to a total order.

2 Applications of the First-Order Compactness Theorem

An undirected graph G is *connected* if every two vertices a and b have a finite path of edges connecting them. For example, the following graph is connected:

$$\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\cdots$$

Call this structure $\mathfrak{G} = (|\mathfrak{G}|; E^{\mathfrak{G}})$. Let $\mathrm{Th}(\mathfrak{G})$ be the set of all sentences σ (in the language of graphs) which are true in $\mathfrak{G}$. This is called the *theory* of $\mathfrak{G}$. Now (temporarily) add to our language two constant symbols a and b. Let Σ be all the sentences in $\mathrm{Th}(\mathfrak{G})$ in addition to the following sentences σ_n (which together say that a and b do not have a finite path connecting them):

$$\begin{aligned}
\sigma_0 :=& \quad a \neq b \\
\sigma_1 :=& \quad \neg Eab \\
\sigma_2 :=& \quad \neg\exists x_1\,(Eax_1 \wedge Ex_1b) \\
\sigma_3 :=& \quad \neg\exists x_1\exists x_2\,(Eax_1 \wedge Ex_1x_2 \wedge Ex_2b) \\
&\vdots \\
\sigma_n :=& \quad \neg\exists x_1 \cdots \exists x_{n-1}\,(Eax_1 \wedge Ex_1x_2 \wedge \ldots \wedge Ex_{n-2}x_{n-1} \wedge Ex_{n-1}b) \\
&\vdots
\end{aligned}$$

Lemma 4. Σ *is satisfiable*

Proof. First we show Σ is finitely satisfiable Take any finite set of sentences $\Sigma' \subseteq \Sigma$. There is some n such that $\Sigma' \subseteq \mathrm{Th}(\mathfrak{G}) \cup \{\sigma_0, \ldots, \sigma_n\}$ and this is satisfiable since it has the following model

$$\underbrace{\underset{a}{\bullet} - \bullet - \cdots - \bullet - \underset{b}{\bullet}}_{n+1 \text{ edges}} - \bullet - \cdots$$

(The constants a and b act as labels for two special nodes in our graph. We use the original graph $\mathfrak{G}$, but label two edges that are more than distance n apart.)

By the compactness theorem, Σ is satisfiable. $\square$

Since Σ is satisfiable, there is a structure $\mathfrak{G}^* = (|\mathfrak{G}^*|; E^{\mathfrak{G}^*}, a^{\mathfrak{G}^*}, b^{\mathfrak{G}^*})$ which satisfies Σ. This a graph with two labels. By the sentences σ_n we have that the labeled nodes $a^{\mathfrak{G}^*}$ and $b^{\mathfrak{G}^*}$ are not connected by a finite path. So the labeled graph $\mathfrak{G}^*$ is not connected.

Now, remove the constant symbols a and b from our language. Now, by $\mathfrak{G}^*$ we mean the same structure, but without the constant symbols. That is $\mathfrak{G}^* = (|\mathfrak{G}^*|; E^{\mathfrak{G}^*})$. So this is a graph which is not connected. It is also still a model of $\mathrm{Th}(\mathfrak{G})$ since these sentences did not use the symbols a and b.

Recall these definitions/facts:

- Isomorphism: $\mathfrak{A} \cong \mathfrak{B}$ iff there is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.
- Elementary equivalence: $\mathfrak{A} \equiv \mathfrak{B}$ iff for all sentences σ, ($\models_{\mathfrak{A}} \sigma$ iff $\models_{\mathfrak{B}} \sigma$).
- $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{A} \equiv \mathfrak{B}$
- A class $\mathcal{C}$ of structures is EC_Δ if there is a set of sentences Γ such that $\mathfrak{A} \in \mathcal{C}$ iff $\mathfrak{A}$ is a model of Γ.

Lemma 5. *$\mathfrak{G}^* \equiv \mathfrak{G}$ but $\mathfrak{G}^* \not\cong \mathfrak{G}$.*

Proof. The structure $\mathfrak{G}^*$ is a model of $\mathrm{Th}(\mathfrak{G})$ so we have

$$\begin{aligned} \models_{\mathfrak{G}} \sigma \quad &\text{implies} \quad \sigma \in \mathrm{Th}(\mathfrak{G}) \\ &\text{implies} \quad \models_{\mathfrak{G}^*} \sigma \end{aligned}$$

and

$$\begin{aligned} \not\models_{\mathfrak{G}} \sigma \quad &\text{implies} \quad \models_{\mathfrak{G}} \neg\sigma \\ &\text{implies} \quad \neg\sigma \in \mathrm{Th}(\mathfrak{G}) \\ &\text{implies} \quad \models_{\mathfrak{G}^*} \neg\sigma \end{aligned}$$

Hence $\mathfrak{G}^* \equiv \mathfrak{G}$.

But $\mathfrak{G}$ is connected and $\mathfrak{G}^*$ is not, so they are not isomorphic. *(Exercise: work out the details!)* $\square$

Lemma 6. *$\mathfrak{G}^*$ shares the following first order properties of $\mathfrak{G}$:*

- *There is exactly one vertex of $\mathfrak{G}^*$ which has degree one (one neighbor). (Call this the* endpoint.

- *Every other vertex of $\mathfrak{G}^*$ which has degree two (two neighbors).*
- *For each n, there is exactly one vertex of distance n from the endpoint.*

Proof. These properties are expressible as first-order sentences. $\square$

Therefore $\mathfrak{G}^*$ must look something like the following:

$$\begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \cdots \\ \cdots \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \cdots \end{array}$$

More specifically it has one component which looks like $\mathfrak{G}$ and a bunch of extra stuff composed of one or more chains of the form

$$\cdots \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \cdots$$

$\mathfrak{G}^*$ is called a *nonstandard* model of $\mathfrak{G}$.

Theorem 7. *The class of connected graphs is not EC_Δ.*

Proof. Assume there is a set Γ of sentences such that $\mathfrak{A} \in \mathcal{C}$ iff $\mathfrak{A}$ is a model of Γ. Then the structure $\mathfrak{G}$ is a model of Γ since it is connected. But so is $\mathfrak{G}$ since $\mathfrak{G}^* \equiv \mathfrak{G}$. However, $\mathfrak{G}^*$ is not connected. $\square$

By a similar proof we can show that the following classes of structures are not EC_Δ.

- the class of finite graphs (resp. groups, rings, fields)
- the class of graphs of finite degree (i.e. each vertex has finitely many neighbors)
- the class of cyclic graphs (resp. cyclic groups)
- the class of fields of finite characteristic
- the class of all p-groups for some finite p