Chapter 15 

Fun with Cardinality

15.1 Introduction and Summary

When it comes to infinity, just how big is it? And is that even a well-posed question—could there be more than one size for infinity to have? Addressing this mountainous question is the focus of our chapter. The vehicle for exploring the sizes of infinite sets is a play (that's Section 15.2) in which the characters encounter and discuss ways of relating infinite sets to each other.

This is followed by (or interleaved with, depending on how you decide to read the chapter) a lot of exploratory questions in Section 15.4. Some of the more challenging ideas are reviewed in Section 15.5, and then of course there are problems to think about. Along the way, we will need to introduce new ideas, new notations, and new sets ... and we will even leave the realm of discrete mathematics, though just for a minute or two.

15.2 Read This! Parasitology, the Play

Perhaps a dramatic reading is in order. You will want ten willing readers (though you can make do with five), and a chalkboard or whiteboard would be handy for investigational pauses. Particularly good pausing moments are marked with $[\star_k]$ and have matching exploratory questions in Sections 15.4.1–15.4.4.

Characters:

CATALOGUER
LAB ASSISTANT
LAB ASSISTANT 2 (LA2)
LAB ASSISTANT 3 (LA3)
LAB ASSISTANT 4 (LA4)
NARRATOR
PROTAGONIST
RECEPTIONIST
STORAGE COORDINATOR
TAXONOMIST

15.2.1 Scene 1: The Storage Coordinator

Characters: LAB ASSISTANT, NARRATOR, PROTAGONIST, RECEPTIONIST, STORAGE COORDINATOR.

Enter RECEPTIONIST, *who stands as at a desk, and* NARRATOR. PROTAGONIST *enters as* NARRATOR *speaks:*

NARRATOR: (*in a declarative tone*) Recently when visiting the friendly neighborhood ducks, our protagonist noticed something odd about the fecal matter in the duck yard and, having recently read Carl Zimmer's *Parasite Rex*, wondered whether a parasite might be to blame. Our protagonist did a quick search and found that not only is there a local parasitology lab, but that this lab is a national center that accepts specimens for examination *and* among its specialities is parasites of waterfowl! We enter the scene as our protagonist arrives at the lab.

RECEPTIONIST: Welcome to Parasite Central! How may I help you?

PROTAGONIST: Hello. I understand that you accept specimens from the public for examination and identification?

RECEPTIONIST: Indeed we do. Have you brought a specimen with you? And if so, what kind of specimen is it, and what is your reason for bringing the specimen to us?

PROTAGONIST: Yes, I have brought a sample of duck fecal matter that I suspect contains parasites of some kind. I was hoping you could tell me what's in there.

RECEPTIONIST: Certainly. Let me page a lab assistant and we'll get that sample taken care of. (*The* RECEPTIONIST *presses a button.*)

Enter LAB ASSISTANT.

RECEPTIONIST: (*to* LAB ASSISTANT) We have a member of the public here who has an avian waterfowl sample to submit.

LAB ASSISTANT: (*to* PROTAGONIST) Excellent. Please accompany me to the sample analysis area.

Exit LAB ASSISTANT, *followed by* PROTAGONIST; *exit* RECEPTIONIST *in a different direction. Soon after, enter* LAB ASSISTANT, *followed by* PROTAGONIST.

LAB ASSISTANT: Let me quickly check your sample under a microscope. (*takes the sample from* PROTAGONIST *and quickly mimes creating a slide and examining it under the microscope*) That's interesting. There are definitely parasites in here, but I don't recognize them as a known species. (*looks up*) We will definitely want to store this sample for further study, and I'll need to consult with one of the senior scientists to find out whether the specimen warrants cataloguing as a probably new species. Would you like to accompany me on these tasks? I could give you a quick tour of part of the lab while I'm doing them.

PROTAGONIST: That would be great. (*They start to walk off.*) Do you get new samples all the time? If so, how do you have room for them all? Or do you not store that many of the samples?

LAB ASSISTANT: Oh, yes, we get new samples all the time. Many of them don't have parasites, but many do and many are new. Luckily, we have infinitely many sample drawers and so storage is not a problem.

Exit LAB ASSISTANT, *followed by* PROTAGONIST; *soon after, re-enter* LAB ASSISTANT, *followed by* PROTAGONIST, *whilst* STORAGE COORDINATOR *enters from another direction.*

LAB ASSISTANT: (*to* PROTAGONIST) This is our storage area.

