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The discrete membrane model on trees

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This talk is based on a joint work with Alessandra Cipriani (TU Delft), Rajat Subhra Hazra (Leiden University) and Rounak Ray (TU/e).

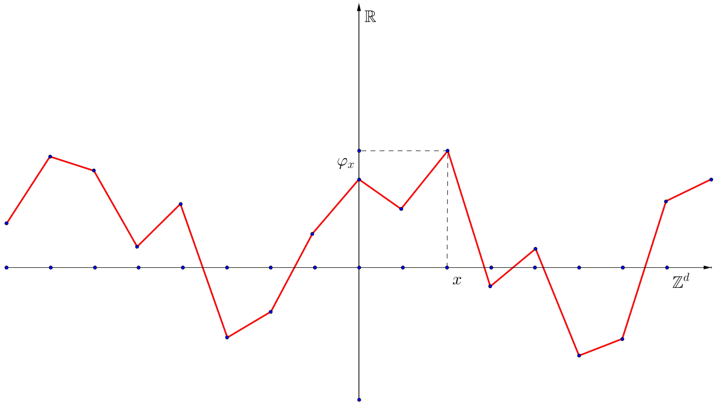
The membrane model on the integer lattice is studied in the literature as a **random interface model for semiflexible polymer or semiflexible membrane**.

What is interface?

What is interface?

- An interface on a graph $\mathcal{G} = (V, E)$ is the graph of a function $\varphi : V \to \mathbb{R}$.

- $\varphi_x := \varphi(x)$ is the height of the interface at the vertex $x \in V$.



Random interface

A random interface $\varphi = (\varphi_x)_{x \in V}$ is determined by a probability measure on the space of height configurations.

Example 1: Discrete Gaussian free field (DGFF) on $\mathbb{Z}^d$

Let $V_n := [-n, n]^d \cap \mathbb{Z}^d$, $n \in \mathbb{N}$.
DGFF on $\mathbb{Z}^d$ with zero boundary conditions outside V_n is defined to be the interface $(\varphi_x)_{x\in\mathbb{Z}^d}$ with the following properties:

- $\varphi_x = 0$, for all $x \in \mathbb{Z}^d \setminus V_n$.
- $(\varphi_x)_{x\in V_n} \sim \mathcal{N}(\mathbf{0}, g_n)$ with

$$g_n(x, y) = \mathbf{E}_n[\varphi_x\varphi_y].$$

- For all $x \in V_n$

$$\begin{cases} -\Delta g_n(x, y) = \delta_x(y), & y \in V_n \\ g_n(x, y) = 0, & y \notin V_n. \end{cases}$$

Here

$$\Delta f(x) := \frac{1}{2d}\sum_{y\sim x}(f(y) - f(x)).$$

Discrete Gaussian free field (DGFF) on $\mathbb{Z}^d$

- DGFF is determined by the probability measure $\mathbf{P}_n$ given by

$$\mathbf{P}_n(\mathrm{d}\varphi) \propto \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d} \varphi_x(-\Delta)\varphi_x\right)\prod_{x\in V_n}\mathrm{d}\varphi_x \prod_{x\in V_n^{\mathrm{c}}} \delta_0(\mathrm{d}\varphi_x).$$

- Define $\Delta_n := (\Delta(x,y))_{x,y\in V_n}$. Then

$$g_n = (-\Delta_n)^{-1}.$$

Discrete Gaussian free field (DGFF) on $\mathbb{Z}^2$

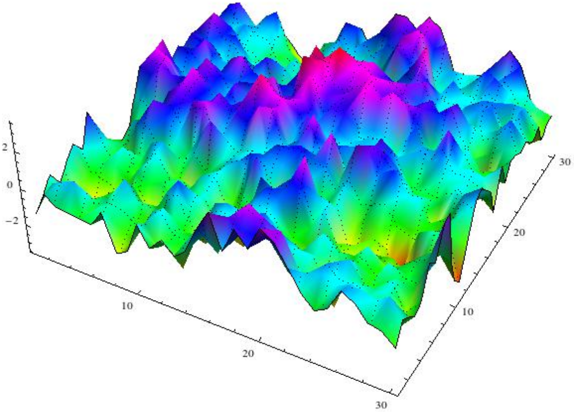


Figure: Discrete Gaussian Free Field on a 30x30 box

Random walk representation for g_n

Let $\mathbb{P}_x$ be the law of a SRW $(S_k)_{k\geq 0}$ started at $x \in \mathbb{Z}^d$. Then

