On the structure and the joint spectrum of a pair of commuting isometries

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Outline

The Hardy Space of the unit disc

- Let $\mathbb{D}$ denote the open unit disc in the complex plane.
- For a Hilbert space $\mathcal{E}$, the Hardy space of $\mathcal{E}$-valued functions on the unit disc in the complex plane is

$$H^2_{\mathbb{D}}(\mathcal{E}) = \{f : \mathbb{D} \to \mathcal{E} \mid f \text{ is analytic and } f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{E}} < \infty\}.$$

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• $H^2_{\mathbb{D}}(\mathcal{E})$ is identifiable $(z^n x \mapsto z^n \otimes x)$ with $H^2_{\mathbb{D}} \otimes \mathcal{E}$ where $H^2_{\mathbb{D}}$ stands for the Hardy space of scalar-valued functions on $\mathbb{D}$.

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• Let $M_z^{\mathcal{E}}$ denotes the multiplication by z operator on $H^2_{\mathbb{D}}(\mathcal{E})$. Under the identification, $M_z^{\mathcal{E}} = M_z \otimes I_{\mathcal{E}}$.

Wold decomposition

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$$V\text{-isometry} \implies V \simeq \begin{pmatrix} M_z \otimes I_{\mathcal{D}} & 0 \\ 0 & W \end{pmatrix}, \text{ for some unitary } W.$$

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- The situation for a pair of commuting isometries is vastly different.

Let (V_1, V_2) be a pair of commuting isometries. Notation: $V = V_1 V_2$

Doubly commuting isometries

• The topic of pair of commuting isometries has been vigorously pursued by C.A. Berger, L.A. Coburn, A. Lebow, M. Słociński, D. Popovici, M.Kosiek, H. Bercovici, R. Douglas, C. Foias, R. Yang, Z. Burdak, Gaspar, J. Sarkar and their collaborators. The above list is by no means exhaustive. One of the early results of M. Słociński is the following.

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Theorem (M. Słociński)

Let (V_1, V_2) be a pair of doubly commuting isometries on a Hilbert space $\mathcal{H}$. Then there exists a unique decomposition

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{pu} \oplus \mathcal{H}_{up} \oplus \mathcal{H}_{uu}$$

where the subspace $\mathcal{H}_{ij}$ reduces both V_1 and V_2 for all $i, j = p, u$. Moreover, V_1 on $\mathcal{H}_{i,j}$ is pure if $i = p$ and unitary if $i = u$ and V_2 is pure if $j = p$ and unitary if $j = u$.

M. Słociński, *On the Wold-type decomposition of a pair of commuting isometries*, Ann. Polon. Math. 37 (1980), 255–262. MR0587496

Theorem (Berger-Coburn-Lebow)

Let (V_1, V_2) be a commuting pair of isometries acting on $\mathcal{H}$. Then, up to unitary equivalence, the Hilbert space $\mathcal{H}$ breaks into a direct sum of reducing subspaces

$$\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_u$$

so that there is a unique (up to unitary equivalence) triple $(\mathcal{E}, P, U)$ of a Hilbert space $\mathcal{E}$, a projection P in $\mathcal{E}$ and a unitary U on $\mathcal{E}$ such that $\mathcal{H}_p = H^2_{\mathbb{D}}(\mathcal{E})$, the functions φ_1 and φ_2 defined on $\mathbb{D}$ by

$$\varphi_1(z) = U^*(P^\perp + zP) \text{ and } \varphi_2(z) = (P + zP^\perp)U,$$

are commuting multipliers in $H^\infty_{\mathbb{D}}(\mathcal{B}(\mathcal{E}))$ and

$$V_i = \begin{pmatrix} M_{\varphi_i} & 0 \\ 0 & V_i|_{\mathcal{H}_u} \end{pmatrix}, \ i = 1, 2,$$

where $V_1|_{\mathcal{H}_u}$ and $V_2|_{\mathcal{H}_u}$ are commuting unitary operators.

The triple $(\mathcal{E}, P, U)$ is called as the BCL triple for (V_1, V_2).

