Strong contrasting diffusivity in general oscillating domains: Homogenization of optimal control problems

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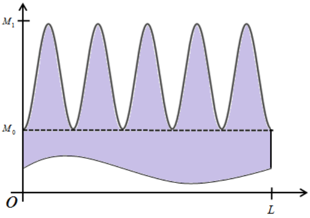
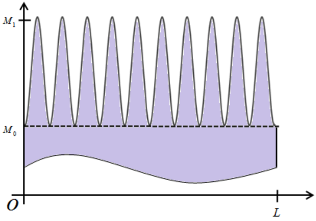
Oscillating domain

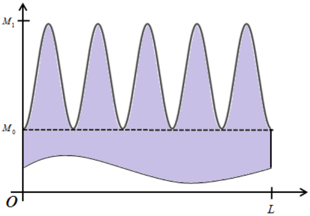


$\varepsilon = \frac{1}{5}$



$\varepsilon = \frac{1}{10}$

Oscillating domain



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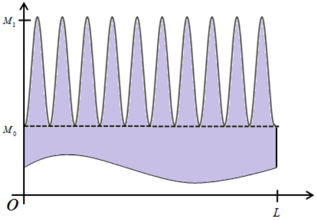

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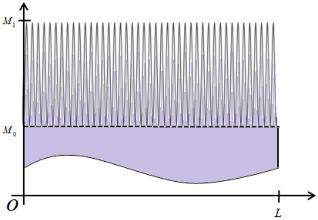
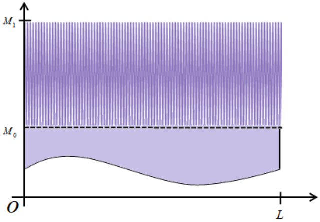


$\varepsilon = \frac{1}{50}$



$\varepsilon = \frac{1}{100}$

Introduction: Strong contrasting diffusivity

- Partial differential equations (PDEs) with strong contrasting diffusivity are appeared in several context such as: modeling of several multi-scale physical problems such as the double porosity model, effective properties of composite material with soft and hard core, effective conductivity of composites made of materials having high and low conductivities, etc.

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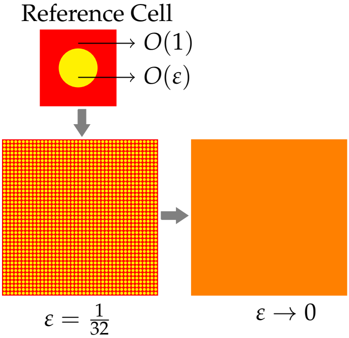

Figure 1: Composite material

- G. P. Panasenko, N. S. Bakhvalov, A. Yu. Kosarev , H. Charef, M. Bellieudand, G. Bouchitté had studied several problem on this kind of composites.

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To be more precised, they have considered domain like the following;

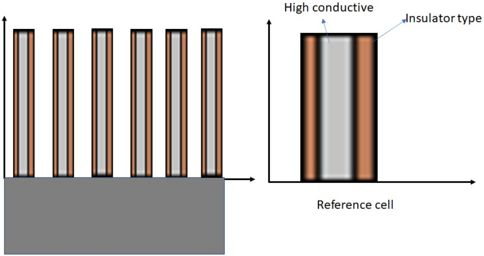

Figure 2: Pillar type oscillating domain

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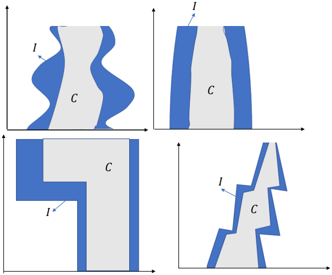

