Bounds for the Bergman kernel and the sup-norm of holomorphic Siegel cusp forms

K Hariram
joint work with by Dr. Soumya Das
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Notations

For a positive integer n,

Definition (Siegel upper-half plane)

$$\mathcal{H}_n := \{ Z = X + iY \in \mathrm{Sym}(n, \mathbb{C}) \mid Y > 0 \} . \tag{1}$$

Definition (Symplectic group)

$$\mathrm{Sp}(n, \mathbb{R}) := \left\{ \gamma \in \mathrm{GL}(2n, \mathbb{R}) \mid \gamma^t J \gamma = J \right\} \tag{2}$$

The matrix $J = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix} \right)$ with 1_n being the identitiy matrix.

If $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ is the block notation with $n \times n$ sub-matrices, then this condition is equivalent to $A^t C = C^t A$, $B^t D = D^t B$ and $A^t D - C^t B = 1_n$.

Definition ($\mathrm{Sp}(n, \mathbb{R})$ action on upper half plane)

$$\gamma \langle Z \rangle := (AZ + B) (CZ + D)^{-1} . \tag{3}$$

Siegel modular forms

Definition (Stroke action of weight k)

For a function $F : \mathcal{H}_n \to \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{R})$,

$$F|_k\gamma(Z) = \det\left(CZ + D\right)^{-k} F(\gamma\left\langle Z \right\rangle). \tag{4}$$

$\Gamma_n := \mathrm{Sp}(n, \mathbb{Z}) \subset \mathrm{Sp}(n, \mathbb{R})$.

Definition (Siegel modular forms)

A holomorphic function $F : \mathcal{H}_n \to \mathbb{C}$ is a Siegel modular form of weight k for Γ_n if

$$F|_k\gamma = F(Z) \text{ for all } \gamma \in \Gamma_n. \tag{5}$$

With the usual convergence at cusps condition if $n = 1$. The space of such functions will be called M_k^n. It is a $\mathbb{C}$-vector space.

$\dim M_k^n = O(k^{n(n+1)/2})$

Fundamental domain and Minkowski reduced set

Definition (Fundamental domain)

$\mathcal{F}_n := \Gamma_n \backslash \mathcal{H}_n$ is the fundamental domain. We will use Siegel's standard one with $|x_{i,j}| \leq 1/2$, Y is Minkowski reduced and $|\det(CZ + D)| \geq 1$ for all $Z \in \mathcal{F}_n$ and C, D lower blocks of any $\gamma \in \Gamma_n$.

The Minkowski reduced set is the set of $Y \in \mathrm{Sym}(n, \mathbb{R})^+$ reduced under the action of $\mathrm{GL}(n, \mathbb{Z}) \ni U$ by $U^t Y U$. Such a $Y = (y_{i,j})$ satisfies

1. $y_{i,i} \leq y_{j,j}$ when $i < j$.
2. $2\,|y_{i,j}| \leq y_{i,i}$.
3. $y_{i,i+1} \geq 0$.

There exists constants $r_n > 1$ such that for the diagonal matrix Y_D formed the the diagonal of Y,

$$Y_D / r_n < Y < r_n Y_D. \tag{6}$$

If $Z \in \mathcal{F}_n$, then for the imaginary part Y, the diagonal entries are bounded below $y_{i,i} > \sqrt{3}/2$. This means $Y \gg 1_n$.

Fourier Series and cusp forms

For any $B \in \mathrm{Sym}(n,\mathbb{Z})$, the element $\left(\begin{smallmatrix} 1_n & B \\ 0 & 1_n \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{Z})$. Thus any $F \in M_k^n$ satisfies $F(Z) = F(Z+B)$. It thus has the Fourier series:

$$F(Z) = \sum\nolimits_{T \in \Lambda_n^*} a_F(T)\, e(\mathrm{tr}(TZ)) \tag{7}$$

where $\Lambda_n^* := \{\, T \in \mathrm{Sym}(n,\mathbb{Q}) \mid 2T \in \mathrm{Sym}(n,\mathbb{Z}), \mathrm{diag}(T) \in \mathbb{Z}, T \geq 0 \,\}$.

Definition (Siegel cusp forms)

$F \in S_k^n \subseteq M_k^n$ is a cusp form if for all $a_F(T) \neq 0$, we have $T > 0$.

We will use Λ_n to denote the strictly positive definite elements of Λ_n^*.