Defect operator

The defect operator is introduced by Guo and Yang[1] for any commuting pair of isometries (V_1, V_2) as

$$C(V_1, V_2) = I - V_1 V_1^* - V_2 V_2^* + V_1 V_2 V_2^* V_1^*.$$

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- $C(V_1, V_2) = P_{\ker V_1^*} - P_{V_2(\ker V_1^*)} = P_{\ker V_2^*} - P_{V_1(\ker V_2^*)}$.
- If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2) then

$$C(V_1, V_2) \simeq (E_0 \otimes (U^* PU - P)) \oplus 0,$$

where E_0 is the projection onto the constants in $H^2_{\mathbb{D}}$.

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$C(V_1, V_2) \geq 0 \iff$ doubly commuting

- He, Qin and Yang[2] characterize (V_1, V_2) with defect positive, negative or zero defect. They also give examples.

[2] W. He, Y. Qin, and R. Yang, *Numerical invariants for commuting isometric pairs*, Indiana Univ. Math. J. 64 (2015), 1-19. MR3320518

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Lemma (W. He et al., A. Maji et al.)

Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

(a) $C(V_1, V_2) \geq 0$.

(b) (V_1, V_2) is doubly commuting.

(c) If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then $U(\text{ran } P) \subseteq \text{ran } P$.

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We provide the structure and the joint spectrum of (V_1, V_2), based on fundamental pairs of isometries consisting of multiplication operators, when the defect operator $C(V_1, V_2)$ is

- zero, negative, positive or
- difference of two mutually orthogonal projections with ranges adding up to $\ker V^*$, where $V = V_1 V_2$.

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- difference of two mutually orthogonal projections with ranges adding up to $\ker V^*$, where $V = V_1 V_2$.

• The fundamental pairs are such that in both cases (of $C(V_1, V_2)$ positive or negative), the joint spectrum of (V_1, V_2) is the whole closed bidisc $\overline{\mathbb{D}^2}$. If the defect operator $C(V_1, V_2)$ is zero, we show that the joint spectrum of (V_1, V_2) is contained in the topological boundary of the bidisc.

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• The joint spectrum of the prototypical pair in the last case, is neither the closed bidisc nor contained inside the topological boundary of the bidisc.

Taylor joint spectrum

• Recall that: If (T_1, T_2) is a pair of commuting bounded operators on $\mathcal{H}$, then for defining the Taylor joint spectrum $\sigma(T_1, T_2)$, one considers the Koszul complex $K(T_1, T_2)$:

$$0 \xrightarrow{\delta_0} \mathcal{H} \xrightarrow{\delta_1} \mathcal{H} \oplus \mathcal{H} \xrightarrow{\delta_2} \mathcal{H} \xrightarrow{\delta_3} 0$$

where $\delta_1(h) = (T_1 h, T_2 h)$ for $h \in \mathcal{H}$ and $\delta_2(h_1, h_2) = T_1 h_2 - T_2 h_1$ for $h_1, h_2 \in \mathcal{H}$. From the way the complex is constructed, $\operatorname{ran} \delta_{n-1} \subseteq \ker \delta_n$.

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• When $\operatorname{ran} \delta_{n-1} = \ker \delta_n$ for all $n = 1, 2, 3$ we say that the Koszul complex $K(T_1, T_2)$ is exact or the pair (T_1, T_2) is non-singular.

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• A pair $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ is said to be in the *joint spectrum* $\sigma(T_1, T_2)$ if the pair $(T_1 - \lambda_1 I, T_2 - \lambda_2 I)$ is singular.

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• $\sigma(T_1, T_2) \subseteq \sigma(T_1) \times \sigma(T_2)$.

Zero defect

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Example (Prototypical)

Let $\mathcal{L}$ be a non-zero Hilbert space and W be a unitary on $\mathcal{L}$. Consider the commuting pair of isometries $(M_z \otimes I_{\mathcal{L}}, I_{H^2_{\mathbb{D}}} \otimes W)$ on $H^2_{\mathbb{D}} \otimes \mathcal{L}$. As $I \otimes W$ is a unitary, the defect $C(M_z \otimes I_{\mathcal{L}}, I_{H^2_{\mathbb{D}}} \otimes W) = 0$. Also, $\sigma(M_z \otimes I_{\mathcal{L}}, I_{H^2_{\mathbb{D}}} \otimes W) = \overline{\mathbb{D}} \times \sigma(W)$.

