Towards a mod p Lubin-Tate theory for GL_2 over totally real fields

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What the talk is about ?

The aim of the talk is to show that the mod p local Langlands correspondence for a totally real field can be realised in the mod p cohomology of the Lubin-Tate towers. The motivation for such a result came from the following theorem of Chojecki.

Chojecki (2015)

Let $\rho : G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \bar{\mathbb{F}}_p$ be a continuous, irreducible and odd Galois representation. Assume that $\rho_p := \rho|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is irreducible, then we have a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant injection

$$\rho \otimes_{\bar{\mathbb{F}}_p} \pi \hookrightarrow \hat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p},$$

where π is the supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to ρ_p under the mod p local Langlands correspondence.

We generalise the above result to the case of totally real fields as follows-

We start by defining notations.

- F be a totally real field of degree $d > 1$
- fix a rational prime p of $\mathbb{Q}$ and a prime $\mathfrak{p}$ of F lying above p
- $G_F := \mathrm{Gal}(\bar{\mathbb{Q}}/F)$
- for any finite place v of F, F_v denotes the completion of F at v
- D be a quaternion algebra over F which splits at exactly one real place (say τ) and non-split at other real places
- further assume that D splits at $\mathfrak{p}$, i.e, $D \otimes_F F_{\mathfrak{p}} = \mathrm{M}_2(F_{\mathfrak{p}})$

Main result

Let $\rho : G_F \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be a continuous, irreducible and totally odd representation such that $\rho_{\mathfrak{p}} := \rho|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/F_{\mathfrak{p}})}$ is irreducible, then assuming a conjecture by Buzzard-Diamond-Jarvis we have a $G_F \times \mathrm{GL}_2(F_{\mathfrak{p}}) \times \prod_{w\in\Sigma\setminus\Sigma'}(D \otimes_F F_w)^\times$-equivariant injection

$$\rho \otimes \pi_{\mathfrak{p}} \otimes (\otimes_{w\in\Sigma\setminus\Sigma'}\pi_w) \hookrightarrow H^1_{\bar{\mathbb{F}}_p,ss}[\mathfrak{m}_\rho]^K,$$

where $\pi_{\mathfrak{p}}$ corresponds to $\rho_{\mathfrak{p}}$ under the conjectural mod p local Langlands correspondence for $\mathrm{GL}_2(F_{\mathfrak{p}})$ (this means $\pi_{\mathfrak{p}}$ is supersingular) and K is some specific compact open subgroup.

Furthermore, assuming a Serre type conjecture we have a $G_F \times \mathrm{GL}_2(F_{\mathfrak{p}}) \times \prod_{w\in\Sigma\setminus\Sigma'}(D \otimes_F F_w)^\times$-equivariant injection

$$\rho \otimes \pi_{\mathfrak{p}} \otimes (\otimes_{w\in\Sigma\setminus\Sigma'}\pi_w) \hookrightarrow \hat{H}^1_{\mathrm{LT},\bar{\mathbb{F}}_p}.$$

- The first step is to show that the mod p local Langlands correspondence can be realised in the mod p cohomology of the supersingular locus of the Shimura curve (this is where we did most of the work).
- The second step is to show that the mod p cohomology of the supersingular cohomology of Shimura curves injects into the mod p cohomology of the Lubin-Tate towers. (This is sort of automatic by definitions and most of the work was already done by Carayol)

In fact we would be spending almost the whole talk to prove the first step and there would be no mention of Lubin-Tate towers at all.
Most of the time the proof of the above theorem follows the strategy of Chojecki except there is one crucial difference in the proof of first step which we will mention at the end of the talk (if time permits).

Shimura curves (Following Carayol)

We define few more notations.

