Transformation formula for the reduced Bergman kernel

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Overview

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Reduced Bergman space

Let $D \subset \mathbb{C}$ be a domain. Define $\mathcal{D}(D)$, space of holomorphic functions f on D such that

1. $f = g'$ for some $g \in \mathcal{O}(D)$,
2. $\int_D |f(z)|^2 dA(z) < \infty$.

We call $\mathcal{D}(D)$ the **reduced Bergman space** of D.

1. A Hilbert space w.r.t. the inner product
$$\langle f, g \rangle = \int_D f(z)\overline{g(z)}\, dA(z)$$
2. For every $\zeta \in D$, the evaluation functional
$$f \mapsto f(\zeta), \qquad f \in \mathcal{D}(D)$$
is continuous.

Reduced Bergman kernel

There exists a unique function $\tilde{K}_D(\cdot,\cdot)$ on $D \times D$ such that

1. $\tilde{K}_D(\cdot,\zeta) \in \mathcal{D}(D)$ when $\zeta \in D$
2. $f(\zeta) = \langle f, \tilde{K}_D(\cdot,\zeta)\rangle$ when $f \in \mathcal{D}(D)$, $\zeta \in D$.

Definition

The function $\tilde{K}_D(\cdot,\cdot)$ is called the *reduced Bergman kernel of D*.

1. $\tilde{K}_D(z,w) = \overline{\tilde{K}_D(w,z)}$ for all $z, w \in D$.
2. $\tilde{K}_D$ is holomorphic in first variable and antiholomorphic in the second variable.

Weighted reduced Bergman kernel

Let μ be positive function on D such that $1/\mu \in L^{\infty}_{loc}(D)$. Define $\mathcal{D}_\mu(D)$, space of holomorphic functions f on D such that

1. $f = g'$ for some $g \in \mathcal{O}(D)$,
2. $\int\limits_D |f(z)|^2 \mu(z)\, dA(z) < \infty$.

We call $\mathcal{D}_\mu(D)$ the **reduced Bergman space** of D **with respect to the weight** μ.

1. A Hilbert space w.r.t. the inner product
$$\langle f, g \rangle_\mu = \int_D f(z)\overline{g(z)}\mu(z)\, dA(z)$$
2. For every $\zeta \in D$, the evaluation functional
$$f \mapsto f(\zeta), \qquad f \in \mathcal{D}_\mu(D)$$
is continuous.

Weighted reduced Bergman kernel

There exists a unique function $\tilde{K}_{D,\mu}(\cdot,\cdot)$ on $D \times D$ such that

1. $\tilde{K}_{D,\mu}(\cdot,\zeta) \in \mathcal{D}_\mu(D)$ when $\zeta \in D$
2. $f(\zeta) = \langle f, \tilde{K}_{D,\mu}(\cdot,\zeta)\rangle_\mu$ when $f \in \mathcal{D}_\mu(D)$, $\zeta \in D$.

Definition

The function $\tilde{K}_{D,\mu}(\cdot,\cdot)$ is called the *reduced Bergman kernel of D with respect to the weight* μ.

1. $\tilde{K}_{D,\mu}(z,w) = \overline{\tilde{K}_{D,\mu}(w,z)}$ for all $z, w \in D$.
2. $\tilde{K}_{D,\mu}$ is holomorphic in first variable and antiholomorphic in the second variable.

Proper correspondences

1. D_1 and D_2 are domains in $\mathbb{C}$ and $\pi_1 : D_1 \times D_2 \longrightarrow D_1$, $\pi_2 : D_1 \times D_2 \longrightarrow D_2$ are canonical projections.
2. V is an analytic subvariety of $D_1 \times D_2$.
3. Consider the multivalued map $f : D_1 \multimap D_2$ given by

$$f(z) = \pi_2 \pi_1^{-1}(z) = \{w : (z, w) \in V\}.$$

Definition

The map f is called a *holomorphic correspondence* and V is called the graph of f. The correspondence f is said to be proper if the projection maps $\pi_1 : V \longrightarrow D_1$ and $\pi_2 : V \longrightarrow D_2$ are proper.

For eg. $V = \{(z, w) \in D_1 \times D_2 : z^m = w^n\}$ for positive integers m and n.