Poincaré Series

Definition (Poincaré Series)

For $k > n+1$ and for each $T \in \Lambda_n$ we define the T-th Poincaré series by the infinite series

$$P^n_{k,T}(Z) = P_T(Z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_n} e\left(\mathrm{tr}(TZ)\right)|_k \gamma \tag{8}$$

where $\Gamma_\infty = \left\{ \left(\begin{smallmatrix} 1_n & S \\ 0_n & 1_n \end{smallmatrix}\right) \in \Gamma_n \;\middle|\; S \in \mathrm{Sym}(n, \mathbb{Z}) \right\}$.

They satisfy the Petersson trace formula,

$$\sum\nolimits_{F \in \mathcal{B}^n_k} |a_F(T)|^2 = c^{-1}_{n,k} \det(T)^{k - \frac{n+1}{2}} a_{P_T}(T) =: p(T) \tag{9}$$

where

$$c_{n,k} = \pi^{n(n-1)/2}(4\pi)^{n(n+1)/4 - nk} \prod\nolimits_{j=1}^{n} \Gamma(k - (n+j)/2). \tag{10}$$

L^2-norm and sup-norm

With $Z = X + iY$, $\mathcal{H}_n$ has the invariant measure $d\mu(Z) = dX dY \det(Y)^{-(n+1)}$. We have the following inner product in S_k^n.

Definition (Petersson inner product)

For $F, G \in S_k^n$

$$\langle F, G\rangle := \int_{\mathcal{F}_n} F(Z)\overline{G(Z)} \det(Y)^k d\mu(Z). \tag{11}$$

With this we define the L^2-norm as $\| F \|_2^2 := \langle F, F\rangle$.

Definition (Sup-norm)

$$\| F \|_\infty := \sup_{Z \in \mathcal{F}_n} |F(Z)| \det(Y)^{k/2}. \tag{12}$$

These norms are independent of the choice of the fundamental domain because $|F(Z)| \det(\Im(Z))^{k/2} = |F(\gamma \langle Z\rangle)| \det(\Im(\gamma \langle Z\rangle))^{k/2}$.

The problem

For an L^2-normalised $F \in S_k^n$, we seek to prove bounds on the growth of $\|F\|_\infty$ with respect to k. In fact, we wish to do this for a sum over an orthonormal basis. Let $\mathcal{B}_k^n$ be an orthonormal basis for S_k^n. In this regard, we define

$$\mathbb{B}_k(Z) := \sum_{F \in \mathcal{B}_k^n} |F(Z)|^2 \det(Y)^k \qquad (= \det(Y)^k B_k(Z,Z)). \tag{13}$$

We will show bounds on $\sup_{Z \in \mathcal{F}_n} \mathbb{B}_k(Z)$. The sum $\mathbb{B}_k(Z)$ is closely related to the Bergman kernel $B_k(Z,W)$ of S_k^n as shown above. The Bergman kernel $B_k(Z,W) = \sum_{F \in \mathcal{B}_k^n} F(Z)\overline{F(W)}$ satisfies

$$G(W) = \langle G(Z), B_k(Z,W) \rangle_Z. \tag{14}$$

Existing results I

The sup-norm problem has its genesis in the question of understanding the mass distribution of the eigenfunctions on a complete Riemannian manifold without boundary. Use of arithmetic methods for this problem was pioneered by [IS95]. For certain compact hyperbolic surfaces:

Theorem (H. Iwaniec and P. Sarnak, 1995)

Let ϕ_j be the eigenfunction of the Laplacian Δ with eigenvalue λ_j. For all j and $\epsilon > 0$

$$\sqrt{\log\log\lambda_j} \ll \|\phi_j\|_\infty \ll_\epsilon \lambda_j^{\frac{5}{24}+\epsilon}. \tag{15}$$

Let us mention some results known in our setting. For elliptic modular forms, [Xia07] has shown the sharp result

Theorem (H. Xia, 2006)

For a holomorphic cuspidal Hecke eigenform f on $\mathrm{SL}(2,\mathbb{Z})$ of weight k,

$$k^{\frac{1}{4}-\epsilon} \ll_\epsilon \|f\|_\infty \ll_\epsilon k^{\frac{1}{4}+\epsilon}. \tag{16}$$

Existing results II

[Blo15] considered the case of Saito-Kurokawa lifts and based on the results obtained made the following remark.

