What is the origin of Group Theory?

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29/3/2020

DATE: 29/03/2020.

Origin of Group Theory

The first formal definition of a group was given by Arthur Cayley in 1854. Of course, this did not come out of the blue! From the work of Lagrange (in 1770), mathematicians had been studying permutation groups. Cauchy and then Galois played with this idea in theory of Algebraic equations. Galois, at the age of seventeen came up with a stunning result (which we will discuss later)

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This definition did not get much attention, unfortunately. However, around this time, George Boole was working on symbolic algebra in logic. At this time, there was a trend to move towards 'axiomatics'. In fact, groups were the first algebraic system to be axiomatised.

In 1854, Cayley wrote a paper entitled, "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$". His definition is:

Definition of set

A set of symbols $1, \alpha, \beta \ldots$ all of them different, and such that the product of any two of them (no matter in what order), or the product of

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any one of them into itself, belongs to the set, is said to be a group..." symbols are associative.

In modern day language,

" A set G, (non empty) is said to be a group, if it satisfies the following properties, $* : G \times G \to G$

i.e, $\forall\ a, b \in G,\ a * b \in G$.

i) $\forall\ a, b, c \in G \quad a*(b*c) = (a*b)*c$

ii) $\exists\ e \in G$ s.t $\ a*e = e*a = a \quad \forall a \in G$

What led Cayley to choose these 3 properties? Why was the property of commutativity $(a*b = b*a)$ not assumed? The answer comes from the work of Lagrange, Abel, Cauchy, Galois.

Before Cayley (BC)

It took about 100 years and the work of brilliant "all rounders" such as Gauss, Klein, Cauchy, Dedekind, Lagrange to make the theory of groups a cohesive theory with a wide range of applications. When we read the history of mathematics, we encounter some of these "all rounders" in diverse areas

For example, C.F. Gauss has contributed to number theory, geometry, probability, astronomy, magnetism etc.

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THEORY OF ALGEBRAIC EQUATIONS

Somewhere in middle school, children are introduced to the idea of variables and their assignments are full of symbols and parameters. We take this idea for granted and assume it has always been around.

Actually the idea of using symbols goes back to 16th century. Vieté wrote an essay called 'Introduction to the Analytic Art' in 1591. He introduced the idea of arbitrary parameters into an

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equation and distinguished these from the equations variables

He used consonants (B, C, D...) to denote parameters and vowels (A, E..) to denote variables. In modern language, we write a quadratic polynomial as ax^2+bx+c. This was written as $BA^2+CA+D=0$.

This was a very powerful idea, as it allowed people to ask import-ant and interesting questions. For example, when does an equation have roots, how many roots, how do roots relate to co-efficients etc.

We can ask these question for equations of degree 3, 4, 5 etc.

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For many decades, people were grappling with the idea of negative roots, complex roots, maximum number of roots etc.

C.F. Gauss showed that every polynomial equation (with real or complex co-efficients) of degree n has atleast 1 root, which may be a real or complex number. In fact, a polynomial of degree n has exactly 'n' roots in the set of complex numbers. This is called the 'Fundamental theorem of Algebra'. This was 1799 doctoral dissertation of Gauss. In this proof, he had made an 'obvious' assumption.

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Later he gave three proofs of FTA. Most proofs require tools from complex analysis/topology.

Coming back to 16-17th century, one of the main questions was to find roots of an equation in terms of co-efficients, using the operations of $+, -, \times, /, \sqrt[n]{\ }$.

In school, we learn how to find roots of quadratic equation,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(The history of this is interesting and goes back to Babylonians!)

We also know the relationship

$x_1 + x_2 = -b/a$, $x_1 . x_2 = c/a$, if

x_1, x_2 are the roots.

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The study of polynomial equations and properties of various number systems happened in parallel. This was also accompanied by axiomatic approach, rigour and development of language of functions.

In the language of functions, if $f(x_1, x_2) = x_1 + x_2$, f is a symmetric function, as $f(x_1, x_2)$ is same as $f(x_2, x_1)$. Similarly, $f(x_1, x_2) = x_1 x_2$ is symmetric.

Also $f(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_3 x_1$ is symmetric.