Figure 3: Typical example of reference cells

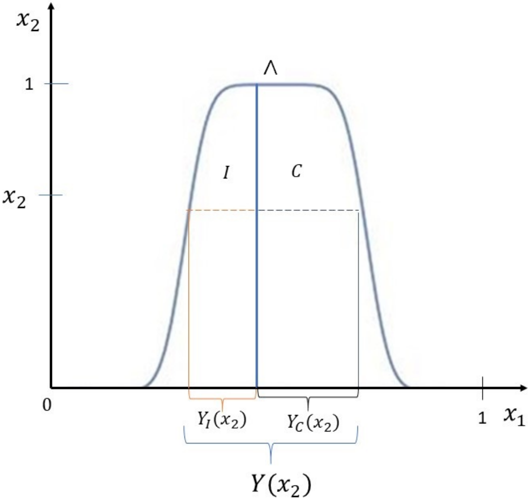

Figure 4: Reference cell

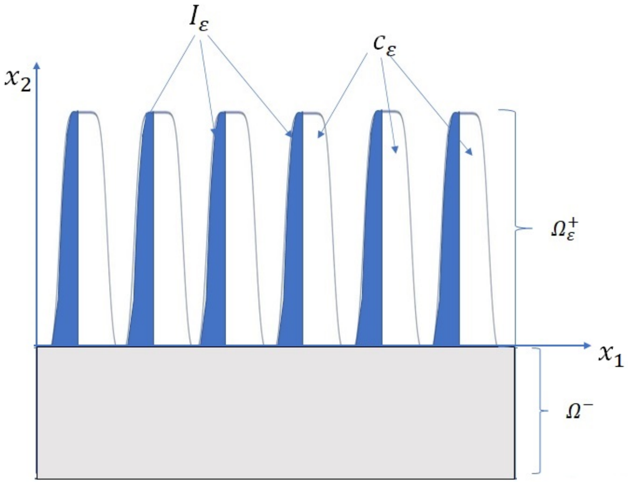

Figure 5: Oscillating domain

Oscillating domain

For $\varepsilon = \frac{1}{m}$ where $m \in \mathbb{Z}^+$, (in fact, one can take any $\varepsilon \to 0$) define

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- The interface between Ω^+ and Ω^- is demoted by γ, which is given by $\gamma = \{(x_1, 0) : \ x_1 \in (0,1)\}$.

Variational problem

- We want to consider the following ε dependent variational problem,

$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon} \right) \nabla u_\varepsilon \nabla \phi + \int_{\Omega_\varepsilon} u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi, \end{cases} \tag{1}$$

for all $\phi \in H^1(\Omega_\varepsilon)$, where $f \in L^2(\Omega)$.

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- Our aim is to analyze the asymptotic behavior of the above variational form as the oscillating parameter $\varepsilon \to 0$.

Unfolding operator and properties

The unfolded domain corresponding to the upper part Ω_ε^+ is given by

- $\Omega^u = \{(x_1, x_2, y_1) : (x_1, x_2) \in \Omega^+, y_1 \in Y(x_2)\}$.

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- $\Omega_C^u = \{(x_1, x_2, y_1) : (x_1, x_2) \in \Omega^+, y_1 \in Y_C(x_2)\}$.

Definition.

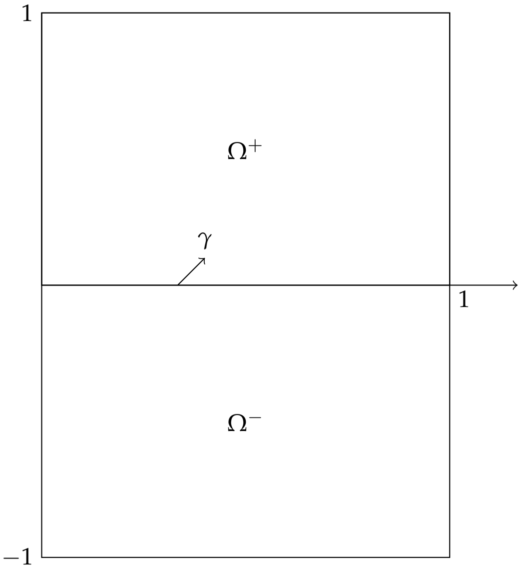
(The unfolding operator) Let $\phi^\varepsilon : \Omega^u \to \Omega_\varepsilon^+$ *be defined as* $\phi^\varepsilon(x_1, x_2, y_1) = \left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right)$. *The* ε*- unfolding of a function* $u : \Omega_\varepsilon^+ \to \mathbb{R}$ *is the function* $u \circ \phi^\varepsilon : \Omega^u \to \mathbb{R}$. *The operator which maps every function* $u : \Omega_\varepsilon^+ \to \mathbb{R}$ *to its* ε*-unfolding is called the unfolding operator. Let the unfolding operator is denoted by* T^ε*, that is,*