Conjecture (V. Blomer, 2015)

Let $F \in S_k^n$ be an L^2-normalised Hecke eigenform. Then,

$$\|F\|_\infty = k^{n(n+1)/8+o(1)} \qquad (\text{as } k \to \infty). \tag{17}$$

A weaker but unconditional result has been obtained in [DS20].

Theorem (S. Das and J. Sengupta, 2020)

Let $F \in S_k^2$ be a L^2-normalised Saito-Korokawa lift of an eigenform on $\mathrm{SL}(2,\mathbb{Z})$. Then L^∞-norm satisfies

$$\|F\|_\infty \ll_\epsilon k^{17/12+\epsilon} \tag{18}$$

Existing results III

When one moves to higher degree, very little is known. The only good result in the literature seems to be that of [BP16], where Siegel-Maaß Hecke eigenforms of weight 0 were considered in a fixed compact subset of $\Gamma_2\backslash\mathcal{H}_2$. Direct approaches to the sup-norm seem hard and so we use tools available for the Bergman kernel. There is a folklore conjecture that $\mathbb{B}_k$ satisfies the following bound.

Conjecture (J. Kramer et al.)

With the above notation and setting, the following is true.

$$\sup_{Z\in\mathcal{H}_n} \mathbb{B}_k(Z) \asymp_n k^{\frac{3n(n+1)}{4}}. \tag{19}$$

Existing results IV

For $n = 1$, this conjecture has tackled by [FJK16] by analysing associated heat kernels. Their method works for any Fuschian group of first kind.

Theorem (J.S. Friedman, J. Jorgenson and J. Kramer, 2016)

$$k^{3/2-\epsilon} \ll_\epsilon \sup_{z \in \mathcal{H}_1} \mathbb{B}_k(z) \ll k^{3/2}. \tag{20}$$

In restrictions to compact sets, we can deduce the following from [CL11]:

Theorem (J.W. Cogdell and W. Luo, 2011)

For a fixed compact subset $\Omega \subset \mathcal{F}_n$,

$$\sup_{Z \in \Omega} \mathbb{B}_k(Z) \ll_{n,\Omega} k^{\frac{n(n+1)}{2}}. \tag{21}$$

Main result

Theorem (S. Das and *)

Let $\epsilon > 0$ be given, put $\ell(n) := 3n(n+1)/4$ and suppose $k > (n+1)^2$ be even and large. Then with the above notation and setting

$$k^{\ell(n)} \ll_n \sup_{Z \in \mathcal{H}_n} \mathbb{B}_k(Z) \ll_{n,\epsilon} \begin{cases} k^{\ell(1)} & (n=1) \\ k^{\ell(2)+\epsilon} & (n=2) \\ k^{(5n-3)(n+1)/4+\epsilon} & (n \geq 3). \end{cases} \tag{22}$$

Therefore when $n = 1, 2$ the first two set of inequalities in (22) prove the conjecture in (19) (up to the ϵ in the upper bound when $n = 2$).

Corollary

Let $F \in S_k^n$ be L^2 normalised. Then

$$\|F\|_\infty \ll_{n,\epsilon} \begin{cases} k^{9/4+\epsilon} & (n=2) \\ k^{(5n-3)(n+1)/8+\epsilon} & (n \geq 3). \end{cases} \tag{23}$$

Using the fact that $\dim S_k^n = O(k^{n(n+1)/2})$,

Corollary

For k even and large enough, there exist L^2 normalized $F_1 \in S_k^n$ such that $\|F_1\|_\infty \gg k^{n(n+1)/8}$.

We have found the following upper bound on the size of $\mathbb{B}_k(Z)$ dependent on $\det(Y)$.

Theorem (S. Das and *)

For $k > (n+1)^2$ and for all $Z \in \mathcal{F}_n$ and some $\epsilon > 0$,

$$\mathbb{B}_k(Z) \ll_{n,\epsilon} k^{\frac{n(n+1)}{2}} (\det(Y)/y_1^n)^{\frac{n+1}{2}+\epsilon} \min\left\{k^{\frac{n(n+1)}{4}}, \det(Y)^{\frac{n+1}{4}}\right\} \tag{24}$$

where y_1 is the smallest diagonal entry of Y.

Amplification result

Further for $n = 2$, when Z is restricted to a fixed compact subset of $\Omega \subset \mathcal{F}_n$ we have improved on the existing results by use of amplification.

