Limits of an increasing sequence of complex manifolds

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Union problem

Suppose a complex manifold M is exhausted by biholomorphic images of a domain $\Omega \subset \mathbb{C}^n$:

$$M_1 \subset M_2 \subset \cdots \subset \cup_{j=1}^{\infty} M_j = M, \quad M_j \cong \Omega.$$

Then it is of interest to describe M in terms of Ω. This problem will be referred to as the *union problem*.

Background

Behnke-Stein [1938]: An important step towards the solution of the Levi problem: If $\Omega \subset \mathbb{C}^n$ is the union of an increasing sequence of domains of holomorphy

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \cup_{j=1}^{\infty} \Omega_j = \Omega,$$

then Ω is a domain of holomorphy.

Fornaess [1976, 1977]: The Bhenke-Stein result is not true for general Stein manifolds: there exists a complex manifold M such that M is the union of an increasing sequence of Stein manifolds

$$M_1 \subset M_2 \subset \cdots \subset \cup_{j=1}^{\infty} M_j = M,$$

but M is not Stein.

Next question: Suppose each M_j satisfies some extra condition. How does that affect the limit M? This is how the union problem arises.

Background

Fornaess and Stout [1977]: Suppose in the union problem $\Omega = \Delta^n$ or $\mathbb{B}^n$ and M is an open subset of a possibly larger taut manifold. Then $M \cong \Omega$.

Taut manifold: A complex manifold M is *taut* if every sequence of holomorphic maps $f_j : \Delta \to M$ has a subsequence that either converges uniformly on compact subsets to a holomorphic map $f : \Delta \to M$ or diverges uniformly on compact subsets.

Kobayashi pseudometric: Let M be a complex manifold. The *Kobayashi pseudometric* on M is the function $F_M : TM \to [0, \infty)$ defined by

$$F_M(p; v) = \inf \left\{ 1/R : R > 0, f : \Delta \to M, f(0) = p, f'(0) = Rv \right\}.$$

The *Kobayashi pseudodistance* on M is the integrated form of F_M. If it is a true distance, then M is called *Kobayashi hyperbolic*. If the Kobayashi distance is complete, M is called *complete hyperbolic*.

Background

Fornaess and Sibony [1981]: Assume that $\Omega/\operatorname{Aut}\Omega$ is compact.

- *Hyperbolic case*—M is Kobayashi hyperbolic. Then $M \cong \Omega$.
- *Non-hyperbolic case*—M is not Kobayshi hyperbolic. Then the dimension of

  $$\{v : F_M(p;v) = 0\} \subset T_pM$$

  is independent of $p \in M$, and is called the Kobayashi corank of M.

  If M has corank one, then there exists a holomorphic retract $Z \subset \Omega$ such that M is biholomorphic to a locally trivial holomorphic fibre bundle over Z with fibre (biholomorphic to) $\mathbb{C}$.

Behrens [1985]: Assume that Ω is strongly pseudoconvex.

- *Hyperbolic case*. M is biholomorphic to Ω or $\mathbb{B}^n$.
- *Non-hyperbolic case*. Same as Fornaess-Sibony.

Union problem—Hyperbolic case

Theorem

Assume that in the union problem, M is hyperbolic.

(i) *If $\Omega \subset \mathbb{C}^n$ is a bounded Levi corank one domain, then M is biholomorphic either to Ω or to a domain of the form*

$$\Omega_\infty = \left\{ z \in \mathbb{C}^n : 2\,\mathrm{Re}\, z_n + P_{2m}(z_1, \overline{z}_1) + \sum_{j=2}^{n-1} |z_j|^2 < 0 \right\},$$

where $m \geq 1$ is a positive integer and $P_{2m}(z_1, \overline{z}_1)$ is a subharmonic polynomial of degree at most $2m$ without any harmonic terms.

(ii) *If $\Omega \subset \mathbb{C}^n$ is a smoothly bounded convex domain then M is biholomorphic either to Ω or to a limiting domain Ω_∞ arising from scaling.*

(iii) *If $n = 2$ and $\Omega \subset \mathbb{C}^2$ is a strongly pseudoconvex polyhedral domain, then M is biholomorphic either to Ω or to Ω_∞, where Ω_∞ is a limiting domain associated to Ω.*

Union problem—Hyperbolic case

Theorem

(iv) *Let* $\Omega = \left\{ z \in \mathbb{C}^n : \frac{1}{2}\left(\sum_{j=1}^n |z_j|^2 + \left| \sum_{j=1}^n z_j^2 \right| \right) < 1 \right\}$ *be the minimal ball or more generally any bounded convex domain whose boundary (is not necessarily smooth but) does not contain nontrivial complex analytic varieties. If* M *is a priori known to be complete hyperbolic, then* $M \cong \Omega$*.*

(v) *If* $n = 2$ *and* $\Omega \subset \mathbb{C}^2$ *is a simply connected domain with generic piecewise* C^∞*-smooth Levi-flat boundary, then* M *is biholomorphic either to* Ω *or to the unit bidisc* $\Delta^2 \subset \mathbb{C}^2$*.*