$$T^\varepsilon : \{u : \Omega_\varepsilon^+ \to \mathbb{R}\} \to \{T^\varepsilon u : \Omega^u \to \mathbb{R}\}$$

is defined by

$$T^\varepsilon u(x_1, x_2, y_1) = u\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y_1, x_2\right) \text{ for all } (x_1, x_2, y_1) \in \Omega^u.$$

Limit domain



Limit function spaces

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- For any $x_2 \in (0,1)$, define $V^{x_2} = \{w \in H^1(Y(x_2)) : w = 0 \ \ in\ Y_C(x_2)\}$ with the following norm

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- $V(\Omega) = \left\{\psi \in L^2(\Omega^u) : \psi = 0\ in\ \Omega_C^u, \frac{\partial\psi}{\partial y_1} \in L^2(\Omega^u)\right\}$ with the following norm

$$\|\psi\|_{V(\Omega)} = \|\psi\|_{L^2(\Omega^u)} + \left\|\frac{\partial\psi}{\partial y_1}\right\|_{L^2(\Omega^u)}.$$

Limit problem

The limit variational problem :find $u = (u^+, u^-) \in H(\Omega)$ such that

$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x) u^+ \phi + \int_{\Omega^-} u^- \phi \\ \qquad\qquad \displaystyle + \int_{\Omega^-} \nabla u^- \nabla \phi = \int_{\Omega^+} \alpha(x) f \phi + \int_{\Omega^-} f \phi, \end{cases}$$

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$$\left\{ \begin{aligned} \int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x) u^+ \phi + \int_{\Omega^-} u^- \phi \\ + \int_{\Omega^-} \nabla u^- \nabla \phi = \int_{\Omega^+} \alpha(x) f\phi + \int_{\Omega^-} f\phi, \end{aligned} \right.$$

for all $\phi \in H(\Omega)$, here $\alpha(x) = \left(|Y(x_2)| - \int_{Y_I(x_2)} \xi dy_1\right)$,where

$$\left\{ \int_{Y(x_2)} \frac{\partial \xi(x_2, y_1)}{\partial y_1} \frac{\partial w(y_1)}{\partial y_1} + \int_{Y(x_2)} \xi(x_2, y_1) w(y_1) = \int_{Y(x_2)} w(y_1), \text{ for all } w \in V^{x_2}. \right.$$

- Now we will show $\alpha(x) = \left(|Y(x_2)| - \int_{Y_I(x_2)} \xi\right) > 0$. By taking $w = \xi$ in the cell problem, we get

$$\int_{Y(x_2)} \left(\left| \frac{\partial \xi}{\partial y_1} \right|^2 + \xi^2 \right) = \int_{Y(x_2)} \xi \Rightarrow \|\xi\|_{L^2(Y_I(x_2))} \leqslant |Y_I(x_2)|^{\frac{1}{2}}$$

$$\begin{aligned} \left(|Y(x_2)| - \int_{Y_I(x_2)} \xi\right) &\geqslant (|Y(x_2)| - |Y_I(x_2)|^{1/2} \|\xi\|_{L^2(Y_I(x_2))}) \\ &\geqslant |Y(x_2)| - |Y_I(x_2)| = |Y_C(x_2)| > \delta. \end{aligned}$$

Theorem (Nandakumaran-Sufian, JDE-2021).

