GOE Fluctuations for Alternating Sign Matrices

Arvind Ayyer,
Indian Institute of Science, Bangalore,

(joint with S. Chhita and K. Johansson, `arXiv:2109.02422`)

IISc–IISER Joint Math Symposium

September 17, 2021

Plan of the Talk

1. Alternating sign matrices
2. Large ASMs
3. Totally symmetric self-complementary plane partitions
4. Gog and magog trapezoids

Alternating Sign Matrices

Definition

An alternating sign matrix (ASM) of order n is an $n \times n$ matrix A with entries in $\{0, \pm 1\}$ such that all row and column sums of A are 1 and nonzero entries in every row and column alternate in sign.

Theorem (conjectured by Mills, Robbins and Rumsey '83, proved by Zeilberger, '95 & Kuperberg, '96)

The number of ASMs of order n is given by

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

The sequence starts $1, 2, 7, 42, 429, \ldots$

An example of size 13

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

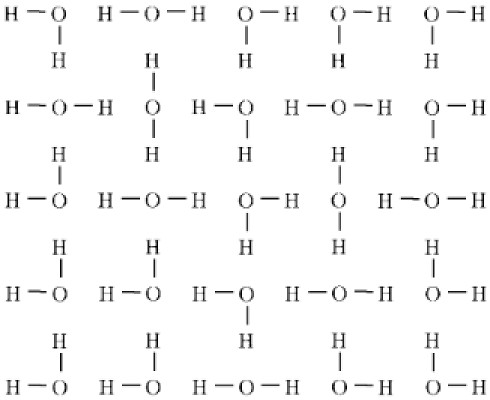
Alternating Sign Matrices of size 3

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},$$

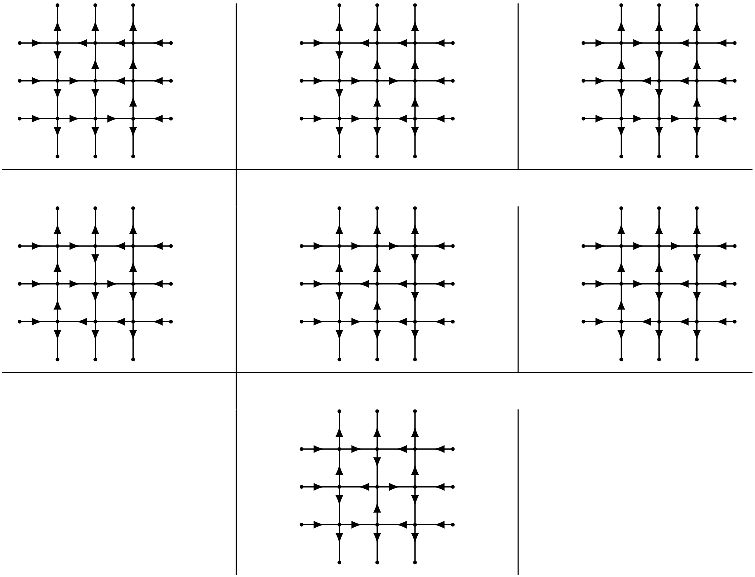
$$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

Six-vertex or Square ice model with domain wall boundary conditions



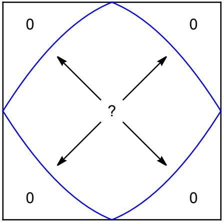
Credit: Bressoud's book

Six-vertex configurations of size 3



Large random ASMs

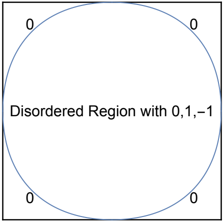
- Use intuition from random tiling models.
- Rescale the ASMs of size n by $n/2$ so that it fits into $[0,2]^2$. We expect to see frozen corners of 0's with a disordered region in the middle, and an Arctic curve separating the two.



- Colomo-Pronko (2010) and Colomo-Sportiello (2016) predicted the limit shape using two different methods.

Large random ASMs

- Use intuition from random tiling models.
- Rescale the ASMs of size n by $n/2$ so that it fits into $[0,2]^2$. We expect to see frozen corners of 0's with a disordered region in the middle, and an Arctic curve separating the two.



