Some presentations of the Hecke algebra of a reductive p-adic gp with I_n-level structure

(Based on joint work with Xuhua He):

The affine Weyl gp:-

Let $(X^*, X_*, R, R^\vee, \Pi)$ be a root system where Π is a set of simple roots.

$$W_0 = \langle s_\alpha \mid \alpha \in \Pi \rangle \text{ - Weyl gp.}$$

Let $\widetilde{W} = X_* \rtimes W_0 = \{ t_\lambda w \mid \lambda \in X_*,\ w \in W_0 \}$

$$\left((t_\lambda w)(t_{\lambda'} w') = t_{\lambda + w\lambda'} w w' \right).$$

$W_a = Q^\vee \rtimes W_0$, $Q^\vee = \mathbb{Z}$-lattice spanned by $R^\vee$.

- W_a is a Coxeter gp. It is generated by $S_a = \{ s_\alpha \mid \alpha \in \Pi \} \cup \{ t_{-\alpha^\vee} s_\alpha \mid \alpha \in \Pi_m \}$

Π_m = max. elements of R wrt Π.

(1) $W_a \trianglelefteq \widetilde{W}$ and $\Omega = X_* / Q^\vee$.

(2) There is a length function on $\widetilde{W}$ extending the one on W_a and Ω is set of

W_a ... of length ...
elements of

Hecke algebras:- The Braid gp $\mathcal{B}_{\widetilde{W}}$ of $\widetilde{W}$ is generated by T_w, $w \in \widetilde{W}$ subject to
$\underline{T_w T_{w'} = T_{ww'}}$ if $l(ww') = l(w) + l(w')$.

$$\mathcal{H}_{\widetilde{W}} = \mathbb{C}\mathcal{B}_{\widetilde{W}} \Big/ \left\langle (T_s - 1)(T_s + \boxed{q_s}) \mid s \in S_a \right\rangle$$

For each $s \in S_a$, let $q_s \in \mathbb{C}$, $q_s = q_{s'}$ whenever s, s' are $\widetilde{W}$ conj.

We will also denote the elements of $\mathcal{H}_{\widetilde{W}}$ as T_w..

Similarly $\left[\mathcal{H}_a = \mathbb{C}\mathcal{B}_{\underline{W_a}} \Big/ \left\langle (T_s - 1)(T_s + q_s) \mid s \in S_a \right\rangle \right.$

The presentation above can be refined

$$\langle T_s \mid s \in S_a \rangle \Big/ \begin{array}{l} (ss')^{m(s,s')} = 1 \\ \underbrace{T_s T_{s'} \cdots}_{m(s,s') \text{ factors}} = T_{s'} T_s \cdots \\ (T_s - 1)(T_s + q_s) = 0 \end{array}$$

Let G be a conn. red. gp over ... field F.

a non-arch local field.
A - max F-split torus.
(F_s - sep closure of F
$\breve{F}$: comp. of max un. extn of F in F_s).
$\phi(G, A)$ - relative roots.
Let $\widetilde{W} = N_G(A)(F) / Z_G(A)(F)_1 \to X_*(T) \rtimes W_0$

where $Z_G(A)(F)_1$ is the unique parahoric subgp of $Z_G(A)(F)$.
Eg: When G is split, $Z_G(A)(F)_1 = A(O)$.
$\widetilde{W}$ is called the Iwahori Weyl gp of $G(F)$.

$\widetilde{W} = N_G(T)(F) / T(O)$ $\quad 1 \to T(F)/T(F)_1 \to \widetilde{W} \to N_G(A)(F)/T(F)$

<u>Fact</u>:- Inside $\widetilde{W}$, there is a subg W_a which is the Iwahori Weyl gp attached to $\underline{G_{sc}(F)}$

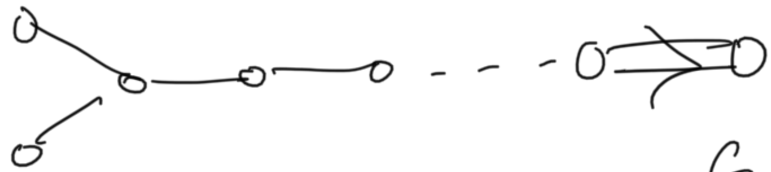
W_a is an affine Weyl gp in the sense discussed. $\exists$ a reduced root system $\Sigma \ni W_a = Q^\vee(\Sigma) \rtimes W_0(\Sigma)$.
... G is split, $\Sigma = \phi(G, A)$

Remark:- When G is split, ...

But when G is non-split, Σ is not nec. $\Phi(G, A)$ but is closely related to it.

Eg: Split B_n: $\Phi(G, A) = \Sigma$

G = odd. orth. gp.



G = q-s ram. unit gp over F

$B_n^\vee$:

$\Phi(G, A)$ is type C

Σ is of type B.

As before we have a length fn ℓ of $\tilde{W}$, extending W_a and $\Omega = \tilde{W}/W_a$ & length 0.

Consider $\mathcal{A} = \mathcal{A}(A, F)$ and let $v \in \mathcal{A}$ be special vertex. Then

$$\mathcal{A} = X_*(A) \otimes \mathbb{R}.$$

Let Φ_{aff} be the set of affine roots of (G, A).

