Phase Transitions and Percolation at Criticality in some Random Graph Models on the Plane

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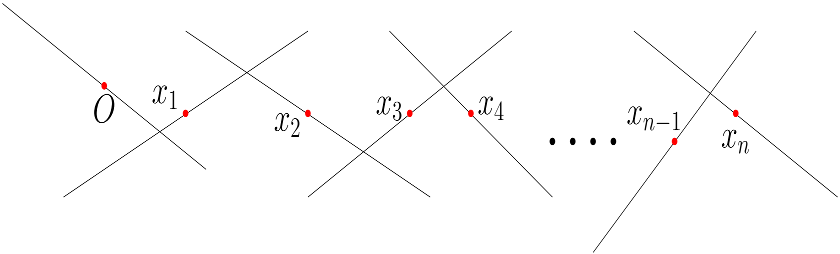
The Percolation Problem

- Percolation refers to the existence of an infinite connected component in a graph.
- A Phase Transition is said to occur if there is an abrupt emergence of some interesting phenomenon.
- We are interested in existence abrupt transition of phase from non-percolation to percolation in certain Geometric Random Graphs on the plane. If so does percolation occur at the critical point?
- What is special about the plane?

Poisson Stick Model

- $\mathcal{P}_\lambda$ be a homogeneous Poisson Point process in $\mathbb{R}^2$ of intensity λ.
- h be a probability density function on $[0, \infty)$.
- Place sticks of independent random lengths distributed according to h with random independent orientation (non-degenerate) with mid points located at points of $\mathcal{P}_\lambda$
- Intersecting lines form a path in the graph (PS_λ).

A path in the Poisson Stick Model



The Poisson Stick Model

- Model was introduced by Roy (1991) with h having bounded support.
- Natural model for a network structure formed by silicon nanowires and carbon and other nanotubes on the surface of substrates.
- Scale invariant spatial networks: Aldous (2014).
- Poisson Line Process.

The Enhanced Random Connection Model

- RCM G_λ (Penrose, 1991): Vertex set is $\mathcal{P}_\lambda$.
- Edge between pairs of nodes located at x, y with probability $g(|x-y|)$.
- The model is non-trivial iff $0 < \int_0^\infty rg(r)\, dr < \infty$.
- iRCM H_λ Deijfen, Hofstad and Hooghiemstra (2013); Deprez, Hazra and Wüthrich (2015), Deprez, and Wüthrich (2019): Edge probability is given by

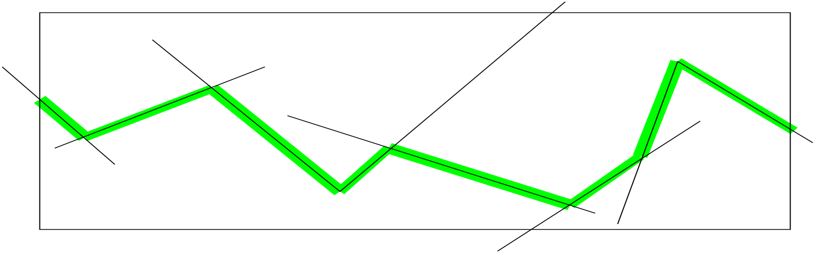
$$g(|x-y|) = 1 - \exp\left(-\frac{\eta W_x W_y}{|x-y|^\alpha}\right)$$

$$P(W > w) = w^{-\beta} 1_{[1,\infty)}(w).$$

- The model is non-trivial iff $\min\{\alpha, \alpha\beta\} > d$.
- Both graphs are *enhanced* to obtain the eRCM G^e_λ and the ieRCM H^e_λ.

Paths in the Enhanced Graph

- Think of the edges in the original graphs as line segments.
- A Path is a continuous curve contained entirely in $\cup_{i=1}^{n} e_i$ for some edges $e_1, e_2, \ldots, e_n$. Paths need not start or end at vertices in the graph.



Random Spatial Networks

- Aldous and Shun (2010): Optimal road network - Trade-off between shortness of route and normalized route length.
- Aldous (2014): Scale invariant spatial networks. Primitives are the routes between points on the plane. Problems of interest - Existence and uniqueness of infinite geodesics, continuity of routes, number of routes.
- The enhanced graph has some similarity with Steiner trees.
- The Poisson line process and Steiner trees were used in Aldous and Kendall (2008) to establish asymptotics of excess route lengths in arbitrary graphs.
- The enhanced models can be thought of as road or pipeline networks where traffic or fluid can switch from one edge to another intersecting one.