- Colomo-Pronko (2010) and Colomo-Sportiello (2016) predicted the limit shape using two different methods.
- This was made rigorous by Aggarwal (2020) confirming the limit shape curves for the ASMs.

Heights from ASMs

- For each ASM of size $n+1$, $A = (a_{i,j})_{1 \le i,j \le n+1}$ construct a path corner sum matrix (PCSM) $C = (c_{i,j})_{1 \le i,j \le n}$ by

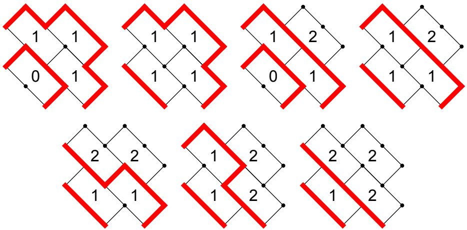
$$c_{i,j} = n - \sum_{\substack{1 \le r \le i \\ 1 \le s \le n+1-j}} a_{r,s}, \quad 1 \le i,j \le n.$$

- This is a height function representation in natural bijection with ASMs.
- In a PCSM $C = (c_{i,j})_{1 \le i,j \le n}$,
    - $c_{i,j} \ge 0$ for $1 \le i,j \le n$,
    - $c_{1,j} \in \{n-1, n\}$ and $c_{i,n} \in \{n-1, n\}$ for $1 \le i,j \le n$,
    - $c_{i,j+1} - c_{i,j}, c_{i,j} - c_{i+1,j} \in \{0,1\}$ for $1 \le i,j \le n$.

Example

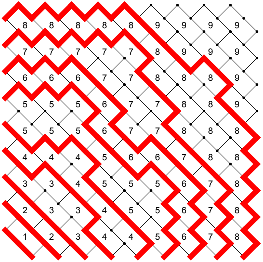
- PCSMs of size 2 are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$



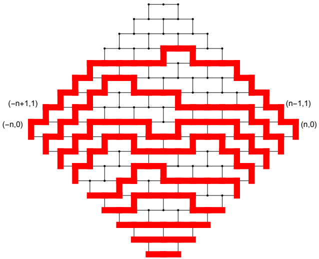
Maximum of the Top Path

Consider the level lines of the heights.



Maximum of the Top Path

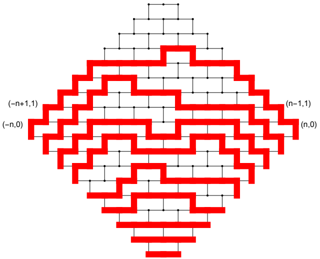
Consider the level lines of the heights.



- Rotate by $\pi/4$ counterclockwise.
- Abscissa marks time and ordinate marks height.

Maximum of the Top Path

Consider the level lines of the heights.

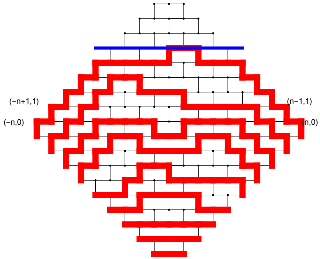


- Rotate by $\pi/4$ counterclockwise.
- Abscissa marks time and ordinate marks height.
- Focus on the highest path from $(-n, 0)$ to $(n, 0)$,

$$T_n = (T_n(-n), \ldots, T_n(-1), T_n(0), T_n(1), \ldots, T_n(n)).$$

Maximum of the Top Path

Consider the level lines of the heights.

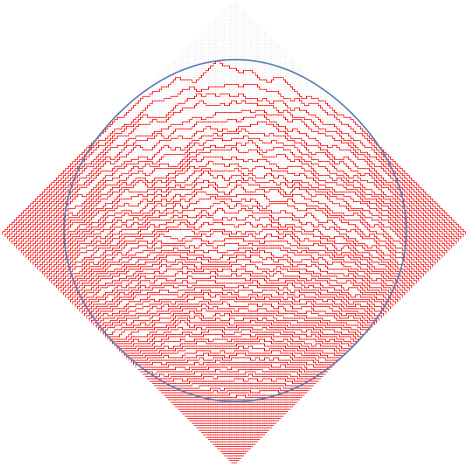


- Rotate by $\pi/4$ counterclockwise.
- Abscissa marks time and ordinate marks height.
- Focus on the highest path from $(-n, 0)$ to $(n, 0)$,

$$T_n = (T_n(-n), \ldots, T_n(-1), T_n(0), T_n(1), \ldots, T_n(n)).$$

- Our theorem concerns fluctuations of $\max(T_n)$.

