WORKSHEET - 1.

1. Important operations

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.

$A \times B = \{(x, y) \mid x \in A, y \in B\}$

$A \oplus B = (A \setminus B) \cup (B \setminus A)$

Some questions that could be asked are

a) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$?

b) $A \setminus (B \cup C) = (A \setminus B) \cup C$?

c) $A \times B = B \times A$?

d) $A \oplus A = A$?

e) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$?

d) $A \oplus (B \setminus C) = (A \oplus B) \setminus (A \oplus C)$?

e) Yes, $A = A \cup B$ if $B \subset A$.

f) $A = B$ if elements of A are same as elements of B.

$A = B$ if and only if $A \subseteq B$ and $B \subset A$.

Proof (a) Let $A = B$ Then clearly $A \subseteq B$ and $B \subseteq A$.

(b) Now conversely suppose $B \subseteq A$ & $A \subseteq B$ we show that $A = B$.

Let $x \in A$., given that $A \subseteq B$, $x \in B$. Similarly $y \in B$, $y \in A$

So $A = B$.

g). Is $A \times (C \cup B) = (A \times C) \cup (A \times B)$

~~Let~~ We use (f). Let LHS $= E$, RHS $= F$

We show that $E \subseteq F$ and $F \subseteq E$.

Let $(x,y) \in E = A \times (C \cup B)$

This means $x \in A$, $y \in C \cup B$.

if $y \in C$ then $(x,y) \in A \times C$

whereas $y \in B$ then $(x,y) \in A \times B$

$\Rightarrow (x,y) \in A \times C$ or $A \times B$

$\Rightarrow (x,y) \in (A \times C) \cup (A \times B) = F$

This shows $E \subseteq F$.

Conversely Let $(x,z) \in F$

So $(x,z) \in A \times C$ or $A \times B$.

If $(x,z) \in A \times C$ then

$(x,z) \in A \times (C \cup B) = E$

Thus $F \subseteq E$.

(b) Counter ex.

$A = \{1,2,3\}$ $B = \{3,4\}$ $C = \{a\}$

$A \setminus B = \{1,2\}$. $B \cup C = \{3,4,a\}$

$A \setminus (B \cup C) = \{1,2\}$ $(A \setminus B) \cup C = \{1,2,a\}$.

So $A \setminus (B \cup C) \neq (A \setminus B) \cup C$.

Q2 $f(A \cup B) = f(A) \cup f(B)$

Let $x \in f(A \cup B)$ ie $x = f(y)$, where $y \in A \cup B$. Since y is in the union it is in A or B. If $y \in A$ then $x = f(y) \in f(A)$

ie ~~$f(A \cup B)$~~ If y is in B

then $x = f(y) \in f(B)$

$\Rightarrow x \in f(A)$ or $f(B)$

$\Rightarrow x \in f(A) \cup f(B)$

Thus $f(A \cup B) \subseteq f(A) \cup f(B)$ — (i).

We observe that if $C \subseteq D$, then $f(C) \subseteq f(D)$

Now, $A \subseteq A \cup B$, $B \subseteq A \cup B$

$f(A) \subseteq f(A \cup B)$

$f(B) \subseteq f(A \cup B)$

$\Rightarrow f(A) \cup f(B) \subseteq f(A \cup B)$ — (ii)

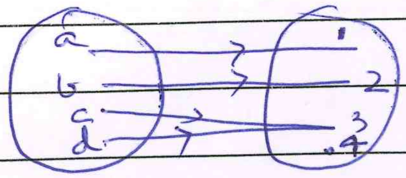
From (i) & (ii) $f(A) \cup f(B) = f(A \cup B)$.

Note that in ~~the~~ 3rd worksheet we asked if same holds for pre-image

Let $f: X \to Y$, $A \subseteq Y$.

Then $f^{-1}(A) = \{x \mid f(x) \in A\}$.

~~$X = \{$~~



$A = \{2, 3\}$

$f^{-1}(A) = \{b, c, d\}$

$f^{-1}(\{4\}) = \phi$

$f^{-1}(\{1\}) = \{a\}$

We can talk of pre-image even without inverse.

We can ask if
$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

Q 2 (2). Is $f(A \cap B) = f(A) \cap f(B)$.

Again note that $A \cap B \subseteq A$, $A \cap B \subseteq B$

so $f(A \cap B) \subset f(A)$

$f(A \cap B) \subseteq f(B)$

$\Rightarrow$ $f(A \cap B) \subseteq f(A) \cap f(B)$ – always.

Consider $f : \mathbb{Z} \to \mathbb{Z}$. $x \to x^2$

$A = \{-1, 0\}$ $B = \{0, 1\}$

$A \cap B = \{0\}$. | $f(A) = f(B) = \{0, 1\}$

$f(A \cap B) = \{0\}$. | $f(A) \cap f(B) = \{0, 1\}$.

so. $f(A \cap B) \neq f(A) \cap f(B)$

If we restrict the same function to
$$f : \{0, 1, 2 \ldots\} \to \mathbb{Z}$$
$$x \to x^2.$$

we will find that $f(A \cap B) = f(A) \cap f(B)$.
so, we need to be careful about specifying domain & codomain.

