Lane-Emden equations with Hardy potential and measure data

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IISc–IISER-Pune Math Symposium
September 18, 2021

We consider the Lane-Emden equations with Hardy potential

$$-L_\mu u = u^p \quad \text{in } \Omega, \tag{E}$$

and the Lane-Emden systems of the form

$$\begin{cases} -L_\mu u = v^p & \text{in } \Omega, \\ -L_\mu v = u^q & \text{in } \Omega, \end{cases} \tag{S}$$

where

$$L_\mu := \Delta + \frac{\mu}{\delta^2}, \quad \delta(x) := \mathrm{dist}(x, \partial\Omega)$$

$0 < \mu < C_H(\Omega)$, $1 < p \le q$ and Ω be a C^2 bounded domain in $\mathbb{R}^N$ ($N \geq 3$).

$C_H(\Omega)$ is the best constant in the Hardy inequality

$$C_H(\Omega)\int_\Omega \frac{\varphi^2}{\delta^2}dx \leq \int_\Omega |\nabla\varphi|^2 dx \quad \forall\varphi \in H_0^1(\Omega).$$

where Ω is any bounded domain with Lipschitz boundary.

- $C_H(\Omega) \in (0, \frac{1}{4}]$
- if Ω is convex then $C_H(\Omega) = \frac{1}{4}$.
- $C_H(\Omega)$ is achieved if and only if $C_H(\Omega) < \frac{1}{4}$.

Marcus-Mizel-Pinchover, Trans. AMS'1998

Brezis-Marcus, Ann. Sc. Norm. Super. Pisa'1997 proved that for every $\mu < \frac{1}{4}$, $\exists! \, \lambda_{\mu,1}$ s.t.

$$\mu = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla u|^2 - \lambda_{\mu,1} u^2) dx}{\int_\Omega \frac{u^2}{\delta^2} dx}$$

and the infimum is achieved i.e., $\lambda_{\mu,1}$ is an eigenvalue of $-L_\mu$. They also proved that $\lambda_{\mu,1}$ is simple, the corresponding eigenfunction $\varphi_{\mu,1}$ is positive and $\lambda_{\mu,1} > 0$ when $\mu < C_H(\Omega)$. Further, by Marcus-Shafrir, Ann. Sc. Norm. Super. Pisa'2000

$$\varphi_{\mu,1} \sim \delta^\alpha,$$

where

$$\alpha := \frac{1 + \sqrt{1 - 4\mu}}{2}.$$

i.,e., $\frac{1}{2} < \alpha < 1$.

When $\mu = 0$, equation (E) with measure boundary data were studied by Bidaut-Véron and Vivier'2000, Bidaut-Véron and Yarur'2002 in which various existence results and apriori estimates were established based on delicate estimate of Green kernel and Poisson kernel of $-\Delta$ in Ω.

BVP with L_μ operator and absorption nonlinearity with measure data

$$-L_\mu + u^p = 0 \quad \text{in} \quad \Omega, \qquad u = \nu \quad \text{on} \quad \partial\Omega$$

has been introduced in Marcus-Nguyen, AIHP Non Linéaire'2017, Math. Ann'2019.

For $\mu \in (0, C_H(\Omega))$, BVP for (E) (i.e., with source nonlinearity) with measure data has been studied in Nguyen, CVPDE'2017.

The condition $\mu \in (0, C_H(\Omega))$ is imposed as Martin kernel K_μ of L_μ in Ω exists in this range.

$$K_\mu(x,y) := \lim_{\Omega \ni z \to y} \frac{G_\mu(x,z)}{G_\mu(x_0,z)}, \quad \forall x \in \Omega,\ y \in \partial\Omega,$$

where $x_0 \in \Omega$ is a fixed reference point and G_μ is Green kernel of L_μ

Moreover, the following two-sided estimate holds

$$K_\mu(x,y) \sim \delta(x)^\alpha |x-y|^{-(N+2\alpha-2)} \quad \forall\, x \in \Omega,\ y \in \partial\Omega.$$

Theorem (Ancona, Ann. Math'1987)

Representation theorem. *For every positive bounded Borel measure ν, the function*

$$\mathbb{K}_\mu[\nu](x) := \int_{\partial\Omega} K_\mu(x,y)d\nu(y) \quad \forall\, x \in \Omega$$

is L_μ-harmonic, i.e., $L_\mu(\mathbb{K}_\mu[\nu]) = 0$ and conversely, $\forall$ positive L_μ harmonic function u, $\exists!$ bounded Borel measure ν s.t. $u = \mathbb{K}_\mu[\nu]$.

