An application of SCP for the p-Laplacian

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Joint work with
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September 18, 2021

Plan of the talk

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- Motivation

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- Shape Optimization Problems

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Such problems have stimulated much mathematical thought.

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In many cases, the functional being minimized depends on the solution of a given partial differential equation defined on the variable domain.

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Mathematically,
For any piecewise smooth simple closed curve C in a plane with arc-length ℓ and enclosing area $A > 0$ we have

$$\ell^2(C) \geq 4\,\pi\,A(C)$$

and equality holds if and only if C is a circle of radius $\sqrt{\frac{A}{\pi}}$.

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- For example, The celebrated Faber-Krahn Theorem: Amongst all domains with fixed volume the ball minimizes the first Dirichlet Eigenvalue of the Laplacian.
- Let $\lambda_1(\Omega)$ denote the first Dirichlet eigenvalue of the Laplacian on a bounded domain Ω in $\mathbb{R}^n$. Then

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

where B is a ball in $\mathbb{R}^n$ such that $Vol(B) = Vol(\Omega)$, and equality holds iff $\Omega = B$.

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where

- $\Omega = B_1 \setminus \bar{B}_0$,
- B_0, B_1 are open (geodesic) balls in a Riemannian manifold (M, g) such that $\overline{B_0} \subset B_1$,
- $\Delta_p u := div\left(\|\nabla u\|^{p-2} \nabla u\right) \quad (1 < p < \infty)$.

$\lambda_1(\Omega)$

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Without loss of generality, an eigenfunction $y_1(\Omega)$ corresponding to $\lambda_1(\Omega)$ can be chosen to be positive and of unit L^p-norm.

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Theorem

The first Dirichlet eigenvalue λ_1 attains its maximum if and only if the balls are concentric.

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Generalizations

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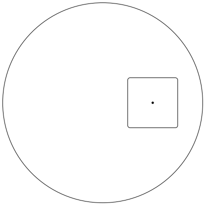
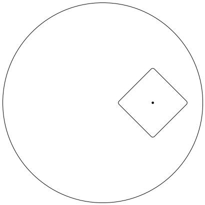
- With Rajesh Mahadevan — Generalized these results for the p-Laplacian (Δ_p) operator, $1 < p < \infty$.

With Souvik Roy — The case $p = 2$ for $\mathcal{F} = \{B \setminus P \subset \mathbb{R}^2\}$ where

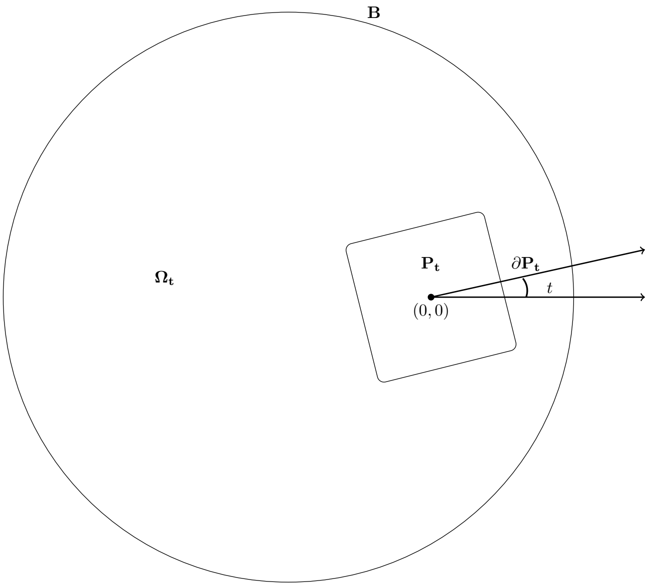
- B is a given disk,
- P is a domain having a $\mathbb{D}_n$ symmetry, n even.
- Partial results were obtained for the n odd case.
- λ_1 is optimum **when** an axis of symmetry of P coincides with a diameter of B.
- Complete characterization of the maximizing and minimizing domains for n even case.
- Monotonicity of λ_1 between the consecutive maximizing and minimizing configurations.



(a) OFF position



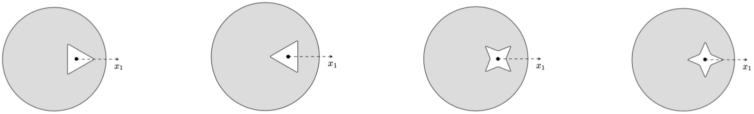
(b) ON position



(c) Configuration at time t

Images

OPTIMAL SHAPES FOR THE FIRST DIRICHLET EIGENVALUE



(A) OFF (B) ON (C) OFF (D) ON

FIGURE 1. OFF and ON configurations: P having $\mathbb{D}_3$ symmetry-(A) and (B), P having $\mathbb{D}_4$ symmetry-(C) and (D).

