Seshadri constants of equivariant vector bundles on toric varieties

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Positivity of line bundles

- “Positivity” of line bundles means that it has “many global sections”.

$$
\begin{tikzcd}
 & \boxed{\textbf{Positivity}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{\textbf{Geometric}} & \boxed{\textbf{Cohomological}} & \boxed{\textbf{Numerical}}
\end{tikzcd}
$$

Positivity of line bundles

Framework: All varieties are nonsingular projective defined over $\mathbb{C}$.

Let $\mathcal{L}$ be a line bundle on X and $s_0, s_1, \ldots s_N$ be a $\mathbb{C}$-basis for $H^0(X, \mathcal{L})$. Then there is the associated **Kodaira map**

$$\phi_{\mathcal{L}} : X \setminus Bs(\mathcal{L}) \longrightarrow \mathbb{P}^N, \text{ defined by } x \longmapsto [s_0(x) : s_1(x) : \ldots : s_N(x)],$$

where $Bs(\mathcal{L}) := \mathbb{V}(s_0) \cap \ldots \cap \mathbb{V}(s_N)$ is the base locus of the line bundle $\mathcal{L}$.

- The line bundle $\mathcal{L}$ is called globally generated if $Bs(\mathcal{L}) = \emptyset$. In addition, if $\phi_{\mathcal{L}}$ defines a closed embedding $\phi_{\mathcal{L}} : X \hookrightarrow \mathbb{P}^N$, then $\mathcal{L}$ is said to be very ample.
- The line bundle $\mathcal{L}$ is called ample if there exists a positive integer m such that $\mathcal{L}^{\otimes m}$ is very ample.

Theorem 1 (Nakai-Moishezon-Kleiman criterion)

Let $\mathcal{L}$ be a line bundle on a projective variety X. Then $\mathcal{L}$ is ample if and only if

$$\mathcal{L}^{\mathit{dim}\ V} \cdot V > 0$$

for every positive dimensional irreducible subvariety $V \subseteq X$.

- A line bundle $\mathcal{L}$ is called numerically effective (nef) if $\mathcal{L} \cdot C \geq 0$ for all irreducible curves C in X.

Theorem 2 (Seshadri criterion for ampleness (1972))

A line bundle $\mathcal{L}$ be on X is ample if and only if for every point $x \in X$ there exists a positive number ε such that

$$\frac{\mathcal{L} \cdot C}{\mathrm{mult}_x C} \geq \varepsilon$$

for all curves C passing through x.

Seshadri constants are introduced by Demailly(1992).

Definition 1

Let $\mathcal{L}$ be a nef line bundle on a complex projective variety X. For a point $x \in X$, the Seshadri constant of $\mathcal{L}$ at x is defined to be

$$\varepsilon(X, \mathcal{L}, x) := \inf_{x \in C} \frac{\mathcal{L} \cdot C}{\mathrm{mult}_x C} .$$

Connection with Fujita conjecture

Let $\mathcal{L}$ be an ample line bundle on X and $n = \dim(X)$.

- If $\varepsilon(X, \mathcal{L}, x) > \frac{n}{n+1}$ for all $x \in X$ then $K_X + (n+1)\mathcal{L}$ is globally generated.
- If $\varepsilon(X, \mathcal{L}, x) > \frac{2n}{n+2}$ for all $x \in X$ then $K_X + (n+2)\mathcal{L}$ is very ample.

Miranda(1993) : too optimistic to conclude Fujita conjecture.

Fix any $\delta > 0$, then there exists a smooth surface X, a point $x \in X$, and an ample line bundle $\mathcal{L}$ on X such that

$$\varepsilon(X, \mathcal{L}, x) < \delta.$$

Note : In the above example the Seshadri constant is small only at a "very special" point.