(vi) *If* M *is a bounded domain in* $\mathbb{C}^n$ *and* Ω *is the symmetrized polydisc, then either* M *is biholomorphic to* Ω *or* M *admits a proper holomorphic correspondence to* Δ^n*, the unit polydisc in* $\mathbb{C}^n$*, with each fibre having cardinality at most* $n!$*.*

Proof

We have

$$M_1 \subset M_2 \subset \cdots \subset \cup_{j=1}^{\infty} M_j = M, \quad \psi_j : M_j \xrightarrow{\cong} \Omega.$$

Fixing $z^0 \in M$, we may assume that $z^0 \in M_j$ for all j, and consider the *orbit* $\{p^j := \psi^j(z^0)\}$.

Case I—$\{p^j\}$ is a relatively compact subset of Ω. Since Ω is taut, $\{\psi^j\}$ has a limit $\psi : M \to \Omega$ and as M is hyperbolic, it follows that ψ is a biholomorphism.

Case II—$\{p^j\}$ has at least one limit point $p^0 \in \partial\Omega$. Suppose Ω is as in (i), (ii), (iii), or (iv).
Step I: Scaling method gives biholomorphisms $A^j : \Omega \to \Omega_j$ such that

$$\Omega_j \to \Omega_\infty, \quad A^j(p^j) =: q_j \to q^0 \in \Omega_\infty.$$

Consider $M_j \xrightarrow{\psi^j} \Omega \xrightarrow{A^j} \Omega_j$, call it $\tilde{\psi}^j$, and note $\tilde{\psi}^j(z^0) \to q_0 \in \Omega_\infty$.

Proof

Since Ω_∞ is taut, $\tilde{\psi}^j$ has a limit, say $\tilde{\psi} : M \to \Omega_\infty$.

Step II: $\tilde{\psi} : M \to \Omega_\infty$ is a biholomorphism.

- A candidate for the inverse of $\tilde{\psi}$ would be a limit of ${\tilde{\psi}^j}^{-1} : \Omega_j \to M_j$.
- $\tilde{\psi}$ is injective: Follows from the stability property

$$\limsup_{j\to\infty} d_{\Omega_j}\big(\tilde{\psi}(z^1), \tilde{\psi}(z^2)\big) \le d_{\Omega_\infty}\big(\tilde{\psi}(z^1), \tilde{\psi}(z^2)\big), \quad z^1, z^2 \in M.$$

- Therefore, $M \cong \tilde{\psi}(M) \subset \Omega_\infty$. Since Ω_∞ is taut, one can show that ${\tilde{\psi}^j}^{-1} : \Omega_j \to M$ has a limit $\tilde{\phi} : \Omega_\infty \to M$ and is the inverse of $\tilde{\psi}$.

Proof

Now suppose Ω is as in (v), i.e., $\partial\Omega$ is generic piecewise smooth Levi flat. Following work of Fu and Wong, one obtains that the c/k-invariant of M with respect to Δ^2 is 1 at z_0:

$$\frac{c_{M_j}(z^0)}{k_M(z^0)} \geq 1 \text{ for all } j \;\Rightarrow \frac{c_M(z_0)}{k_M(z_0)} \geq 1 \Rightarrow \frac{c_M(z_0)}{k_M(z_0)} = 1,$$

and hence $M \cong \Delta^2$.

Finally, suppose Ω is as in (vi), i.e., the symmetrized polydisc. Let $\pi : \Delta^n \to \Omega$ be the symmetrization map and pick $\lambda^j \in \pi^{-1}(p^j)$. Consider

$$M_j \xrightarrow{\psi_j} \Omega \xrightarrow{\pi^{-1}} \Delta^n \xrightarrow{H_j} \Delta^n$$

By a result of Klingenberg and Pinchuk, this sequence of proper correspondances is normal and hence after passing to a subsequence converges to a proper correspondance from M to Δ^n.

Union problem—Non-hyperbolic case

Theorem

Assume that in the union problem, M is a non-hyperbolic manifold. Then under any of the hypothesis as in Theorem 1 (i)-(iii),

Then the dimension of $\{v : F_M(p; v) = 0\} \subset T_pM$ is independent of $p \in M$. If M has corank one, then M is biholomorphic to a locally trivial holomorphic fibre bundle over a retract Z of Ω or that of a limiting domain Ω_∞ with fibre (biholomorphic to) $\mathbb{C}$.

Proof

Recall $\psi^j : M_j \xrightarrow{\cong} \Omega$, and $p^j = \psi^j(z^0)$, and we have two cases to consider

Case I—$\{p^j\}$ is a relatively compact subset of Ω.
By arguments as in Fornaess–Sibony, M is biholomorphic to a fibre bundle over a retract of Ω with fibre $\mathbb{C}$.

Case II—$\{p^j\}$ has at least one limit point $p^0 \in \partial\Omega$.