Generalizations

With Mrityunjoy Ghosh —

- λ_1 is optimum **only when** an axis of symmetry of P coincides with a diameter of B.
- Complete characterization of the maximizing and minimizing domains for each $n \geq 2$.
- Monotonicity of λ_1 between the consecutive maximizing and minimizing configurations
- We rule out the possibilty of the nodal lines for a second eigenfunction having a dihedral symmetry of same order as that of P.

The key highlights:

- For $\frac{3}{2} < p < \infty$, we obtain the strict monotonicity of λ_1.
- We use a strong comparison principle due to Sciunzi in the proof. This paper gives a direct and simpler proof for this strict monotonicity as compared to the one available in [Anoop-Sasi-Bobkov]. This proof also extends results obtained in [Anisa-Mahadevan].
- For $1 < p \leq \frac{3}{2}$ we obtain the non-strict monotonicity of λ_1 on the perturbed domains using the weakcomparison principle due to [Chorwadwala-Mahadevan-Toledo].
- Monotonicity results for λ_1 implies for $\frac{3}{2} < p < \infty$ that the nodal set of a second eigenfunction can not possess a dihedral symmetry of the same order as that of P.
- To the best our knowledge, this is the first result regarding the geometry of the nodal set of a second eigenfunction for doubly connected planar domains other than the domains bounded by two spheres.
- We also prove the conjecture posed in [Chorwadwala-Roy] for the n odd and $p = 2$ case.

Key Steps in the Proof

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$$\lambda_1'(t) = -(p-1) \int_{\partial P_t} \left| \frac{\partial u_t}{\partial \eta_t}(x) \right|^p \langle \eta_t, v \rangle (x) \, dS.$$

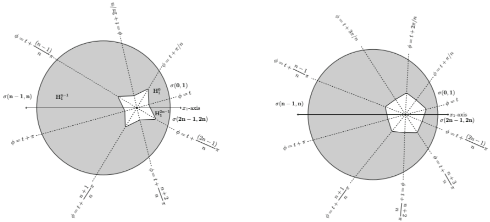
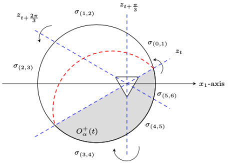


Fig. 7 Sectors of Ω_t for $n = 4, 5$



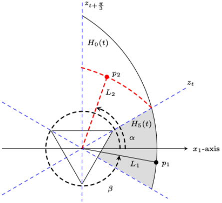


FIGURE 2. (A) Sector pairings for n odd. (B) Containment of sectors which lie on either side of the x_1-axis as in Lemma 3.1.

(3) Maximum Principles and Comparison Principles

Lemma of Hopf for the linear case:

Lemma of Hopf for the linear case:
Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$. Let $u \in C^1(\overline{\Omega})$ satisfy

$$L(u) := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) \geq 0 \text{ in } \Omega,$$

where L is a uniformly elliptic operator on Ω and $a_{ij} \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Suppose that $u \leq M$ in Ω and $u(x_0) = M$ for some $x_0 \in \partial\Omega$ such that the interior sphere condition is satisfied at x_0. Then

$$\frac{\partial u}{\partial \eta}(x_0) > 0, \text{ unless } u \equiv M \text{ in } \Omega,$$

where η denotes the unit outward normal to Ω on $\partial\Omega$.

Strong Comparison Principle for the p-Laplacian from [Sciunzi2014]

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$. Let g be a locally Lipschitz function such that $g(x) > 0$ for $x > 0$. Let $u, v \in C^1(\overline{\Omega})$ satisfy

$$-\Delta_p u - g(u) \leq -\Delta_p v - g(v) \ \text{ in } \Omega.$$

Assume that either u or v is a non-negative solution of $-\Delta_p w = g(w)$ for $\frac{2N+2}{N+2} < p < \infty$. Then if $u \leq v$ in a connected subdomain $\Omega' \subset \Omega$, it follows that

$$u < v \ \text{ in } \Omega', \ \text{ unless } u \equiv v \text{ in } \Omega'.$$

A version of the Weak Comparison Principle from [Chorwadwala-Mahadevan-Toledo]:

A version of the Weak Comparison Principle from [Chorwadwala-Mahadevan-Toledo]:

Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 2$. Let $u, v \in C^1(\overline{\Omega})$ be two non-negative weak solutions of $-\Delta_p w = \lambda w^{p-1}$ on Ω for some p, $1 < p < \infty$. Then if $u \leq v$ on $\partial\Omega$,

$$u \leq v \text{ on } \Omega.$$

Furthermore, if $x_0 \in \partial\Omega$ be such that $u(x_0) = 0 = v(x_0)$, then $\frac{\partial v}{\partial \eta}(x_0) \leq \frac{\partial u}{\partial \eta}(x_0)$.