Characterization

The following lemma follows from W. He et al.[3] and A. Maji et al. [4]

Lemma

Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

(a) $C(V_1, V_2) = 0$.

(b) $\ker V_1^*$ and $\ker V_2^*$ are orthogonal and their direct sum is $\ker V^*$.

(c) If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then $\operatorname{ran} P$ reduces U.

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Structure for the case $C(V_1, V_2) = 0$

Theorem (D. Popovici)

We have

$$\mathcal{H} = (H^2_{\mathbb{D}} \otimes \mathcal{E}_1) \oplus (H^2_{\mathbb{D}} \otimes \mathcal{E}_2) \oplus \mathcal{K}$$

and in this decomposition,

$$V_1 = \begin{pmatrix} M_z \otimes I_{\mathcal{E}_1} & 0 & 0 \\ 0 & I_{H^2_{\mathbb{D}}} \otimes U_2^* & 0 \\ 0 & 0 & W_1 \end{pmatrix}, \; V_2 = \begin{pmatrix} I_{H^2_{\mathbb{D}}} \otimes U_1 & 0 & 0 \\ 0 & M_z \otimes I_{\mathcal{E}_2} & 0 \\ 0 & 0 & W_2 \end{pmatrix},$$

up to unitarily equivalence, for some unitary U_i on $\mathcal{E}_i, i = 1, 2$ and commuting unitaries W_1, W_2 on $\mathcal{K}$.

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- The *Hardy space of* $\mathbb{C}$*-valued functions on the bidisc* $\mathbb{D}^2$ is

$$H^2_{\mathbb{D}^2} = \{f : \mathbb{D}^2 \to \mathbb{C} \mid f \text{ is analytic and } f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n$$

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• Let M_{z_1} and M_{z_2} denotes the multiplication by the co-ordinate functions z_1 and z_2 on $H^2_{\mathbb{D}^2}$ respectively.

Negative defect

Example (Fundamental)

Let $U : H^2_{\mathbb{D}^2} \to H^2_{\mathbb{D}^2}$ be the unitary defined by

$$U(z_1^{m_1} z_2^{m_2}) = \begin{cases} z_1^{m_1+2} z_2^{m_2} & \text{if } m_1 \geq m_2, \\ z_1^{m_1+1} z_2^{m_2-1} & \text{if } m_1 + 1 = m_2, \\ z_1^{m_1} z_2^{m_2-2} & \text{if } m_1 + 2 \leq m_2. \end{cases} \tag{1}$$

on the orthonormal basis $\{z_1^{m_1} z_2^{m_2}\}_{m_1, m_2 \geq 0}$.

• Let $\tau_1 := U^* M_{z_1}$ and $\tau_2 := M_{z_2} U$. The pair (τ_1, τ_2) is called as the fundamental isometric pair of negative defect.

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- Let $\tau_1 := U^* M_{z_1}$ and $\tau_2 := M_{z_2} U$. The pair (τ_1, τ_2) is called as the fundamental isometric pair of negative defect.
- The unitary U defined in (1) commutes with $M_{z_1 z_2}$. That proves commutativity of τ_1 and τ_2.
- $\ker(\tau_1^*) = \overline{\text{span}}\{z_2^2, z_2^3, z_2^4, \dots\}$ and $\tau_2(\ker(\tau_1^*)) = \overline{\text{span}}\{z_2, z_2^2, z_2^3, \dots\}$.

Thus, we have

$$C(\tau_1, \tau_2) = P_{\ker(\tau_1^*)} - P_{\tau_2(\ker(\tau_1^*))} = -P_{\text{span}\{z_2\}} \leq 0.$$

Proposition

$$\sigma(\tau_1, \tau_2) = \overline{\mathbb{D}^2}$$

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The following lemma follows from W. He et al.

Lemma (Characterization)

Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

(a) $C(V_1, V_2) \leq 0$ and $C(V_1, V_2) \neq 0$.