$$g_n(x,y) := \mathbb{E}_x \left[\sum_{k=0}^{\tau_{V_n}-1} \mathbb{1}_{[S_k=y]} \right]$$

where

$$\tau_{V_n} := \inf\{k \geq 0 : S_k \in V_n^{\mathrm{c}}\}.$$

Infinite volume DGFF

The infinite volume measure

$$\mathbf{P} := \lim_{n \to \infty} \mathbf{P}_n$$

exists if $d \geq 3$.

$\mathbf{P}$ is the law of a centered Gaussian field $(\varphi_x)_{x \in \mathbb{Z}^d}$ with covariance matrix

$$g(x, y) := \mathbf{E}[\varphi_x \varphi_y] = \mathbb{E}_x \left[\sum_{k=0}^{\infty} \mathbb{1}_{[S_k = y]} \right]$$

Example 2: Membrane model(MM) on $\mathbb{Z}^d$

MM on $\mathbb{Z}^d$ with zero boundary conditions outside V_n is defined to be the interface $(\varphi_x)_{x\in\mathbb{Z}^d}$ with the following properties:

- $\varphi_x = 0$, for all $x \in \mathbb{Z}^d \setminus V_n$.
- $(\varphi_x)_{x\in V_n} \sim \mathcal{N}(\mathbf{0}, G_n)$ with

$$G_n(x,y) = \mathbf{E}_n[\varphi_x\varphi_y].$$

- For all $x \in V_n$

$$\begin{cases} \Delta^2 G_n(x,y) = \delta_x(y), & y \in V_n \\ G_n(x,y) = 0, & y \notin V_n. \end{cases}$$

Membrane model(MM) on $\mathbb{Z}^d$

- MM is determined by the probability measure $\mathbf{P}_n$ given by

$$\mathbf{P}_n(\mathrm{d}\varphi) \propto \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\varphi_x \Delta^2 \varphi_x\right)\prod_{x\in V_n}\mathrm{d}\varphi_x \prod_{x\in V_n^{\mathrm{c}}}\delta_0(\mathrm{d}\varphi_x)$$

- Define $\Delta_n^2 := (\Delta^2(x,y))_{x,y\in V_n}$. Then

$$G_n = (\Delta_n^2)^{-1}.$$

- Note that

$$\Delta_n^2 \neq (\Delta_n)^2.$$

Membrane model(MM) on $\mathbb{Z}^2$

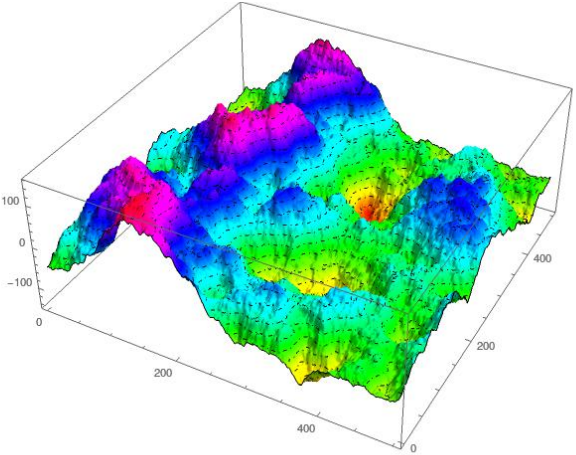


Figure: Membrane model on a 500x500 box

The MM on $\mathbb{Z}^d$ has been studied in the literature by Bolthausen, Buchholz, Caravenna, Chiarini, Cipriani, D., Deuschel, Ding, Hazra, Müller, Kurt, Roy, Sakagawa, Schweiger, Zeitouni, ...

