Finite element analysis of the constrained Dirichlet boundary control problem governed by the Diffusion Equation

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Outline

- Dirichlet Boundary Control Problem

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- FE analysis of the Constrained Dirichlet Boundary Control Problem

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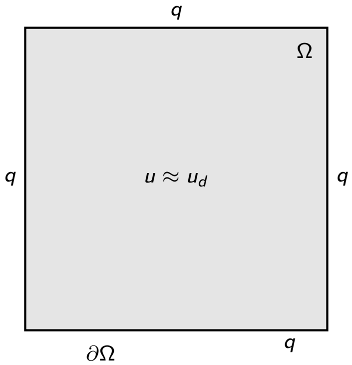
Dirichlet Boundary Control Problem

To find an appropriate control q on the boundary $\partial\Omega$ of the domain Ω so that the state u will be close to the given state $u_d \in L^2(\Omega)$ inside Ω.



Dirichlet Boundary Control Problem

The Dirichlet boundary optimal control problem consists of finding a state u and a control q such that

$$J(u,q) = \min_{(y,p)} J(y,p),$$

subject to the PDE,

$$\begin{aligned} -\Delta y &= f \quad \text{in} \quad \Omega, \\ y &= p \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where J is the quadratic functional defined by

$$J(y,p) := \frac{1}{2}\|y - u_d\|^2_{L^2(\Omega)} + \frac{\rho}{2}\|p\|^2_Q, \quad \rho > 0,$$

Q is an appropriate control trial space and u_d is a given function.

Approach 1

- One of the approaches is to find the control from the space $Q = L^2(\partial\Omega)$ (Casas and Raymond, 2006).
- The difficulty in this approach is that the trace of the state y is an $L^2(\partial\Omega)$ function. Therefore the standard weak formulation for y can not be used.
- The difficulty is addressed by introducing the ultra weak formulation (transposition method) using some dual problem.
- This approach requires the higher regularity of the dual solution but in general the regularity of the dual problem is limited.
- This controls induce some layers in the state variable at the corners of the domain.

Approach 2

- A recent approach is based on the energy space $Q = H^{\frac{1}{2}}(\partial\Omega)$ (Steinbach et. al. 2014)
- For any $p \in H^{\frac{1}{2}}(\partial\Omega)$ there exists a unique harmonic extension $u_p \in H^1(\Omega)$, such that

$$\begin{aligned} -\Delta u_p &= 0 \quad \text{in } \Omega \\ u_p &= p \quad \text{on } \partial\Omega. \end{aligned}$$

- Also we have the following norm equivalence:

$$\|p\|_{H^{\frac{1}{2}}(\partial\Omega)} \equiv \|u_p\|_{H^1(\Omega)} .$$

- Using Green's formula, and the above norm equivalence motivates the use of semi-norm

$$|p|^2_{H^{\frac{1}{2}}(\partial\Omega)} := \left\langle \frac{\partial u_p}{\partial n}, p \right\rangle_{\partial\Omega} = \int_\Omega |\nabla u_p|^2.$$

Approach 2

- By introducing the Steklov-Poincaré operator $S : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$, $Sp := \frac{\partial u_p}{\partial n}$, we can rewrite the seminorm as

$$|p|^2_{H^{\frac{1}{2}}(\partial\Omega)} = \langle Sp, p\rangle_{\partial\Omega}.$$

- Hence, define the quadratic cost functional J by

$$J(y,p) := \frac{1}{2}\|y - u_d\|^2_{L^2(\Omega)} + \frac{\rho}{2}\langle Sp, p\rangle_{\partial\Omega}.$$

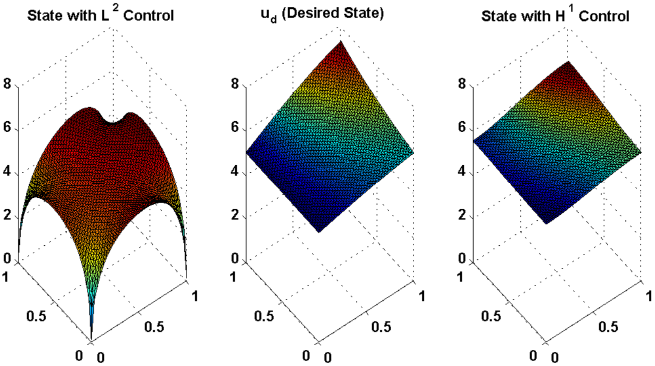
- Unfortunately, in this approach the numerical scheme will be very expensive as it requires a Poisson solve for each trial function of discrete control space.

