Towards Cluster Duality for $\mathrm{LG}(n, 2n)$ and $\mathrm{OG}(n+1, 2n+1)$

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Introduction

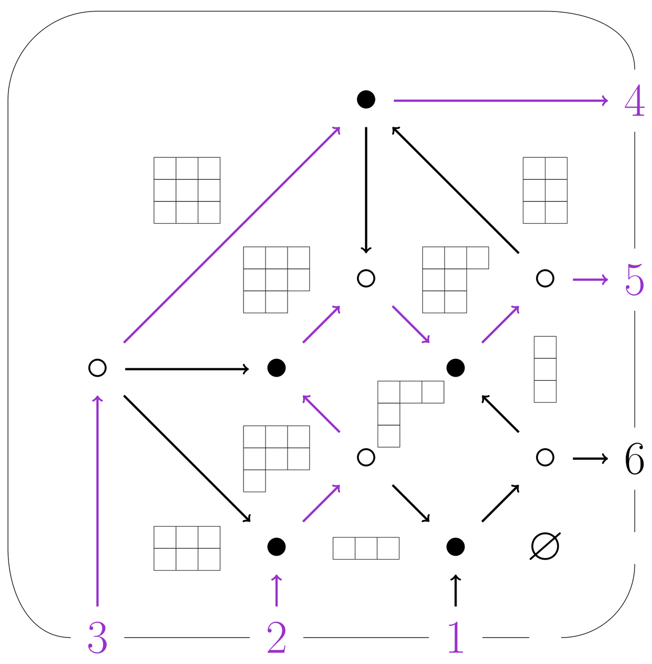
In [RW19], Rietsch and Williams associate dual cluster structures to the Langlands dual Grassmannians $X = \mathrm{Gr}(n-k, n)$ and $X^\vee = \mathrm{Gr}(k, n)$. They use these cluster structures to define two families of polytopes – **Newton-Okounkov (NO) bodies** and **superpotential polytopes** – indexed by *seeds* in the cluster structures, and one of their main results is that these two families coincide. A crucial step in this work is to check that these two polytopes coincide for a particular seed, which they call the *rectangles seed*.

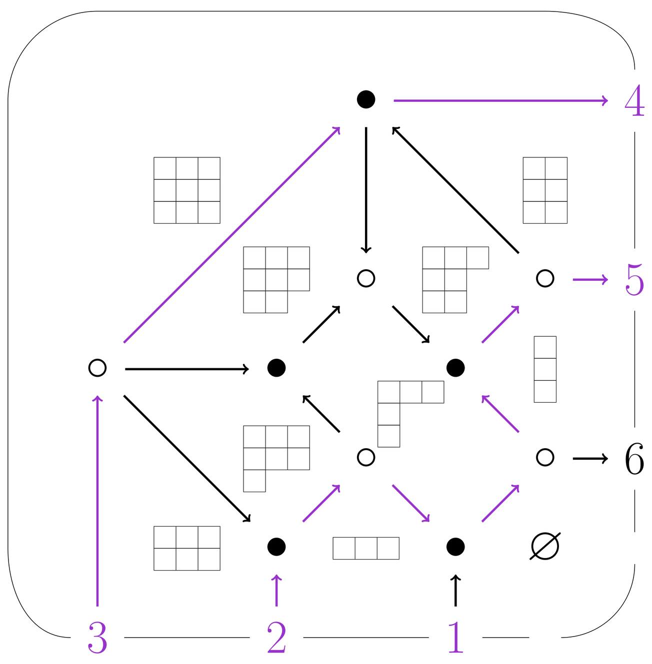
We extend the work of Rietsch and Williams to the Langlands dual Lagrangian Grassmannians $X = \mathrm{LG}(n, 2n)$ (type C) and orthogonal Grassmannians $X^\vee = \mathrm{OG}(n+1, 2n+1)$ (type B) by identifying a particular seed, which we call the **co-rectangles seed**, for a cluster structure on X. We then show that the corresponding Newton-Okounkov body is unimodularly equivalent to a polytope defined from a superpotential defined on $X^\vee$.

Plabic Graphs and NO-Bodies

One of the main combinatorial tools in Rietsch and Williams is the **plabic graph**, originally defined by Postnikov in [Pos06]. A **plabic graph** is a *pla*nar *bic*olored graph embedded in a disk with boundary vertices $1, \dots N$. In [Kar18], Karpman defined a symmetric analogue of plabic graphs, having an even number $N = 2n$ of boundary vertices, and a diameter d of the bounding disk with endpoints between $n, n+1$ and $2n, 1$ such that no vertex lies on d and reflection through d gives the same graph with colors switched.

Each plabic graph gives a way to express regular functions on X as Laurent polynomials in the face labels of the graph, and we can use the combinatorics of **flows** to do this. A flow is a vertex-disjoint collection of n paths starting and ending on the boundary, and the *weight* of a flow is the product over all paths in the flow of the product of face labels to the left of that path. We give in the following two figures the plabic graph (underlying undirected graph) with a perfect orientation as well as all flows from $\{1, 2, 3\}$ to $\{1, 4, 5\}$.





In this example, we compute that the Plücker coordinate $P_{\{1,4,5\}}$ has expansion

$$P_{\{1,4,5\}} = (x^2_{(3,3,3)} x^2_{(3,3)} x^2_{(3,3,1)} x_{(3,3,2)})(1 + x_{(3,1,1)})$$

in the torus coordinates. Using this, we define a **valuation** ν_n sending a Plücker coordinate to the exponent vector of the (lexicographically) minimal term of its torus expansion. In the example above, this is the monomial $x^2_{(3,3,3)} x^2_{(3,3)} x^2_{(3,3,1)} x_{(3,3,2)}$.

