ON THE STABILITY OF HOLOMORPHIC DISCS ATTACHED TO AN N-SPHERE IN $\mathbb{C}^N$

Purvi Gupta

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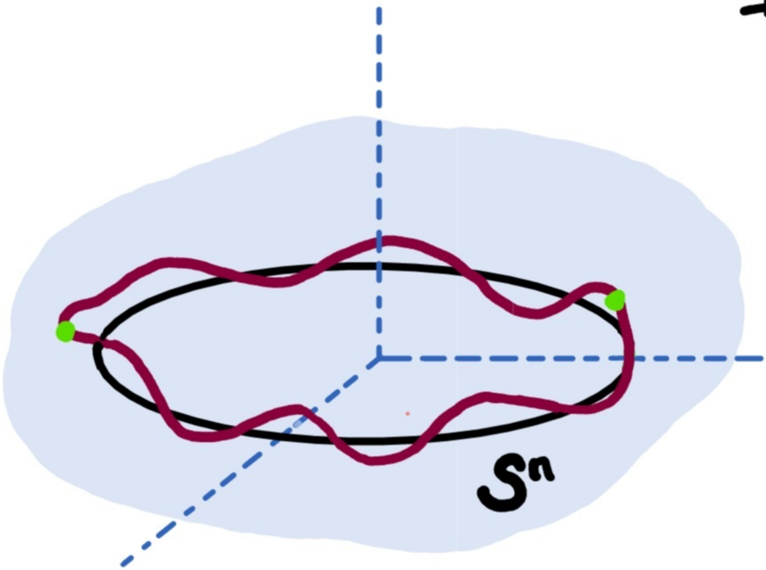

[Joint with Chloe U. Wawrzyniak]

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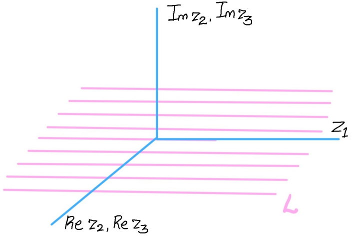

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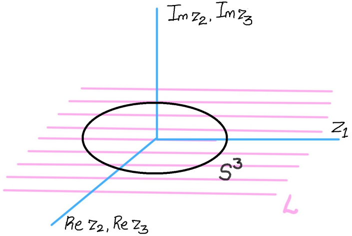

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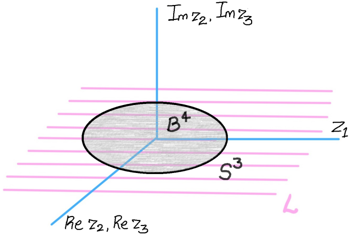

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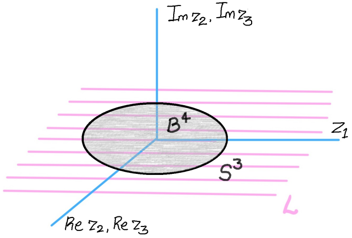

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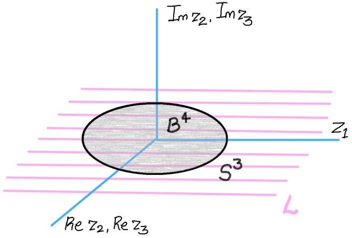

For $K^{\text{cpt.}} \subset \mathbb{C}^n$,

$$\widehat{K}^{hol.} \text{``}\cong\text{''} \operatorname{spec}(\mathcal{O}(K)) \qquad \text{and} \qquad \widehat{K}^{poly.} \cong \operatorname{spec}(\mathcal{P}(K)),$$

where

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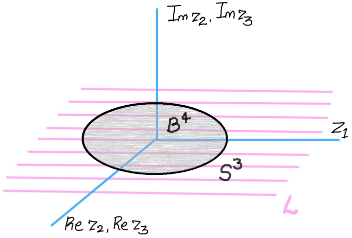

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Q. Does a "small" perturbation of $\mathbf{S}^n$ also bound a foliated $(n+1)$-ball in $\mathbb{C}^n$? If yes, does the "Levi-flat filling" coincide with some spectrum of the perturbed sphere?

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Significance (Wermer, Bishop, etc.). For S as above,

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Graph version in $\mathbb{C}^3$.

$$S = \{(z, f(z), g(z)) \in \mathbb{C} \times \mathbb{C}^2 : z \in S^1\}.$$

Q. When do f and g extend holomorphically to $\mathbb{D}$?

The complex Plateau problem: the general case

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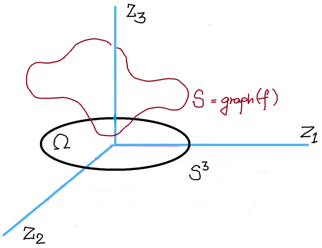
Graph version in $\mathbb{C}^3$ ($k = 2$).

