Combinatorial games on multi-type Galton-Watson trees

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Where it all began

- Galton-Watson Games, by Alexander Holroyd and James Martin.
- They studied combinatorial games on rooted Galton-Watson trees.

Galton-Watson trees:

- Start with the root ϕ. Let it have X_0 children where $X_0 \sim \chi$.
- If $X_0 = m$ for some $m \neq 0$, then name these children $v_1, \ldots, v_m$. Let v_i have X_i children, where $X_1, \ldots, X_m$ are i.i.d. χ.
- Continue thus. The (random) rooted tree obtained is denoted $\mathcal{T}$.

For the purpose of the games: $\mathcal{T}$ will be visualized as a directed graph: an edge $\{u, v\}$ between parent u and child v will be directed from u to v and denoted (u, v).

Games that Holroyd-Martin analyzed

- A token placed at an *initial* vertex of $\mathcal{T}$.
- Two players take turns to move the token along directed edges.

Three different games studied:

1. Normal games: The players are denoted P1 and P2. Whichever player fails to move the token first, loses.
2. Misère games: The players are denoted P1 and P2. Whichever player fails to move the token first, wins.
3. Escape games: The players are denoted *Stopper* and *Escaper*. If either player fails to move the token, Stopper wins. Else Escaper wins. **Note:** No draw is possible in this game.

My set-up: multi-type Galton-Watson trees

- Let $\sigma_v \in \{\text{blue}, \text{red}\}$ denote colour of each vertex v in $\mathcal{T}$.
- Each vertex v gives birth independent of all else; offspring distribution depends on σ_v only:

$$\mathbf{P}[v \text{ has } m \text{ blue and } n \text{ red children} \big| \sigma_v = \text{blue}] = \chi_b(m,n),$$
$$\mathbf{P}[v \text{ has } m \text{ blue and } n \text{ red children} \big| \sigma_v = \text{red}] = \chi_r(m,n).$$

- Directed edge (u,v) *monochromatic* if $\sigma_u = \sigma_v$ and *non-monochromatic* if $\sigma_u \neq \sigma_v$.

My version of the games

- Players P1 (respectively Stopper) and P2 (respectively Escaper) take turns to move the token along directed edges.
- P1 / Stopper allowed to move token only along monochromatic edges.
- P2 / Escaper allowed to move token only along non-monochromatic edges.

The outcomes of the games are decided via the same rules as before:

- Normal game: Whoever fails to move the token, loses the game.
- Misére game: Whoever fails to move the token, wins the game.
- Escape game: Stopper wins if either player is unable to move the token. Else Escaper wins. Draw not possible.

Analysis of the normal game: defining subsets

- $\mathrm{NW}_{1,b}$ set of blue vertices v such that, if v initial vertex and P1 plays first round, P1 wins. Likewise, define $\mathrm{NW}_{1,r}$.
- $\mathrm{NL}_{1,b}$ set of blue vertices v such that, if v initial vertex and P1 plays first round, P1 loses. Likewise, define $\mathrm{NL}_{1,r}$.
- Similarly, define $\mathrm{NW}_{2,b}$, $\mathrm{NW}_{2,r}$, $\mathrm{NL}_{2,b}$ and $\mathrm{NL}_{2,r}$.
- For $n \in \mathbb{N}$, define $\mathrm{NW}_{1,b}^{(n)} \subset \mathrm{NW}_{1,b}$ comprising v such that if v initial vertex and P1 plays first round, game lasts $< n$ rounds. Set $\mathrm{NW}_{1,b}^{(0)} = \emptyset$.
- Likewise, define $\mathrm{NW}_{1,r}^{(n)}$, $\mathrm{NL}_{1,b}^{(n)}$, $\mathrm{NL}_{1,r}^{(n)}$, $\mathrm{NW}_{2,b}^{(n)}$, $\mathrm{NW}_{2,r}^{(n)}$, $\mathrm{NL}_{2,b}^{(n)}$, and $\mathrm{NL}_{2,r}^{(n)}$.

Analysis of the normal game: defining probabilities

- $\mathrm{nw}_{1,b}^{(n)} = \mathbf{P}\left[\phi \in \mathrm{NW}_{1,b}^{(n)} \middle| \sigma_\phi = \text{blue}\right]$.
- $\mathrm{nw}_{1,r}^{(n)} = \mathbf{P}\left[\phi \in \mathrm{NW}_{1,r}^{(n)} \middle| \sigma_\phi = \text{red}\right]$.
- Likewise, define $\mathrm{n}\ell_{1,b}^{(n)}$, $\mathrm{n}\ell_{1,r}^{(n)}$, $\mathrm{nw}_{2,b}^{(n)}$, $\mathrm{nw}_{2,r}^{(n)}$, $\mathrm{n}\ell_{2,b}^{(n)}$ and $\mathrm{n}\ell_{2,r}^{(n)}$.

- Define $\mathrm{nw}_{1,b} = \mathbf{P}\left[\phi \in \mathrm{NW}_{1,b} \middle| \sigma_\phi = \text{blue}\right]$.
- Define $\mathrm{nw}_{1,r} = \mathbf{P}\left[\phi \in \mathrm{NW}_{1,r} \middle| \sigma_\phi = \text{red}\right]$.
- Likewise, define $\mathrm{n}\ell_{1,b}$, $\mathrm{n}\ell_{1,r}$, $\mathrm{nw}_{2,b}$, $\mathrm{nw}_{2,r}$, $\mathrm{n}\ell_{2,b}$ and $\mathrm{n}\ell_{2,r}$.