- $M_j \xrightarrow{\psi^j} \Omega \xrightarrow{A^j} \Omega_j$, denoted by $\tilde{\psi}^j$, has a limit $\tilde{\psi} : M \to \Omega_\infty$.
- $\Omega_j \xrightarrow{{\tilde{\psi}^j}^{-1}} M_j \xrightarrow{\tilde{\psi}} \Omega_\infty$, denoted by $\tilde{\alpha}^j$, has a limit $\tilde{\alpha} : \Omega_\infty \to \Omega_\infty$. Also,

$$\tilde{\alpha} \circ \tilde{\psi}(z) = \lim_{j\to\infty} \tilde{\alpha}^j \circ \tilde{\psi}^j(z) = \lim_{j\to\infty} \tilde{\psi} \circ \tilde{\phi}^j \circ \tilde{\psi}^j(z) = \tilde{\psi}(z),$$

  for all $z \in M$. Define $\tilde{Z} = \{w \in \Omega_\infty : \tilde{\alpha}(w) = w\} \supset \tilde{\psi}(M)$

Proof

- One can check that $\tilde{Z} = \tilde{\psi}(M)$, $\tilde{Z}$ is a closed connected submanifold of Ω_∞, the mapping $\tilde{\alpha}$ is a holomorphic retractions from Ω_∞ onto $\tilde{Z}$, and the mapping $\tilde{\psi}$ has constant rank.
- Also, as $\tilde{\psi}^j$ are biholomorphisms and hence Kobayashi isometries,

$$F_{M_j}(p,v) = F_{\Omega_j}\big(\tilde{\psi}^j(p), d\tilde{\psi}^j(p)v\big) \Rightarrow F_M(p,v) = F_{\Omega_\infty}\big(\tilde{\psi}(p), d\tilde{\psi}(p)v\big)$$

  and as Ω_∞ is hyperbolic,

$$F_M(p,v) = 0 \text{ iff } d\tilde{\psi}(p)v = 0.$$

  Therefore, $\dim\{v \in T_pM : F_M(p,v) = 0\} =$ nullity $d\tilde{\psi}(p)$ is constant.
- If corank is 1, arguments of Fornaess-Sibony yields that M is biholomorphic to a fibre bundle over $\tilde{Z}$ with fibre $\mathbb{C}$.

Consequences

Corollary

A C^2-smooth strongly pseudoconvex domain $D \subset \mathbb{C}^n$ cannot be exhausted by the symmetrized polydisc.

Corollary

Assume that in the union problem, $n = 2$ and $\Omega \subset \mathbb{C}^2$ is a C^∞-smoothly bounded strongly convex domain. If M is hyperbolic then $M \cong \Omega$ or $\mathbb{B}^2$. If M is non-hyperbolic and the corank of F_M is one, then $M \cong \Delta \times \mathbb{C}$.

Proof. In the non-hyperbolic case, the retract Z is one-dimensional and so by Lempert's work,

$$Z = f(\Delta) \text{ where } f : \Delta \to \Omega \text{ is a Kobayashi extremal.}$$

Thus f is a complex geodesic and hence an embedding. Thus $Z \cong \Delta$ and hence Z admits solutions to both the Cousin problems. It follows that $M \cong Z \times \mathbb{C} \cong \Delta \times \mathbb{C}$.

Retracts of $\Delta \times \mathbb{B}^{n-1}$

Theorem

Let Z be a holomorphic retract of $\Delta \times \mathbb{B}^n$. Assume that Z contains the origin, $Z \neq \{0\}$ and $Z \neq \Delta \times \mathbb{B}^n$. Then Z is given as one of the following:

(i) *Z is the graph of a $\mathbb{B}^n$-valued holomorphic mapping of Δ,*

(ii) *Z is the graph of a Δ-valued holomorphic function over a complex linear subspace of $\mathbb{B}^n$,*

(iii) *Z is the intersection of a linear subspace with $\Delta \times \mathbb{B}^n$ of complex dimension at least two.*

Proof

The idea is to examine how T_0Z intersects $\partial(\Delta \times \mathbb{B}^n)$. For example, suppose $T_0Z \cap \partial(\Delta \times \mathbb{B}^n) \subset \partial\Delta \times \partial\mathbb{B}^n$.

- Write $L = T_0Z \cap (\Delta \times \mathbb{B}^n)$.
- Since $\partial L \subset \partial\Delta \times \partial\mathbb{B}^n$, the projection $\pi_1 : L \to \Delta$ is proper.
- Since proper holomorphic maps do not decrease dimension, $\dim L = 1$ and so $\dim Z = 1$.
- Since π_1 is linear, the fibres of π_1 are singletons and so $\pi_1 : L \to \Delta$ is a biholomorphism. Thus

  $$L = \{(w, \beta_1(w), \dots, \beta_n(w)) : w \in \Delta\}$$

  where β_i's are linear.
- Show that $L = Z$.

Thank you!