STORAGE COORDINATOR: Are you bringing me a sample? I must warn you, all of the drawers are full at the moment.

PROTAGONIST: But I thought you had infinitely many sample drawers . . . ?

STORAGE COORDINATOR: We do. But as samples get analyzed, it's often realized that we don't need them anymore and so drawers free up regularly. At the moment, all of them are full.

LAB ASSISTANT: Well, we need to put this sample *somewhere*. Should I just leave it on the desk here until a drawer is vacated?

STORAGE COORDINATOR: Oh, no, we can't do that. What if more samples come in before drawers are available? They could pile up, and imagine what a mess we would have—both literally and logistically. No, hang on and let me think for a moment . . . do you have a favorite number? We can move all the samples from that number onwards up a drawer, and then we can put your sample in the drawer with your favorite number.

PROTAGONIST: (*aside, to* LAB ASSISTANT) Infinitely many field biologists will be a lot for a few lab assistants to handle. Aren't you going to need to help with managing these people? Maybe I should just go ...

LAB ASSISTANT: (*aside, to* PROTAGONIST) It's not a problem. We have infinitely many lab assistants here.

STORAGE COORDINATOR: (*sets down the phone*) Whew. Perhaps you overheard that we have a bit of a situation. Let me get you on your way.

NARRATOR: The storage coordinator grabs a slip of paper and a pair of linked tags.

PROTAGONIST: I don't mean to delay things further, but how will you manage all infinitely many field biologists at once?

STORAGE COORDINATOR: It'll be a bit complicated, but totally doable. I'll leave the first specimen in the first drawer. Then I'll put two specimens from the first field biologist in the second and third drawers. I'll move the specimen that was in the second drawer to the fourth drawer, and the specimen from the third drawer to the ninth drawer. Then two more specimens from the first field biologist will go in drawers 5 and 6, and two specimens from the second field biologist will go in drawers 7 and 8. The specimens from those drawers will get moved down further ... are you with me so far?

PROTAGONIST: I think so. You've moved the old specimens up by some number of drawers, and you've started placing specimens from the first and second field biologists. So far you've filled the first nine drawers. But you'll have to work in specimens from more field biologists, right?

STORAGE COORDINATOR: Yes, and that happens in the next round of placements. Two more specimens from the first field biologist will go in drawers 10 and 11, two more specimens from the second field biologist will go in drawers 12 and 13, and two specimens from the third field biologist will go in drawers 14 and 15. And so forth and so on, including specimens from an additional field biologist in each round of placements, until every specimen has been placed in a drawer or moved to a new drawer. [⋆4]

Anyway, (*making some notes on the form and attaching one of the two tags*) here is your receipt and the tag that identifies your specimen. The matching tag stays with the specimen, so you can be sure upon retrieval that you got the

specimen you submitted. Have a lovely day and enjoy your time at Parasite Central.

All characters exit.

15.2.2 Scene 2: The Taxonomist

Characters: LAB ASSISTANT, PROTAGONIST, TAXONOMIST.

LAB ASSISTANT *enters, followed by* PROTAGONIST; *from another direction,* TAXONOMIST *enters.*

TAXONOMIST: Greetings! Do you have some slides for me?

LAB ASSISTANT: Yes, I have *a* slide for you. I am also showing the collector of the specimen around part of our lab.

TAXONOMIST: Lovely—(*to* PROTAGONIST) Let me tell you a little bit about what we do here. (*gestures to the room*) I examine slides of waterfowl parasites and note their characteristics. With this information, I classify the parasites; if they are rare, or if they are different enough to be considered a new species, I store the slides in this array. The bulk of my job is deciding where in the "family tree" of waterfowl parasites new species should go.

(*to* LAB ASSISTANT) The slide, please?

LAB ASSISTANT *mimes handing a slide to* TAXONOMIST, *who mimes examining it under a microscope.*

TAXONOMIST: That's interesting … hmmm … (*looks up*) I'm going to need to do a little bit more work with this one before I can tell you whether it needs to be taken to Cataloguing. Could you take our guest to get some coffee or something for a few minutes? I should have an answer for you relatively soon. Oh, but I'll need your receipt for now, so I can make notes on it for potential cataloguing purposes.

PROTAGONIST: I'm confused. You need the receipt to take notes? I thought it was for tax purposes …

TAXONOMIST: Oh, it *is*, as far as you are concerned. But before you leave with it, we will have preliminary notes recorded on it and a copy will be stored either here or with Cataloguing, depending on what the eventual fate of the sample and slide are.