Objectives

- Determine the appropriate function space for the control and attach a meaning to the energy cost functional.
- Derive the optimality system and discretize using finite elements.
- Discuss the error estimates and numerical experiments.

Numerical Comparison

Discrete states and the desired sate u_d:



Constrained Dirichlet Boundary Control Problem

$$\underset{y\in Q, p\in Q_{ad}}{\text{Minimize}}\, J(y,p) := \frac{1}{2}\|y-u_d\|^2_{0,\Omega} + \frac{\rho}{2}\|\nabla p\|^2_{0,\Omega} \tag{1}$$

subject to the PDE,

$$\begin{aligned} -\Delta y &= f \quad \text{in} \quad \Omega, \\ y &= 0 \quad \text{on} \quad \Gamma_D, \\ y &= p \quad \text{on} \quad \Gamma_C, \end{aligned}$$

$$Q := \{p \in H^1(\Omega) : \gamma_0(p) = 0 \text{ on } \Gamma_D\}.$$

and

$$Q_{ad} := \{p \in Q : q_a \leq \gamma_0(p) \leq q_b \text{ a.e. on } \Gamma_C\}, \text{with } \; q_a \leq 0 \leq q_b \text{ are constants.}$$

The set $\Omega \subset \mathbb{R}^2$, be a bounded polygonal domain with boundary $\partial\Omega$ consists of two non-overlapping open subsets $|\Gamma_D| > 0$ and $|\Gamma_C| > 0$ with $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_C$. The interior force $f \in L^2(\Omega)$ and $u_d \in L^2(\Omega)$ is a given target function.

Existance and Uniqueness/Optimality System

Theorem 1 (Existance and Uniqueness of the Solution).

There exists a unique solution of the control problem (1).

Proposition 2 (**Continuous Optimality System).**

The unique solution $(u, q) \in Q \times Q_{ad}$ for the Dirichlet control problem (1) and, there exists a unique adjoint state $\phi \in V$ satisfies following:

$$
\begin{aligned}
u &= w + q \quad w \in V, \\
a(w, v) &= \ell(v) - a(q, v) \quad \forall v \in V, \\
a(v, \phi) &= (u - u_d, v) \quad \forall v \in V, \\
\rho\, a(q, p - q) &\geq a(p - q, \phi) - (u - u_d, p - q) \quad \forall p \in Q_{ad}.
\end{aligned}
$$

Control satisfies simplified Signorini problem

Remark 3.

The optimal control q is the weak solution of the following simplified Signorini problem

$$\begin{aligned} -\rho\Delta q &= 0 \quad \text{in} \quad \Omega, \\ q &= 0 \quad \text{on} \quad \Gamma_D, \\ q_a \le q &\le q_b \quad \text{on} \quad \Gamma_C, \end{aligned}$$

further the following holds for almost every $x \in \Gamma_C$:

$$\begin{aligned} &\text{if } q_a < q(x) < q_b \quad \text{then} \quad \Big(\rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\Big)(x) = 0, \\ &\text{if } q_a \le q(x) < q_b \quad \text{then} \quad \Big(\rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\Big)(x) \ge 0, \\ &\text{if } q_a < q(x) \le q_b \quad \text{then} \quad \Big(\rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\Big)(x) \le 0. \end{aligned}$$

Regularity Assumption/FE Discretization

- We assume that the solution $u \in H^{\frac{3}{2}+\epsilon}(\Omega)$, $\phi \in H^{\frac{3}{2}+\epsilon}(\Omega)$ and $q \in H^{\frac{3}{2}+\epsilon}(\Omega)$ where $0 < \epsilon \leq \frac{1}{2}$.
- Define the discrete state space $V_h \subset V$ by

$$V_h := \{v_h \in V : v_h|_T \in \mathbb{P}_1(T) \ \ \forall T \in \mathcal{T}_h\}.$$

- The discrete control space $Q_h \subset Q$ by

$$Q_h := \{q_h \in Q : q_h|_T \in \mathbb{P}_1(T), \ \ \forall T \in \mathcal{T}_h\}.$$

- The discrete admissible set of controls is defined by

$$Q_{ad}^h := \{q_h \in Q_h : q_a \leq q_h(z) \leq q_b \text{ for all } z \in \mathcal{V}_h^C\}.$$