Definition. Using the valuation defined above, we define $\delta_n = \mathrm{Conv}(\nu(P))$, where P denotes the set of Plücker coordinates for $\mathrm{LG}(n, 2n)$ (in correspondence with Young diagrams modulo transpose).

We record the valuations here, in the coordinates (, , , , ,):

$I \in \binom{[6]}{3}$	$\nu_n(p_I)$	$I \in \binom{[6]}{3}$	$\nu_n(p_I)$
123	$(0,0,0,0,0,0)$	$146 = 245$	$(1,2,1,2,1,2)$
124	$(0,0,0,0,0,1)$	$156 = 345$	$(1,3,1,3,2,2)$
$125 = 134$	$(0,1,0,1,1,1)$	236	$(2,2,1,2,1,1)$
$126 = 234$	$(1,1,1,2,1,1)$	246	$(2,2,1,2,1,2)$
135	$(0,2,0,2,1,1)$	$256 = 346$	$(2,3,1,3,2,2)$
$136 = 235$	$(1,2,1,2,1,1)$	356	$(2,4,1,4,2,2)$
145	$(0,2,0,2,1,2)$	456	$(2,4,1,4,2,3)$

Note that every row can be written as an integer linear combination of the rows indexed by 124, 125, 126, 135, 136, 236 (in Young diagrams: , , , , ,). This is a general fact which we prove on the way to our main theorem.

Superpotential polytopes

A **Landau-Ginzburg** model for X is a space $X^\vee$ and a rational function W on $X^\vee$ such that the quantum cohomology ring $QH^*(X)$ of X is isomorphic to the Jacobi ring $k[\mathrm{Dom}(W)]/(\partial W)$ of W. The function W is often called a **superpotential**.

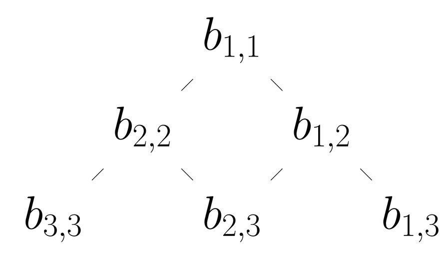
Given an algebraic torus $\mathbb{T}$ inside $\mathrm{Dom}(W)$, we obtain a polytope from W as follows. Restricting W to $\mathbb{T}$, we obtain a Laurent polynomial expansion of W in the coordinates of $\mathbb{T}$. We then "*tropicalize*" this expression in the following sense. For each monomial of the Laurent polynomial expansion, replace "$*$" with "$+$" and "$\div$ with $-$, and set the resulting linear expression greater than or equal to 0.

In [PR13], Pech and Rietsch constructed a Landau-Ginzburg model, and computed its restriction to a particular algebraic torus sometimes called a *Lusztig torus*:

$$a_{11} + a_{12} + a_{13} + a_{22} + a_{23} + a_{33} + \frac{q}{a_{11}a_{12}a_{13}} + \frac{q}{a_{11}a_{12}a_{23}} + \frac{q}{a_{11}a_{22}a_{23}} + \frac{q}{a_{11}a_{22}a_{33}}.$$

From this, we obtain the inequalities (left)

$$\begin{aligned} A_{ij} &\geq 0 \\ 1 - A_{11} - A_{12} - A_{13} &\geq 0 \\ 1 - A_{11} - A_{12} - A_{23} &\geq 0 \\ 1 - A_{11} - A_{22} - A_{23} &\geq 0 \\ 1 - A_{11} - A_{22} - A_{33} &\geq 0 \end{aligned}$$



which can easily be seen to define a bounded polytope. In general it's straightforward to recognize this polytope as the **order polytope** of the poset $\mathcal{P}_n$ (right) on the elements $\{b_{i,j} \mid 1 \leq i \leq j \leq n\}$, with the cover relations $b_{i,j} > b_{i+1,j+1}$ and $b_{i,j} > b_{i,j+1}$. Order and chain polytopes were initially studied by Stanley in [Sta86].

Definition. Using the tropicalization procedure above, we define Γ_n to be the intersection of the half-spaces corresponding to monomials of W.

Proposition. *Γ_n has $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$ many vertices, and volume equal to $\deg(X)$. Equivalently, $\mathcal{P}_n$ has C_{n+1} antichains, and $\deg(X)$ linear extensions.*

Main Results and Future Work

Theorem. *δ_n and Γ_n are unimodularly equivalent.*

Proof Sketch. The vertices of Γ_n are in bijection with the Plücker coordinates of $\mathrm{LG}(n, 2n)$, because both are counted by C_{n+1}. We find a concrete bijection which is determined by sending the singleton antichains to Young diagrams which are the complement of a hook partition. (These were exactly the partitions we observed earlier.) This defines a map from Γ_n to δ_n, and it is a hard linear algebra check that it is unimodular. $\square$

By general considerations, δ_n is contained in the NO body associated to our valuation ν_n, and the volume of a NO body is equal to the degree of the variety to which it is associated. Combining this with our main theorem, we obtain the following.

Corollary. *δ_n is a Newton-Okounkov body for X, and the Plücker coordinates form a Khovanskii basis with respect to ν_n.*

There has so far been no easily accessible cluster structure on $X^\vee$, so we could only obtain a unimodular equivalence rather than an equality. In future work, we will give a concrete cluster structure on the orthogonal Grassmannians, and complete the full cluster duality story. Furthermore, many of the Young diagram combinatorics have connections to quantum cohomology in type A, and it would be interesting to investigate in type C.

References

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