For every $\varepsilon > 0$, let u_ε be the unique solution to the considered variational problem. Let $H(\Omega)$ and V^{x_2} be defined as earlier and $u = (u^+, u^-) \in H(\Omega)$ be the unique solution of the limit variational form.

Theorem (Nandakumaran-Sufian, JDE-2021).

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$$u_\varepsilon^- \rightharpoonup u^- \text{ weakly in } H^1(\Omega^-),$$

$$\begin{cases} \widetilde{u_\varepsilon^+} \rightharpoonup |Y(x_2)|u^+ + \int_{Y_I(x_2)} (f - u^+)\xi(x_2, y_1)dy_1 \\ \chi^+_{C_\varepsilon} \widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup 0, \quad \chi^+_{C_\varepsilon} \widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup |Y_C(x_2)| \dfrac{\partial u^+}{\partial x_2} \\ \varepsilon\chi^+_{I_\varepsilon} \widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup (f - u^+) \int_{Y_I(x_2)} \dfrac{\partial \xi}{\partial y_1} dy_1, \quad \varepsilon\chi^+_{I_\varepsilon} \widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup 0 \\ \text{weakly in } L^2(\Omega^+) \end{cases}$$

as $\varepsilon \to 0$.

Control on C_ε

For $\theta_\varepsilon \in L^2(C_\varepsilon)$ consider the cost functional

$$J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_{C_\varepsilon} |\theta_\varepsilon|^2$$

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The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(C_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon)\}. \tag{2}$$

- We will use the characterization of optimal control $\bar{\theta}_\varepsilon$ by introducing the adjoin state $\bar{v}_\varepsilon$ which is the solution of the following variational form

$$\left\{\begin{array}{l} \text{find } \bar{v}_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla \bar{v}_\varepsilon \nabla \phi + \bar{v}_\varepsilon \phi = \int_{\Omega_\varepsilon} (\bar{u}_\varepsilon - u_d)\phi, \end{array}\right. \tag{3}$$

for all $\phi \in H^1(\Omega_\varepsilon)$.

- We will use the characterization of optimal control $\bar{\theta}_\varepsilon$ by introducing the adjoin state $\bar{v}_\varepsilon$ which is the solution of the following variational form

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for all $\phi \in H^1(\Omega_\varepsilon)$.

Theorem.

Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (2) *and $\bar{v}_\varepsilon$ be the unique solution of* (3). *Then $\bar{\theta}_\varepsilon$ is characterized by*

$$\bar{\theta}_\varepsilon = -\chi_{C_\varepsilon} \frac{1}{\beta} \bar{v}_\varepsilon. \tag{4}$$

Limit optimal control

Cost functional: For $\theta \in L^2(\Omega^+)$

$$J(u,\theta) = \frac{1}{2}\int_{\Omega^+}\int_{Y(x_2)} \left|(1-\xi)u^+ + f\xi - u_d\right|^2 + \frac{1}{2}\int_{\Omega^-} |u^- - u_d|^2 + \frac{\beta}{2}\int_{\Omega^+} |Y_C(x_2)||\theta|^2$$

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Limit state equation:

$$\begin{cases} \text{find } u \in H(\Omega), \text{ such that,} \\ \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x) u^+ \phi + \int_{\Omega^-} u\phi + \int_{\Omega^-} \nabla u^- \nabla \phi \\ \qquad\qquad \displaystyle = \int_{\Omega^+} \alpha(x) f\phi + \int_{\Omega^-} f\phi + \int_{\Omega^+} |Y_C(x_2)|\theta\phi, \\ \text{for all } \phi \in H(\Omega). \end{cases}$$

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- $J(\bar{u},\bar{\theta}) = \inf\{J(u,\theta)\}$

Adjoint equation:

$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial \bar{v}^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x) \bar{v}^+ \phi + \int_{\Omega^-} \bar{v}^- \phi + \int_{\Omega^-} (\nabla \bar{v}^- \nabla \phi \\ \displaystyle = \int_{\Omega^+} \left[\left(\int_{Y(x_2)} (1-\xi)^2 dy_1 \right) \bar{u}^+ - \alpha(x) u_d + \left(\int_{Y_I(x_2)} (\xi - \xi^2) dy_1 \right) f \right] \phi \\ \displaystyle \qquad + \int_{\Omega^-} (\bar{u}^- - u_d) \phi. \end{cases}$$

Adjoint equation:

$$\left\{\begin{aligned} &\int_{\Omega^+} |Y_C(x_2)| \frac{\partial \bar{v}^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x)\bar{v}^+\phi + \int_{\Omega^-} \bar{v}^-\phi + \int_{\Omega^-} (\nabla \bar{v}^- \nabla \phi \\ &= \int_{\Omega^+} \left[\left(\int_{Y(x_2)} (1-\xi)^2 dy_1\right)\bar{u}^+ - \alpha(x)u_d + \left(\int_{Y_I(x_2)} (\xi - \xi^2) dy_1\right) f\right]\phi \\ &\qquad + \int_{\Omega^-} (\bar{u}^- - u_d)\phi. \end{aligned}\right.$$

- Optimal control is given by $\bar{\theta} = -\frac{1}{\beta}\bar{v}^+$

Control on I_ε

For $\theta_\varepsilon \in L^2(I_\varepsilon)$, consider the following L^2-cost functional

$$J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_{\mathrm{I}_\varepsilon} |\theta_\varepsilon|^2,$$

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$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{\mathrm{C}_\varepsilon} + \varepsilon^2 \chi_{\mathrm{I}_\varepsilon}\right) \nabla u_\varepsilon \nabla \phi + u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi + \int_{\Omega_\varepsilon} \chi_{\mathrm{I}_\varepsilon} \theta_\varepsilon \phi, \end{cases} \tag{5}$$

for all $\phi \in H^1(\Omega_\varepsilon)$.

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$$J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_{\mathrm{I}_\varepsilon} |\theta_\varepsilon|^2,$$

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for all $\phi \in H^1(\Omega_\varepsilon)$. The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(I_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon) : \ (u_\varepsilon, \theta_\varepsilon) \text{ satisfies (5)}\}. \tag{6}$$

Theorem (Characterization).

Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ *be the optimal solution to the optimal control problem* (6) *and* $\bar{v}_\varepsilon$ *be the unique solution of the adjoint state. Then* $\bar{\theta}_\varepsilon$ *can be written as* $\bar{\theta}_\varepsilon = -\chi_{I_\varepsilon} \dfrac{1}{\beta} \bar{v}_\varepsilon$.

Partial scales separation

Reduced cost functional: The L^2-cost functional reduces to

$$J(u, u_{11}, \theta, \theta_1) = \frac{1}{2}\int_{\Omega^+}\int_{Y(x_2)}((1-\xi)u^+ + \xi f + u_{11} - u_d)^2 + \int_{\Omega^-}(u^- - u_d)^2 + \frac{\beta}{2}\int_{\Omega^+}\int_{Y(x_2)}(\theta+\theta_1)^2$$

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Reduced state equation: The state $(u,u_{11}) \in H(\Omega)\times V(\Omega)$ satisfies the following system,

$$\begin{cases} \displaystyle\int_{\Omega^+}|Y_C(x_2)|\frac{\partial u^+}{\partial x_2}\frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+}\alpha(x)u^+\phi^+ + \int_{\Omega^-}\nabla u^-\nabla\phi^- + \int_{\Omega^-}u^-\phi \\ \displaystyle\quad = \int_{\Omega^+}\int_{Y(x_2)}((1-\xi)f + (1-\xi)(\theta+\theta_1))\phi^+ + \int_{\Omega^-}f\phi^-, \\ \displaystyle\int_{\Omega^u}\frac{\partial u_{11}}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u}u_{11}\phi_1 = \int_{\Omega^u}(\theta+\theta_1)\phi_1, \end{cases}$$