Theorem (S. Das and *)

Let $F \in S_k^2$, $\|F\|_2 = 1$ and F be an eigenfunction of all Hecke operators. Assume further that F is orthogonal to the space of Saito-Kurokawa lifts. Then $\| F \|_{\infty,\Omega} \ll_\Omega k^{3/2-\eta}$ for some absolute constant $\eta > 0$.

Lower bounds

To prove the lower bounds in the main theorem (22), we first establish a connection between $\sup \mathbb{B}_k(Z)$ and the 1_n-th Fourier coefficient of the Poincaré series P_{1_n}. For any $Y_0 \in \mathrm{Sym}(n, \mathbb{R})^+$ we show

$$\sup_{Z \in \mathcal{F}_n} \mathbb{B}_k(Z) \geq \det(Y_0)^k \exp\left(\mathrm{tr}(-4\pi Y_0)\right) \sum_{F \in \mathcal{B}_k^n} |a_F(1_n)|^2. \qquad (25)$$

This relates to $a_{P_{1_n}}(1_n)$ by Petersson trace formula. For even k sufficiently larger than n, we get the non-vanishing of these Fourier coefficients from [KST11]. Evaluating the above at $Y_0 = (k/(4\pi))1_n$ gives us

$$\sup_{Z \in \mathcal{F}_n} \mathbb{B}_k(Z) \gg_n k^{3n(n+1)/4} \qquad (26)$$

Upper bounds method-1 (Fourier series) I

To prove the upper bound in (22) in the general case, we will look at two methods. One from the Fourier expansion of F and another from an expression for the Beregman kernel. Any bound on the Fourier coefficients of the Poincaeé series should give us bounds of the form

$$p(T)^{1/2} = \left(\sum_{F \in \mathcal{B}_k^n} |a_F(T)|^2 \right)^{1/2} \ll_n \frac{(4\pi)^{nk/2} k^{\alpha} \det(T)^{k/2-\beta}}{\Gamma(k)^{n/2}} \tag{27}$$

where α, β are some real parameters. We first define a function $q_k(Y) : \mathrm{Sym}(n, \mathbb{R})^+ \to \mathbb{C}$ as

$$q_k(Y) := \det(Y)^{k/2} \sum_{T \in \Lambda_n} p(T)^{1/2} \exp\left(-2\pi \operatorname{tr}(TY)\right). \tag{28}$$

Using Cauchy-Schwarz, one can observe that $\mathbb{B}_k(Z) \le q_k(Y)^2$.

Upper bounds method-1 (Fourier series) II

Proposition

Suppose $p(T)$ satisfies bounds of the form (27). Then for Y Minkowski reduced, q_k satisfies

$$q_k(Y) \ll_n \left(\frac{k^n}{\det(Y)}\right)^{(n+1)/2-\beta} k^{\frac{n}{4}+\alpha} + \exp\left(-c_0 k^{\epsilon}\right)\det(Y)^{-(n+1)/2}. \tag{29}$$

Additionally, if we restrict $Y \gg 1_n$ we have the better bound

$$q_k(Y) \ll_{n,\epsilon} \left(\frac{k^n}{\det(Y)}\right)^{(n+1)/4-\beta} k^{\frac{n}{4}+\alpha+\epsilon} + \exp\left(-c_0 k^{\epsilon}\right)\det(Y)^{-(n+1)/2}. \tag{30}$$

Here $c_0 > 0$ depends only on n and $0 < \epsilon < 1$. Finally, if the largest diagonal entry $y_n > nk\, r_n/(2\pi)$ we have the decay:

$$q_k(Y) \ll_n \exp\left(-c_0 y_n\right)\det(Y)^{-(n+1)/2}. \tag{31}$$

Upper bounds method-1 (Fourier series) III

We prove this by performing the Fourier series like sum and observing that only those T contribute significantly for which all eigenvalues of TY are in an interval $k/(4\pi) \pm O(k^{1/2+\epsilon})$. This then becomes a counting problem on T and we will call such a set $\mathcal{C}_Y$. Simple computation with trace of TY gives us the first bound (29). With the further assumption that Y is bounded below, we get a count of the form

$$\#\mathcal{C}_Y \ll_n \left(\frac{k^n}{\det(Y)} \right)^{(n+1)/4} \tag{32}$$

and the second bound (30). When $\mathcal{C}_Y$ becomes empty, we get exponential decay.