Proper correspondences

Note that,
there exist analytic sub-varieties V_1 and V_2 of D_1 and D_2 respectively, and positive integers m and n such that

1. $\pi_1\pi_2^{-1}$ is locally given by m holomorphic maps on $D_2 \setminus V_2$ which we will denote by $\{F_i\}_{i=1}^m$.
2. $\pi_2\pi_1^{-1}$ is locally given by n holomorphic maps on $D_1 \setminus V_1$ which we will denote by $\{f_i\}_{i=1}^n$,

Remark:

1. V_1 and V_2 are discrete subsets of D_1 and D_2 respectively.
2. When $n = 1$, f is a proper holomorphic map with multiplicity m and $\{F_i\}_{i=1}^m$ denote the local inverses of f.

Transformation formula

Theorem

Let D_1 and D_2 be bounded domains in $\mathbb{C}$. If $f : D_1 \multimap D_2$ is a proper holomorphic correspondence, then the reduced Bergman kernels $\tilde{K}_j$'s associated with D_j's, $j = 1, 2$, transform according to

$$\sum_{i=1}^{n} f_i'(z)\tilde{K}_2(f_i(z), w) = \sum_{j=1}^{m} \tilde{K}_1(z, F_j(w))\overline{F_j'(w)},$$

for all $z \in D_1$ and $w \in D_2$, where f_i's and F_j's, and the positive integers m, n are as above.

As a corollary, if $f : D_1 \to D_2$ is a proper holomorphic map, we get

$$f'(z)\tilde{K}_2(f(z), w) = \sum_{j=1}^{m} \tilde{K}_1(z, F_j(w))\overline{F_j'(w)},$$

for all $z \in D_1$ and $w \in D_2$, where m is the multiplicity of f and F_j's are local inverses of f.

Steps

1. For $u \in L^2(D_2)$ and $v \in L^2(D_1)$, define maps Γ_1 and Γ_2 by:

$$\Gamma_1(u) = \sum_{i=1}^{n} f_i'(u \circ f_i) \quad \text{and} \quad \Gamma_2(v) = \sum_{j=1}^{m} F_j'(v \circ F_j).$$

2. $\Gamma_1(u) \in L^2(D_1)$ for all $u \in L^2(D_2)$ and $\Gamma_2(v) \in L^2(D_2)$ for all $v \in L^2(D_1)$.
3. Γ_i's, $i = 1, 2$ are bounded linear maps.
4. $\langle \Gamma_1 u, v \rangle_1 = \langle u, \Gamma_2 v \rangle_2$ for all $u \in L^2(D_2)$ and $v \in L^2(D_1)$.
5. We have

$$\Gamma_1(\mathcal{D}(D_2)) \subset \mathcal{D}(D_1) \quad \text{and} \quad \Gamma_2(\mathcal{D}(D_1)) \subset \mathcal{D}(D_2).$$

6. Set $\tilde{\Gamma}_1 := \Gamma_1|_{\mathcal{D}(D_2)}$ and $\tilde{\Gamma}_2 := \Gamma_2|_{\mathcal{D}(D_1)}$.

Steps

In the final step,

1. For $w \in D_2 \setminus V_2$, define
$$G(z) = \sum_{j=1}^{m} \tilde{K}_1(z, F_j(w))\overline{F_j'(w)}, \quad z \in D_1.$$
2. We have $G \in \mathcal{D}(D_1)$.
3. For an arbitrarily chosen $v \in \mathcal{D}(D_1)$

$$\begin{aligned}
\langle v, G\rangle_1 &= \sum_{j=1}^{m} F_j'(w) \langle v, \tilde{K}_1(\cdot, F_j(w))\rangle_1 \\
&= \sum_{j=1}^{m} F_j'(w)\, v(F_j(w)) \\
&= (\tilde{\Gamma}_2 v)(w) \\
&= \langle \tilde{\Gamma}_2 v, \tilde{K}_2(\cdot, w)\rangle_2 \\
&= \langle v, \tilde{\Gamma}_1(\tilde{K}_2(\cdot, w))\rangle_1
\end{aligned}$$

Steps

1. $\langle v, G\rangle_1 = \langle v, \tilde{\Gamma}_1(\tilde{K}_2(\cdot, w))\rangle_1$ for every $v \in \mathcal{D}(D_1)$.
2. Therefore, $G = \tilde{\Gamma}_1(\tilde{K}_2(\cdot, w))$.
3. Thus, we have proved that

$$\sum_{i=1}^{n} f_i'(z)\tilde{K}_2(f_i(z), w) = \sum_{j=1}^{m} \tilde{K}_1(z, F_j(w))\overline{F_j'(w)}.$$

   for $z \in D_1$, $w \in D_2 \setminus V_2$.
4. Since LHS is anti-holomorphic in w and V_2 is discrete, the points in V_2 are removable singularities of RHS. Hence, the formula holds everywhere by continuity.