(b) $C(V_1, V_2)$ is the negative of a non-zero projection.

(c) If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then $U(\operatorname{ran} P^{\perp}) \subsetneq \operatorname{ran} P^{\perp}$.

On the structure of the negative defect case

Theorem ($C(V_1, V_2) \leq 0$ and $C(V_1, V_2) \neq 0$.)

Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Then there is a non-trivial closed subspace $\mathcal{L} \subsetneq \mathcal{E}$ such that, up to a unitary equivalence

$$\mathcal{E} = (l^2(\mathbb{Z}) \otimes \mathcal{L}) \oplus \mathcal{E}_2, \quad H^2_{\mathbb{D}}(\mathcal{E}) = (H^2_{\mathbb{D}}(l^2(\mathbb{Z})) \otimes \mathcal{L}) \oplus H^2_{\mathbb{D}}(\mathcal{E}_2)$$

and

$$M_{\varphi_i} = \begin{array}{c c} \begin{array}{cc} H^2_{\mathbb{D}}(l^2(\mathbb{Z})) \otimes \mathcal{L} & H^2_{\mathbb{D}}(\mathcal{E}_2) \end{array} & \\ \begin{pmatrix} M_{\psi_i} \otimes I_{\mathcal{L}} & 0 \\ 0 & M_{\varphi_i}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)} \end{pmatrix} & \begin{array}{c} H^2_{\mathbb{D}}(l^2(\mathbb{Z})) \otimes \mathcal{L} \\ H^2_{\mathbb{D}}(\mathcal{E}_2) \end{array} \end{array}, \quad i = 1, 2$$

$$C(M_{\varphi_1}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)}, M_{\varphi_2}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)}) = 0,$$

where $\psi_1, \psi_2 : \mathbb{D} \to \mathcal{B}(l^2(\mathbb{Z}))$ are the multipliers associated to the BCL triple $(l^2(\mathbb{Z}), p_-, \omega)$, viz. $\psi_1(z) = \omega^*(p_-^{\perp} + zp_-)$, $\psi_2(z) = (p_- + zp_-^{\perp})\omega$, where p_- is the projection onto $\overline{\text{span}}\{e_n : n < 0\}$ and ω is the bilateral shift in $l^2(\mathbb{Z})$.

Proof: From the lemma $U(\operatorname{ran} P^{\perp}) \subsetneq \operatorname{ran} P^{\perp}$. Take

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Lemma

The pair (M_{ψ_1}, M_{ψ_2}) is jointly unitarily equivalent to (τ_1, τ_2).

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Lemma

The pair (τ_1, τ_2) does not have any non-trivial joint reducing subspace.

Structure

Theorem ($C(V_1, V_2) \leq 0$ and $C(V_1, V_2) \neq 0$)

There is a non-trivial subspace $\mathcal{L} \subsetneq \ker V^*$ such that, up to unitary equivalence,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$$

where $\mathcal{H}_0 = H^2_{\mathbb{D}^2} \otimes \mathcal{L}$ and in this decomposition

$$V_i = \begin{pmatrix} \tau_i \otimes I_{\mathcal{L}} & 0 \\ 0 & V_i|_{\mathcal{H}_0^\perp} \end{pmatrix}, i = 1, 2 \text{ and } C(V_1|_{\mathcal{H}_0^\perp}, V_2|_{\mathcal{H}_0^\perp}) = 0,$$

where the dimension of $\mathcal{L}$ is same as the dimension of the range of $C(V_1, V_2)$. Moreover,

$$\sigma(V_1, V_2) = \overline{\mathbb{D}^2}.$$

Example

Definition

The fundamental isometric pair of positive defect is the pair (M_{z_1}, M_{z_2}) of multiplication by the coordinate functions on $H^2_{\mathbb{D}^2}$.

- The joint spectrum $\sigma(M_{z_1}, M_{z_2})$ is the whole bidisc $\overline{\mathbb{D}^2}$. Indeed, every point in the open bidisc is a joint eigenvalue for $(M^*_{z_1}, M^*_{z_2})$.

Structure

The following theorem also follows from Z. Burdak et al. [3].