Guiding problems on Seshadri constants

Note that for any $x \in X$,

$$0 \leq \varepsilon(X, \mathcal{L}, x) \leq \sqrt[n]{\mathcal{L}^n}.$$

Some of the guiding problems on Seshadri constants involve:

Computing Seshadri constants

Giving bounds on them

Checking if they are irrational

(Nagata Conjecture -1958)

Some known results

- Let $\mathcal{L}$ be an **ample and globally generated** line bundle on a variety X, then

$$\varepsilon(X, \mathcal{L}, x) \geq 1$$

for all $x \in X$.

- Ein-Lazarsfeld (1993): Let X be a smooth projective surface, and $\mathcal{L}$ be an **ample** line bundle on X. Then

$$\varepsilon(X, \mathcal{L}, x) \geq 1$$

for "very general point" $x \in X$.

- **Characterization of $\mathbb{P}^n$:** The only smooth projective variety with ample tangent bundle is $\mathbb{P}^n$ (Mori (1979)).

Let X be a smooth Fano variety of dimension n. Then

$$X = \mathbb{P}^n \iff \varepsilon(X, -K_X, x) \geq n+1 \text{ for some } x \in X,$$

(Bauer-Szemberg (2009)).

Example 2

- Consider the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$, then for every $x \in \mathbb{P}^n$

$$\varepsilon(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), x) = 1.$$

Take a line C passing through x, then the ratio $\frac{\mathcal{L} \cdot C}{\mathrm{mult}_x C} = 1$. So $\varepsilon(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), x) \leq 1$.

The equality follows from Bézout's theorem:

$$\mathcal{L} \cdot C \geq \mathrm{mult}_x C.$$

Aim: To compute Seshadri constants for vector bundles.

Seshadri constant for vector bundles

X : nonsingular complex projective variety, $\mathcal{E}$: vector bundle on X

$\pi : \mathbb{P}(\mathcal{E}) \to X$: projectivized bundle associated to $\mathcal{E}$

$\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$: tautological line bundle on $\mathbb{P}(\mathcal{E})$

A vector bundle $\mathcal{E}$ on X is ample (resp. nef) if the tautological line bundle ξ is ample (resp. nef) on the projectivized bundle $\mathbb{P}(\mathcal{E})$.

Definition 3 (Hacon (2000), Fulger-Murayama (2021))

The Seshadri constant of a nef vector bundle $\mathcal{E}$ at $x \in X$ is defined to be

$$\varepsilon(X, \mathcal{E}, x) := \inf_{C \subset \mathbb{P}(\mathcal{E})} \frac{\xi \cdot C}{\mathrm{mult}_x \pi_* C},$$

where the infimum is taken over all curves C on $\mathbb{P}(\mathcal{E})$ that meet $\pi^{-1}(x)$ but not completely contained in $\pi^{-1}(x)$.

Some known results

- Let $\mathcal{E}$ be an **ample and globally generated** vector bundle on a smooth complex projective curve X, then for all $x \in X$

$$\varepsilon(X, \mathcal{E}, x) \geq 1.$$

- Hacon (2000): Let $\mathcal{E}$ be a nef vector bundle on a smooth complex projective curve X, then for all $x \in X$

$$\varepsilon(X, \mathcal{E}, x) = \mu_{min}(\mathcal{E}),$$

  where $\mu_{min}(\mathcal{E})$ denotes the smallest slope of any quotient bundle of $\mathcal{E}$. Here slope of the vector bundle $\mathcal{E}$ is $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}$.

- Fulger-Murayama (2021): If $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r$ is a nef vector bundle on a variety X, then for any $x \in X$

$$\varepsilon(X, \mathcal{E}, x) = \min_{1 \leq i \leq r} \{\varepsilon(X, \mathcal{E}_i, x)\}.$$

- $\mathcal{E}$ **semistable discriminant zero** nef vector bundle of rank r on a variety X, then for all $x \in X$,

$$\varepsilon(X, \mathcal{E}, x) = \frac{1}{r}\, \varepsilon(X, \det(\mathcal{E}), x).$$

- **Another Characterization of $\mathbb{P}^n$:** Let X be a smooth Fano variety of dimension n with nef tangent bundle. Then

$$X = \mathbb{P}^n \Longleftrightarrow \varepsilon(X, \mathscr{T}_X, x) > 0 \text{ for some } x \in X,$$

(Fulger-Murayama (2021)).