- $F_{\mathfrak{p}}$ be the completion of F at $\mathfrak{p}$, $\mathcal{O}_{\mathfrak{p}}$ be its ring of integers, k be the reside field of $\mathcal{O}_{\mathfrak{p}}$ (finite extension of $\mathbb{F}_p$)
- $\mathbb{A}_f$ be the finite adeles of F
- let $G = \mathrm{Res}_{F/\mathbb{Q}}(D^{\times})$

Indexed by any compact open subgroup K of $G(\mathbb{A}_f)$ we have a projective system of compact Riemann surfaces given by

$$M_K(G)(\mathbb{C}) := G(\mathbb{Q}) \setminus G(\mathbb{A}_f) \times (\mathbb{C} - \mathbb{R})/K.$$

Shimura defined a canonical model $M_K(G)$ of $M_K(G)(\mathbb{C})$ over F. In fact these $M_K(G)$ constitutes projective system of complete smooth algebraic curves over F.

Let Γ be the restricted direct product of $(D \otimes_F F_v)^{\times}$ at all the finite places $v \neq \mathfrak{p}$. Let $K = K_{\mathfrak{p}^n} \times H$, for H a compact open subgroup of Γ and let $M_{n,H} := M_{K_{\mathfrak{p}^n} \times H}$, where $K_{\mathfrak{p}^n} = \{g \in \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) : g \equiv I_2 \bmod \mathfrak{p}^n\}$.

We are interested in the mod p reduction of Shimura curves so it is necessary to work with its integral model. Carayol showed that for sufficiently small H there exists an integral model of $M_{n,H}$ over $\mathcal{O}_{\mathfrak{p}}$ which we again denote by $M_{n,H}$. Note that the curve $M_{n,H}$ does not have a moduli description as in the case of modular curve, nevertheless one can define ordinary points and supersingular points for the mod p reduction of the curve $M_{n,H}$.

Ordinary and Supersingular points of $M_{n,H}$

Let $\bar{M}_{n,H}$ be the mod p reduction of $M_{n,H}$ and x be a geometric point of $\bar{M}_{n,H}$. Then Carayol has attached to x an $\mathcal{O}_{\mathfrak{p}}$-divisible module E_x. By the classification given by Drinfeld for the height h divisible $\mathcal{O}_{\mathfrak{p}}$-module over $\bar{k}$, we have following two possibilities for E_x:

1. $E_x = \Sigma_1 \times (F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}})$, we call x to be ordinary in this case,
2. $E_x = \Sigma_2$, we call x to be supersingular in this case,

where Σ_h is the unique (upto isomorphism) formal $\mathcal{O}_{\mathfrak{p}}$-module of height h. Carayol also showed that the $\bar{k}$-scheme $\bar{M}_{n,H}$ is a proper connected curve which is smooth outside the set of supersingular points. Let $a \in \mathrm{P}^1(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^{\mathrm{n}}\mathcal{O}_{\mathfrak{p}})$, let $\bar{M}^{\mathrm{ord}}_{n,H}$ to be the set of all ordinary points of $M_{n,H}$ and $\bar{M}^{\mathrm{ord}}_{n,H,a} := \bar{M}_{n,H,a} \cap \bar{M}^{\mathrm{ord}}_{n,H}$. Hence $\bar{M}^{\mathrm{ord}}_{n,H}$ is the disjoint union of $\bar{M}^{\mathrm{ord}}_{n,H,a}$, i.e.,

$$\bar{M}^{\mathrm{ord}}_{n,H} = \sqcup_{a \in \mathrm{P}^1(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n\mathcal{O})} \bar{M}^{\mathrm{ord}}_{n,H,a}.$$

The curves $\bar{M}_{n,H,a}$ is permuted by the action of $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n)$ and hence it permutes $\bar{M}^{\mathrm{ord}}_{n,H,a}$.

Let $M_{n,H}^{an}$ denote the analytification of $M_{n,H}$ which is a Berkovich space. Let $\pi : M_{n,H}^{an} \to \bar{M}_{n,H}$ be the reduction map. We define the ordinary and supersingular part of $M_{n,H}^{an}$ as

$$M_{n,H}^{\mathrm{ord}} = \pi^{-1}(\bar{M}_{n,H}^{\mathrm{ord}}), \qquad M_{n,H}^{\mathrm{ss}} = \pi^{-1}(\bar{M}_{n,H}^{\mathrm{ss}}),$$