Discrete Optimality System

The discrete control problem finds $(w_h, q_h, \phi_h) \in V_h \times Q_{ad}^h \times V_h$ such that

$$\begin{aligned} u_h &= w_h + q_h, \\ a(w_h, v_h) &= \ell(v_h) - a(q_h, v_h) \quad \forall v_h \in V_h, \\ a(v_h, \phi_h) &= (u_h - u_d, v_h) \quad \forall v_h \in V_h, \\ \rho\, a(q_h, p_h - q_h) &\geq a(p_h - q_h, \phi_h) - (u_h - u_d, p_h - q_h) \quad \forall p_h \in Q_{ad}^h. \end{aligned}$$

Error Analysis

- Introduce the projections $P_h w$ and $\bar{P}_h\phi$ such that

$$a(P_h w, v_h) = \ell(v_h) - a(q, v_h) \quad \forall v_h \in V_h,$$

and

$$a(v_h, \bar{P}_h\phi) = (u - u_d, v_h) \quad \forall v_h \in V_h.$$

- Using Galerkin orthogonality, we have

$$\left\|\nabla(\phi - \bar{P}_h\phi)\right\|_{0,\Omega} \leq Ch^{\frac{1}{2}+\epsilon} \|\phi\|_{\frac{3}{2}+\epsilon,\Omega},$$

and

$$\|\nabla(P_h w - w)\|_{0,\Omega} \leq Ch^{\frac{1}{2}+\epsilon} \|w\|_{\frac{3}{2}+\epsilon,\Omega}.$$

Error Analysis

Lemma 4.

There holds,

$$\|\nabla(q-q_h)\|_{0,\Omega}^2 + \|u-u_h\|_{0,\Omega}^2 \leq C\Big(a(q,p_h-q) - a(p_h-q,\phi) + (u-u_d,p_h-q)\Big) + C\left(\|\nabla(q-p_h)\|_{0,\Omega}^2 + \|\nabla(\phi-\bar{P}_h\phi)\|_{0,\Omega}^2 + \|P_hw-w\|_{0,\Omega}^2\right) + C\|p_h-q\|_{0,\Omega}^2,$$

for all $p_h \in Q_{ad}^h$.

- The right-hand side of the estimate in the above lemma contains terms corresponding to the interpolation errors except for the term in the first parentheses.
- How much order of convergence do we expect for $\|\nabla(q-q_h)\|_{0,\Omega}^2$?
- $\|\nabla(q-q_h)\|_{0,\Omega}^2 \lesssim h^{1+2\epsilon}$ (given $q \in H^{\frac{3}{2}+\epsilon}(\Omega)$).
- All the last four terms gives the expected order of convergence $(O(h^{1+2\epsilon}))$.

Error Analysis

- Using the integration by parts, we find that

$$\begin{aligned}\rho a(q, p_h - q) - a(p_h - q, \phi) + (u - u_d, p_h - q) &= \int_{\Gamma_C} \left(\rho \frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}\right)(p_h - q)ds \\ &\leq \left\| \rho \frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n} \right\|_{0,\Gamma_C} \|p_h - q\|_{0,\Gamma_C} \\ &\lesssim h^{1+\epsilon} (\text{not optimal}).\end{aligned}$$

- The lack of order of convergence for the Signorini solution had been a popular and challenging issue for nearly one and a half decades, Hilld et al. 2015 made this improvement.

Error Analysis

- For a fixed triangle T which shares an edge with Γ_C, define

$$S_{NC} = \{x \in T \cap \Gamma_C : \; q_a < q(x) < q_b\},$$

and

$$S_C = \{x \in T \cap \Gamma_C : \; q(x) = q_a\} \cup \{x \in T \cap \Gamma_C : \; q(x) = q_b\}.$$

- Since $q|_{\Gamma_C}$ belongs to $H^{1+\epsilon}(\Gamma_C)$, Sobolev embedding theorem ensures that $q|_{\Gamma_C}$ is a continuous function on Γ_C.
- The sets S_C and S_{NC} are measurable because they are inverse images of Borel sets by a continuous function. We denote by $|S_C|$, $|S_{NC}|$ their measure in $\mathbb{R}$.

Error Analysis

Lemma 5.