PROTAGONIST: Okay. Here's the receipt. Do you need the tag as well? ... (*drifts off*) ... that's odd; it says 134. But I thought we were told it was going to be
[⋆5] number 59. Oh, well.

TAXONOMIST: Thanks. The tag isn't necessary at this point. If I've already finished with your slide when you return, I'll be storing it in holder $(147, 2032)$ in the array. That one's empty at the moment.

PROTAGONIST: As in the hundred-and-forty-seventh holder in the two-thousand-and-thirty-second row? You must have a lot of slides here!

TAXONOMIST: It's more like the two-thousand-and-thirty-second holder in the hundred-and-forty-seventh aisle. These coordinates are just like (x, y) coordinates of a point in the plane. And yes, we have a lot of slides here. Infinitely many, in fact.

PROTAGONIST: With two coordinates to count with, you must have room for infinitely many more than the Storage area has!

TAXONOMIST: Actually, I have exactly the same amount of room as Storage has. The only reason I use two coordinates is because my laboratory area is a different shape than the Storage area—when we first moved in, I numbered my holders in the array to match the drawers in Storage. It was like this, as shown in Figure 15.1: (*gestures*) holder 1 was in the corner here, holder 2 to the right of it, and then diagonally up and left was holder 3, holder 4 was just above it, and then I kept numbering by going diagonally back until I was at the right of holder 2, and so forth. But then over time, I got tired of zig-zagging back and forth from the corner to find a slide and decided to use array numbering instead. Besides, when we started, there were the same number of slides as samples, and now there are *way* more samples than slides—not everything we need to store is studied by me, but everything I study has a corresponding
[⋆6] sample in Storage.

LAB ASSISTANT: Before we go, I should warn you that a large influx of specimens arrived earlier today.

TAXONOMIST: How large are we talking?

LAB ASSISTANT: Well, over in Storage we heard that infinitely many field biologists had arrived with infinitely many samples each.

TAXONOMIST: Oh. Great.

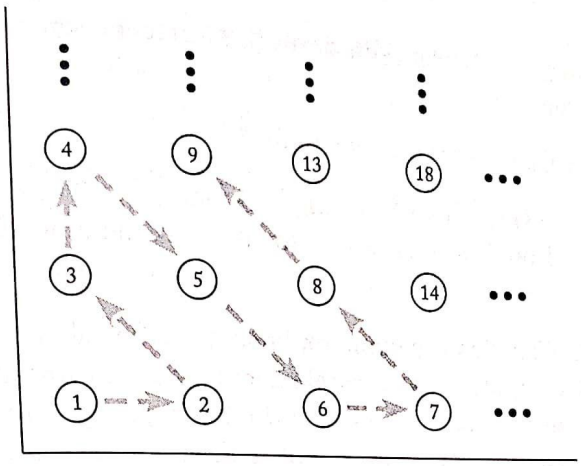


Figure 15.1. The old numbering of slide holders in Taxonomy.

LAB ASSISTANT: Will this be a problem? I mean, should we take a loooong coffee break, or . . . ?

TAXONOMIST: Don't worry. It's not like Storage will process all of them at once, and besides, there will probably not be very many that get slides passed to me; I just get the avian waterfowl slides, and even with infinitely many samples, there could be finitely many of those. And it's not like I could analyze more than one slide at a time anyway, so whoever comes in while I'm working can just wait. Or . . . I could pretend I'm on break and then no one will disturb me . . . just don't worry about it.

LAB ASSISTANT: Okay. We'll see you later.

PROTAGONIST: (*to* TAXONOMIST) Thanks for explaining your slide storage system to me! It was lovely viewing your area and hearing about your work.

All exit, LAB ASSISTANT *and* PROTAGONIST *in a different direction from* TAXONOMIST.

15.2.3 Scene 3: The Café

Characters: LAB ASSISTANT, LAB ASSISTANT 2 (LA2), LAB ASSISTANT 3 (LA3), NARRATOR, PROTAGONIST.

Enter LAB ASSISTANT *followed by* PROTAGONIST; *from another direction, enter* LAB ASSISTANT 2 (LA2) *and* LAB ASSISTANT 3 (LA3), *who sit down.*

NARRATOR: There is a counter with stools that stretches across the room. All the stools are occupied.

LAB ASSISTANT: Can I get you some coffee?

PROTAGONIST: (*looking around, staring*) ... Sure. ... There are a *lot* of people here. And I don't see anywhere to sit ... is there anywhere besides the counter?

