Quotients of algebraic varieties by tori

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Plan

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- In case $X = \operatorname{Spec} A$ is an affine variety, A admits a grading by $\mathbb{Z}^n$.

Cox rings

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Theorem (Cox)

Let $Z \subset X = X_\Sigma$ be the subvariety defined by the ideal

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$$(\textit{Graded } S\textit{-modules}) \longrightarrow (\textit{Quasi-coherent } \mathcal{O}_X\textit{-modules}); \quad F \mapsto \widetilde{F}$$

This functor is exact and essentially surjective when X is simplicial.

Definition

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Like a toric variety, T-varieties can also be described combinatorially.

Combinatorial description of a T-variety

- (Reference: Altmann, Hausen, Süss, 2008). The description of a T-variety has two parts: (1) a non-combinatorial part involving an $n-k$-dimensional semi-projective variety Y, that is projective over an affine scheme and (2) a fan of "polyhedral divisor"s on Y.

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- There is a notion of a face of a polyhedral divisor obtained by taking faces of the constituent polyhedras which correspond to open embeddings.
- This allows one to generalize the usual notion of a fan of cones to that of a fan of polyhedral divisors.

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- $S_{(f)}$ is the degree zero part of S_f and let $D_+(f) = \operatorname{Spec} S_{(f)} \subset \operatorname{Quot}(S)$,
- Define $\operatorname{Proj} S = \bigcup_{f \in S \text{ relevant}} \operatorname{Spec} S_{(f)} \subset \operatorname{Quot}(S)$, which by construction is a scheme.

Coherent sheaves and line bundles

Theorem (Mallick,)

Let D be a finitely generated free abelian group, $S = \bigoplus_{d \in D} S_d$ be a D-graded ring and $\operatorname{Proj} S$ be the corresponding multihomogeneous space. One can define shifted modules $\mathcal{O}_X(d)$ as usual. Further we have $\Gamma(X, \mathcal{O}_X(d)) \cong A_d$ and $\mathcal{O}_X(d)$ is a reflexive sheaf.
Moreover, if d is such that $d \in D_f$ for every relevant element $f \in A$, then $\mathcal{O}_X(d)$ is a line bundle.

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Remark

This recovers a similar theorem on weighted projective spaces.

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There is a birational morphism $\varphi\colon Y \longrightarrow \operatorname{Proj} A$. Furthermore, $\mathcal{D}$ can be described in terms of the shifted modules over $\operatorname{Proj} A$.

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Remark

These in turn are birational to the Perling's construction of $\operatorname{tProj}(A)$.

A condition of isomorphism

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Example: products of weighted projective spaces.

Thank you!

References

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