Let, h_e be the length of the edge $T \cap \Gamma_C$ and $|S_C|$, $|S_{NC}|$ are defined as above. Assume that $|S_C| > 0$, $|S_{NC}| > 0$. The following L^2 and L^1 -estimates hold for σ_n and q':

$$\|\sigma_n\|_{0,T\cap\Gamma_C} \leq \frac{1}{|S_{NC}|^{1/2}} \; h_e^{\frac{1}{2}+\epsilon} \; |\sigma_n|_{\epsilon,T\cap\Gamma_C},$$

$$\|\sigma_n\|_{L^1(T\cap\Gamma_C)} \leq \frac{|S_C|^{1/2}}{|S_{NC}|^{1/2}} \; h_e^{\frac{1}{2}+\epsilon} \; |\sigma_n|_{\epsilon,T\cap\Gamma_C},$$

$$\left\|q'\right\|_{0,T\cap\Gamma_C} \leq \frac{1}{|S_C|^{1/2}} \; h_e^{\frac{1}{2}+\epsilon} \; |q'|_{\epsilon,T\cap\Gamma_C},$$

$$\left\|q'\right\|_{L^1(T\cap\Gamma_C)} \leq \frac{|S_{NC}|^{1/2}}{|S_C|^{1/2}} \; h_e^{\frac{1}{2}+\epsilon} \; |q'|_{\epsilon,T\cap\Gamma_C},$$

where $\sigma_n := \rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}$ and q' is the (tangential) distribution derivative of q on $T \cap \Gamma_C$.

Error Analysis

Lemma 6.

There holds

$$|a(q, p_h - q) - a(p_h - q, \phi) + (u - u_d, p_h - q)| \leq Ch^{1+2\epsilon}\left(\|q\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|\phi\|^2_{\frac{3}{2}+\epsilon,\Omega}\right).$$

- We have seen

$$\rho a(q, p_h - q) - a(p_h - q, \phi) + (u - u_d, p_h - q) = \int_{\Gamma_C} \sigma_n (p_h - q) ds. \quad (3)$$

- Set $\sigma_n := \rho\frac{\partial q}{\partial n} - \frac{\partial \phi}{\partial n}$, and choose $p_h = \mathcal{I}_h q \in Q_h$ where $\mathcal{I}_h$ is the Lagrange interpolation operator.
- The right hand side of (3) reads as and equals

$$\int_{\Gamma_C} \sigma_n(\mathcal{I}_h q - q) ds = \sum_{T \in \mathcal{T}_h} \int_{T\cap\Gamma_C} \sigma_n(\mathcal{I}_h q - q) ds.$$

Error Analysis

- It remains to estimate the following:

$$\int_{T\cap\Gamma_C} \sigma_n(\mathcal{I}_h q - q)ds \quad T \in \mathcal{T}_h. \tag{4}$$

- Let T be a fixed triangle and h_e be the length of the edge $e = T \cap \Gamma_C$ and obviously $|S_C| + |S_{NC}| = h_e$. There arises two cases:
  (i) either S_C or S_{NC} have measure zero,
  (ii) both of S_C and S_{NC} have positive measure.
- For the case (i), it is easy to check that the integral term in (4) vanishes. So we need to estimate (4) for the case (ii).
- We will derive two estimates of the same error term (4): The first one depending on $|S_{NC}|$ and the second one depending on $|S_C|$.

Error Analysis

Estimate of (4) depending on S_{NC}:

$$\int_{T\cap\Gamma_C} \sigma_n(\mathcal{I}_h q - q)ds \leq \|\sigma_n\|_{0,T\cap\Gamma_C} \|\mathcal{I}_h q - q\|_{0,T\cap\Gamma_C}$$

$$\leq C\frac{1}{|S_{NC}|^{\frac{1}{2}}} h_e^{\frac{3}{2}+2\epsilon}\left(|\sigma_n|^2_{\epsilon,T\cap\Gamma_C} + |q'|^2_{\epsilon,T\cap\Gamma_C}\right).$$

Estimate of (4) depending on S_C:

$$\int_{T\cap\Gamma_C} \sigma_n(\mathcal{I}_h q - q)ds \leq C\frac{1}{|S_C|^{\frac{1}{2}}} h_e^{\frac{3}{2}+2\epsilon}\left(|\sigma_n|^2_{\epsilon,T\cap\Gamma_C} + |q'|^2_{\epsilon,T\cap\Gamma_C}\right).$$

Error Analysis

- Note that either $|S_{NC}|$ or $|S_C|$ is greater than or equal to $h_e/2$ and by choosing the appropriate above estimate, we obtain

$$\int_{T\cap\Gamma_C} \sigma_n(\mathcal{I}_h q - q)ds \leq C h_e^{1+2\epsilon}\left(|\sigma_n|^2_{\epsilon,T\cap\Gamma_C} + |q'|^2_{\epsilon,T\cap\Gamma_C}\right).$$

- By summation and using the trace theorem, we get

$$\int_{\Gamma_C} \sigma_n(\mathcal{I}_h q - q)ds \leq C\ h^{1+2\epsilon}\left(\|q\|^2_{\frac{3}{2}+\epsilon,\Omega} + \|\phi\|^2_{\frac{3}{2}+\epsilon,\Omega}\right).$$

$\square$

Theorem 7 (Error estimate of control variable).

