Introduction to Mathematical Thinking 2

April 13, 2020

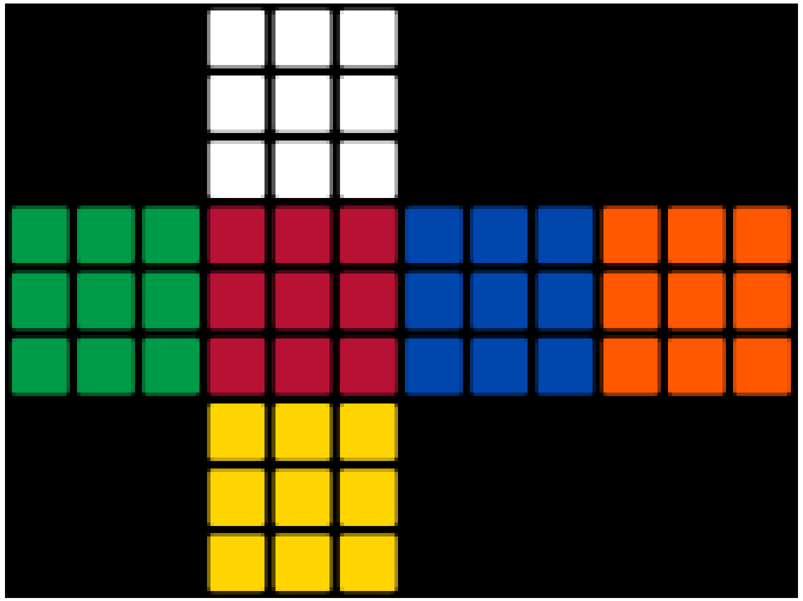
Groups and Symmetry

Let us start our discussion with two games

Rubik's cube:

The cube has six faces each with different colour. Each side has 6 squares of the same colour. When we rotate the face by 90 degrees, the colours of the smaller squares get scrambled. The puzzle requires us to unscramble and get it back to original configuration.

Rubik's cube



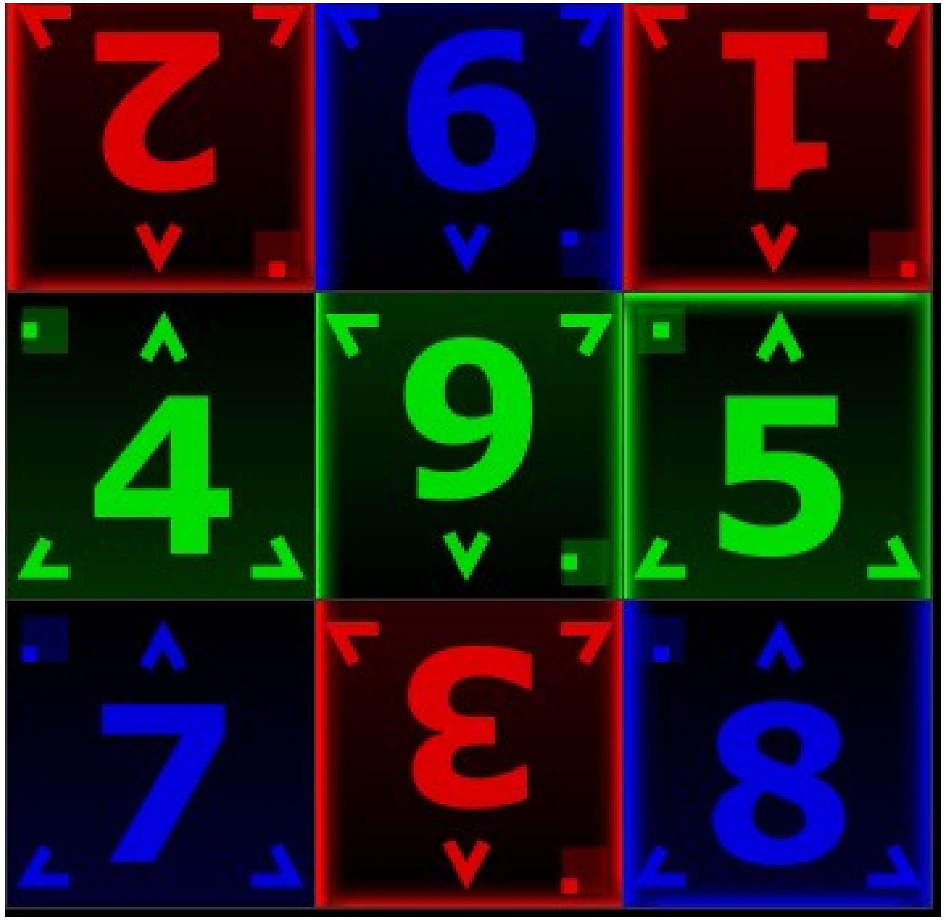
How many ways?