LAB ASSISTANT: Well, this is a common break time for lab assistants, so it's not surprising that there are a lot of people here. (*gazes toward where the counter would be, where* LA2 *and* LA3 *are sitting*) I'm pretty sure that while there are infinitely many people at the counter, there are also infinitely many lab assistants who aren't here. I don't see anyone whose name begins with "J," for example, and there are infinitely many of those. Anyway, we can sit at the counter. (*to* LA2 *and* LA3) Could you make room for two? I have a guest here who brought in a specimen.

LA2: Sure! (*to invisible people seated on the left*) Could you move over, please? We have a new arrival over here.

LA3: (*to invisible people seated to the right*) Please move down a seat; pass on the message.

LAB ASSISTANT: (*to* LA2, *while sitting down just to* LA2*'s right*) Has coffee been poured recently?

LA2: No, I think it will be soon, though. I heard some mugs clinking a moment ago.

NARRATOR: A server comes by, setting down steaming mugs of coffee while walking past.

LAB ASSISTANT: (*to* LA2) I usually like to have two cups. You?

LA2: Not usually, but today I'm very tired. (*turns to the left*) Could you please give me your coffee? And the coffee given to the person on your left?

LAB ASSISTANT: (*to* PROTAGONIST) You didn't want two cups of coffee, did you?

PROTAGONIST: No ... but I would like cream for my coffee. And is it seriously okay to make those people to the left give up their coffee for you?

LAB ASSISTANT: Cream should be along in a minute; also, sugar. And the people to our left will get coffee from the people on their left. We're used to passing the mugs around before drinking.

NARRATOR: A server comes by, setting down dishes of creamer and sugar cubes while walking past. [⋆7]

LA3: (*to* PROTAGONIST) Could you pass me the dish of creamer and sugar cubes? We don't have enough over here.

LA2: (*overhearing*) Extra creamer is being passed up this way, hang on a moment … here.

LA2 *hands a dish of creamer to* LAB ASSISTANT, *who hands it to* PROTAGONIST, *who takes two creamers before passing it to* LA3.

PROTAGONIST: A snack would be lovely … can we order anything? … from anyone?

NARRATOR: A server walks by with an armload of cookies but doesn't set any down.

LAB ASSISTANT: We'll have to ask around. No one stops long enough for us to order drinks or snacks. (*to* LA2) Any cookies down that way? (*to* LA3) Do you see any cookies?

LA2: I'll ask. I'm pretty hungry myself.

LA3: Let me see ….

LA2: (*to invisible people seated on the left*) Do you have any cookies?

LA3: (*to invisible people seated on the right*) Could we … Hey! You down there, stop hoarding all the cookies! (*pauses to listen*) It doesn't matter if you want infinitely many cookies, you can still send a few our way! (*turning to* PROTAGONIST *and shaking head*) Some people … [⋆8]

LA2: (*to* LAB ASSISTANT) What group meeting do you have today?

LAB ASSISTANT: That's a good question. I'll have to check the schedule; it changes so often. You?

LA2: I'm lucky—for today I'm assigned to a group that's discussing my primary project!

LAB ASSISTANT: Wow, *nice*. Is it a big group?

LA2: Not too big. I think there are under 50 people.

LAB ASSISTANT: That's good. Last week I was in a group with 30,000,000 people, and it was tough for anyone to hear each other during the meeting.

PROTAGONIST: How often do you get to work on your primary project?

LAB ASSISTANT: We work on our primary projects whenever we have no other specific duties—dealing with sample intake, assisting senior scientists, and so on. But it's rare to have group meetings about our primary projects, because we are but lowly lab assistants.

PROTAGONIST: Oh, so most of the group meetings are about senior scientists' projects?

LAB ASSISTANT: No ... they're about other lab assistants' primary projects. It's part of a workflow experiment. Every lab assistant has his or her own primary project, but until recently we worked in relative isolation on our projects. One of the heads of the lab thought that some collaboration would increase research productivity on lab assistant projects and so asked the lab's operations researcher to figure out what would be best. But it's not a straightforward problem—What size discussion group is best? Should discussion groups be formed of lab assistants with close expertise to the project topic, or lab assistants who know very little and so bring fresh thoughts, or a mixture of the two? So the operations researcher decided we would try having every possible collection of lab assistants meet and have everyone take notes on the meetings, and then see what works best.

PROTAGONIST: Wow. That's a lot of meetings.