Transformation formula

Theorem

Let D_1 and D_2 be bounded domains in $\mathbb{C}$ and ν be a positive measurable function on D_2 such that $1/\nu \in L^{\infty}_{loc}(D_2)$. If $f : D_1 \longrightarrow D_2$ is a proper holomorphic map, then the weighted reduced Bergman kernels $\tilde{K}_1^{\nu\circ f}$ and $\tilde{K}_2^{\nu}$ associated with D_1 and D_2 respectively, transform according to

$$f'(z)\tilde{K}_2^{\nu}(f(z), w) = \sum_{k=1}^{m} \tilde{K}_1^{\nu\circ f}(z, F_k(w))\overline{F_k'(w)},$$

for all $z \in D_1$ and $w \in D_2$, where m is the multiplicity of f and F_k's are the local inverses of f.

The proof techniques are similar.

Application

Theorem

Suppose D is a bounded domain in $\mathbb{C}$ whose associated reduced Bergman kernel is a rational function. Then every proper holomorphic mapping $f : D \to \mathbb{D}$ must be rational.

1. We denote the reduced Bergman kernel of $\mathbb{D}$ and D by K and $\tilde{K}$ respectively.
2. For fixed $w \in \mathbb{D}$ and $\alpha \in \mathbb{N} \cup \{0\}$, define a linear functional Λ on the Bergman space of D i.e. $A^2(D)$ by

$$\Lambda(h) = \partial^\alpha \left(\sum_{k=1}^{m} F_k'(h \circ F_k) \right)(w),$$

where $\partial^\alpha = \frac{\partial^\alpha}{\partial z^\alpha}$ denotes the standard holomorphic differential operator of order α.

Proof

Lemma (Bell)

Let $\xi_1, \xi_2, \ldots, \xi_q$ denote the points in $f^{-1}(w)$. There exist a positive integer s and constants $c_{l,\beta}$ such that:

$$\Lambda(h) = \sum_{l=1}^{q} \sum_{\beta \le s} c_{l,\beta} \partial^{\beta} h(\xi_l)$$

for all $h \in A^2(D)$.

Proof of the application:

1. Let $f^{-1}(0) = \{\zeta_1, \cdots, \zeta_q\}$.
2. Consider the linear functional Λ on $A^2(D)$, as defined above, corresponding to $\alpha = 1 \in \mathbb{N} \cup \{0\}$ and $0 \in \mathbb{D}$, i.e.

$$\Lambda(h) = \frac{\partial}{\partial w} \left(\sum_{k=1}^{m} F_k'(h \circ F_k) \right)(0).$$

Proof

1. We have a positive integer $s > 0$ and constants $c_{l,\beta}$ such that
$$\frac{\partial}{\partial w}\left(\sum_{k=1}^{m} F_k'(h \circ F_k)\right)(0) = \sum_{l=1}^{q}\sum_{\beta \leq s} c_{l,\beta}\partial^{\beta} h(\zeta_l)$$
for all $h \in A^2(D)$.
2. On differentiating the transformation formula for the reduced Bergman kernels under f with respect to $\bar{w}$ and setting $w = 0$, we get
$$\begin{aligned} f'(z)\frac{\partial}{\partial \bar{w}}K(f(z),0) &= \overline{\frac{\partial}{\partial w}\left(\sum_{k=1}^{m} \overline{\tilde{K}(z, F_k(\cdot))} F_k'(\cdot)\right)(0)} \\ &= \sum_{l=1}^{q}\sum_{\beta \leq s} \bar{c}_{l,\beta}\,\bar{\partial}^{\beta}\tilde{K}(z,\zeta_l), \end{aligned}$$

Proof

1. Since $\tilde{K}$ is a rational function, $f'(z)\frac{\partial}{\partial \bar{w}}K(f(z),0)$ is therefore a rational function in z.
2. Similarly, taking $\alpha = 0$ proves that $f'(z)K(f(z),0)$ is a rational function in z.
3. Note that for the disc $\mathbb{D}$, the reduced Bergman kernel is equal to the Bergman kernel as $\mathcal{D}(\mathbb{D}) = A^2(\mathbb{D})$. Therefore,
$$K(z,w) = \frac{1}{\pi}\frac{1}{(1-z\bar{w})^2}.$$
4. So, $\frac{\partial}{\partial \bar{w}}K(z,0) = \frac{2z}{\pi}$ and $K(z,0) = \frac{1}{\pi}$. Thus,
$$\frac{f'(z)\frac{\partial}{\partial \bar{w}}K(f(z),0)}{f'(z)K(f(z),0)} = 2f(z).$$
5. Hence, f is a rational function.

References

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