where $\bar{M}_{n,H}^{\mathrm{ss}}$ is the complement of $\bar{M}_{n,H}^{\mathrm{ord}}$ in $\bar{M}_{n,H}^{an}$. Hence $M_{n,H}^{an} = M_{n,H}^{\mathrm{ord}} \sqcup M_{n,H}^{\mathrm{ss}}$. The action of $g \in \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n\mathcal{O})$ on $\bar{M}_{n,H}^{\mathrm{ord}}$ is such that it permutes the curves $\bar{M}_{n,H,a}^{\mathrm{ord}}$ for $a \in \mathbb{P}^1(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n\mathcal{O})$. Let $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{P}^1(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n\mathcal{O})$ and B_n denote the Borel subgroup of upper triangular matrices in $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n\mathcal{O})$ which stabilizes $\bar{M}_{n,H,b}^{\mathrm{ord}}$. For $a \in \mathbb{P}^1(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n)$, define $M_{n,H,a}^{\mathrm{ord}} = \pi^{-1}(\bar{M}_{n,H,a}^{\mathrm{ord}})$. Hence $H^1_{M_{n,H}^{\mathrm{ord}}}(M_{n,H}, \bar{\mathbb{F}}_p)$ is a representation of B_n. And hence we can write

$$H^1_{M_{n,H}^{\mathrm{ord}}}(M_{n,H}, \bar{\mathbb{F}}_p) \simeq \mathrm{Ind}_{B_n}^{\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n)} H^1_{M_{n,H}^{\mathrm{ord}}}(M_{n,H}, \bar{\mathbb{F}}_p).$$

Completed Cohomology of Shimura Curves

The notion of completed cohomology was introduced by Emerton. Completed cohomology of Shimura curves is interesting because conjecturally it realises mod p local Langlands correspondence (this would be clear when we state Buzzard-Diamond-Jarvis conjecture). Emerton has shown that the completed cohomology of modular curves realizes mod p local Langlands correspondence.

Let $M_{K_{\mathfrak{p}} \times K^{\mathfrak{p}}}$ be the Shimura curve $M_K(G)$ for the level $K = K_{\mathfrak{p}} \times K^{\mathfrak{p}}$.
From now on we assume that there is only one prime $\mathfrak{p}$ of F lying above p.
The completed cohomology for the Shimura curve M_K is defined as

$$H^m(M(K^{\mathfrak{p}}), \bar{\mathbb{F}}_p) := \varinjlim_{K_{\mathfrak{p}}} H^m(M^{an}_{K_{\mathfrak{p}} \times K^{\mathfrak{p}}}, \bar{\mathbb{F}}_p),$$

it has a commuting smooth action of $G(\mathbb{Q}_p) = \mathrm{GL}_2(F_{\mathfrak{p}})$ and a continuous action of G_F.

Buzzard-Diamond-Jarvis conjecture

Let U be an open compact subgroup of $D_f^\times := (D \otimes \mathbb{A}_f)^\times$. Let $\rho : G_F \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be an irreducible continuous totally odd representation and let Σ be the finite set of finite places w of F such that w divides p or D is ramified at w or ρ is ramified at w or $\mathrm{GL}_2(\mathcal{O}_{F,w}) \nsubseteq U$. For finite places v of F at which D splits and $\mathrm{GL}_2(\mathcal{O}_{F,v}) \subset U$ define the Hecke operators $T_v = \left[\mathrm{GL}_2(\mathcal{O}_{F,v}) \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F,v})\right]$ and $S_v = \left[\mathrm{GL}_2(\mathcal{O}_{F,v}) \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F,v})\right]$ which are elements of $\mathrm{End}_{\bar{\mathbb{F}}_p}(H^1(U, \bar{\mathbb{F}}_p))$ and where ϖ denotes a uniformizer of $\mathcal{O}_{F,v}$.

Let $\mathbf{T}^{\Sigma}(U)$ denote the commutative $\bar{\mathbb{F}}_p$-subalgebra of $\mathrm{End}_{\bar{\mathbb{F}}_p}(S^D(U))$ generated by T_v and S_v for all $v \notin \Sigma$. Let $\mathfrak{m}_\rho$ be the maximal ideal of $\mathbf{T}^{\Sigma}(U)$ corresponding to ρ, i.e., generated by the operators

$$T_v - S_v \mathrm{tr}(\rho(\mathrm{Frob}_v)) \quad \text{and} \quad \mathbf{N}(v) - S_v \det(\rho(\mathrm{Frob}_v))$$

for all $v \notin \Sigma$. Also, by taking the limit over all compact open set U of $D^\times$, $H^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}_\rho] = \varinjlim_U H^1(U, \bar{\mathbb{F}}_p)[\mathfrak{m}_\rho]$ becomes a representation of $D_f^\times$.

