Stiefel-Whitney Classes of Representations

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G a group, say all its irreducible complex representations (π, V) are classified. (Typically some parametrization, e.g., partitions for S_n, weights of a maximal torus for Lie groups.)

What next?

Question: How do you take duals $(\pi^\vee, V^\vee)$?

Question: Which π are self-dual? $(\pi \cong \pi^\vee)$

Suppose (π, V) is self-dual, irreducible.

Then there is a nondegenerate G-invariant bilinear form $(\ ,\ )$ on V which is either symmetric or antisymmetric.

So $\pi : G \to \mathrm{Isom}(V, (\ ,\ ))$.

Question: Which one is it?

Spinoriality and Chirality

Suppose (π, V) is self-dual, orthogonal. May regard $\pi : G \to \mathrm{O}(V)$.

There is a nontrivial double cover $\rho : \mathrm{Pin}(V) \to \mathrm{O}(V)$.

Question: Does π lift to $\mathrm{Pin}(V)$?

Related **Question:** What is $\det \pi$?

$$G \xrightarrow{\pi} \mathrm{O}(V) \xrightarrow{\det} \{\pm 1\}$$

Cohomological Interpretation

The representation $\pi : G \to \mathrm{O}(V)$ induces a pullback extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to E_\pi \to G \to 1,$$

where

$$\begin{aligned} E_\pi &= \mathrm{Pin}(V) \times_{\mathrm{O}(V)} G \\ &= \{(p, g) \in \mathrm{Pin}(V) \times G \mid \rho(p) = \pi(g)\}. \end{aligned}$$

Associated to E_π is an "extension class" $w_2(\pi) \in H^2(G, \mathbb{Z}/2\mathbb{Z})$.

Say $\det \pi = 1$.
Then, π is spinorial iff E_π is a trivial extension iff $w_2(\pi) = 0$.

First Cohomology

We are considering $\mathbb{Z}/2\mathbb{Z}$ as a trivial G-module, so there is an isomorphism

$$w_1 : \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H^1(G, \mathbb{Z}/2\mathbb{Z})$$

When π is orthogonal, $\det \pi : G \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

Define $w_1(\pi) = w_1(\det \pi)$.

Stiefel-Whitney Classes

For every orthogonal representation of G, there are Stiefel-Whitney Classes (SWCs)

$$w_k(\pi) \in H^k(G, \mathbb{Z}/2\mathbb{Z})$$

for each nonnegative integer k, satisfying the following properties:

1. $w_k(\pi) = 0$ if $k > \deg \pi$.
2. $w_0(\pi) = 1$.
3. w_1, w_2 are as defined above.
4. $w(\pi_1 \oplus \pi_2) = w(\pi_1) \cup w(\pi_2)$.
5. If $G' \xrightarrow{f} G \xrightarrow{\pi} \mathrm{GL}(V)$, then $w(\pi \circ f) = f^*(w(\pi))$.

Here $f^* : H^k(G, \mathbb{Z}/2\mathbb{Z}) \to H^k(G', \mathbb{Z}/2\mathbb{Z})$.

Symmetric Groups I

Let $G = S_n$. Irreducible reps indexed by $\lambda \vdash n$.
Have "Specht modules" π_λ.
Two linear characters $1, \mathrm{sgn}$.

Theorem (A.Ayyer, A.Prasad, Sp. 2017)

100% of Specht modules have trivial determinant.

(Actually gave explicit formula.) More precisely,

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid \det \pi_\lambda = 1\}}{p(n)} = 1,$$

where $p(n)$ is the number of partitions of n.

Symmetric Groups II

In fact, all representations of symmetric groups are orthogonal.

Theorem (J. Ganguly, Sp. 2020)

100% of Specht modules are spinorial.