LAB ASSISTANT: Yeah, we have at least one in the schedule every day. And it's kind of random because the operations researcher is still trying to figure out the whole schedule, and so we don't know very far in advance when we're meeting about what with whom. But to answer your original question, some lab assistants don't get group meetings about their primary projects. They do get the notes from the meetings about their projects, though. I suspect that mostly meetings like that are a waste of time, but we'll see.

LA3: I heard that the operations researcher ran into
recently, and it might

LAB ASSISTANT: Really? What's going on?

LA3: The operations researcher made a list of all possible groups of lab assistants, right? And every group has to meet about someone's primary project because they have to have something to talk about, and it would be silly to have them talk about something that has nothing to do with the lab. But rumor has it that there is a group with no project, and this is throwing the operations researcher for a loop.

PROTAGONIST: Can't the operations researcher just pick a project no one has met about yet?

LA3: No, it's super-weird. It has to do with the particular group. The operations researcher came up with every possible group, and one of those is the group of lab assistants who are not in the group that discusses their primary project. And apparently there's no project for that group to discuss.

PROTAGONIST: I still don't see why the operations researcher can't just pick a project.

LAB ASSISTANT: Let's pretend that this particular group is going to discuss my primary project. I would love to be there, not that I'm bitter about not having had a group talk about my project yet, or anything .... Anyway, I want to be in the group. But every lab assistant in this group is *not* in the group that discusses his or her primary project. So I can't be in the group. And *that* means I am not in the group that discusses my primary project, which means I *am* in the special group, which is discussing my primary project. It just goes 'round and 'round. [*9]

PROTAGONIST: ... Uhhh....

LA3: Well, I also heard that the operations researcher is considering expanding the focus of these groups; the problematic group might be told to discuss the functioning of this café, for example. That's not a lab assistant's primary project. But there might be other issues that come up.

LAB ASSISTANT: (*to* PROTAGONIST) Did you ever get the cookie you wanted? We should be getting back to Taxonomy to check on your slide. You still have your tag?

PROTAGONIST: Yeah, somewhere in there I got a cookie. The tag is in my pocket ... (*pulls tag out of pocket*) Here. Wait a minute! The tag says 20,449, and I'm *sure* it was a much smaller number before. Wasn't it supposed to be 59?

LAB ASSISTANT: Oh, the tags are quantum entangled. Remember how it was originally one of a pair? They're linked, so when the Storage Coordinator moves a sample to a new drawer, the storage tag gets relabeled and that relabels
[*10] the tag you have here.

PROTAGONIST: (*to* LA2 *and* LA3) It was nice to meet you. Good luck with your meetings and projects!

LA2 *and* LA3 *wave as* LAB ASSISTANT *and* PROTAGONIST *exit, then exit themselves.*

15.2.4 Scene 4: Cataloguing

Characters: CATALOGUER, LAB ASSISTANT, LAB ASSISTANT 4 (LA4), NARRATOR, PROTAGONIST, TAXONOMIST.

LAB ASSISTANT *and* PROTAGONIST *enter from one direction and* TAXONOMIST *enters from another direction.*

TAXONOMIST: Ah! You're back! I have news—the avian waterfowl parasites on your slide are likely an unknown species, which means that they need to be catalogued. That means you will take the receipt and the sample tag down to Cataloguing, where a catalogue number will be assigned, a copy will be made of the receipt, and the tag will be stored with the catalogue information. Here. (*hands them the receipt*)

PROTAGONIST: Thanks!

All exit. Then LAB ASSISTANT *and* PROTAGONIST *enter from one direction and* LAB ASSISTANT 4 (LA4) *enters from another direction.*

LA4: Whoa! You can't go through here.

LAB ASSISTANT: Why not? We need to take some information to Cataloguing.

LA4: All infinitely many specimens in the Liver Fluke Lab need to be collected for transport and we are about to start. So you will need to wait.

PROTAGONIST: ... Are we going to be waiting forever?

LA4: No. Just hang on a minute. (*backs away*)

NARRATOR: As PROTAGONIST and LAB ASSISTANT watch, a dolly stacked with boxes is brought across the hall and then brought back empty. This process takes about 30 seconds. In the next 15 seconds, the dolly passes across the hall full and returns empty. Again a load of boxes passes across, but in one-eighth of a minute. Continuing to pass back and forth twice as fast with each trip, the dolly becomes a blur, and about five seconds later the hall is still.

LA4: (*coming forward again*) Okay, all clear. You can pass through.