BDJ

Let F be a totally real field and let

$$\rho : G_F := \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$$

be a continuous, irreducible and totally odd representation. Then the $\bar{\mathbb{F}}_p$ representation $H^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}_\rho]$ of $G_F \times D_f^\times$ is isomorphic to a restricted tensor product

$$H^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}_\rho] \cong \rho \otimes \ (\otimes'_w \pi_w)$$

where π_w is a smooth admissible representation of $D_w^\times$ such that

- if w does not divide p and D splits at w, then π_w is the representation attached to $\rho_w := \rho|_{G_{F_w}}$ by the modified modulo ℓ-local Langlands correspondence, otherwise its one or two dimensional representation of $D_w^\times$
- if w divides p, then $\pi_w \neq 0$; moreover if both F and D are unramified at w and σ is any irreducible representation of $\mathrm{GL}_2(\mathcal{O}_{F_w})$, then $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_w})}(\sigma, \pi_w) \neq 0$ if and only if $\sigma \in W(\rho_w)$, where $W(\rho_w)$ is a certain set of Serre weights associated to ρ_w.

Buzzard-Diamond-Jarvis Conjecture is a theorem in the case of modular curves (proved by Emerton) but in case of $F \neq \mathbb{Q}$, it is still a conjecture mainly because mod p local Langlands correspondence has not been established for $\mathrm{GL}_2(L)$ where $L \neq \mathbb{Q}_p$ is a finite extension of $\mathbb{Q}_p$. In the above conjecture for finite places $w \nmid p$ if D splits at w, then π_w (which is a mod p representation of $\mathrm{GL}_2(F_w)$) is given by the modified mod l local Langlands correspondence as given by Emerton-Helm.
We recall the modified mod p local Langlands correspondence for $\mathrm{GL}_2(E)$ given by Emerton and Helm where E is a finite extension of $\mathbb{Q}_l$. Let p and l be distinct primes.

Modified mod p local Langlands correspondence

$\rho : G_E \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be a continuous Galois representation.

- If ρ^{ss} is not a twist of $1 \oplus \omega$, then $\pi(\rho)$ is an irreducible representation of $\mathrm{GL}_2(E)$ (uniquely determined).
- If ρ^{ss} is a twist of $1 \oplus \omega$, then we can assume that $\rho = 1 \oplus \omega$ (as LLC is compatible with twists). In the Banal case ($q \not\equiv \pm 1 \mod \mathrm{p}$), we get
  1. If ρ is non-split, then $\pi(\rho) = \mathrm{St} \otimes (|\;^{-}\;| \circ \mathrm{det})$.
  2. If ρ is split, then $\pi(\rho)$ is given by the unique non-split exact sequence

$$0 \to \mathrm{St} \otimes (|\;^{-}\;| \circ \mathrm{det}) \to \pi(\rho) \to (|\;^{-}\;| \circ \mathrm{det}) \to 0.$$

- Let $\rho^{\mathrm{ss}} = 1 \oplus \omega$ and $q \equiv -1 \bmod p$. We have following three possibilities.
    1. if ρ is split, then $\pi(\rho) = \mathrm{env}(\pi_1)$.
    2. If ρ is the non-split extension of ω by 1, then $\pi(\rho)$ is given by the unique non-split extension

$$0 \to \pi_1 \to \pi(\rho) \to |\,\bar{}\,| \circ \det \to 0.$$

    3. If ρ is the non-split extension of 1 by ω, then $\pi(\rho)$ is given by the unique non split extension

$$0 \to \pi_1 \to \pi(\rho) \to 1 \to 0.$$

- Let $\rho^{\mathrm{ss}} = 1 \oplus 1$ and $q \equiv 1 \bmod p$ (hence $\omega = 1$), we have following two possibilities of $\pi(\rho)$.
    1. If ρ is split, then $\pi(\rho)$ is isomorphic to the universal extension of 1 by St (and thus has length three).
    2. If ρ is non-split, then $\pi(\rho)$ corresponds to the non-split extension of 1 by St.