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$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2} \frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+} \alpha(x) u^+ \phi^+ + \int_{\Omega^-} \nabla u^- \nabla \phi^- + \int_{\Omega^-} u^- \phi \\ \quad = \displaystyle\int_{\Omega^+} \int_{Y(x_2)} ((1-\xi)f + (1-\xi)(\theta + \theta_1))\phi^+ + \int_{\Omega^-} f \phi^-, \\ \displaystyle\int_{\Omega^u} \frac{\partial u_{11}}{\partial y_1} \frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} u_{11} \phi_1 = \int_{\Omega^u} (\theta + \theta_1) \phi_1, \end{cases}$$

- $J(\bar{u}, \bar{u}_{11}, \bar{\theta}, \bar{\theta}_1) = \inf\{J(u, u_{11}, \theta, \theta_1)\}$

- **Reduced adjoint state:** The adjoint state $(\bar{v}, \bar{v}_{11}) \in H(\Omega) \times V(\Omega)$ satisfies the following system,

$$\begin{cases} \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial \bar{v}^+}{\partial x_2} \frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+} \alpha(x) \bar{v}^+ \phi^+ + \int_{\Omega^-} \nabla v^- \nabla \phi^- + \int_{\Omega^-} u^- \phi^- \\ \qquad = \displaystyle\int_{\Omega^-} (\bar{u}^- - u_d) \phi^- + \int_{\Omega^+} \int_{Y(x_2)} \left[(1-\xi)^2 \bar{u}^+ + \xi(1-\xi) f \right] \phi^+ \\ \qquad\qquad + \displaystyle\int_{\Omega^+} \int_{Y(x_2)} \left[(1-\xi) \bar{u}_{11} - (1-\xi) u_d \right] \phi^+, \\ \displaystyle\int_{\Omega^u} \frac{\partial \bar{v}_{11}}{\partial y_1} \frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} \bar{v}_{11} \phi_1 = \int_{\Omega^u} [(1-\xi) \bar{u}^+ + \xi f + \bar{u}_{11} - u_d] \phi_1, \end{cases}$$

for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$.

- **Reduced adjoint state:** The adjoint state $(\bar{v}, \bar{v}_{11}) \in H(\Omega) \times V(\Omega)$ satisfies the following system,

$$
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\displaystyle\quad = \int_{\Omega^-} (\bar{u}^- - u_d) \phi^- + \int_{\Omega^+} \int_{Y(x_2)} \left[(1-\xi)^2 \bar{u}^+ + \xi(1-\xi) f \right] \phi^+ \\
\displaystyle\qquad + \int_{\Omega^+} \int_{Y(x_2)} \left[(1-\xi) \bar{u}_{11} - (1-\xi) u_d \right] \phi^+, \\
\displaystyle\int_{\Omega^u} \frac{\partial \bar{v}_{11}}{\partial y_1} \frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} \bar{v}_{11} \phi_1 = \int_{\Omega^u} [(1-\xi) \bar{u}^+ + \xi f + \bar{u}_{11} - u_d] \phi_1,
\end{cases}
$$

  for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$.
- The optimal control is given by $\bar{\theta} + \bar{\theta}_1 = -\frac{1}{\beta} [(1-\xi) \bar{v}^+ + \bar{v}_{11}]$ in Ω_{I}^u.

Remark

In the above variational problem we have considered the contrasting diffusive coefficients as 1 and ε^2. In fact, we can consider the coefficient of the form $O(1)$ and α_ε^2, where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. According to the limit $k = \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\varepsilon}$, we will get three different limit problems for, $k = 0, k = \infty$ and $k \in (0, \infty)$.

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Thank you for your attention!