LAB ASSISTANT *and* PROTAGONIST *cross and exit. Then* LA4 *exits. After a short pause,* LAB ASSISTANT *and* PROTAGONIST *enter from one direction and* CATALOGUER *enters from another direction.* [*11]

CATALOGUER: Hello. How may I help you?

LAB ASSISTANT: We've been informed by the avian waterfowl parasites taxonomist that the specimen brought in by this guest represents a new species, and so the specimen needs to be assigned a number in the catalogue.

CATALOGUER: Excellent! That should be no problem. May I see the receipt? I'll need the sample tag as well.

PROTAGONIST: Here you go. (*hands over the receipt and sample tag*) I've noticed that every department at Parasite Central seems to have its own way of organizing information. How do you do it here?

CATALOGUER: It's very simple. We create a catalogue entry for every identified species. The entries are numbered and have all the taxonomic and descriptive information reported by other departments. We can look up species by entry number or we can search the catalog by key words, though that's less useful as sometimes there are infinitely many results for a search.

PROTAGONIST: So when you get a large number of new species, you renumber?

CATALOGUER: (*runs the receipt through a scanner*) Oh, no. We'd never be able to find anything again! Besides, we don't just have the species identified at Parasite Central in our catalogue. We have every parasite species ever identified anywhere in our catalogue. One way we know a species is new is if its description doesn't match any of the entries in the catalogue.

PROTAGONIST: I'm confused. How do you avoid the renumbering? Do you have finitely many catalogue entries? I thought you said there were infinitely many search results ....

CATALOGUER: We have infinitely many entries. But we don't renumber; we just find a number that hasn't been used yet. Here, let me show you how this works with the species you brought in. First, I'll pull up a few catalogue entries for you to see. *(types and then rotates the computer monitor so* PROTAGONIST *can see it)* Notice that at the top of the entry is the entry number. You can't see the whole thing on screen because it has more digits than will fit. And then below it is the description.

PROTAGONIST: *(squinting a bit)* Wait, is that a decimal point in front of the entry number? Or is it just a symbol in front of the number?

CATALOGUER: Yes, that's a decimal point—every entry number has a value between 0 and 1. Anyway, let's say I have a list of entry numbers and I need a new one. I just write down a decimal point, and then I go through the list. I write down a digit that is different from the first digit of the first number, and then I write down a digit that is different from the second digit of the second number, and I continue in this fashion. When I've gotten through the list, I know that I have a new number because it is different in at least one place from each number on the list. Easy peasy.

PROTAGONIST: Wouldn't it be even easier to use regular numbers instead of the numbers between 0 and 1?

CATALOGUER: No, because there wouldn't be enough of them unless I renumbered all the time. Imagine this: I number all of my current catalogue entries using regular counting numbers. The new entry I made a moment ago can't get a regular counting number because all of those are already in use. The decimals are much better. Anyway, here's your receipt back. The tag will be [•12] attached to a copy of the receipt.

PROTAGONIST: Thanks so much. It's exciting to have a description of a new species!

LAB ASSISTANT: Come on, I'll walk you back to the reception area. Or did you take a shuttle to the building? It might be closer to leave through the café.

All characters exit. Soon LAB ASSISTANT *and* PROTAGONIST *enter from one direction and* LA2 and LA4 *enter from another direction and sit as at a café counter.*

NARRATOR: Again, there is a counter with stools that stretches across the room. And again, all the stools are occupied.

PROTAGONIST: It's still crowded in here. Is the café always full?

LAB ASSISTANT: Not always. It's a popular break time, so most of the lab assistants will be here now. Let me see . . . (*looks around*) In fact, I think that all but finitely many of the lab assistants are here. That must mean that Storage is caught up with all the specimens brought in by the infinitely many field biologists. [★13]

PROTAGONIST: Thanks so much for the tour, and for letting me shadow your work with my sample. I'll have to tell my friendly neighborhood ducks that they've contributed to science.

LAB ASSISTANT: You're welcome! All in a day's work. Well, it's not *every* day that I get to handle a new species. So thank you, too. The door to the outside is this way . . .

Both characters exit.

NARRATOR: The End.

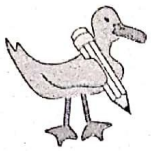
15.3 How Big Is Infinite?

Way back in Section 1.3, we gave the notation $|A|$ as the number of elements in a set A, and in Section 2.2, we explained that this is the cardinality of A. That all makes sense for finite sets, but what do we do with infinite sets? Do we just say $|A| = \infty$ and leave the matter at that? And how do we know whether two infinite sets have the same cardinality, or have the same infinite size?