We computed:

$$w_2(\pi) = \frac{1}{2}(\deg\pi - \chi_\pi((12)(34)))w_2(\pi_n) + \left[\frac{1}{2}(\deg\pi - \chi_\pi((12)))\right] e_{\mathrm{cup}},$$

with π_n the standard representation, and

$$e_{\mathrm{cup}} = w_1(\mathrm{sgn}) \cup w_1(\mathrm{sgn}).$$

Symmetric Groups III

Character values of representations of S_n are integers.
Fix a permutatation $\sigma \in S_k$, consider it in S_n for $n \geq k$. Also fix $d \in \mathbb{N}$.

Theorem (J. Ganguly, A. Prasad, Sp. 2020)

For 100% of $\lambda \vdash n$, the character $\chi_\lambda(\sigma)$ is a multiple of d.

Note: The spinoriality statement follows from the case $k = 2$, $d = 8$, together with the formula for $w_2(\pi)$.

Theorem (S. Bhalerao, J. Ganguly, Sp., in preparation)

Fix a positive integer k. For 100% of $\lambda \vdash n$, we have $w_k(\pi_\lambda) = 0$.

(On the other hand, if $w(\pi) = 1$, then π is trivial.)

Lie Groups I

G connected reductive Lie group, e.g. $G = \mathrm{SL}_n$.

Irreducible representations of G correspond to highest weights λ, which ranges over a cone in a certain lattice. Write "π_λ".

- π_λ is self-dual iff $w_0\lambda = -\lambda$, where w_0 is the longest Weyl group element.
- π_λ is orthogonal iff $\langle \lambda, 2\delta^\vee \rangle$ is even, where $\delta^\vee = \sum_{\alpha>0} \alpha^\vee$.
- If π is orthogonal, then $\det \pi = 1$ by connectedness.

Spinoriality and a Question of Dipendra

Question: (D. Prasad) Does every two-fold cover of G arise as the pullback of an orthogonal representation?

R. Joshi and I give a spinoriality criterion for π_λ (2020). In particular, we enumerated the (simple) G with the property that every orthogonal representation is spinorial.

Some of these, e.g., $G = \mathrm{PSO}_8$ have fundamental group with even order. These give a negative answer to the Question.

Lie Groups III

J. Ganguly and R. Joshi (2021) extend the spinoriality criterion to $G = \mathrm{O}_n(\mathbb{R})$.

Recently we have been looking at Chern classes $c_k(\pi_\lambda) \in H^{2k}(BG, \mathbb{Z})$, defined for any representation.
(BG is the classifying space for G)

Theorem (R. Joshi, Sp., in preparation)

The function $\lambda \mapsto c_k(\pi_\lambda)$ is polynomial. Moreover $c_2(\pi) = 0$ iff π is trivial.

Finite General Linear Group

How about $G = \mathrm{GL}(n, q)$? Exclude some small cases:

$$(n, q) \in \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 4), (4, 2)\}$$

Let $a_1 = \mathrm{diag}(-1, 1, \ldots, 1)$, and put

$$m_\pi = \frac{1}{2}(\deg \pi - \chi_\pi(a_1)).$$

Theorem (R. Joshi, Sp., 2021)

Let π be an orthogonal representation of G.

- *If $q \equiv 1 \mod 4$, then π is spinorial iff m_π is divisible by 4.*
- *If $q \equiv 3 \mod 4$, then π is spinorial iff $m_\pi \equiv 0$ or $3 \mod 4$.*

SWCs for Finite Groups of Lie Type

N. Malik has already presented our work on SWCs for $G = \mathrm{SL}(n, q)$, when $n = 2$ or n odd.

Note that $w_1(\pi) = w_2(\pi) = 0$, since G is perfect and every double cover is trivial. First nonvanishing is $w_4(\pi)$.

J. Ganguly and R. Joshi are computing the total SWC for $G = \mathrm{GL}(n, q)$.

Goal: Compute all $w_k(\pi)$ in terms of character values of π.

For your attention:

Thank You!