Infinite volume MM

The infinite volume measure

$$\mathbf{P} := \lim_{n\to\infty} \mathbf{P}_n$$

exists if $d \geq 5$.
$\mathbf{P}$ is the law of a centered Gaussian field $(\varphi_x)_{x\in\mathbb{Z}^d}$ with covariance matrix

$$G(x,y) := \mathbf{E}[\varphi_x\varphi_y] = \Delta^{-2}(x,y)$$

Random walk representation for G

$$
\begin{aligned}
G(x,y) &= \sum_{z\in\mathbb{Z}^d} (-\Delta)^{-1}(x,z)(-\Delta)^{-1}(z,y) \\
&= \mathbb{E}_{x,y}\left[\sum_{k,\ell=0}^{\infty} \mathbb{1}_{[S_k=\tilde{S}_\ell]}\right]
\end{aligned}
$$

where S_k and $\tilde{S}_k$ are two independent SRW started at x and y respectively.

Random walk representation for G

$$
\begin{aligned}
G(x,y) &= \sum_{z\in\mathbb{Z}^d} (-\Delta)^{-1}(x,z)(-\Delta)^{-1}(z,y) \\
&= \mathbb{E}_{x,y}\left[\sum_{k,\ell=0}^{\infty} \mathbb{1}_{[S_k=\tilde{S}_\ell]}\right] \\
&= \sum_{k=0}^{\infty}(k+1)\mathbb{P}_x(S_k=y)
\end{aligned}
$$

What about G_n?

Guess:

$$G_n(x, y) = \mathbb{E}_{x,y}\left[\sum_{k=0}^{\tau_{V_n}} \sum_{\ell=0}^{\tilde{\tau}_{V_n}} \mathbb{1}_{[S_k = \tilde{S}_\ell]}\right]?$$

What about G_n?

$$\overline{G_n}(x,y) := \mathbb{E}_{x,y}\left[\sum_{k=0}^{\tau_{V_n}} \sum_{\ell=0}^{\tilde{\tau}_{V_n}} \mathbb{1}_{[S_k=\tilde{S}_\ell]}\right]$$

(Kurt, '08) In the bulk

$$\sup_{x,y} |G_n(x,y) - \overline{G_n}(x,y)| = O(n^{4-d}).$$

A new random walk representation for G_n

- Define

$$\tau_i := \inf\{k > \tau_{i-1} : S_k \in V_n^c\},\ \tau_{-1} := -1$$

$$M_j := \prod_{i=0}^{j} (\tau_i - \tau_{i-1} - 1),\ M_{-1} := 1$$

- (Vanderbei, '84) For $d \geq 3$

$$G_n(x,y) = \lim_{t\to\infty} \mathbb{E}_x \left[\sum_{j=0}^{\eta_t} (-1)^j M_{j-1} \sum_{k=\tau_{j-1}}^{\tau_j - 1} (k - \tau_{j-1}) \, \mathbb{1}_{[S_k = y]} \right],$$

where $\{\eta_t\}_{t\geq 0}$ be a Poisson process with parameter 1 which is independent of the random walk S_k.

MM on regular tree

- Let $\mathbb{T}_m$ be an m-regular infinite tree, that is, it is a rooted tree with the root o having m-children and each of the children thereafter has $m-1$ children.

- We consider

$$V_n := \{x \in \mathbb{T}_m : d(o,x) \le n\},\ n \in \mathbb{N}.$$

- The MM on $\mathbb{T}_m$ is defined similarly as on $\mathbb{Z}^d$. Here Δ is defined by

$$\Delta f(x) = \frac{1}{m} \sum_{y \sim x} (f(y) - f(x)).$$

MM on regular tree

MM on $\mathbb{T}_m$ with zero boundary conditions outside V_n is defined to be the interface $(\varphi_x)_{x \in \mathbb{T}_m}$ with the following properties:

- $\varphi_x = 0$, for all $x \in \mathbb{T}_m \setminus V_n$.
- $(\varphi_x)_{x \in V_n} \sim \mathcal{N}(\mathbf{0}, G_n)$ with

$$G_n(x, y) = \mathbf{E}_n[\varphi_x \varphi_y].$$

- For all $x \in V_n$

$$\begin{cases} \Delta^2 G_n(x, y) = \delta_x(y), & y \in V_n \\ G_n(x, y) = 0, & y \notin V_n. \end{cases}$$