Definition 4

A toric variety X: A normal complex variety which contains a torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset such that:

$$\begin{tikzcd} T \times T \arrow[r] \arrow[d, hook] & T \arrow[d, hook] \\ T \times X \arrow[r] & X \end{tikzcd}$$

Example 5

- $(\mathbb{C}^*)^n$, $\mathbb{C}^n$ and $\mathbb{P}^n$.

Theorem 3 (Fundamental theorem for toric varieties)

The category of toric varieties is equivalent to the category of fans.

$$X_\Delta \longleftrightarrow \Delta_X .$$

Combinatorial Data: $M = \mathrm{Hom}(T, \mathbb{C}^*)$,

$N = M^\vee$, fan Δ in $N \otimes \mathbb{R} \cong \mathbb{R}^n$.

- Cone $\sigma \in \Delta \rightsquigarrow$ affine variety U_σ, distinguished point $x_\sigma \in U_\sigma$.

- x_σ is a torus fixed point $\Leftrightarrow \sigma \in \Delta$ is n-dimensional.

- 1-dimensional cone $\rho \in \Delta$ $\rightsquigarrow$ invariant divisors D_ρ.

- $(n-1)$-dimensional cone $\tau \in \Delta$ $\rightsquigarrow$ invariant curves $V(\tau) \cong \mathbb{P}^1$.

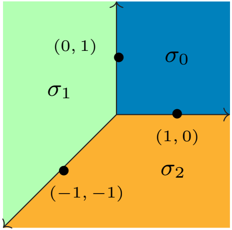

Figure: Fan of $\mathbb{P}^2$

Toric vector bundle

Definition 6

A T-equivariant vector bundle or toric vector bundle: A vector bundle $\pi : \mathcal{E} \to X$ on X with a lift of the action of T on the total space $\mathcal{E}$ in such a way that:

1. the projection map π is equivariant, and
2. the torus T acts linearly on the fibers.

Example 7

line bundle, tangent bundle, cotangent bundle

Klyachko's classification theorem

$\mathcal{E}$: rank r toric vector bundle on X, $E = \mathcal{E}(1_T)$: the fiber at $1_T \in T \subset X$.

Klyachko (1990)

$$\mathcal{E} \longleftrightarrow E \supset \ldots \supset E^{\rho}(i) \supset E^{\rho}(i+1) \supset \ldots \mathbf{0},$$

indexed by invariant prime divisors satisfying certain compatibility conditions.

For each cone $\sigma \in \Delta$, there is a decomposition into eigenspaces as follows

$$E = \bigoplus_{u \in \mathbf{u}(\sigma)} L_u^{\sigma},$$

where $\mathbf{u}(\sigma) \subset M$ is the associated characters of the toric vector bundle $\mathcal{E}$.

(Hering-Mustaţă-Payne (2010)): $\{\mathbf{u}(\sigma)\}_{\sigma}$ determines the restriction of $\mathcal{E}$ to the invariant curves.

Seshadri constant for toric vector bundle

- X toric variety; $x \in X$ a **torus fixed point** and $\mathcal{E}$ a nef toric vector bundle on X. Then

$$\varepsilon(X, \mathcal{E}, x) = \min \{\mu_{min}(\mathcal{E}|_C) \mid x \in C \text{ and } C \text{ is an invariant curve}\}$$

(Hering-Mustaţă-Payne (2010)).

Goal: To compute Seshadri at arbitrary points.

Recall: to compute Seshadri constant at $x \in X$, we need to compute the ratios

$$\frac{\xi \cdot C}{\mathrm{mult}_x \pi_* C}, \quad \text{for all } C \subset \mathbb{P}(\mathcal{E}).$$

Key ingredient: the description of the Mori cone $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{E}))$: the closed cone of curves of the projectived bundle $\mathbb{P}(\mathcal{E})$.