With Σ, the null spaces of the affine fn $\Sigma + \mathbb{Z}$ are identical to those of Φ_{aff}, Weyl chamber

$\Rightarrow$ set of

A choice of $\mathcal{C} \subseteq A \longleftrightarrow$ finite simple roots Π

A choice of $\mathfrak{a}$ $\longleftrightarrow$ set of affine simple roots

$\downarrow$ alcove

$\Pi \cup \Pi_m$

$\downarrow$

$S_a \rightsquigarrow W_a$

Let $\mathcal{I}$ be the Iwahori subgp of $G(F)$ associated to $\mathfrak{a}$.

$G(F) \cong \mathcal{I} \backslash \widetilde{W} / \mathcal{I}$

$\longrightarrow$ Iwahori Weyl gp.

$\mathcal{H}(G(F), \mathcal{I})$ is Iwahori Hecke alg.

When G is semi, s.c, $\mathcal{H} = \mathcal{H}_a$ and then we have the two presentations discussed earlier.

For $\mathcal{H}_a$:

Generators $f_{\dot{w}} = 1_{\mathcal{I}\dot{w}\mathcal{I}}$, $w \in \widetilde{W}$

Relations: • $f_{\dot{w}} * f_{\dot{w}'} = f_{\dot{w}\dot{w}'}$ if $l(w) + l(w') = l(ww')$

• $f_s * f_s = (q_s - 1) f_s + q_s f_1$

$q_s = q^{\check{l}(s)}$

residue field

q - card of residue field of $\tilde{W}_{\mathcal{F}}^{\vee}$ $G(F)$

$\underline{\check{\ell}(s)}$ $\to$ length function of $\tilde{W}_{\mathcal{F}}^{\vee}$

$\tilde{W} \hookrightarrow \tilde{W}_{\mathcal{F}}^{\vee}$

$\tilde{W} = \boxed{\tilde{W}_{\mathcal{F}}^{\vee}}$

$\langle f_i \rangle$

The gp I admits a nice descending filtration $\hat{I}_m$, $m \geq 1$

$\underline{\mathcal{H}(G(F), I_m)}$.

Thm (G-H): The Hecke algebra $\mathcal{H}(G(F), I_n)$, $n \geq 1$ is generated by $1_{I_n g I_n}$, $g \in G(F)$ subject to:

Relations: Let $\pi : G(F) \to \tilde{W}$
$g \to w$ if $g \in IwI$

Then

(0) $I_n g I_n = I_n g' I_n$ then $1_{I_n g I_n} = 1_{I_n g' I_n}$

(1) If $\ell(\pi(gg')) = \ell(\pi(g)) + \ell(\pi(g'))$

then $1_{I_n g g' I_n} = 1_{I_n g I_n} * 1_{I_n g' I_n}$.

(2) If $\pi(g) = s \in S_a$, then

$$1_{I_n g I_n} * 1_{I_n g' I_n} = \begin{cases} q^{\check{\ell}(s)} 1_{I_n g g' I_n} & \text{if } \pi(gg') = \pi(g \\ q^{\check{\ell}(s)} \sum 1_{I_n u g g' I_n} & \text{if} \end{cases}$$

$u \in P_{s,n}/I_n$ $\pi(gg') \leq \pi(g')$

P_s is the parahoric subgp associated to s.

$\downarrow$

$P_{s,n}$ - nth Moy-Prasad filtration.

To refine the presentation:

$\widetilde{W} = N_a(A)(F) / Z_a(A)(F)_1$.

For $w \in \widetilde{W}$, can choose $g_w \in N_a(A)(F)$

~~the~~ $Z_a(A)(F)_1 \subseteq I$

For $n \geq 1$, this doesn't hold,

so we are looking for reps

$n_s \mid s \in S_a$ such that

(1) Coxeter relations $(s s')^{m(s,s')} = 1$

$\Rightarrow n_s n_{s'} \cdots = n_{s'} n_s \cdots$

(2) n_s^2.

<u>Tits</u>: For W_0, and S_0 - finite simple reflection.

reps satisfying (1) exist.

Further $n_s^2 = \alpha^\vee(-1)$, $s = s_\alpha$.

$1 \longrightarrow S_2 \longrightarrow T_0 \longrightarrow W_0 \longrightarrow 1$

elem. abelian

2 gp

Question:– Does the Tits of Iwahori Weyl gp exist?

Thm (G–H):– Assume $\widetilde{T}$ of $\widehat{W}$ exists the Hecke algebra $\mathcal{H}(G(F), I_n)$ admits a finer presentation $n_s, n_\tau \in G(F)$

(1) $\mathbb{1}_{I_n n_s I_n}$, $s \in S_a$

(2) $\mathbb{1}_{I_n n_\tau I_n}$ $\tau \in \Omega$.

(3) $1_{I_n g I_n}$, $g \in I/I_n$.

subject Coxeter relations.

On the existence of $\widetilde{T}$:

Thm (G–H): If G splits over $\breve{F}$, then $\widetilde{T}$ of $\widehat{W}$ over F exists

$$1 \to \underline{\underline{\widetilde{S}_2}} \to \underline{\widetilde{T}} \to \underline{\widehat{W}} \to 1.$$

$\downarrow$

2-gp

Remark:– $\widetilde{T}$ does not exist in general

For a wildly ramified unitary gp, $\widetilde{T}$ does not exist.