Consider the following Dirichlet Boundary Value Problem:

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

where,

- $\Omega = B \setminus P \subset \mathbb{R}^2$,
- B is a bounded open disk in $\mathbb{R}^2$,
  and
- P is a compact simply connected subset of $\mathbb{R}^2$ such that
  (a) the boundary ∂P is a simple closed $\mathcal{C}^2$ curve in $\mathbb{R}^2$,
  (b) P is invariant under the action of the dihedral group D_n, n even.
  (c) $Area(P) = A$, $A > 0$ fixed,
  (d) the distance $d(\underline{o}, x)$ between the 'center' $\underline{o}$ of P and a point $x \in \partial P$ is monotonic as a function of the argument ϕ in a sector delimited by two consecutive axes of symmetry of P,
  (e) $\rho(P) \subset B \; \forall \rho \in D_n$,
  (f) the center of P is different from the center of B.

- For $t \in \mathbb{R}$, let $\rho_t \in SO(2)$ denote the rotation in $\mathbb{R}^2$ about $\underline{o} = (0,0)$ in the anticlockwise direction by an angle t, that is, $\rho_t(\zeta) = e^{it}\zeta$ for $\zeta \in \mathbb{C} \cong \mathbb{R}^2$,
- For $t \in [0, 2\pi)$, let $P_t := \rho_t(P)$ and $\Omega_t := B \setminus P_t$,
- $\mathcal{F} := \{\Omega_t \mid t \in [0, 2\pi)\}$.
- **Goal:**
- To find Ω_{min} such that $\lambda_1(\Omega_{min}) = \min_{\Omega \in \mathcal{F}} \lambda_1(\Omega)$
- and to find Ω_{max} such that $\lambda_1(\Omega_{max}) = \max_{\Omega \in \mathcal{F}} \lambda_1(\Omega)$.

The ON and OFF configurations

- Let C_1 and C_2 denote the 'incircle' and the 'circumcircle' of P_t respectively.
- $V_{in} := \partial P_t \cap C_1$ called 'inner vertices', $V_{out} := \partial P_t \cap C_2$ called 'outer vertices'
- A radius of C_1 containing an inner vertex of P_t will be called an 'inradius'. Similarly, a radius of C_2 containing an outer vertex of P_t will be called an 'circumradius of P_t'
- For P_t such that $\overline{co(C_2(P_t))} \subset B$, we say

OFF P_t is in an OFF position w.r.t. B if an inradius of P_t is along a diameter of B

ON P_t is in an ON position w.r.t. B if a circumradius of P_t is along a diameter of B.

Theorem

The fundamental Dirichlet eigenvalue $\lambda_1(\Omega_t)$ for $\Omega_t \in \mathcal{F}$ is optimal precisely for those $t \in [0, 2\pi)$ for which an axis of symmetry of P_t coincides with a diameter of B.

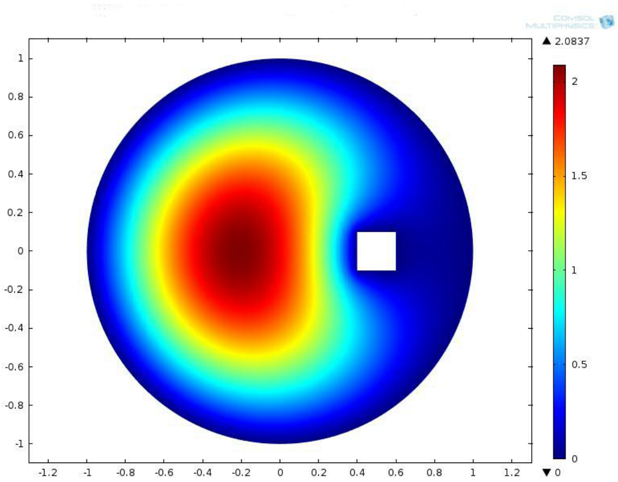
The maximizing configurations are the ones corresponding to those $t \in [0, 2\pi)$ for which P_t is in an ON position w.r.t. B,

and the minimizing configurations are the ones corresponding to those $t \in [0, 2\pi)$ for which P_t is in an OFF position w.r.t. B.