Important result

Proposition

Let $\pi = \pi(\rho)$ be the mod p admissible representation of $\mathrm{GL}_2(E)$ associated by the modified mod p local Langlands correspondence given by Emerton-Helm to a continuous Galois representation $\rho : G_E \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$. Then there exists an open compact subgroup K of $\mathrm{GL}_2(\mathcal{O}_E) \subset \mathrm{GL}_2(E)$ such that $\dim_{\bar{\mathbb{F}}_p} \pi^K = 1$.

For the proof of above proposition we did the case by case analysis of the modified mod l local Langlands correspondence as above and explicitly figured out K.

Proof of first step

$$\cdots \to H^1_{M^{\mathrm{ord}}_{n,H}}(M^{an}_{n,H}, \bar{\mathbb{F}}_p) \to H^1(M^{an}_{n,H}, \bar{\mathbb{F}}_p) \to H^1(M^{\mathrm{ss}}_{n,H}, \bar{\mathbb{F}}_p) \to H^2_{M^{\mathrm{ord}}_{n,H}}(M^{an}_{n,H}, \bar{\mathbb{F}}_p) \to \cdots$$

Taking direct limit over all compact open subgroup $K_{\mathfrak{p}} \subset \mathrm{GL}_2(F_{\mathfrak{p}})$ in the above exact sequence we get the following exact sequence

$$\cdots \xrightarrow{f_0} H^1_{M(K^{\mathfrak{p}})^{\mathrm{ord}}}(M(K^{\mathfrak{p}})^{an}, \bar{\mathbb{F}}_p) \xrightarrow{f_1} H^1(M(K^{\mathfrak{p}})^{an}, \bar{\mathbb{F}}_p) \xrightarrow{f_2} H^1(M(K^{\mathfrak{p}})^{\mathrm{ss}}, \bar{\mathbb{F}}_p) \xrightarrow{f_3} H^2_{M(K^{\mathfrak{p}})^{\mathrm{ord}}}(M(K^{\mathfrak{p}})^{an}, \mathbb{F}_p) \to \cdots$$

From this we get the following exact sequence

$$0 \to \mathrm{Im}\ f_1 \to H^1(M(K^{\mathfrak{p}})^{an}, \bar{\mathbb{F}}_p) \to \mathrm{Im}\ f_2 \to 0,$$

where $\mathrm{Im}\ f_1 \simeq H^1_{M(K^{\mathfrak{p}})^{\mathrm{ord}}}(M(K^{\mathfrak{p}})^{an}, \bar{\mathbb{F}}_p)/\ker(f_1)$.

Let $G = \mathrm{GL}_2(F_\mathfrak{p})$, applying $\mathrm{Hom}_G(\pi_\mathfrak{p}, \_)$ to the above sequence where $\pi_\mathfrak{p}$ corresponds to $\rho_\mathfrak{p}$ under the conjectural mod p LLC for $\mathrm{GL}_2(F_\mathfrak{p})$, we get

$$0 \to \mathrm{Hom}_G(\pi_\mathfrak{p}, H^1_{M(K^\mathfrak{p})^{\mathrm{ord}}}(M(K^\mathfrak{p})^{an}, \bar{\mathbb{F}}_p)/\ker(f_1)) \to \mathrm{Hom}_G(\pi_\mathfrak{p}, H^1(M(K^\mathfrak{p})^{an}, \bar{\mathbb{F}}_p)) \to \mathrm{Hom}_G(\pi_\mathfrak{p}, \mathrm{Im}\ f_2) \cdots$$