- The original (3×3×3) Rubik's Cube has eight corners and twelve edges. There are 8! (40,320) ways to arrange the corner cubes. Each corner has three possible orientations, although only seven (of eight) can be oriented independently; the orientation of the eighth (final) corner depends on the preceding seven, giving 37 (2,187) possibilities. There are 12!/2 (239,500,800) ways to arrange the edges, restricted from 12! because edges must be in an even permutation exactly when the corners are. ……………… The number of ways is 43,252,003,274,489,856,000. (From the wikipedia page on Rubiks cube)
- How do we figure out the solutions?

Observations

- Rule 1. There is a predefined list of actions that never changes.
- Rule 2. Every action is reversible.
- Rule 3. Every action is deterministic.
- Rule 4. Any sequence of consecutive actions is also an action.

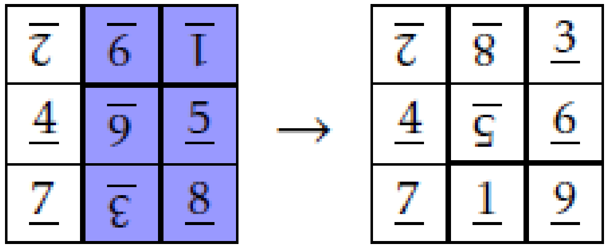
SPIN POSSIBLE

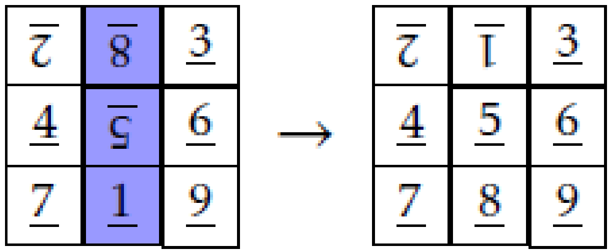


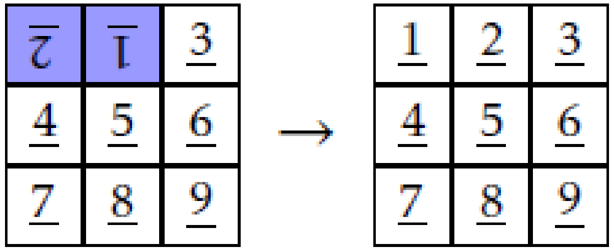
https://www.kongregate.com/games/spinpossible

Spinpossible

- The objective is to unscramble and get the numbers 1 to 9 in the right order facing up.
- In this game, we are allowed to spin a tile or large mxn rectangle about a point.







Spin 3x 3.

(1) **Generating Set:** The set of spins generates $\text{Spin}_{3\times 3}$. That is, every net action from $\text{Spin}_{3\times 3}$ corresponds to a word consisting of spins.†

(2) **Closure:** The composition of any two net actions from $\text{Spin}_{3\times 3}$ results in a net action from $\text{Spin}_{3\times 3}$.

(3) **Associative:** The composition of net actions from $\text{Spin}_{3\times 3}$ is associative.