Q 2 (3) Can we say $f(A \setminus B) = f(A) \setminus f(B)$

Again look at previous example

$A \setminus B = \{-1\}$. $f(A \setminus B) = \{1\}$.

$f(A) = f(B) = \{0, 1\}$. $f(A) \setminus f(B) = \emptyset$

Is this happening because f is not 1-1 ?

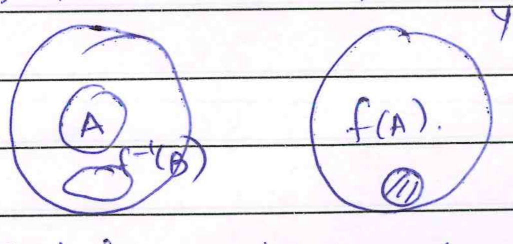
We can ask the question
If f is 1-1 function
Is $f(A \setminus B) = f(A) - f(B)$

let $y \in f(A) \setminus f(B)$.
Since f is 1-1, $\exists$ a unique x such that $f(x) = y$ and $x \in A$. Now.
$y \notin f(B)$ so x cannot be in B.
so $x \in A \setminus B$.
$\therefore$ $y = f(x) \in f(A \setminus B)$.
$\Rightarrow$ $f(A) \setminus f(B) \subseteq f(A \setminus B)$ — (i)

The other way let $z \in f(A \setminus B)$
Then $z = f(a)$ where $a \in A \setminus B$
and there is only one such a.
$z = f(a) \in f(A)$
and $z = f(a) \notin f(B)$
$\Rightarrow$ $z \in f(A) \setminus f(B)$

Here we ~~don't~~ ~~require~~ do we require that f is onto ?

Q 2(4). $f : X \to Y$.



$f(A)$ is image of A
$f^{-1}(B)$ is pre-image of B.
Can we say, pre-image of image of A is same as A
$f^{-1}(f(A)) = A$

Can we say.

$$f\,(f^{-1}(B)) = B.$$

i.e. Image of pre-image of B is B.

Look at example on page 4.

$A = \{-1, 0\}$.

$f(A) = \{0, 1\} = B$

$f^{-1}(f(A)) = \{-1, 0, 1\}$.

Here $A \subset f^{-1}(f(A))$.

$f\,(f^{-1}(B)) = B$ – in this case.

Can you prove it?

Q 3 (2) Let $X = Y = Z$

$f : X \to X \xrightarrow{g} X$.

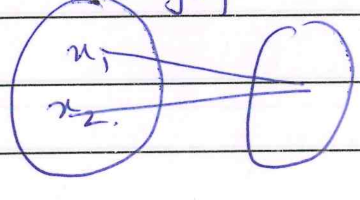
Let f and g be 1-1 functions

<u>Claim</u> $g \circ f$ is 1-1.

Suppose not, then $\exists$ x_1 & x_2

$x_1 \neq x_2$ but $g(f(x_1)) = g(f(x_2))$

$g \circ f$



This means

$z = g(y_1) = g(y_2)$

Since g is 1-1 $\Rightarrow y_1 = y_2$.

$\Rightarrow f(x_1) = f(x_2)$.

But f is 1-1 $\Rightarrow x_1 = x_2$

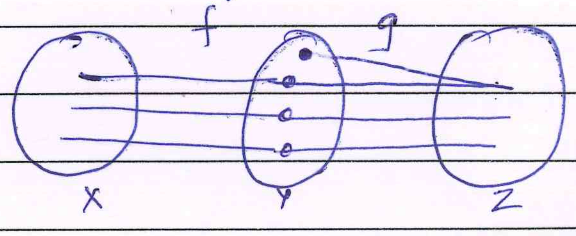
3 (3) Let f and g be onto functions then $g \circ f = h$ is onto?

Suppose h is not onto function
Then $\exists\, x \in X$ s.t. $x \notin h(X)$
x is not in image of h.

But x is in image of g
$\exists\, y$ s.t. $g(y) = x$.
Also f is onto $\exists\, z$ s.t. $f(z) = y$.
$\Rightarrow$ $g(f(z)) = x$
$\Rightarrow$ h is onto.

Now suppose we have info about the composite, what can we say about the individual functions.

Suppose $g \circ f = h$ is $1-1$
Does it force g to be $1-1$.

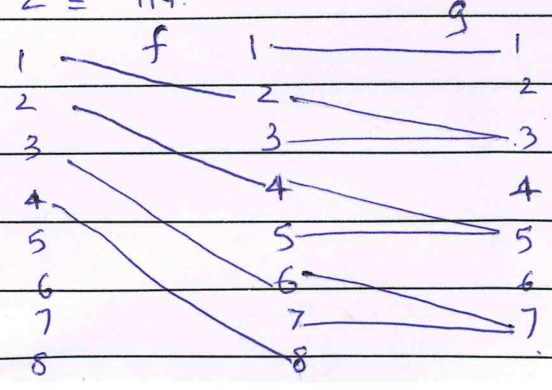


$g \circ f$ is $1-1$
But g is not $1-1$
Here $Y \neq X$.

Here $Y \neq X$.

Consider the following example.

$X = Y = Z = \mathbb{N}$.



g is not $1-1$
$g \circ f$ is $1-1$.