Theorem

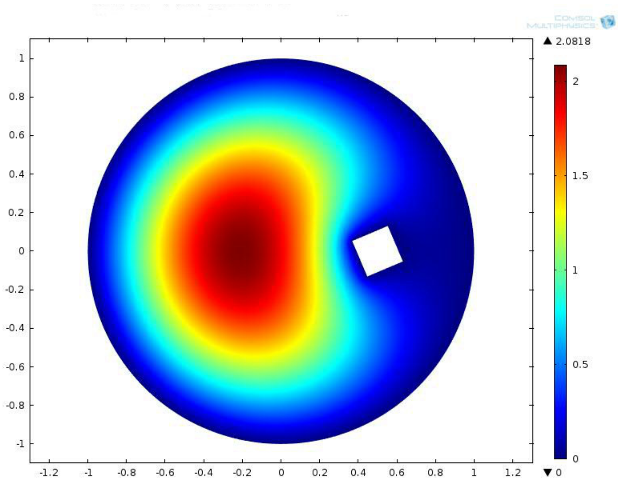
For each $k = 0, 1, 2, \ldots, , 2n-1$, $\lambda_1' \left(k \frac{\pi}{n}\right) = 0$.

For each $t \in (0, \frac{\pi}{n})$, $\lambda_1'(t) > 0$.

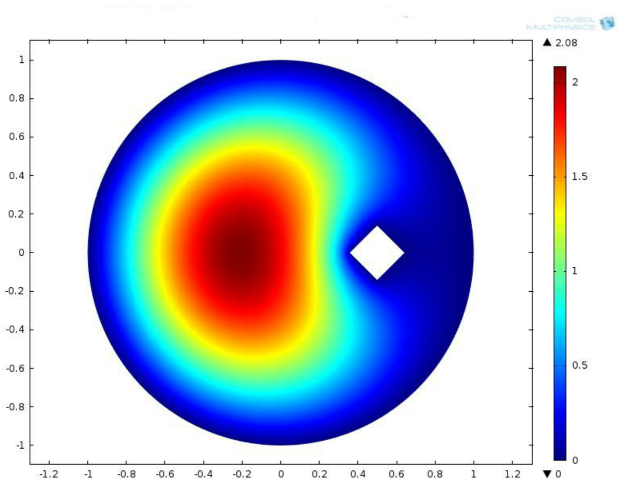


(a) Initial OFF position

Numerical Evidence

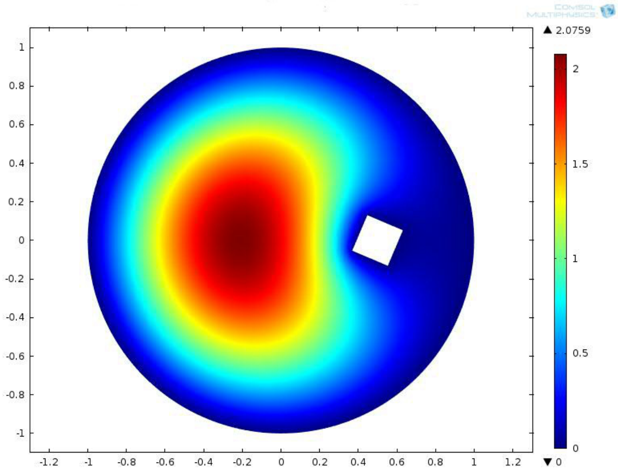


(b) Intermediate Position



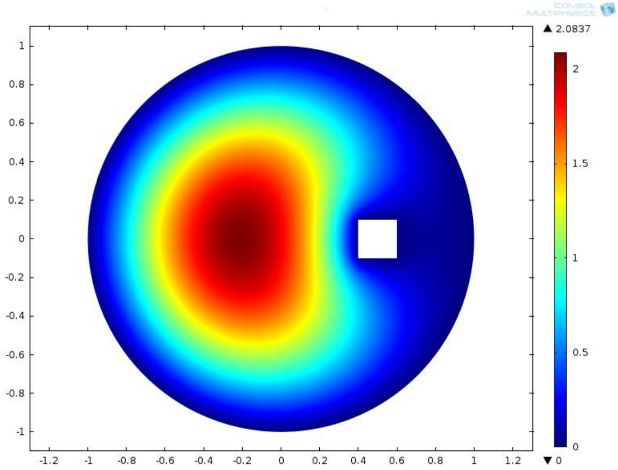
(c) ON position

Figure: Simulations of ON, OFF and intermediate positions of the square.



(a) Another intermediate position

Figure: Simulations of ON, OFF and intermediate positions of the square.



(a) OFF position

θ	λ	Configuration
0	7.5735	OFF
$\pi/8$	7.5739	–
$\pi/4$	7.5742	ON
$3\pi/8$	7.5739	–
$\pi/2$	7.5735	OFF

Table: Variation of λ_1 with rotation of the square P by an angle θ from the initial configuration.

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Thank You!