Theorem ($C(V_1, V_2) \geq 0$ and $C(V_1, V_2) \neq 0$)

There is a non-trivial Hilbert space $\mathcal{L} \subsetneq \ker V^*$ such that, up to unitary equivalence,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp},$$

where $\mathcal{H}_0 = H^2_{\mathbb{D}^2} \otimes \mathcal{L}$. In this decomposition

$$V_i = \begin{pmatrix} M_{z_i} \otimes I_{\mathcal{L}} & 0 \\ 0 & V_i|_{\mathcal{H}_0^{\perp}} \end{pmatrix}, \; i = 1, 2 \text{ and } C(V_1|_{\mathcal{H}_0^{\perp}}, V_2|_{\mathcal{H}_0^{\perp}}) = 0.$$

Moreover, the dimension of $\mathcal{L}$ is the same as the dimension of the range of $C(V_1, V_2)$. In particular, $\sigma(V_1, V_2) = \overline{\mathbb{D}^2}$.

Prototypical Example

- $C(V_1, V_2) = P_{\ker V_1^*} - P_{V_2(\ker V_1^*)} = P_{\ker V_2^*} - P_{V_1(\ker V_2^*)}$.

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• This section deals with the case when the defect operator is the difference of two mutually orthogonal projections whose ranges together span the kernel of $(V_1 V_2)^*$.

Example (Prototypical)

Let $\mathcal{L}$ be a Hilbert space and W be a unitary on $\mathcal{L}$. Consider the pair of commuting isometries $(M_z \otimes I, M_z \otimes W)$ on $H^2_{\mathbb{D}} \otimes \mathcal{L}$.

Joint spectrum

The following lemma follows from the polynomial spectral mapping theorem.

Lemma

If $\mathcal{L} \neq \{0\}$, then the joint spectrum of $(M_z \otimes I_{\mathcal{L}}, M_z \otimes W)$ is

$$\sigma(M_z \otimes I_{\mathcal{L}}, M_z \otimes W) = \{z(1, \alpha) : z \in \overline{\mathbb{D}}, \alpha \in \sigma(W)\}.$$

Characterization

Theorem

The following are equivalent:

(a) The defect operator $C(V_1, V_2)$ is a difference of two mutually orthogonal projections P_1, P_2 with $\operatorname{ran} P_1 \oplus \operatorname{ran} P_2 = \ker V^*$.

(b) $\operatorname{ran} V_1 = \operatorname{ran} V_2$.

(c) If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then $U(\operatorname{ran} P) = \operatorname{ran} P^\perp$ (or equivalently $U(\operatorname{ran} P^\perp) = \operatorname{ran} P$).

Structure

Theorem

There exist Hilbert spaces $\mathcal{L}$ and $\mathcal{K}$ such that up to unitarily equivalence

$$\mathcal{H} = (H^2_{\mathbb{D}} \otimes \mathcal{L}) \oplus \mathcal{K}$$

and in this decomposition,

$$V_1 = \begin{pmatrix} M_z \otimes I_{\mathcal{L}} & 0 \\ 0 & W_1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} M_z \otimes W & 0 \\ 0 & W_2 \end{pmatrix},$$

for some unitary W on $\mathcal{L}$ and commuting unitaries W_1, W_2 on $\mathcal{K}$.

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Corollary

Let (V_1, V_2) be a pair of commuting isometries with $\operatorname{ran} V_1 = \operatorname{ran} V_2$. If $V_1 V_2$ is pure, then both V_1 and V_2 are pure.

Connection on joint spectrum

Relation between the joint spectrum of the commuting isometries and the joint spectra of the associated multipliers at every point of $\mathbb{D}$:

Theorem

Let (V_1, V_2) be a pure pair of commuting isometries on $\mathcal{H}$. Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Then in all the four cases we have,

$$\sigma(V_1, V_2) = \overline{\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z))}.$$

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Open problem

We have dealt with the case of the general defect operator to the extent when it is the difference of mutually orthogonal projections summing up to the projection onto the kernel of V^*. To describe the structure and the joint spectrum of a pair of commuting isometries involving multiplication operators with a general defect operator is an open question.

* * * * *

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THANK YOU!