Mori Cone

X: toric variety; $\mathcal{E}$: toric vector bundle on X; $l_1, \ldots, l_m$: invariant curves in X.

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}|_{l_i}) \arrow[r, hook, "\eta_i"] \arrow[d, "\pi_i"'] & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi"] \\ l_i \arrow[r, hook] & X \end{tikzcd}$$

- Since $\mathbb{P}(\mathcal{E}|_{l_i})$ is a toric variety, there is an invariant fiber curve Σ_i and invariant section curve Ω_i such that $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{E}|_{l_i}) = \text{Cone}(\Sigma_i, \Omega_i)$.

Proposition 8 (Hering-Mustață-Payne (2010))

Take $C_0 := \eta_i(\Sigma_i)$ and $C_i := \eta_i(\Omega_i)$, then the Mori cone is given by

$$\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{E})) = \Big\{ a_0 C_0 + \cdots + a_m C_m \mid a_i \in \mathbb{R}_{\geq 0} \text{ for } i = 0, \ldots, m \Big\}.$$

In particular, $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{E})$ is a polyhedral cone.

Theorem 9 (Dasgupta- __ - Aditya)

*Let $\mathcal{E}$ be a "**nice**" nef equivariant vector bundle of rank r on the projective space $X = \mathbb{P}^n$ $(n \geq 2)$. Then for any point $x \in X$, we have*

$$\varepsilon(\mathcal{E}, x) = \min_{1 \leq i \leq m} \{\mu_{min}(\mathcal{E}|_{l_i})\}.$$

Example 10

- Uniform bundle: a bundle of splitting type $(a_1, \ldots, a_r)$, i.e., for any line $l \subset \mathbb{P}^n$, we have

$$\mathcal{E}|_l \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r).$$

  - $\mathscr{T}_{\mathbb{P}^n}$ is a uniform bundle with splitting type (2,1, …,1), hence for any $x \in \mathbb{P}^n$ the Seshadri constant is given by

$$\varepsilon(\mathscr{T}_{\mathbb{P}^n}, x) = 1.$$

Hirzebruch surface

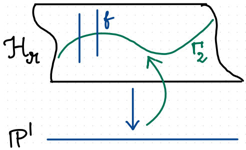

Figure: $\mathcal{H}_{c_{1,2}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(c_{1,2}))$

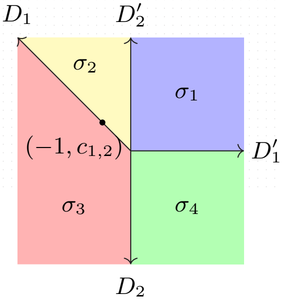

Figure: Fan for $\mathcal{H}_{c_{1,2}}$

- We have $D_1 \equiv D_1' \equiv f$,
  $D_2' \equiv D_2 - c_{1,2} D_1 \equiv \Gamma_2$
- The Picard group is
  $\text{Pic}(X) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2$.
- The Nef cone is
  $\text{Nef}(X) = \mathbb{R}_{\geq 0}D_1 \oplus \mathbb{R}_{\geq 0}D_2$.
- The Mori cone is
  $\overline{\text{NE}}(X) = \mathbb{R}_{\geq 0}\Gamma_2 \oplus \mathbb{R}_{\geq 0}f$.

Theorem 11 (Dasgupta- __ - Aditya)

Let $\mathcal{E}$ be an equivariant nef vector bundle of rank r on the Hirzebruch surface $X_2 = \mathcal{H}_{c_{1,2}}$ satisfying the following conditions:

$$\mu_{min}(\mathcal{E}|_{D_1}) = \mu_{min}(\mathcal{E}|_{D'_1}) \text{ and } \mu_{min}(\mathcal{E}|_{D_2}) \geq \mu_{min}(\mathcal{E}|_{D_1}).$$

Then for any $x \in X_2$, the Seshadri constant is given by:

$$\varepsilon(X_2, \mathcal{E}, x) = \begin{cases} min\{\mu_{min}(\mathcal{E}|_{D_1}),\ \mu_{min}(\mathcal{E}|_{D'_2})\}, & \text{if } x \in \Gamma_2, \\ \mu_{min}(\mathcal{E}|_{D_1}), & \text{if } x \notin \Gamma_2. \end{cases}$$

Seshadri constants of line bundles on Hirzebruch surfaces have been computed by Syzdek (2005), García (2006), Hanumanthu-Mukhopadhyay (2017).