Simulation of a uniform ASM of size 100



Random matrix limit laws

- A random real symmetric matrix belongs to the Gaussian orthogonal ensemble (GOE) if the diagonal and upper-triangular entries are independently chosen from $N(0,2)$ and $N(0,1)$ respectively.
- Let λ_{max} denote the largest eigenvalue of such a matrix.
- The limiting CDF, given by

$$F_1(s) = \lim_{n\to\infty} \mathbb{P}\left(\frac{\lambda_{\text{max}} - 2\sqrt{n}}{n^{-1/6}} \leq s\right),$$

  is known as the GOE Tracy-Widom distribution.
- There is a similar definition for random Hermitian matrices in the Gaussian unitary ensemble (GUE), and $F_2(s)$ is the limiting CDF, known as the GUE Tracy-Widom distribution.

Universality?

These random matrix theory limit laws appear in many other models analyzed by using determinantal point processes. Some examples include

- $\Delta = 0$ six-vertex model with domain wall boundary conditions (equivalent to uniformly random domino tilings of the Aztec diamond).
- Directed last passage percolation in 2D with geometric weights.
- Polynuclear Growth Models.
- TASEP with parallel and sequential updates.

GOE kernel

- Let $\mathrm{Ai}(x)$ denote the Airy function, that is,

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \mathrm{d}t\ \cos\left(\frac{t^3}{3} + xt\right),$$

which converges for all real x.

GOE kernel

- Let $\mathrm{Ai}(x)$ denote the Airy function, that is,

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \mathrm{d}t \, \cos\left(\frac{t^3}{3} + xt\right),$$

  which converges for all real x.
- Introduce the following 2 by 2 block kernel

$$\mathrm{K}_{\mathrm{GOE}}(x,y) = \begin{pmatrix} K^{11}_{\mathrm{GOE}}(x,y) & K^{12}_{\mathrm{GOE}}(x,y) \\ K^{21}_{\mathrm{GOE}}(x,y) & K^{22}_{\mathrm{GOE}}(x,y) \end{pmatrix}$$

GOE block kernel

$$K^{11}_{\mathrm{GOE}}(x,y) = \frac{1}{4}\int_0^\infty \mathrm{d}\lambda\ \big(\mathrm{Ai}(x+\lambda)\mathrm{Ai}'(y+\lambda) - \mathrm{Ai}'(x+\lambda)\mathrm{Ai}(y+\lambda)\big),$$

$$K^{12}_{\mathrm{GOE}}(x,y) = \int_0^\infty \mathrm{d}\lambda\ \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) + \frac{1}{2}\mathrm{Ai}(x)\int_0^\infty \mathrm{d}\lambda\ \mathrm{Ai}(y-\lambda)$$

$$K^{21}_{\mathrm{GOE}}(x,y) = -K^{12}_{\mathrm{GOE}}(y,x),$$

$$\begin{aligned} K^{22}_{\mathrm{GOE}}(x,y) = {} & \int_0^\infty \mathrm{d}\lambda \int_\lambda^\infty \mathrm{d}\mu\ \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\mu) - \mathrm{Ai}(x+\mu)\mathrm{Ai}(y+\lambda) \\ & - \int_0^\infty \mathrm{d}\mu\ \mathrm{Ai}(x+\mu) + \int_0^\infty \mathrm{d}\mu\ \mathrm{Ai}(y+\mu) - \mathrm{sgn}(x-y). \end{aligned}$$

Fredholm Pfaffian

- The Pfaffian of a $2k \times 2k$ anti-symmetric matrix A is given by

$$\mathrm{Pf}(A) = \frac{1}{2^k k!} \sum_{\sigma \in \mathcal{S}_{2k}} \mathrm{sgn}(\sigma) A_{\sigma(1),\sigma(2)} \cdots A_{\sigma(2k-1),\sigma(2k)},$$

where $\mathcal{S}_{2k}$ is the set of permutations of $\{1, \ldots, 2k\}$.