Lemma: $\lim_{n\to\infty} \mathrm{nw}_{1,b}^{(n)} = \mathrm{nw}_{1,b}$. Similar results hold for the other sequences.

Probability generating functions

- For $x, y \in [0,1]$, let

$$G_b(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \chi_b(m,n).$$

- For $x, y \in [0,1]$, let

$$G_r(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \chi_r(m,n).$$

- Define

$$G_b^{(1)}(x) = G_b(x,1),\ G_b^{(2)}(x) = G_b(1,x),$$
$$G_r^{(1)}(x) = G_r(x,1),\ G_r^{(2)}(x) = G_r(1,x).$$

A glimpse of the recursions

Vertex $v \in \mathrm{NW}_{1,b}^{(n+1)}$ if

- $\sigma_v = $ blue, and
- v has at least one blue child u such that, if u is the initial vertex and P2 plays first round, P2 loses in less than n rounds, i.e. $u \in \mathrm{NL}_{2,b}^{(n)}$.

$$\begin{aligned} \mathrm{nw}_{1,b}^{(n+1)} &= \sum_{m_1=1}^{\infty} \sum_{m_2=0}^{\infty} \left\{1 - \left(1 - \mathrm{n}\ell_{2,b}^{(n)}\right)^{m_1}\right\} \chi_b(m_1, m_2) \\ &= 1 - G_b^{(1)}\left(1 - \mathrm{n}\ell_{2,b}^{(n)}\right). \end{aligned}$$

Likewise,

$$\mathrm{nw}_{1,r}^{(n+1)} = 1 - G_r^{(2)}\left(1 - \mathrm{n}\ell_{2,r}^{(n)}\right).$$

A glimpse of the recursions

Vertex $v \in \mathrm{NL}_{1,b}^{(n+1)}$ if

- $\sigma_v = \text{blue}$, and
- either v has no blue child, or each blue child u of v is such that, if u is the initial vertex and P2 plays first round, P2 wins in less than n rounds, i.e. $u \in \mathrm{NW}_{2,b}^{(n)}$.

$$\begin{aligned}
\mathrm{n}\ell_{1,b}^{(n+1)} &= \sum_{m_2=0}^{\infty} \chi_b(0, m_2) + \sum_{m_1=1}^{\infty} \sum_{m_2=0}^{\infty} \left(\mathrm{nw}_{2,b}^{(n)}\right)^{m_1} \chi_b(m_1, m_2) \\
&= G_b^{(1)}\left(\mathrm{nw}_{2,b}^{(n)}\right).
\end{aligned}$$

Likewise,

$$\mathrm{n}\ell_{1,r}^{(n+1)} = G_r^{(2)}\left(\mathrm{nw}_{2,r}^{(n)}\right).$$

What the recursions lead to

- We have $\mathrm{nw}_{1,b}^{(n+4)} = H_1\left(\mathrm{nw}_{1,b}^{(n)}\right)$, where

$$H_1(x) = 1 - G_b^{(1)}\left(1 - G_b^{(2)}\left(1 - G_r^{(2)}\left(1 - G_r^{(1)}(x)\right)\right)\right).$$

  Taking limit as $n \to \infty$, conclude that $\mathrm{nw}_{1,b}$ a fixed point of H_1.
- In fact, using monotonically increasing properties of H_1, we can say that $\mathrm{nw}_{1,b}$ is the minimum fixed point of H_1 in $[0,1]$.
- We have $\mathrm{nw}_{2,b}^{(n+4)} = H_2\left(\mathrm{nw}_{2,b}^{(n)}\right)$, where

$$H_2(x) = 1 - G_b^{(2)}\left(1 - G_r^{(2)}\left(1 - G_r^{(1)}\left(1 - G_b^{(1)}(x)\right)\right)\right).$$

  Taking limit as $n \to \infty$, conclude that $\mathrm{nw}_{2,b}$ a fixed point of H_2.
- As above, $\mathrm{nw}_{2,b}$ is the minimum fixed point of H_2 in $[0,1]$.

A special case: bi-type binary Galton-Watson tree

- Given $\sigma_v =$ blue, v has no child with probability p_0, two blue children with probability p_{bb}, two red children with probability p_{rr}, and one red and one blue child with probability p_{br}.
- Given $\sigma_v =$ red, v has no child with probability q_0, two blue children with probability q_{bb}, two red children with probability q_{rr}, and one red and one blue child with probability q_{br}.
- Let $\mathrm{nd}_{i,b}$ denote the probability, conditioned on $\sigma_\phi =$ blue, that if ϕ is the initial vertex and Pi plays first round, the game ends in a draw, for $i = 1, 2$. Likewise, define $\mathrm{nd}_{i,r}$ for $i = 1, 2$.

Lemma

- $\mathrm{nd}_{i,b} = \mathrm{nd}_{i,r} = 1$ *for* $i = 1, 2$ *if* $p_{\mathrm{br}} = q_{\mathrm{br}} = 1$.
- *In all other cases,* $\mathrm{nd}_{i,b} = \mathrm{nd}_{i,r} = 0$ *for* $i = 1, 2$.

Similar conclusions hold for draw probabilities in misère games and win probabilities for Escaper in escape games.