Assignment 1 - Unit 3

Introduction to Mathematical Thinking-2, 13 April 2020

1. Check that the following sets form a group. Mention the binary operation that has been used.

   a. $2\mathbb{Z} = \{m|m = 2n, n \in \mathbb{Z}\}$
   b. $3\mathbb{Z} = \{m|m = 3n, n \in \mathbb{Z}\}$
   c. $S = \{a + b\sqrt{2}|a, b \in \mathbb{Z}\}$
   d. $S = \{a + b\sqrt{2}|a, b \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{2})$
   e. $S = \{a + b\sqrt{2}|a, b \in \mathbb{R}\}$
   f. $S = \{a + ib|a, b \in \mathbb{Z}\}$

2. Show that in $\mathbb{Q}(\sqrt{2})$, every non zero element has multiplicative inverse.

3. Let S_n=$\{z|z^n = 1, z \in \mathbb{C}\}$. Show that S is a group under multiplication. Show that the union of all S_n is also a group.

4. Show that $\forall a, b \in G$, $(ab)^{-1} = b^{-1}.a^{-1}$, where G is a group.

5. Recall the definition of $\mathbb{Z}_m$ is the collection of congruence classes mod m. There are two possible group operations, addition and multiplication.
   a. Check that this forms a group under addition. What is the identity element here?
   b. When does $\mathbb{Z}_m$ form a group?. Hint- Check for m prime and m composite. Write down the table for small m to see what is going on.

6. Is $\mathbb{Z}_{20}$ a group under multiplication? Look at subset $\{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$. Is it a group? Check the following subsets. $\{\bar{1}\bar{9}, \bar{1}3, \bar{1}7\}$. $\{\bar{1}.\bar{3}, \bar{7}, \bar{9}, \bar{1}1, \bar{1}3, \bar{1}7, \bar{1}9\}$. Now study the set $\mathbb{Z}_{14}$. Is it a group under multiplication?. What about its subset $\{\bar{1}, \bar{3}, \bar{5}\}$, $\{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$, $\{\bar{1}, \bar{7}, \bar{1}3\}$

7. Let G be a group. We define order of a, 0(a) as the smallest n such that if $a^n = e$, where e is the identity of the group. Show that the order of any element a and its inverse are the same. Also show that ab and ba have the same order.

8. Let G be a group. Let o(a)=m, o(b)=n and ab=ba. Show that o(ab)=mn.

9. If o(a)=18. That is 18 is the smallest natural number such that $a^{18} = e$. What is the order of the following elements: a^2, a^3, a^4...? What is the pattern here?

10. Show that $a^k = e$ if and only if $(bab^{-1})^k = e$.

11. G is said to be an abelian group if all the elements commute with each other, that is ab=ba for all elements a, b in G. The set of integers, rational, reals are all abelian under addition. Let G be a finite abelian group $\{g_1, ...g_n\}$. Show that the product of these elements $g = g_1.g_2.....g_n$ has order 2.

12. Let G be a group. Show the following.
    a. The identity element is unique. b.Every element has a unique inverse. c. $(a^{-1})^{-1} = a$.

13. Make a Cayley table to show the group structure of a set with three elements, $\{e, a, b\}$. Also make the Cayley table for the group $\mathbb{Z}_3$, under addition. Are the two table same or different. If they are same, in what sense?

14. Make Cayley table and show that there are two different groups of order 4. What is the method you used to distinguish these two groups?

15. Let G be a group. If a, b are in G, then there exists a unique element in the group say x, such that $a.x = b$. Similarly there exisits a unique y such $y.a = b$.

16. Let G be a group and for all a, b in G we have $(ab)^2 = a^2.b^2$. Then show that G is abelian. Also show that if G is abelian, then $(ab)^n = a^n.b^n$.