Phase Transition

- Recall that the half length density in the Poisson stick model, the connection function in the eRCM and the ieRCM are given respectively by $h, g : [0, \infty) \to [0, 1]$ and

$$g(|x-y|) = 1 - \exp\left(-\frac{\eta W_x W_y}{|x-y|^\alpha}\right),$$

$$P(W > w) = w^{-\beta} 1_{[1,\infty)}(w); \qquad \beta > 0.$$

Theorem

A phase transition in λ occurs in the

(i) *eRCM G^e_λ if $0 < \int_0^\infty r^3 g(r)\, dr < \infty$. ($0 < \int_0^\infty rg(r)\, dr < \infty$)*

(ii) *ieRCM H^e_λ if $\alpha > 4$ and $\alpha\beta > 8$. ($\alpha > 2, \alpha\beta > 4$)*

(iii) *PS_λ if $0 < \int_0^\infty \ell^2 h(\ell)\, d\ell < \infty$. (h bounded support)*

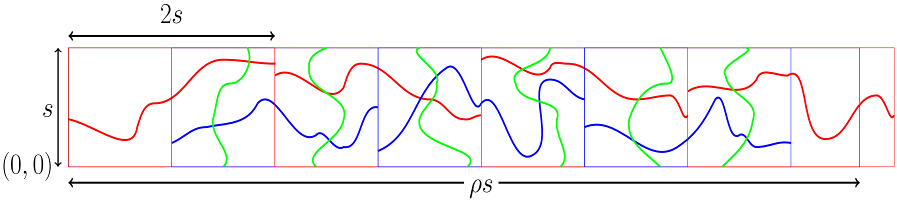
The RSW Lemma

- A rather useful result for percolation models on the plane.
- If the probability of crossing squares of side lengths at least one is uniformly bounded away from zero then so is the probability of crossing a rectangle along the longer side.
- Continuity of the percolation function.
- Non-triviality of the box-crossing probability at criticality/critical window. Equality of critical intensities. Noise sensitivity.

The RSW Lemma

- Bernoulli bond percolation on $\mathbb{Z}^2$: Russo; Seymour and Welsh (1978).
- Occupied, Vacant component in a Poisson Boolean model with bounded radius, Poisson stick model with bounded stick length: Roy (1990, 1991).
- Poisson Voronoi percolation: Tassion (2016).
- Unbounded radii in the Poisson Boolean model – Sharpness of phase transition, non-percolation at criticality: Ahlberg, Tassion, Teixeira (2018).

The RSW Lemma



The RSW Lemma

Theorem

Suppose the following conditions hold.

(I) *eRCM* G_λ^e: $g(r) = O(r^{-c})$ *as* $r \to \infty$ *with* $c > 4$.

(II) *ieRCM* H_λ^e: $\min\{\alpha, \alpha\beta\} > 4$.

(III) PS_λ: $h(\ell) = O(\ell^{-c})$ *with* $c > 3$.

Then the following conclusions hold for all the three graphs G_λ^e, H_λ^e *and* PS_λ.

(i) *If* $\inf_{s \geq 1} C_s(1) > 0$ *then for any* $\rho \geq 1$, $\inf_{s \geq 1} C_s(\rho) > 0$.

(ii) *If* $\lim_{s \to \infty} C_s(1) = 1$ *then for any* $\rho \geq 1$, $\lim_{s \to \infty} C_s(\rho) = 1$.

(iii) *The percolation function is continuous.* (*iRCM* H_λ: $\alpha \in (2, 4)$, $\alpha\beta > 4$)

Proof of RSW

- The FKG Inequality: Combine the FKG for the finite discrete percolation model with that for square integrable functionals on a PPP: For A, B increasing

$$P(A \cap B) \geq P(A)P(B)$$

- Square Root Trick: For $A_1, A_2, \ldots, A_k$ increasing events

$$\max P(A_i) \geq 1 - \left(1 - P\left(\cup_{i=1}^{k} A_i\right)\right)^{\frac{1}{k}}.$$

- Tassion proved the RSW by leveraging the above two results and the symmetries: translation, rotation and reflection, correlation decay.
- For the Poisson-Boolean model due to the lack of symmetry one needs to prove a stronger version of the RSW Lemma to show equality of the critical intensities.

Proof of RSW

- In the Poisson-Voronoi case, if there are enough points in an annulus, then there is no cell crossing the annulus.
- In the Poisson Boolean model one needs $(2 + \epsilon)$-moment of the radius distribution to be finite.

Theorem

For any $s > 0$ let M_s be the length of the longest edge in G_λ intersecting the box $B_s = [-s, s]^2$. Suppose that the connection function g satisfies $g(r) = O(r^{-c})$ as $r \to \infty$. Then for any $c > 4$, $t > 0$ and $\tau > \frac{2}{c-2}$ we have $P\left(M_{ts} > s^{\tau}\right) \to 0$ as $s \to \infty$.

Non-Percolation at Criticality