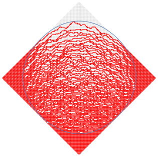
- The GOE Tracy–Widom distribution is defined through a Fredholm Pfaffian by

$$\begin{aligned} F_1(s) =& \mathrm{Pf}(\mathbb{J} - \mathrm{K}_{\mathrm{GOE}})_{L^2(s,\infty)} \\ =& 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_s^{\infty} \mathrm{d}x_1 \cdots \int_s^{\infty} \mathrm{d}x_k \, \mathrm{Pf}(\mathrm{K}_{\mathrm{GOE}}(x_i, x_j))_{1 \le i,j \le k}, \end{aligned}$$

where

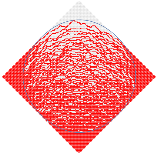
$$\mathbb{J}(x, y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbb{I}_{x=y}.$$

Main Theorem



Introduce the constants
$\alpha = 2 - \sqrt{3}$ and $c_0 = \frac{1}{2 \cdot 3^{1/6}}$.

Main Theorem

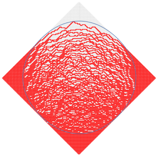


Introduce the constants $\alpha = 2 - \sqrt{3}$ and $c_0 = \frac{1}{2 \cdot 3^{1/6}}$.

Theorem (A.-Chhita-Johansson (2021+))

$$\lim_{n \to \infty} \mathbb{P}\left[\frac{\max(T_n) - (1-\alpha)n}{c_0 n^{\frac{1}{3}}} \leq s\right] = F_1(s).$$

Main Theorem



Introduce the constants
$\alpha = 2 - \sqrt{3}$ and $c_0 = \frac{1}{2 \cdot 3^{1/6}}$.

Theorem (A.-Chhita-Johansson (2021+))

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\max(T_n) - (1-\alpha)n}{c_0 n^{\frac{1}{3}}} \leq s\right] = F_1(s).$$

Conjecture (A.-Chhita-Johansson (2021+))

After rescaling, T_n converges to the Airy-2 process. In particular

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{T_n(0) - (1-\alpha)n}{c_0 (4n)^{\frac{1}{3}}} \leq s\right] = F_2(s).$$

Strategy of the Proof

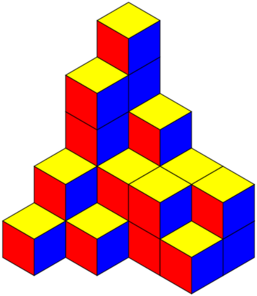
- Use Zeilberger's results to relate statistics of ASMs to statistics of TSSCPPs.

Strategy of the Proof

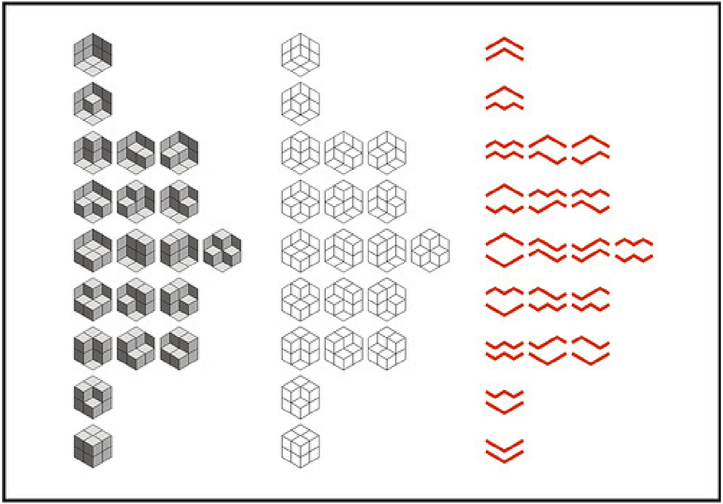
- Use Zeilberger's results to relate statistics of ASMs to statistics of TSSCPPs.
- Use formulas for correlations in TSSCPPs from A.-Chhita (2021+).

Plane partitions

- A plane partition is a three-dimensional generalisation of a Young diagram.
- Place boxes in a room so that they are aligned in a corner.



