# Gersten's conjecture for Milnor K-theory

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# Milnor K-theory of fields

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2. (Steinberg relation) $\{a, 1 - a\} = 0$ for $a \in F^*$.
3. If $a_1, \cdots, a_n \in F^*$ and if $a_1 + \cdots + a_n$ is either 0 or 1, then $\{a_1, \cdots, a_n\} = 0$ in $K_n^{\mathrm{M}}(F)$.

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This is used to define the differential map in the Gersten's complex for Milnor K- theory.

## Some computations:

- For a finite field $F$, $K_n^{\mathrm{M}}(F) = 0$ for $n > 1$.
- For a number field $F$, $K_n^{\mathrm{M}}(F) = (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}$ for $n > 2$, where $r_1$ is the number of real embeddings $F \hookrightarrow \mathbb{R}$.
- For a local field $F$, with finite residue field, $K_n^{\mathrm{M}}(F)$ is uniquely divisible for $n > 2$ and
$$K_2^{\mathrm{M}}(F) = \mu_\infty(F) \oplus div$$
where $\mu_\infty(F)$ is the group of roots of unity in $F$ and div is uniquely divisible.

**Theorem (Milnor)**

*There is a (split) exact sequence*

$$0 \to K^{\mathrm{M}}_{n+1}(F) \to K^{\mathrm{M}}_{n+1}(F(T)) \xrightarrow{\partial} \bigoplus_{p(T)} K^{\mathrm{M}}_{n}(F[T]/(p(T))) \to 0$$

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Write $E = F[T]/(p(T))$ for a monic irreducible polynomial $p(T)$.

$$\begin{tikzcd}
K^{\mathrm{M}}_{n+1}(F(T)) \arrow[rr, "\partial"] \arrow[d, "\partial^{-1/T}"] & & \bigoplus_{p(T)} K^{\mathrm{M}}_{n}(F[T]/(p(T))) \arrow[ll, bend right, "s"'] \arrow[dll, "Nm_{E/F}"] \\
K^{\mathrm{M}}_n(F) & &
\end{tikzcd}$$

# Milnor's conjecture/ Theorem by Voevodsky et al.

**Theorem (Voevodsky et al.)**

*Let $F$ be a field of characteristic $\neq 2$. The maps below are isomorphisms for all $n \geq 0$.*

$$\begin{array}{ccc} \{a_1, \cdots, a_n\} & \longrightarrow & (a_1) \cup \cdots \cup (a_n) \end{array}$$

$$\begin{tikzcd} K_n^{\mathrm{M}}(F)/2 \arrow[r, "\text{Galois symbol}"] \arrow[d, "\text{Pfister forms}"] & H^n(F, \mathbb{Z}/2(n)) \\ I_F^n / I_F^{n+1} & \end{tikzcd}$$

$$\begin{tikzcd} \{a\} \arrow[d] \\ < 1, -a > \end{tikzcd}$$

$H^n(F, \mathbb{Z}/2(n))$ denotes the weight $n$-Motivic cohomology of $F$ with coefficients in $\mathbb{Z}/2$.
$I_F$ is the fundamental ideal in the Witt ring of $F$ of quadratic forms of even rank. $I_F^n$ denotes its $n^{th}$ power.

## Gersten's complex for Milnor K-theory

(Kato 1986) Let $X$ be an excellent scheme of dimension $d$. Gersten's/Rost's cycle complex for $X$ is defined as follows.

$$0 \to \bigoplus_{\eta \in X^{(0)}} K_n^{\mathrm{M}}(\kappa(\eta)) \xrightarrow{d} \bigoplus_{x \in X^{(1)}} K_{n-1}^{\mathrm{M}}(\kappa(x)) \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{x \in X^{(d)}} K_{n-d}^{\mathrm{M}}(\kappa(x)) \to 0$$

where the differential $d$ is defined using the residue maps and the norm maps defined earlier.
Let us denote the cohomology of the complex by $A^p(X, K_n^{\mathrm{M}})$.

Let $A$ be a regular local ring of dimension $d$.

# Gersten's conjecture

Let $A$ be a regular local ring of dimension $d$. The Gersten's conjecture states that the complex

$$0 \to K_n^{\mathrm{M}}(Frac(A)) \xrightarrow{d} \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A^{(1)}} K_{n-1}^{\mathrm{M}}(\kappa(\mathfrak{p})) \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A^{(d)}} K_{n-d}^{\mathrm{M}}(\kappa(\mathfrak{p})) \to 0$$

is exact in degrees $\geq 1$, for each $n \geq 0$. Here $\operatorname{Spec} A^{(p)}$ denotes the set of codimension $p$ points of $\operatorname{Spec} A$.

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The conjecture above in its most general form is largely open till now.

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**Theorem (Lüders 2020)**

*Let $X$ be the spectrum of a smooth local ring over an excellent dvr, then the complex*

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Here for a dvr $R$, $X$ is the Spectrum of a local ring of a smooth scheme over $\mathrm{Spec}\, R$ at a point. This uses the works of M. Rost on "Cycle modules" and a Chinese remainder theorem trick.

# Generalization

We generalize the Lüders theorem to the case where the base $S$ is the Spectrum of a regular Henselian domain and $X$ is the Henselization of a smooth $S$-scheme at a point.