There holds

$$\|\nabla(q-q_h)\|_{0,\Omega} + \|u-u_h\|_{0,\Omega} \leq C(h^{\frac{1}{2}+\epsilon}\|q\|_{\frac{3}{2}+\epsilon,\Omega} + h^{\frac{1}{2}+\epsilon}\|\phi\|_{\frac{3}{2}+\epsilon,\Omega} + h^{1+2\epsilon}\|w\|_{\frac{3}{2}+\epsilon,\Omega}).$$

Error Analysis

Theorem 8 (Error estimate of state and adjoint state variable).

There holds

$$\|\nabla(u-u_h)\|_{0,\Omega}+\|\nabla(\phi-\phi_h)\|_{0,\Omega}\leq C\Big(h^{\frac{1}{2}+\epsilon}\,\|q\|_{\frac{3}{2}+\epsilon,\Omega}+h^{\frac{1}{2}+\epsilon}\,\|\phi\|_{\frac{3}{2}+\epsilon,\Omega}$$
$$+\,h^{\frac{1}{2}+\epsilon}\,\|w\|_{\frac{3}{2}+\epsilon,\Omega}\Big).$$

Numerical Experiment

For the numerical experiment we slightly modify the cost functional J, denoted by $\tilde{J}$, by

$$\tilde{J}(w,p) = \frac{1}{2}\|w - u_d\|_{0,\Omega}^2 + \frac{\rho}{2}\|\nabla(p - q_d)\|_{0,\Omega}^2, \quad w \in Q,\ p \in Q_{ad}.$$

where q_d is a given function.

Example 9.

For this example, the domain Ω is taken to be the unit square $(0,1) \times (0,1)$, $\Gamma_C = (0,1) \times \{0\}$, $\Gamma_D = \partial\Omega \backslash \Gamma_C$ and set $\alpha = 1$, $q_a = -0.10$, $q_b = 0.25$. We choose the data of the problem as follows. Choose the state to be $u(x,y) = x(1-x)(1-y)\exp(y)$, the adjoint state to be $\phi(x,y) = \sin^2(\pi x)\sin^2(\pi y)$ and the control to be $q(x,y) = x(1-x)(1-y)\exp(y)$. Then compute $f = -\Delta u$ and $u_d = u + \Delta\phi$. The choice of ϕ leads to $q_d = q$. We choose u and q in such a way that $u = q$ on Γ_C and $u = q = 0$ on Γ_D.

Numerical Experiment

Table 1: Errors and orders of convergence (order) for the Example 9.

h	$\|\nabla(u-u_h)\|_{0,\Omega}$	order	$\|\nabla(q-q_h)\|_{0,\Omega}$	order	$\|\nabla(\phi-\phi_h)\|_{0,\Omega}$	order
0.250000	0.202248	———-	0.206402	———-	1.281567	——-
0.125000	0.103894	0.961003	0.104789	0.977964	0.668652	0.938581
0.062500	0.052462	0.985768	0.052614	0.993963	0.345636	0.952001
0.031250	0.026311	0.995583	0.026335	0.998458	0.175148	0.980672
0.015625	0.013167	0.998700	0.013171	0.999612	0.087964	0.993595
0.007812	0.006585	0.999627	0.006586	0.999902	0.044041	0.998053

Conclusions

- The controls from the $L^2(\partial\Omega)$ space induce some layers in the state at the corners of the domain.
- The energy approach with Steklov-Poincaré map for defining the Dirichlet control problem is realized to be quite expensive.
- The new approach, which we have introduced seems an attractive alternative and it is computationally efficient. It provides sufficiently smooth control and the state.
- The new framework allows to consider the energy functional with gradients of the state

$$J(y,p) = \frac{1}{2}\|\nabla(y-u_d)\|^2_{0,\Omega} + \frac{\rho}{2}\|\nabla p\|^2_{0,\Omega}.$$

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Thank you for your attention!