Plane partitions in a $2 \times 2 \times 2$ room



Symmetry classes

All	(PP)
(Vertically) Symmetric	(SPP)
Cyclically symmetric	(CSPP)
Totally symmetric	(TSPP)
Self-complementary	(SCPP)
Transpose = complement	(TCPP)
Symmetric and self-complementary	(SSCPP)
Cyclically symmetric and transpose = complement	(CSTCPP)
Cyclically symmetric and self-complementary	(CSSCPP)
Totally symmetric and self-complementary	(TSSCPP)

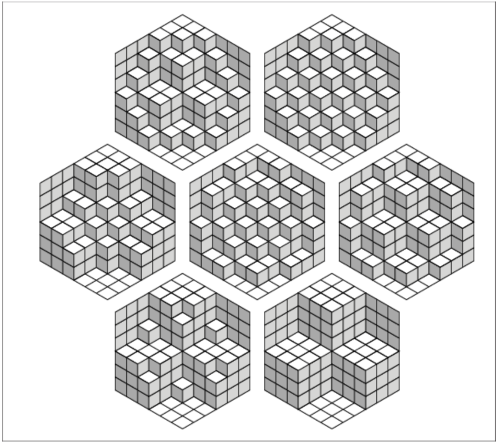
Enumeration

- The number of plane partitions for each of this symmetry classes is a nice product formula.

Theorem (Andrews, '92)

The number of TSSCPPs in a $(2n) \times (2n) \times (2n)$ box is A_n.

Example of TSSCPPs for $n = 3$



Credit: Bressoud's book

ASMs ≡ Gog triangles

Definition

A monotone triangle or gog triangle of order n is a triangular array $(g_{i,j})_{1 \le j \le i \le n}$ of positive integers such that

- $g_{i,j} \le g_{i-1,j} \le g_{i,j+1}$ whenever all the entries are defined,
- $g_{i,j} < g_{i,j+1}$ whenever both entries are defined,
- $1 \le g_{i,j} \le n+1$, for $1 \le j \le i \le n$.

$$\begin{array}{ccc} & 1 & \\ 1 & & 2 \end{array} \qquad \begin{array}{ccc} & 1 & \\ 1 & & 3 \end{array} \qquad \begin{array}{ccc} & 2 & \\ 1 & & 2 \end{array} \qquad \begin{array}{ccc} & 2 & \\ 1 & & 3 \end{array}$$

$$\begin{array}{ccc} & 2 & \\ 2 & & 3 \end{array} \qquad \begin{array}{ccc} & 3 & \\ 1 & & 3 \end{array} \qquad \begin{array}{ccc} & 3 & \\ 2 & & 3 \end{array}$$

TSSCPPs ≡ Magog Triangles

Definition

A magog triangle of order n is a triangular array

$$(m_{i,j})_{1\leq j\leq i\leq n}$$

of positive integers such that

- $m_{i,j} \leq m_{i+1,j+1}$ whenever both entries are defined,
- $m_{i,j} \geq m_{i+1,j}$ whenever both entries are defined,
- $m_{i,j} \leq j+1$ for all valid i,j.

$$\begin{array}{ccc} & 2 & \\ 2 & & 3 \end{array} \qquad \begin{array}{ccc} & 2 & \\ 2 & & 2 \end{array} \qquad \begin{array}{ccc} & 1 & \\ 1 & & 2 \end{array} \qquad \begin{array}{ccc} & 2 & \\ 1 & & 2 \end{array}$$

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array} \qquad \begin{array}{ccc} & 1 & \\ 1 & & 3 \end{array} \qquad \begin{array}{ccc} & 2 & \\ 1 & & 3 \end{array}$$

ASM theorem

- Kuperberg's later proof of the ASM conjecture used the connection to the six-vertex model.
- He used a determinantal formula for the partition function of the six-vertex model due to Izergin and Korepin.

ASM theorem

- Kuperberg's later proof of the ASM conjecture used the connection to the six-vertex model.
- He used a determinantal formula for the partition function of the six-vertex model due to Izergin and Korepin.
- Zeilberger's original proof used constant term identities to prove that ASMs and TSSCPPs are equinumerous.
- However, Zeilberger proved something stronger ...