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**Theorem (——— 2021)**

*Let $S = \operatorname{Spec} R$ for $R$ a regular local Henselian domain of Krull dimension $\geq 1$. Let $X$ be an essentially smooth Henselian local $S$-scheme of dimension $d$, then*

$$A^p(X, K_n^{\mathrm{M}}) = 0$$

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# Sketch of the proof of Lüders' theorem

**Theorem (Lüders 2020)**

*Let $X$ be the spectrum of a smooth local ring over an excellent dvr $R$, then the complex*

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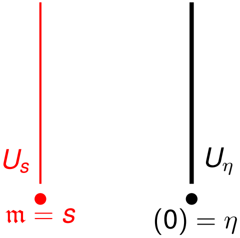
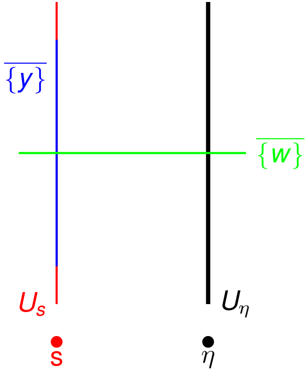
*is exact in degrees* $> 0$.

Let $S = \operatorname{Spec} R$. $X = \operatorname{Spec} \mathcal{O}_{V,v}$, for a smooth $S$-scheme $V$. Given a Zariski nbd $U$ of $v$ in $V$, we can think of $U$ as follows.



# Sketch




There are two types of irreducible closed subschemes of positive codimension. Ones which are vertical and others which are horizontal.

# Sketch

$$A^p(X, K_n^{\mathrm{M}}) = \varinjlim_{(U,u)} A^p(U, K_n^{\mathrm{M}})$$

where the colimit runs over $(U, u)$ such that $U$ is a Zariski neighbourhood of $v$ in $V$.

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$$A^p(U, K_n^{\mathrm{M}}) = \varinjlim_{Y \subset U} A_Y^p(U, K_n^{\mathrm{M}})$$

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**Idea:**

- Every cohomology class in degree $p > 0$ is represented by a class which is supported on closed subschemes which are flat over the base $S$ (horizontal).

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**Idea:**

- Every cohomology class in degree $p > 0$ is represented by a class which is supported on closed subschemes which are flat over the base $S$ (horizontal).
- Given a closed subscheme $Y$ in $U$ flat (horizontal) over $S$, there is a smaller Zariski neighbourhood $U'$ of $v$ in $U$ such that the map

$$A^p_{Y \times_U U'}(U', K_n^{\mathrm{M}}) \to A^p(U', K_n^{\mathrm{M}})$$

is 0. (This uses a technical result, Gillet-Levine's version of Geometric Presentation lemma).

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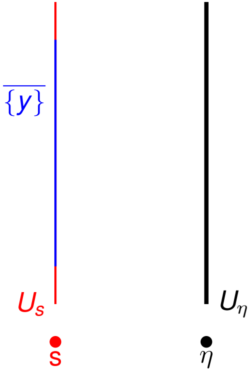


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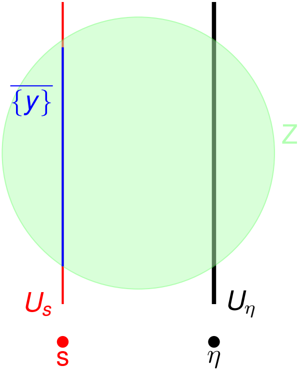


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(Saito-Sato) Find an integral closed subscheme $Z$ of codimension $p-1$ containing $\overline{\{y\}}$ such that $Z$ is regular at $y$ and $Z \cap V_\eta \neq \emptyset$.

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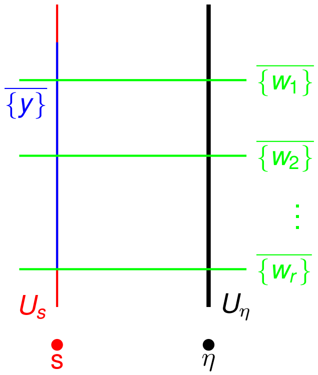


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(Saito-Sato) Find a closed subscheme $Z$ of codimension $p-1$ containing $\overline{\{y\}}$ such that $Z$ is regular at $y$ and $Z \cap U_\eta \neq \emptyset$.

Now codimension 1 points of $Z$ are $y$ and the points $w$ such that $\overline{\{w\}}$ are horizontal.

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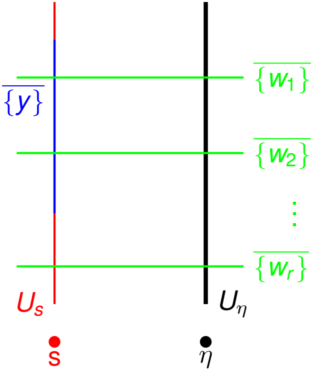


Now using Chinese remainder theorem, one constructs a Milnor K-theory symbol $\beta$ on $\kappa(Z)$ such that

$$d(\beta) = \{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_{n-p}\} + \sum_{\overline{\{w_i\}}} \gamma_i.$$

This finishes the argument for the first step.

Lüders, M., On the relative Gersten conjecture for Milnor K-theory in the smooth case, arXiv:2010.02622 (2020).

Pawar, R., A remark on relative Gersten's complex for Milnor K-theory, arXiv:2105.06962.

Thank you!