On the growth of cuspidal cohomology of GL_4 by symmetric cube transfer

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- Consider the group of idèles $\mathbb{A}_{\mathbb{E}}^{\times}$.
- Using automorphic induction, every character of $\mathbb{A}_{\mathbb{E}}^{\times}$ yields an automorphic form ϕ for $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$.
- It is a mass form if $\mathbb{E}$ is real quadratic field and a holomorphic modular form if $\mathbb{E}$ is imaginary quadratic field.

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- Also, for a fixed weight and level structure of $GL_2(\mathbb{A}_{\mathbb{Q}})$, the space of cusp forms has finite dimension.
- One can ask, how much of cuspidal cohomology of $GL_2(\mathbb{A}_{\mathbb{Q}})$ is obtained by automorphic induction.

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Theorem 1 (Ambi)

Let $k \geq 1, \epsilon \in (0,1)$. Then,

$$|C_k(N)| \ll_{k,\epsilon} N \cdot N'^{(1+\epsilon)} \qquad \text{as } N \longrightarrow \infty$$

where N' is the product of all distinct prime factors of N.

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- Similarly, one can ask, how much of cuspidal cohomology of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$ is obtained by symmetric square transfer.
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- Hence we estimate the cuspidal cohomology of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ which is obtained by symmetric cube transfer from $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to a specific level structure.

The level structure

Consider the algebraic group GL_m. For a finite prime p of $\mathbb{Q}$, let $\mathbb{Z}_p$ be the ring of p-adic integers.

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$$K_p^m(n) := \{x = (x_{i,j})_{m\times m} \in \mathrm{GL}_m(\mathbb{Z}_p) : x_{m,k} \in p^n\mathbb{Z}_p,\ 1 \leq k < m,\\ x_{m,m} - 1 \in p^n\mathbb{Z}_p\}.$$

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Let $N = \prod_{i=1}^{r} p_i^{n_i}$. Define a compact open subgroup $K_f^m(N) = \prod_p K_p$ of $\mathrm{GL}_m(\mathbb{A}_f)$ where

$$K_p = \begin{cases} K_{p_i}^m(n_i) & \text{if } p \mid N \text{ i.e., } p = p_i \\ \mathrm{GL}_m(\mathbb{Z}_p) & \text{if } p \nmid N \end{cases}$$

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$K_f^m(N) \subseteq \mathrm{GL}_m(\mathbb{A}_f)$ is called the level structure corresponding to N.

Conductor of a representation

- Let $(\rho, \mathcal{H}) = (\bigotimes_{p \leq \infty} \rho_p, \bigotimes_{p \leq \infty} \mathcal{H}_p)$ be an irreducible automorphic representation of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$.

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  For each finite prime p, the conductor of ρ_p is defined to be the smallest integer $c(\rho_p) \geq 0$ such that the set $\mathcal{H}_p^{K_p^m(c(\rho_p))}$ consisting of all $K_p^m(c(\rho_p))$-fixed vectors of $\mathcal{H}_p$ is non-zero.

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- The conductor of ρ is defined as

$$N_\rho = \prod_{p<\infty} p^{c(\rho_p)}.$$

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  We write $\pi \in \mathrm{Coh}(\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}}), \mu, K_f)$ if for the relative Lie algebra cohomology,

$$H^*(\mathfrak{gl}_4(\mathbb{R}), \mathbb{R}_+^* \cdot \mathrm{SO}_4(\mathbb{R}), \pi_\infty \otimes M_{\mu,\mathbb{C}}) \otimes \pi_f^{K_f} \neq 0.$$

Theorem 2 (Theorem 6.1, [KS])

Let $\mathbb{F}$ *be a number field and* π *be a automorphic cuspidal representation of* $\mathrm{GL}_2(\mathbb{A}_\mathbb{F})$*. Then sym*$^3(\pi)$ *is automorphic representation of* $\mathrm{GL}_4(\mathbb{A}_\mathbb{F})$*.*

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In particular, if $\mathbb{F} = \mathbb{Q}$ and π is the automorphic cuspidal representation attached to a non-dihedral holomorphic form of weight ≥ 2, then sym$^3(\pi)$ is cuspidal.

- Let $E_k(N)$ denote the set of cuspidal automorphic representations of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ corresponding to the level structure $K_f^4(N)$ and highest weight ν_k which are obtained by symmetric cube transfer of cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ of highest weight $\lambda_k = \left(\frac{k}{2} - 1, 1 - \frac{k}{2}\right)$.

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Theorem 3 (Bhagwat,....)

Let $k \geq 2$ be an even integer and $p \geq 3$ be a prime. Then

$$|E_k(p^n)| \gg_k p^n \qquad \text{as } n \to \infty$$

where the implied constant depends on k.

Lemma 4

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Then the following holds:

$$\Pi \in Coh(GL_4(\mathbb{A}_{\mathbb{Q}}), \nu_k, K_f^4(c(\Pi)),$$

$$\nu_k = \left(3\left(\frac{k}{2} - 1\right),\ \frac{k}{2} - 1,\ 1 - \frac{k}{2},\ 3\left(1 - \frac{k}{2}\right)\right)$$

We now give an outline of the proof of theorem 3.

proof

Let $\pi = \otimes_{q \leq \infty} \pi_q$ be an cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to the level structure $K_f^2(p^n), n \geq 1$.

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Step 1: Let $c(\pi_p), c(\Pi_p)$ be the conductor of the representations of π_p and $\Pi_p = \mathrm{sym}^3(\pi_p)$ respectively. Then, for odd prime p and taking π_p to be supercuspidal, we have, $c(\pi_p) \leq c(\Pi_p) \leq 2c(\pi_p)$.

Proof(continued).

Step 2: By Theorem 2, either $\mathrm{sym}^3(\pi) \in \cup_{(j\geq 1)} E_k(p^j)$ or π is obtained by automorphic induction.

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We will count the contribution of supercuspidal representations π_p to $\cup_{(j\geq 1)} E_k(p^j)$.

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Using this and the relation between the conductors, we get

$$\{ \text{ supercuspidal within } \oplus_{1\leq i\leq n} S_k^{\mathrm{new}}(\Gamma(p^i))\backslash C_k(p^n)\} \subseteq E_k(p^{2n}).$$

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We now use Theorem 1 and the dimension formula of the space of newforms to get the lower bound.

Using the fact that $\dim_{\mathbb{C}} S_k(\Gamma_1(N)) \sim_k N^2$, we have the following upper bound of $|E_k(p^n)|$:

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Corollary 5

For $p \geq 3$,

$$|E_k(p^n)| \ll_k p^{2n} \qquad \text{as } n \to \infty.$$

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Thank you