The measure ν s.t. $u = \mathbb{K}_\mu[\nu]$ is called the L_μ boundary measure of u.

- If $\mu = 0$, then this ν is equivalent to the classical measure boundary trace of u.

Definition (Measure boundary trace)

Let $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 1$. We say that u possesses an M-boundary trace on Ω if there exists $\nu \in \mathfrak{M}(\partial\Omega)$ s.t. for every uniform C^2 exhaustion D_n and every $h \in C(\bar{\Omega})$

$$\int_{\partial D_n} u\lfloor \partial D_n h dS \to \int_{\partial\Omega} h d\nu.$$

Here $u\lfloor \partial D_n$ the Sobolev trace.

- If $0 < \mu < C_H(\Omega)$, $\forall \nu \in \mathfrak{M}_+(\partial\Omega)$, measure boundary trace of $\mathbb{K}_\mu[\nu]$ is 0.

$\rightsquigarrow$ the classical notion of boundary trace no longer plays an important role in describing the boundary behavior of L_μ-harmonic function or solutions of (E).

$\rightsquigarrow$ We need to introduce the notion of L_μ boundary trace which is defined as follows

Let D_n be a uniformly Lipschitz exhaustion of Ω with $x_0 \in D_n$ for all large n and $P_\mu^{D_n}$ and $\omega_\mu^{x_0,D_n}$ denote the Poisson kernel of L_μ and the harmonic measure of L_μ in D_n (relative to the fixed reference point x_0) respectively, i.e.,

$$d\omega_\mu^{x_0,D_n} = P_\mu^{D_n}(x_0,\cdot)dS \quad \text{on} \quad \partial D_n.$$

Definition (L_μ boundary trace)

A non-negative Borel function u defined in Ω has an L_μ boundary trace $\nu \in \mathfrak{M}(\partial\Omega)$ if

$$u\lfloor \partial D_n d\omega_\mu^{x_0,D_n} \rightharpoonup \nu$$

i.e.,

$$\lim_{n\to\infty} \int_{\partial D_n} uhd\omega_\mu^{x_0,D_n} = \int_{\partial\Omega} hd\nu \quad \forall\, h \in C(\bar{\Omega})$$

for all uniformly Lipschitz exhaustion D_n of Ω.

Lemma (Marcus, PAFA'2020)

If D_n is a uniformly Lipschitz exhaustion of Ω then for every positive L_μ harmonic function $u = \mathbb{K}_\mu[\nu]$,

$$u\lfloor \partial D_n d\omega_\mu^{x_0, D_n} \rightharpoonup \nu,$$

i.e., L_μ boundary trace of $\mathbb{K}_\mu[\nu]$ is ν.

For every positive Radon measure τ in Ω, define

$$\mathbb{G}_\mu[\tau](x) := \int_\Omega G_\mu(x,y) d\tau(y).$$

- L_μ boundary trace of $(\mathbb{G}_\mu[\tau]) = 0$.
- Let u be a positive L_μ-superharmonic function. Then there exist $\nu \in \mathfrak{M}_+(\partial\Omega)$ and $\tau \in \mathfrak{M}_+(\Omega, \delta^\alpha)$ s.t.

$$u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu].$$

Another equivalent definition of trace:

A nonnegative function $u \in W^{1,p}_{loc}(\Omega)$ is said to have a normalized boundary trace $\nu \in \mathfrak{M}_+(\partial\Omega)$ if

$$\lim_{\beta \to 0} \beta^{\alpha-1} \int_{\Sigma_\beta} \left| u(x) - \mathbb{K}_\mu[\nu](x) \right| dS(x) = 0,$$

where $\Sigma_\beta := \{x \in \Omega : \delta(x) = \beta\}$.

We denote normalized boundary trace as tr^*.