(4) **Identity:** There is an identity in $\text{Spin}_{3\times 3}$ whose corresponding net action is "do nothing".

(5) **Inverses:** Every net action from $\text{Spin}_{3\times 3}$ has an inverse net action in $\text{Spin}_{3\times 3}$. Composing a net action and its inverse results in the identity.

(6) The composition of two net actions from $\text{Spin}_{3\times 3}$ may or may not commute.

Combining things

- Just like we combined two spins or two rotations we can combine two objects at a time. We are familiar with combining numbers by adding or multiplying them. This leads to the idea of binary operations.
- A binary operation on a set A is a function from $A \times A$ into A. For

each (a , b) in $A \times A$, we denote the element $(a;b)$ via $a * b$. If the context is clear, we may abbreviate $a * b$ as ab.

- Please note that the effect of combining two ‘things’ in A should lead back to an element of A.
- Give examples of binary examples.
- Are there ways of combining elements of A, for which the result is not an element of the set A?

Binary operations

- The operations of addition, subtraction and multiplication are binary operations on the real numbers. All three are also binary operations on the integers. However, subtraction is not a binary operation on the natural numbers.
- There are mathematical objects that are not numbers. For example functions, matrices, permutations, sets etc. Figure out how to combine them to get something in the same set.

Properties of binary operations

Let A be a nonempty set and let * be a binary operation on A.

- (a) We say that * is associative if and only if $(a * b) * c = a * (b * c)$ for all a, b, c in A.
- (b) We say that * is commutative if and only if $a * b = b * a$ for all a, b in A.
- Question : Let a*b=a.b+1, where a, b are real numbers. Is * a binary operation. If so is it commutative, is it associative?
- Question: Check associativity for some of examples given in previous slide

Identity

- We saw that in both the games , there was an action which was “do nothing”. Combining any action A with do nothing results in the action A.
- Mathematically we call such an element as identity. i.e,

  a*e=a, e*a=a, for every element a in the set G.

Question: Go back to examples and check whether there is an identity for the operation *.

Inverse

- In the Rubik's cube, for every action A , there was a way of undoing the action A to get back to the same configuration.
- Mathematically , if for every a in G, we can find another element b in G such $a*b=e=b*a$. We call this element b as the inverse of a.
- Remember that for sets G where there are many binary operations, we must specify the operation. For example, in real number system, the element 2 has additive inverse as -2, multiplicative inverse is ½.

Groups

A group $(G; *)$ is a set G together with a binary operation such that the following axioms hold.

- (1) The operation $*$ is associative.
- (2) There is an element e in G such that for all g in G, $e * g = g * e = g$. We call e the identity.
- (3) Corresponding to each g in G, there is an element g' in G such that $g * g' = g' * g = e$. In this case, g' is said to be an inverse of g.

The order of G, denoted $|G|$, is the cardinality of the set G. If $|G|$ is finite, then we say that G has finite order. Otherwise, we say that G has infinite order.

Examples of Groups

- The set of integers, under addition is a group.
- The set of rationals, under addition is a group.
- The set of integers under multiplication is not a group. The operation satisfies closure, associativity, identity but not existence of inverse.
- The set of 2x2 matrices under addition is a group.
- The set of permutation of 3 objects is a group. The * operation is nothing but the composition operation. Permutation is nothing but a function.

Cayley tables

For a finite set, we can pictorially show the result of binary operation.

*	a	b	c
a	b	c	b
b	a	c	b
c	c	b	a

Is this a group?

Cayley Table

*	a	b
a	a	b
b	b	a

Cayley Table

*	a	b	c
a	a	b	c
b	b		
c	c		

Resources

- http://danaernst.com/teaching/mat411f16/materials/
-
- http://www.math.clemson.edu/~macaule/classes/m19_math4120/

- Visual Group Theory-Nathan Carter. Mathematics Association America. Classroom resource materials. 2009.