Upper bounds method-2 (Bergman Kernel) I

The Bergman kernel has an alternate natural expression which allows us to write $\mathbb{B}_k(Z)$ as follows

$$\mathbb{B}_k(Z) = 2^{-1} a(n,k) \sum_{\gamma \in \Gamma_n} h_\gamma(Z)^k \tag{33}$$

where

$$h_\gamma(Z) = \frac{\det(Y)}{\det\left(\frac{Z-\gamma\overline{Z}}{2i}\right)\det(C\overline{Z}+D)} \tag{34}$$

$$a(n,k) := 2^{-n(n+3)/2}\pi^{-n(n+1)/2}\prod_{v=1}^{n}\frac{\Gamma(k-\frac{v-1}{2})}{\Gamma(k-\frac{v+n}{2})} \ll_n k^{\frac{n(n+1)}{2}}. \tag{35}$$

We will carry out this summation by breaking γ into

$$\gamma = \left(\begin{smallmatrix} 1_n & S \\ 0 & 1_n \end{smallmatrix}\right)\begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix}\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right). \tag{36}$$

Upper bounds method-2 (Bergman Kernel) II

So we have

$$\sum_{\gamma\in\Gamma_n} h_\gamma(Z)^k = (2i)^{nk}\det(Y)^k$$

$$\sum_{\{C,D\}} \det(CZ+D)^{-k} \sum_{\substack{U\in \mathrm{GL}(n,\mathbb{Z})\\ S\in \mathrm{Sym}(n,\mathbb{Z})}} \det\left(Z - U\overline{\widetilde{\gamma}\langle Z\rangle}U^t - S\right)^{-k} \quad (37)$$

We rely on 3 lemmas to carry out this summation.

Lemma

$$\sum_{S\in \mathrm{Sym}(n,\mathbb{Z})} \det(Z+S)^{-k} \ll_n \begin{cases} k^{\frac{n(n+1)}{4}}\det(Y)^{-k} & (Z\in\mathcal{H}_n) \\ \det(Y)^{-k+\frac{n+1}{4}} & (Z\in\mathcal{F}_n). \end{cases} \quad (38)$$

Upper bounds method-2 (Bergman Kernel) III

Lemma

For $k > (n+1)^2$ and $Y, \widetilde{Y} \in \mathrm{Sym}(n,\mathbb{R})^+$, with λ_1 as the smallest eigenvalue of Y, we have for any $\epsilon > 0$:

$$\sum_{U \in GL_n(\mathbb{Z})} \det(Y + U\widetilde{Y}U^t)^{-k}$$
$$\ll_{n,\epsilon} 2^{-nk} (\det(Y)\det(\widetilde{Y}))^{-\frac{k}{2}} \left(\frac{\det(\widetilde{Y})}{\lambda_1^n} \right)^{\frac{n+1}{2}+\epsilon}. \qquad (39)$$

Lemma

The summation $\sum_{\{C,D\}} |\det(CZ+D)|^{-s} = \mathbb{E}_s(Z)$ is uniformly bounded for $s = n+1+\epsilon$ when $Z \in \mathcal{F}_n$.

Using these lemmas, we prove our second theorem about $\mathbb{B}_k(Z)$.

$n = 2$ case I

[Kit84] finds bounds for the T-Fourier coefficients of P_Q in terms of T. We do bookkeeping to keep track of dependence on k, Q as well and arrive at

$$a_{P_T}(T) \ll O(1) + k^{-2/3} \det(T)^{1+\delta}. \tag{40}$$

Moreover, the counting argument present in the Fourier series method can be done more precisely. We can actually calculate the eigenvalues of TY in terms of its entries. This has been done in [Blo15] to get the following where $y_1 \leq y_2$ are the diagonal entries of Y:

$$\#\mathcal{C}_Y \ll k^{\epsilon} \left(\frac{k^{3/2}}{\det(Y)^{3/2}} + \frac{k}{y_1^{3/2} y_2^{1/2}} + \frac{k^{1/2}}{y_1} + 1 \right). \tag{41}$$

We can get a piece-wise bound on $\mathbb{B}_k(Z)$ by treating each term individually and then taking the maximum for a given value of $\det(Y)$.