Now consider the BVP of the form

$$\begin{cases} -L_\mu u = u^p \text{ in } \Omega, \\ \mathrm{tr}^*(u) = \rho\nu \text{ on } \partial\Omega, \end{cases} \qquad (P_\rho)$$

where ρ is a positive parameter and $\nu \in \mathfrak{M}_+(\partial\Omega)$ with norm 1.

Definition (Weak solution of (P_ρ))

(i) A function u is called a weak solution of (E) if $u^p \in L^1_{loc}(\Omega)$ and (E) is satisfied in the sense of distributions in Ω.
(ii) A function u is called a weak solution of (P_ρ) if u is a weak solution of (E) and has boundary trace $\rho\nu$.

Theorem (Nguyen, CVPDE'2017)

The following statements are equivalent:

(i) u is a positive weak solution of (P_ρ).

(ii) $u^p \in L^1(\Omega, \delta^\alpha)$ and $u = \mathbb{G}_\mu[u^p] + \mathbb{K}_\mu[\rho\nu]$.

(iii) $u \in L^1(\Omega, \delta^{\alpha-1})$, $u^p \in L^1(\Omega, \delta^\alpha)$ and

$$-\int_\Omega u L_\mu \phi dx = \int_\Omega u^p \phi dx - \int_\Omega \mathbb{K}_\mu[\rho\nu] L_\mu \phi dx \quad \forall\, \phi \in X(\Omega),$$

where $X(\Omega) := \left\{\psi \in C^2(\Omega) : \delta^{1-\alpha} L_\mu \psi \in L^\infty(\Omega),\ \delta^{-\alpha}\psi \in L^\infty(\Omega)\right\}$.

$$N_\mu := \frac{N+\alpha}{N+\alpha-2}.$$

Theorem (Nguyen, CVPDE'2017)

Let $\rho > 0$, $p > 1$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$.

I. Subcritical case: $p \in (1, N_\mu)$. There exists $\rho^* \in (0, \infty)$ s.t.

(i) *If $\rho \in (0, \rho^*]$ then problem (P_ρ) admits a minimal positive weak solution $\underline{u}_{\rho\nu}$. Moreover,*

$$C^{-1}\rho\mathbb{K}_\mu[\nu] \le \underline{u}_{\rho\nu} \le C\rho\mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega,$$

where $C > 0$ (independent of ρ). If, in addition, $\{\rho_n\}_{n\ge1}$ be such that $0 < \rho_n \uparrow \rho^*$ then

$$\underline{u}_{\rho_n\nu} \uparrow \underline{u}_{\rho^*\nu} \quad \text{in} \quad L^1(\Omega;\delta^{\alpha-1}) \quad \text{and} \quad L^p(\Omega;\delta^\alpha).$$

(ii) If $\rho > \rho^*$ then (P_ρ) does not admit any positive solution.

II. Supercritical case: $p \geq N_\mu$. For every $\rho > 0$ and $z \in \partial\Omega$, there is no positive weak solution of (P_ρ) with $\nu = \delta_z$, where δ_z is the Dirac mass concentrated at $z \in \partial\Omega$.

Multiplicity result for (P_ρ)

Theorem (B.,-Mukherjee-Nguyen '2021)

Assume $N > 3$, $p \in (1, N_\mu)$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. Let $\rho^* > 0$ be the threshold value as in previous theorem. Then

(i) for any $\rho \in (0, \rho^*)$, problem (P_ρ) admits a second positive weak solutions u such that $u > \underline{u}_{\rho\nu}$, where $\underline{u}_{\rho\nu}$ is the minimal solution of (P_ρ), constructed in previous theorem.

(ii) (P_ρ) admits a unique positive solution when $\rho = \rho^*$.

In order to construct the 2nd solution, we first consider an auxiliary problem

$$\begin{cases} -L_\mu u = (\underline{u}_{\rho\nu} + u^+)^p - \underline{u}_{\rho\nu}^p & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \qquad (\widetilde{P}_\rho)$$

and we construct a positive variational solution v_ρ of $(\widetilde{P}_\rho)$ using mountain pass theorem when $\rho \in (0, \rho^*)$.