Gog trapezoids

Definition

An (n,k)-gog trapezoid is a trapezoidal array $(g_{i,j})_{1 \le i \le n, 1 \le j \le \min(i,k)}$ of positive integers such that

- $g_{i,j} \le g_{i-1,j} \le g_{i,j+1}$ whenever all the entries are defined,
- $g_{i,j} < g_{i,j+1}$ whenever both entries are defined,
- $g_{i,j} \le n+1$ for all valid i,j.

An example of an $(6,3)$-gog trapezoid is

$$
\begin{array}{ccccccccc}
 & & & & & 6 & & & \\
 & & & & 5 & & 6 & & \\
 & & & 3 & & 5 & & 7 & \\
 & & 3 & & 4 & & 6 & & \\
 & 2 & & 3 & & 5 & & & \\
2 & & 3 & & 4 & & & &
\end{array}
$$

Magog trapezoids

Definition

An (n, k)-magog trapezoid is a trapezoidal array $(m_{i,j})_{1 \le i \le n, \max(1, i-k+1) \le j \le i}$ of positive integers such that

- $m_{i,j} \le m_{i+1,j+1}$ whenever both entries are defined,
- $m_{i,j} \ge m_{i+1,j}$ whenever both entries are defined,
- $m_{i,j} \le j + 1$ for all valid i, j.

An example of a $(6, 3)$-magog trapezoid is

$$
\begin{array}{ccccccc}
 & & 1 & & & & \\
 & 1 & & 2 & & & \\
1 & & 1 & & 2 & & \\
 & 1 & & 2 & & 4 & \\
 & & 1 & & 2 & & 5 \\
 & & & 1 & & 3 & & 7
\end{array}
$$

Zeilberger's proof

Theorem (Zeilberger, '96)

For $n \geq 1$ and $0 \leq k \leq n$, the number of (n, k)-gog trapezoids equals the number of (n, k)-magog trapezoids.

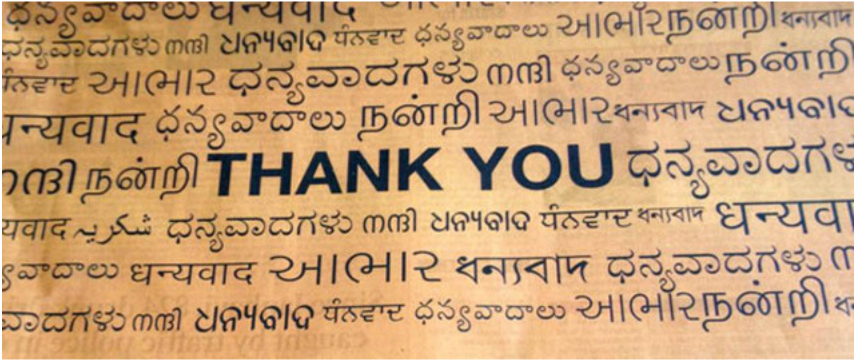
n											
1						2					
2					5		7				
3				14		35		42			
4			42		219		387		429		
5		132		1594		4862		7007		7436	
6	429		12935		76505		166296		210912		218348

Remainder of the proof

- Use Zeilberger's theorem to relate $\max(T_n)$ to an appropriate random variable in TSSCPPs.
- Define a Pfaffian point process on $[0, n] \cap \mathbb{Z}$ using a dimer model formulation of TSSCPPs.
- Perform careful asymptotic analysis of the Pfaffian kernel above and show that it converges to the GOE kernel.

References

- David Bressoud, *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*, Cambridge University Press, Cambridge, (1999).
- George Andrews, *Plane partitions. V. The TSSCPP conjecture*, J. Combin. Theory Ser. A **66** (1994), no. 1, 28–39.
- Doron Zeilberger, *Proof of the alternating sign matrix conjecture*, Electron. J. Combin. **3** (1996), no. 2, Research Paper 13, approx. 84 pp.
- Greg Kuperberg, *Another proof of the alternating-sign matrix conjecture*, Internat. Math. Res. Notices **1996** (1996), no. 3, 139–150.
- A. Ayyer and S. Chhita, *Correlations in totally symmetric self-complementary plane partitions*, `arXiv:2012.12623`, to appear in *Trans. Lond. Math. Soc.*.


THANK YOU