We answered this question in Chapter 3—long before we even asked it! Remember the Facts of Section 3.2? The pertinent one here is . . .

Fact 3. If there is a bijective map from A to B, then $|A| = |B|$.

Well, there you have it. We never said that this was only true for finite sets, and so it gives us a way to determine when two infinite sets have the same size: we consider two sets to be the same size if we can put them in one-to-one correspondence.



15.4 Try This: Investigating the Play

There is a lot of mathematics embedded in Section 15.2, and our purpose here is to find, expose, and figure out that mathematics. The questions that follow correspond to the $[\star_k]$ markers scattered throughout Section 15.2.

15.4.1 Questions about Sample Storage

$[\star_1]$. "... do you have a favorite number? We can move all the samples from that number onwards up a drawer, and then we can put your sample in the drawer with your favorite number ... And there is no last sample, because there are infinitely many."

(a) To what set does the drawer numbering system correspond?

(b) Moving the samples can be described by a function. What is it? Make sure to specify the domain and target of the function as well as the defining rule.

(c) Is the function you just defined a bijection? If so, prove it; if not, explain why not.

(d) Consider the following statements: $\infty = \infty + 1$; $\infty < \infty + 1$. Argue that exactly one of these statements could be true.

$[\star_2]$. Eight additional samples arrive and so the Storage Coordinator cannot guarantee that the Protagonist's sample will remain in drawer 59. What two sets seem to have the same cardinality here? Define a function to verify that their cardinality is, in fact, the same.

$[\star_3]$. "Every sample is in a numbered drawer. I'll just move every sample to the drawer with twice the current number, and then all the odd-numbered drawers will be free."

(a) What is the sample-moving function proposed by the storage coordinator?

(b) Make and prove a conjecture about cardinalities of sets. Is there a bijection hiding here?

$[\star_4]$. Infinitely many field biologists arrive, each with infinite

specimen that was in the second drawer to the fourth drawer, and the specimen from the third drawer to the ninth drawer. Then two more specimens from the first field biologist will go in drawers 5 and 6, and two specimens from the second field biologist will go in drawers 7 and 8. The specimens from those drawers will get moved down further ... two more specimens from the first field biologist will go in drawers 10 and 11, two more specimens from the second field biologist will go in drawers 12 and 13, and two specimens from the third field biologist will go in drawers 14 and 15. And so forth and so on, including specimens from an additional field biologist in each round of placements, until every specimen has been placed in a drawer or moved to a new drawer."

(a) What is the sample-moving function described in this quotation?

(b) What is the algorithm the Storage Coordinator is using to place new samples?

(c) Will there actually be enough room for all of the new samples? Explain.

(d) What monstrous cardinality fact is hiding here? Make and justify (but don't attempt to prove) a conjecture.

(e) Did you expect that the set of drawers and the set of new samples would have the same cardinality? Why or why not?

15.4.2 More Questions about Sample Storage

[⋆5]. "... that's odd; [the sample tag] says 134. But I thought we were told it was going to be number 59."

Assuming the sample was originally going to get the number 59, what does this tell us about how many of the constantly arriving specimens the storage coordinator has processed?

[⋆6]. "... holder 1 was in the corner here, holder 2 to the right of it, and then diagonally up and left was holder 3, holder 4 was just above it, and then I kept numbering by going diagonally back until I was at the right of holder 2, and so forth. But then over time, I got tired of zig-zagging back and forth from the corner to find a slide and decided to use array numbering instead. Besides, when we started, there were the same number of slides as samples, and now there are *way* more samples than slides—not everything we need to store is studied by me, but everything I study has a corresponding sample in Storage."

(a) Add array numbering to Figure 15.1. What bijection does this suggest (in particular, what sets are involved)? And what interesting cardinality fact can you thereby deduce?

(b) Find a surjection from array numbering to the rational numbers $\mathbb{Q}$. (Why is this not an injective map?) What surprising and interesting cardinality fact does this suggest?

(c) The taxonomist says that there are more samples than slides, but you know that there are $|\mathbb{N}|$ samples (because the sample drawers are numbered) and $|\mathbb{N}|$ slides (because these slide holders used to be numbered). What gives? (How can both statements be true?)

15.4.3 Questions about Café Conversations

[⋆7]. Lab Assistant and LA2 each want to have two cups of coffee. LA2 turns to the left and asks, "Could you please give me your coffee? And the coffee given to the person on your left?" Lab Assistant explains that "... the people to our left will get coffee from the people on their left."