Since $\pi_\mathfrak{p}$ is a supersingular representation it cannot occur as a sub-quotient of $H^1_{M(K^\mathfrak{p})^{\mathrm{ord}}}(M(K^\mathfrak{p})^{an}, \bar{\mathbb{F}}_p)$. We get the following injection

$$\mathrm{Hom}_G(\pi_\mathfrak{p}, H^1(M(K^\mathfrak{p})^{an}, \bar{\mathbb{F}}_p)) \hookrightarrow \mathrm{Hom}_G(\pi_\mathfrak{p}, \mathrm{Im}\ f_2) \hookrightarrow \mathrm{Hom}_G(\pi_\mathfrak{p}, H^1(M(K^\mathfrak{p})^{\mathrm{ss}}, \bar{\mathbb{F}}_p)),$$

taking direct limit over all compact open subgroup $K^\mathfrak{p}$ we get

$$\mathrm{Hom}_G(\pi_\mathfrak{p}, H^1_{\bar{\mathbb{F}}_p}) \hookrightarrow \varinjlim_{K^\mathfrak{p}} \mathrm{Hom}_G(\pi_\mathfrak{p}, \mathrm{Im}\ f_2) \hookrightarrow \mathrm{Hom}_G(\pi_\mathfrak{p}, H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}).$$

Taking $[\mathfrak{m}_\rho]$-torsion we get

$$\mathrm{Hom}_G(\pi_\mathfrak{p}, H^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}_\rho]) \hookrightarrow \mathrm{Hom}_G(\pi_\mathfrak{p}, H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}[\mathfrak{m}_\rho]).$$

Tensoring both sides by $\pi_\mathfrak{p}$ over $\bar{\mathbb{F}}_p$ and using BDJ conj. we have

$$\pi_\mathfrak{p} \otimes_{\bar{\mathbb{F}}_p} \mathrm{Hom}_G(\pi_\mathfrak{p}, \rho \otimes (\otimes'_w \pi_w)) \hookrightarrow \pi_\mathfrak{p} \otimes_{\bar{\mathbb{F}}_p} \mathrm{Hom}_G(\pi_\mathfrak{p}, H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}[\mathfrak{m}_\rho]).$$

Now $\rho \otimes (\otimes'_{w \nmid p} \pi_w) \hookrightarrow \mathrm{Hom}_G(\pi_{\mathfrak{p}}, \rho \otimes (\otimes'_w \pi_w))$ and using evaluation map

$$\mathrm{ev} : \pi_{\mathfrak{p}} \otimes \mathrm{Hom}_G(\pi_{\mathfrak{p}}, H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}[\mathfrak{m}_\rho]) \to H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}[\mathfrak{m}_\rho], \quad (v, f) \mapsto f(v)$$

we get that $\pi_{\mathfrak{p}} \otimes_{\bar{\mathbb{F}}_p} \mathrm{Hom}_G(\pi_{\mathfrak{p}}, H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}[\mathfrak{m}_\rho]) \hookrightarrow H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}[\mathfrak{m}_\rho]$ since $\pi_{\mathfrak{p}}$ is irreducible representation of G over $\bar{\mathbb{F}}_p$. Hence we have the following injection

$$\rho \otimes \pi_{\mathfrak{p}} \otimes (\otimes'_{w \nmid p} \pi_w) \hookrightarrow H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}[\mathfrak{m}_\rho].$$

Let $\Sigma' = \{w \in \Sigma \mid D \text{ splits at } w\}$ and put $K = K_{\Sigma'} K^{\Sigma}$ where $K^{\Sigma} = \Pi_{w \notin \Sigma} \mathrm{GL}_2(\mathcal{O}_w)$ and $K_{\Sigma'} = \Pi_{w \in \Sigma'} K_w$ where we choose K_w using above proposition, i.e., $\dim_{\bar{\mathbb{F}}_p} \pi_w^{K_w} = 1$. Taking K-invariance we have

$$\rho \otimes \pi_{\mathfrak{p}} \otimes (\otimes_{w \in \Sigma \setminus \Sigma'} \pi_w) \hookrightarrow H^1_{\bar{\mathbb{F}}_p, \mathrm{ss}}[\mathfrak{m}_\rho]^K,$$

as $G_F \times \mathrm{GL}_2(F_{\mathfrak{p}}) \times \Pi_{w \in \Sigma \setminus \Sigma'} (D \otimes_F F_w)$-representation.

Thank you.