Definition

An element $u \in H_0^1(\Omega)$ is said to be a variational solution of $(\widetilde{P}_\rho)$ if

$$\int_\Omega \nabla u \cdot \nabla \phi dx - \mu \int_\Omega \frac{u\phi}{\delta^2} dx = \int_\Omega \left[(u^+ + \underline{u}_{\rho\nu})^p - \underline{u}_{\rho\nu}^p\right] \phi dx \quad \forall \phi \in H_0^1(\Omega).$$

Then we prove that v_ρ is a weak solution of

$$\begin{cases} -L_\mu v = (\underline{u}_{\rho\nu} + v)^p - \underline{u}_{\rho\nu}^p & \text{in } \Omega, \\ \mathrm{tr}^*(v) = 0 \quad \text{in } \partial\Omega. \end{cases}$$

Put $u = v_\rho + \underline{u}_{\rho\nu}$
$\rightsquigarrow$ u is a weak solution of (P_ρ).

Next, we consider the system of the form

$$\begin{cases} -\Delta u - \dfrac{\mu}{\delta^2} u = v^p & \text{in } \Omega, \\ -\Delta v - \dfrac{\mu}{\delta^2} v = u^q & \text{in } \Omega, \\ u = \rho\nu, \ v = \sigma\tau & \text{on } \partial\Omega, \end{cases} \tag{$S_{\rho,\sigma}$}$$

where ρ, σ are positive parameters, ν, τ are Borel measures on $\partial\Omega$ with $\|\mu\| = 1 = \|\nu\|$, $0 < p \leq q < N_\mu$, $pq \neq 1$.

Theorem (Gkikas-Nguyen, JDE'2019)

Assume, $\mathbb{K}_\mu[\tau+\nu] \in L^q(\Omega, \delta^\alpha)$. *Then system* $(S_{\rho,\sigma})$ *has a weak solution* $(\underline{u}_{\rho\nu}, \underline{v}_{\sigma\tau})$ *for any* ρ, σ *small enough if* $pq > 1$ *and for any* $\rho > 0$, $\sigma > 0$ *if* $pq < 1$.

Moreover, $(\underline{u}_{\rho\nu}, \underline{v}_{\sigma\tau})$ *is the minimal positive weak solution of* $(S_{\rho,\sigma})$ *in the sense that if* $(\tilde{u}, \tilde{v})$ *is another weak solution of* $(S_{\rho,\sigma})$ *such that* $\tilde{u}$, $\tilde{v} > 0$, *then* $\tilde{u} \geq \underline{u}_{\rho\nu}$ *and* $\tilde{v} \geq \underline{v}_{\sigma\tau}$ *a.e. in* Ω.

Multiplicity for system $(S_{\rho,\sigma})$

Theorem (B.,-Mukherjee-Nguyen'2021)

Assume $N \geq 3$, ρ, $\sigma > 0$ *and* $1 < p \leq q < N_\mu$. *If* $N = 3$, *we assume in addition that* $q < \frac{4p}{p+1}$. *Let* $0 \leq \nu$, $\tau \in L^r(\partial\Omega)$, *for some* $r > \frac{N-1}{1-\alpha}$, *with* $\|\nu\|_{L^r(\partial\Omega)} = \|\tau\|_{L^r(\partial\Omega)} = 1$. *There exists* $t^* > 0$ *such that if* $\max\{\rho, \sigma\} < t^*$ *then system* $(S_{\rho,\sigma})$ *admits a second solution* (u, v) *such that*

$$u > \underline{u}_{\rho\nu} \quad \text{and} \quad v > \underline{v}_{\sigma\tau} \quad \text{in} \quad \Omega,$$

where $(\underline{u}_{\rho\nu}, \underline{v}_{\sigma\tau})$ *is the minimal solution of* $(S_{\rho,\sigma})$, *constructed as in previous theorem.*

Open questions

- Can we relax ν, $\tau \in L^r(\partial\Omega)$ in the above multiplicity proof for system $(S_{\rho,\sigma})$?
- Instead does the above multiplicity result for system $(S_{\rho,\sigma})$ hold for $\tau, \nu \in \mathfrak{M}^+(\partial\Omega)$ with $\mathbb{K}_\mu[\tau + \nu] \in L^q(\Omega, \delta^\alpha)$?

Thank you