$n = 2$ case II

Lemma

Let $\eta := \log_k(\det(Y))$. *For* $n = 2$, *we have* $\mathbb{B}_k(Z) \ll k^{2w_2(\eta)+\epsilon}$ *where*

$$w_2(\eta) = \begin{cases} \frac{35}{12} - \frac{5\eta}{4} & (0 \le \eta \le \frac{1}{2}) \\ \frac{29}{12} - \frac{\eta}{4} & (\frac{1}{2} \le \eta \le 1) \\ \frac{35}{12} - \frac{3\eta}{4} & (1 \le \eta \le \frac{4}{3}) \\ \frac{9}{4} - \frac{\eta}{4} & (\frac{4}{3} \le \eta \le \frac{3}{2}) \\ \frac{3}{4} + \frac{3\eta}{4} & (\frac{3}{2} \le \eta \le 2) \end{cases}. \tag{42}$$

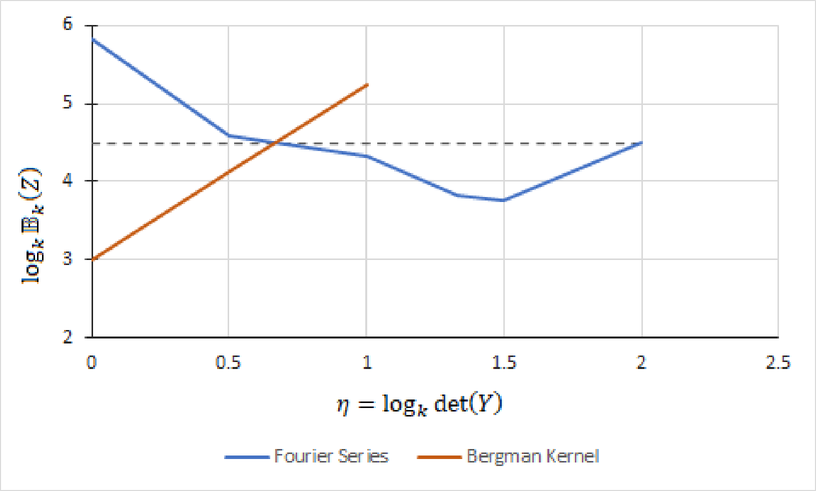
And directly from the Bergman kernel we have

$$\mathbb{B}_k(Z) \ll_\epsilon k^3 \det(Y)^{9/4+\epsilon}. \tag{43}$$

The Bergman kernel bound is better when $\eta < 2/3$. Otherwise we use the Fourier series piece-wise bound. At the intersection we get the bound $\mathbb{B}_k(Z) \ll_\epsilon k^{9/2+\epsilon}$ When $\eta \ge 2$, we have exponential decay.

$n = 2$ case III

Comparing the bounds.



Amplification I

For a fixed compact set Ω and when $n = 2$, $\sup_{Z \in \Omega} \mathbb{B}_k(Z) \ll_\Omega k^3$. We can improve this by means of amplification. For this we borrow the amplifier constructed in [BP16]. This amplifier relies on the fact that

$$|\lambda(p,F)| + \frac{1}{p^{3/2}} \left|\lambda(p^2,F)\right| + \frac{1}{p^{9/2}} \left|\lambda(p^4,F)\right| \gg p^{3/2}. \tag{44}$$

We choose F to be a Hecke eigenform that is orthogonal to the space of Saito-Kurokawa lifts. The above inequality implies at least one of the above 3 eigenvalues is large. Similar to [CL11], our bound relies on counting the number of γ for which $\gamma \langle Z \rangle$ is close to Z. We perform this counting, taking inspiration from [DS15].

Applications

Our results imply that the supremum of $\mathbb{B}_k(Z)$ must lie high in the cusps.

$$\mathbb{B}_k(Z_0) = \sup_{Z \in \mathcal{F}_n} \mathbb{B}_k(Z) \implies k^{n/3} \ll_n \det(\Im(Z_0)) \ll_n k^n. \qquad (45)$$

The bounds on the Bergman kernel can be used to get bounds on the Fourier coeffients. We can show (27) is satisfied for $(\alpha, \beta) = ((5n^2 - 3)/8, 0)$ which we restate here.

$$\left(\sum_{F \in \mathcal{B}_k^n} |a_F(T)|^2 \right)^{1/2} \ll_n \frac{(4\pi)^{nk/2} k^{\alpha} \det(T)^{k/2-\beta}}{\Gamma(k)^{n/2}} \qquad (46)$$

Resnikoff and Saldaña [RSn74] make a conjecture that $\beta = (n+1)/4$. This along with the expected value of α from Poincaré series is strong enough to show the conjectured bound for all n up to an ϵ.
Some of these methods can be used for congruence subgroups of Γ_n but our frequent reliance on a lower bound for Y when $Z \in \mathcal{F}_n$ will likely cause trouble.

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Thank You