RW representation for G_n

We consider $m \geq 3$. Then

$$G_n(x,y) = \lim_{t\to\infty} \mathbb{E}_x \left[\sum_{j=0}^{\eta_t} (-1)^j M_{j-1} \sum_{k=\tau_{j-1}}^{\tau_j - 1} (k - \tau_{j-1})\, \mathbb{1}_{[S_k = y]} \right]$$

We write

$$G_n(x,y) = \mathbb{E}_x \left[\sum_{k=0}^{\tau_0 - 1} (k+1)\, \mathbb{1}_{[S_k = y]} \right] + E_n(x,y)$$

$$|E_n(x,y)| \leq C\, d(x, V_n^{\mathrm{c}})\, d(y, V_n^{\mathrm{c}})\, (m-1)^{-\max\{d(x,V_n^{\mathrm{c}}), d(y,V_n^{\mathrm{c}})\}}$$

Infinite volume measure

Theorem (Cipriani,D.,Hazra,Ray, '21)

Let $m \geq 3$. The infinite volume measure

$$\mathbf{P} := \lim_{n \to \infty} \mathbf{P}_n$$

exists. $\mathbf{P}$ *is the law of a centered Gaussian field* $(\varphi_x)_{x \in \mathbb{Z}^d}$ *with covariance matrix*

$$G(x, y) := \mathbf{E}[\varphi_x \varphi_y] = \mathbb{E}_x \left[\sum_{k=0}^{\infty} (k+1) \, \mathbb{1}_{[S_k = y]} \right]$$

For any $x, y \in \mathbb{T}_m$

$$G(x,y) \asymp d(x,y)(m-1)^{-d(x,y)}$$

For $x, y \in \mathbb{Z}^d$

$$G(x,y) \asymp \|x-y\|^{4-d}, \ d \geq 5$$

$$g(x,y) \asymp \|x-y\|^{2-d}, \ d \geq 3$$

Convergence of Maximum for the infinite volume MM

We define

$$b_n := \sqrt{G(o,o)}\left[\sqrt{2\log|V_n|} - \frac{\log\log|V_n| + \log(4\pi)}{2\sqrt{2\log|V_n|}}\right], \quad a_n := G(o,o)b_n^{-1}.$$

Theorem (Cipriani,D.,Hazra,Ray, 21)

For any $\theta \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\max_{x\in V_n}\varphi_x - b_n}{a_n} \le \theta\right) = \exp(-\mathrm{e}^{-\theta}).$$

Convergence of maximum for 'normalized' finite volume MM

We define

$$B_n := \sqrt{2\log|V_n|} - \frac{\log\log|V_n| + \log(4\pi)}{2\sqrt{2\log|V_n|}}, \quad A_n := B_n^{-1}.$$

Theorem (Cipriani,D.,Hazra,Ray, '21)

Let m be large and define $\psi_x = \varphi_x / \sqrt{G_n(x,x)}$ *for* $x \in V_n$*. Then for any* $\theta \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbf{P}_n\left(\frac{\max_{x\in V_n}\psi_x - B_n}{A_n} \leq \theta\right) = \exp(-\mathrm{e}^{-\theta}).$$

Convergence of expected maximum

Theorem (Cipriani,D.,Hazra,Ray, '21)

$$\lim_{n\to\infty} \frac{\mathbf{E}_n\left[\max_{x\in V_n} \varphi_x\right]}{\sqrt{2\log|V_n|}} = \sqrt{G(o,o)}.$$

Proof idea for convergence of maximum

For $\theta \in \mathbb{R}$, let

$$u_n(\theta) := a_n\theta + b_n,$$
$$I_x := \mathbb{1}_{[\varphi_x > u_n(\theta)]}$$
$$W_n := \sum_{x \in V_n} I_x, \ \lambda_n := E[W_n]$$

We prove

$$\mathrm{d}_{TV}(W_n, \mathrm{Poi}(\lambda_n)) \to 0$$
$$\mathrm{d}_{TV}(\mathrm{Poi}(\lambda_n), \mathrm{Poi}(\mathrm{e}^{-\theta})) \to 0$$

For the first convergence we use Poisson approximation obtained by Holst and Janson.

Future plan

- Entropic repulsion for MM on tree
- MM on general graphs
- MM on random graphs.

References

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Thank you