Example 12

Consider the tangent bundle $\mathcal{E} = \mathscr{T}_{X_2}$ on the Hirzebruch surface X_2. Then the associated filtrations $(E, \{E^i(j)\}_{i=1,\dots,4;\, j\in\mathbb{Z}})$ are given by:

$$E^i(j) = \begin{cases} \mathbb{C}^2 & j \leqslant 0 \\ \text{Span } (v_i) & j = 1 \\ 0 & j > 1 \end{cases}.$$

$\mathcal{E} \otimes \mathcal{O}(D)$ is nef, where $D = a_1 D_1 + a_2 D_2$, $a_1 \geq c_{1,2}$, $a_2 \geq 0$.

$$(\mathcal{E} \otimes \mathcal{O}(D))|_{D_1'} = \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_2), (\mathcal{E} \otimes \mathcal{O}(D))|_{D_1} = \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_2),$$

$$(\mathcal{E} \otimes \mathcal{O}(D))|_{D_2'} = \mathcal{O}_{\mathbb{P}^1}(a_1 - c_{1,2}) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_1),$$

$$(\mathcal{E} \otimes \mathcal{O}(D))|_{D_2} = \mathcal{O}_{\mathbb{P}^1}(a_1 + c_{1,2}\, a_2 + c_{1,2}) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1 + c_{1,2}\, a_2 + 2).$$

The Seshadri constant is given by

$$\varepsilon(\mathcal{E} \otimes \mathcal{O}(D), x) = \begin{cases} \min\{a_1 - c_{1,2},\, a_2\}, & \text{if } x \in \Gamma_2, \\ a_2, & \text{if } x \notin \Gamma_2. \end{cases}$$

Bott towers

Bott towers are a particular class of nonsingular projective toric varieties. They were constructed by Grossberg-Karshon (1994).

For an integer $n \geq 0$, a **Bott tower of height** n

$$X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = \{\text{point}\}$$

is defined inductively as an iterated $\mathbb{P}^1$–bundle so that

$$X_k = \mathbb{P}(\mathcal{O}_{X_{k-1}} \oplus \mathcal{L}_{k-1})$$

for a line bundle $\mathcal{L}_{k-1}$ over X_{k-1}.

So $X_1 = \mathbb{P}^1$ and X_2 is a Hirzebruch surface and so on.

Theorem 13 (Biswas-Dasgupta-Hanumanthu-__)

Let $x \in X_n$. We have constructed smooth curves $\Gamma_n, \Gamma_n^{(2)}, \ldots, \Gamma_n^{(n)}$ in X_n, which generate the Mori cone of X_n. The Seshadri constant of a nef line bundle $\mathcal{L}$ is given as follows:

$$\varepsilon(X_n,\, \mathcal{L},\, x) \,=\, \min_{\mathrm{i}} \left\{ \mathcal{L} \cdot \Gamma_n^{(i)} \;\middle|\; x \in \Gamma_n^{(i)} \right\}.$$

Theorem 14 (Dasgupta- __ - Aditya)

Let $\mathcal{E}$ be an equivariant nef vector bundle of rank r on X_3 satisfying "certain" conditions. Then the Seshadri constants of $\mathcal{E}$ at any $x \in X_3$ are given by

$$\varepsilon(X_3, \mathcal{E}, x) = \min_{\mathrm{i}} \left\{ \mu_{min}(\mathcal{E}|_{\Gamma_3^{(i)}}) \;\middle|\; x \in \Gamma_3^{(i)} \right\}.$$

Thank You