The idea of symmetry is very important in geometry.

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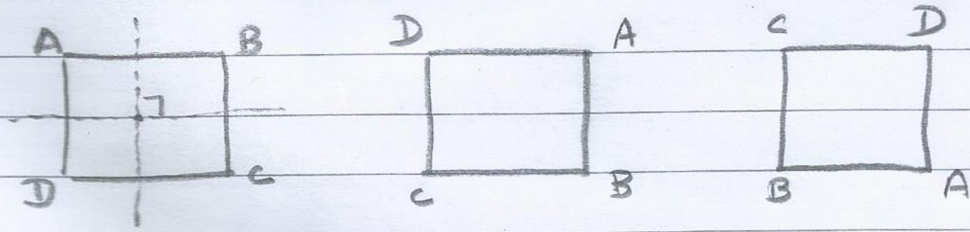
For example, a square remains the same when turned by 90° around the centre.



We notice that rotation by 180°, 270°, 360° about the origin do not change the square. We call 90° rotation a symmetry. A very important area of mathematics was developed by Felix Klein in Germany.

We will come back to the idea of symmetries and how to capture 'how much' symmetry a 'object' has. This idea of 'capturing' led to the beginnings of group theory, which in turn

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turn, led to development of 'Abstract Algebra. These developments were accompanied by growth in 'Number Theory', 'Analysis', 'Topology', 'Algebraic Geometry', 'Differential Geometry' etc.

There were four broad areas that led to the modern definition of Group.

i) Classical Algebra (Lagrange, 1770)

ii) Number theory (Gauss 1801)

iii) Geometry (Klein, 1874)

iv) Analysis (Lie, 1874, Poincare & Klein 1876)

Here, we will discuss the developments in classical Algebra. In 1770, Joseph Louis Lagrange wrote a paper which explored solutions of quintics.

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Lagrange was a brilliant mathematician and astronomer. He contributed to the areas of analysis, number theory, classical and celestial mechanics. Many things are named after him. For example, Lagrange multipliers, Euler-Lagrange equations, Lagrangian points, Lagrangian mechanics etc. He proved that every natural number can be written as sum of four squares. He was also involved in development of metric system of measurements.

Coming back to this particular work. He wanted to find a general method to find roots of degree n-polynomial using method of using resolvent equations.

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This method of finding resolvent equations did not work for degree $n \geq 5$. However it gave a general method for $n < 5$, through a single method. This idea was further taken up by Cauchy and Galois.

Let $f(x)$ be a polynomial of degree "n". and the roots be $x_1, x_2, \dots x_n$.. Pick a rational function of roots and co-efficients $R(x_1, x_2 \dots x_n)$ in a prescribed manner When the roots are permuted, we get different values $y_1, y_2 \dots y_k$. The resolvent equation is $g(x) = (x-y_1)(x-y_2)\dots(x-y_k)$ The co-efficients of $g(x)$ are symmetric functions of $x_1, x_2 \dots x_n$, and hence

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they are polynomials in the elementary symmetri functions of $x_1, x_2 \ldots x_n$. Lagrange showed that k divides $n!$. We will study this as Lagrange's theorem. For example, if $f(x)$ is of degree 4 and we take $R(x_1, x_2, x_3, x_4)$ as $x_1x_2 + x_3x_4$. There are 24 permutations but we get only 3 distinct values y_1, y_2 and y_3. Thus the resolvent equation is a cubic, of degree 1 less than the degree of original degree 4-polynomial.

This led to considerable work by Cauchy and Abel. In 1829, Evariste Galois submitted a paper to the French Academy. This was a breakthrough. Sadly, his paper was not understood very well.

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In 1831, Galois submitted a revised version of the paper. He was only 19 then. He died in 1832 in a tragic duel. He was a fire-brand liberal. It is not clear whether the duel was on account of his politics or over a failed love affair.

On the night before the duel, May 29, 1832, Galois wrote a letter to Chevalier and included 3 manuscripts.

This work is now called Galois theory, in which he founded group theory, established that fields are fundamental in study of solvability of equations. His theorem gave necessary and sufficient condition for a polynomial to be solved by radicals.

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