(a) The stools in the café stretch all the way across the room, left and right. What set would be convenient to use in numbering the stools?

(b) What map on coffee cups is described (that will give Lab Assistant and LA2 two cups of coffee each)?

[⋆8]. "It doesn't matter if you want infinitely many cookies, you can still send a few our way!"

How is it possible for one person to have infinitely many cookies and still send some down?

[⋆9]. "LA3: The operations researcher made a list of all possible groups of lab assistants, right? And every group has to meet about someone's primary project .... The operations researcher came up with every possible group, and one of those is the group of lab assistants who are not in the group that discusses their primary project. And apparently there's no project for that group to discuss. ...

"LAB ASSISTANT: Let's pretend that this particular group is going to discuss my primary project ... every lab assistant in this group is *not* in the group that discusses his or her primary project. So I can't be in the group. And *that* means I am not in the group that discusses my primary project.

which means I *am* in the special group, which is discussing my primary project. It just goes 'round and 'round."

(a) LA3 and Lab Assistant's dialogue suggests a function from the set of groups of lab assistants to the set of projects. What is that function? Describe/define it verbally and using function-and-set-theoretic notation.

(b) In turn, this suggests a function from the set of groups of lab assistants to the set of lab assistants. What is *that* function? (Please call it f.)

(c) Consider the group of lab assistants who are not in the group that discusses their primary project. Define this group g using set-theoretic notation.

(d) What is $f(g)$?

(e) Is $f(g) \in g$? Explain.

(f) Explain why there cannot be a bijection between the set of groups of lab assistants and the set of lab assistants.

(g) What is the relationship between the groups of lab assistants and the lab assistants themselves?

(h) What cardinality conclusion does this present?

[$\star_{10}$]. "Wait a minute! The tag says 20,449, and I'm *sure* it was a much smaller number before. Wasn't it supposed to be 59?"

Assuming again that the sample was originally going to get the number 59, what does this tell us about how many of the constantly arriving specimens the storage coordinator has processed?

15.4.4 Indiscrete Questions

[$\star_{11}$]. The infinitely many specimens in the Liver Fluke Lab are collected for transport: "... a dolly stacked with boxes is brought across the hall and then brought back empty. This process takes about 30 seconds. In the next 15 seconds, the dolly passes across the hall full and returns empty. Again a load of boxes passes across, but in one-eighth of a minute. Continuing to pass back and forth twice as fast with each trip, the dolly becomes a blur, and about five seconds later the hall is still."

(a) How long does it take for all of the boxes to be taken across the hall?

(b) How many trips does the dolly make?

(c) Write the duration of each trip as a fraction of the total time, and add these up. Rewrite the result using summation notation to conclude an interesting fact.

[$\star_{12}$]. The Cataloguer avoids relabeling as follows: "... every entry number has a value between 0 and 1 ... Let's say I have a list of entry numbers and I need a new one. I just write down a decimal point, and then I go through the list. I write down a digit that is different from the first digit of the first number, and then I write down a digit that is different from the second digit of the second number, and I continue in this fashion. When I've gotten through the list, I know that I have a new number because it is different in at least one place from each number on the list."

(a) What is the set of entry numbers?

(b) How does its cardinality compare to that of $\mathbb{N}$? Explain.

[$\star_{13}$]. "In fact, I think that all but finitely many of the lab assistants are here. That must mean that Storage is caught up with all the specimens brought in by the infinitely many field biologists."

Are there more lab assistants in the café now than there were earlier?

15.5 How High Can We Count?

Hey! You! Don't read this unless you have worked through the problems in Section 15.4. I mean it!

We can only count at a finite pace (see Section 15.6 for other possibilities), so the literal answer to the question raised in the title of this section is "to some large finite number." But considering the question more figuratively, we have a pretty good idea of how many natural numbers there are, and thus $|\mathbb{N}|$ is a yardstick against which we can measure the cardinalities of other sets. However, as we observed in Sections 15.2 and 15.4, not all infinite sets have the same cardinality, so it is not useful to say $|\mathbb{N}| = \infty$ (even though that statement is true). Instead, we say $|\mathbb{N}| = \aleph_0$ (pronounced *aleph-naught* or *aleph-null*) so that we have a name for this particular size of infinity. We also say, because $\mathbb{N}$ is the set of counting numbers, that any set with size $\aleph_0$ is *countable*. (Indeed, any set with size larger than $\aleph_0$ is *uncountable*.)