S^3: unit sphere in $\mathbb{C}^2 \subset \mathbb{C}^3$.

$$S = \{(z, f(z)) \in \mathbb{C}^2 \times \mathbb{C} : z \in S^3 \subset \mathbb{C}^2\}$$

Q. When does F extend holomorphically to Ω?

A. When f is CR on S^3.



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- *Analytic discs attached (ADA) to S* : continuous mappings $f : \overline{\mathbb{D}} \to \mathbb{C}^n$ such that

  (a) $f \in \operatorname{hol}(\mathbb{D})$,
  (b) $f(b\mathbb{D}) \subset S$.

Holomorphic discs attached to surfaces in $\mathbb{C}^2$

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Assumption 2. There exist continuous vector fields $X_1, X_2 : b\mathbb{D} \to \mathbb{C}^2$ such that

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Idea. Nonlinear Riemann–Hilbert problem.

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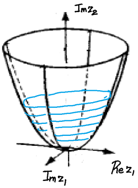

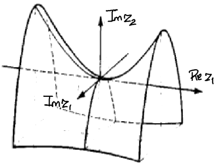

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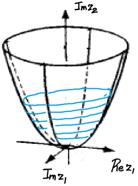

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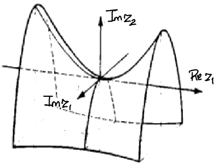

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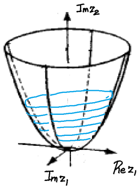

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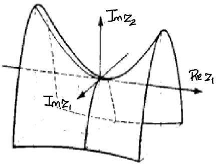

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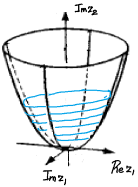

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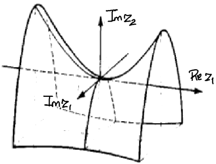

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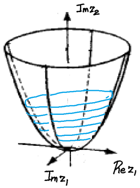

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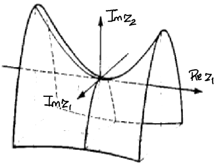

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Local Bishop Problem ($n = 2$). For ε small, $\widehat{S \cap B_p(\varepsilon)}^{hol.}$ is

- a 3-manifold M foliated by Bishop discs near p,
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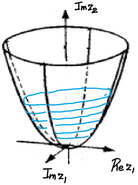

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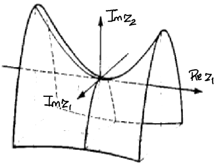

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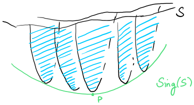
Settled by Kenig–Webster, Webster–Moser, Moser, Huang–Krantz.

The global Bishop problem

1. Elliptic singularities and Bishop discs generalize to n-dim. $S \subset \mathbb{C}^n$.
2. CR singularities are not isolated when $n \geq 3$.
3. The local Bishop problem for $n \geq 3$ was settled by Huang.



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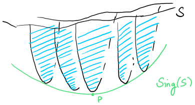

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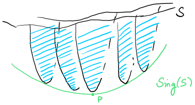

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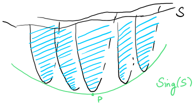

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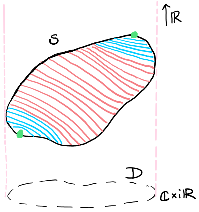

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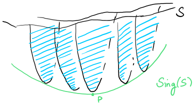

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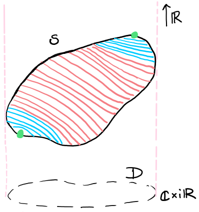

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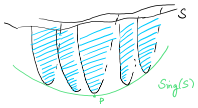

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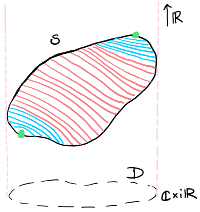

The Main Result

$\mathbf{S}^n = \{||z|| = 1,\ \operatorname{Im} z_2 = \cdots = \operatorname{Im} z_n = 0\}$, $\mathbf{B}^{n+1} = \{||z|| \leq 1,\ \operatorname{Im} z_2 = \cdots = \operatorname{Im} z_n = 0\}$

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Theorem (G.-Wawrzyniak, Adv. Math., 2021)

Given $\delta > 0$, there is an $\varepsilon = \varepsilon_\delta > 0$ such that for $k >> 1$, if

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then $(I + \varphi)\mathbf{S}^n := \mathbf{S}^n_\varphi$ bounds a $\mathcal{C}^{k'}$-smooth $(n+1)$-dimensional submanifold M in $\mathbb{C}^n$ s.t.

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(c) *If $\mathbf{S}^n_\varphi$ is $\mathcal{C}^\infty(\mathcal{C}^\omega)$-smooth, then so is M (upto its boundary).*

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Main Challenges. Nondiscrete $\operatorname{Sing}(S)$ + too many discs + M is of high codimension.

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Step 4. For real-analytic $\mathbf{S}^n_\varphi$, complexify.

Thank you.