An algorithm to recognise hyperbolic manifolds

Tejas Kalelkar,
Indian Institute of Science Education and Research, Pune
(Joint work with Advait Phanse and Sriram Raghunath)

18th September, 2021

Question (Homeomorphism problem)

Given the combinatorial data of two (simplicial) triangulations K_1 and K_2 of manifolds M and N, is there an algorithm to determine whether M and N are homeomorphic?

Question (Homeomorphism problem)

Given the combinatorial data of two (simplicial) triangulations K_1 and K_2 of manifolds M and N, is there an algorithm to determine whether M and N are homeomorphic?

- (Markov) Homeomorphism problem is unsolvable for dimension $n > 3$.

Question (Homeomorphism problem)

Given the combinatorial data of two (simplicial) triangulations K_1 and K_2 of manifolds M and N, is there an algorithm to determine whether M and N are homeomorphic?

- (Markov) Homeomorphism problem is unsolvable for dimension $n > 3$.
- (Haken, Hemion, Matveev) The problem can be solved for 3-manifolds using a bounded Haken hierarchy. Start with topological triangulations of the manifolds, cut inductively along incompressible surfaces to get cell decompositions, combinatorially compare the cell decompositions. [Difficult to implement. No complexity bound.]

Question (Homeomorphism problem)

Given the combinatorial data of two (simplicial) triangulations K_1 and K_2 of manifolds M and N, is there an algorithm to determine whether M and N are homeomorphic?

- (Markov) Homeomorphism problem is unsolvable for dimension $n > 3$.
- (Haken, Hemion, Matveev) The problem can be solved for 3-manifolds using a bounded Haken hierarchy. Start with topological triangulations of the manifolds, cut inductively along incompressible surfaces to get cell decompositions, combinatorially compare the cell decompositions. [Difficult to implement. No complexity bound.]
- (Jaco, Bessieres, Besson, Boileau, Porti, Manning, Sela etc, Scott-Short) The problem can be solved for closed triangulated 3-manifolds using geometrisation. [Difficult to implement. No complexity bound.]

Question (Homeomorphism problem)

Given the combinatorial data of two (simplicial) triangulations K_1 and K_2 of manifolds M and N, is there an algorithm to determine whether M and N are homeomorphic?

- (Markov) Homeomorphism problem is unsolvable for dimension $n > 3$.
- (Haken, Hemion, Matveev) The problem can be solved for 3-manifolds using a bounded Haken hierarchy. Start with topological triangulations of the manifolds, cut inductively along incompressible surfaces to get cell decompositions, combinatorially compare the cell decompositions. [Difficult to implement. No complexity bound.]
- (Jaco, Bessieres, Besson, Boileau, Porti, Manning, Sela etc, Scott-Short) The problem can be solved for closed triangulated 3-manifolds using geometrisation. [Difficult to implement. No complexity bound.]
- (Kuperberg) The computational complexity (in dim 3) is bounded by a bounded tower of exponentials in the number of tetrahedra.

- Let K be a triangulation of an n-manifold M and let D be a disk subcomplex of K which is simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a Pachner move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

- Let K be a triangulation of an n-manifold M and let D be a disk subcomplex of K which is simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a Pachner move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

Pachner moves in dimension 2

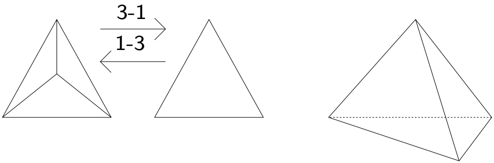


Figure: 3-1 and 1-3 Pachner moves

Pachner Moves

- Let K be a triangulation of an n-manifold M and let D be a disk subcomplex of K which is simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a Pachner move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

Pachner moves in dimension 2

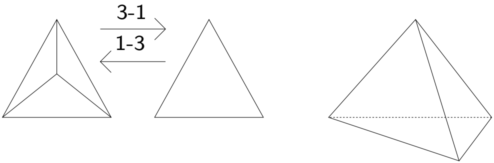


Figure: 3-1 and 1-3 Pachner moves

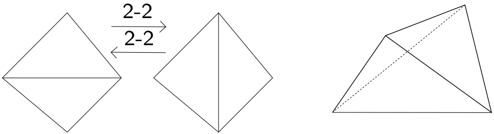
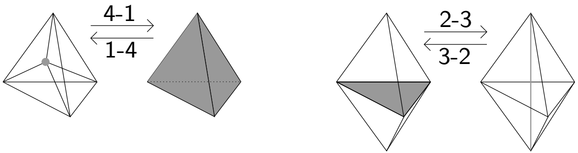


Figure: 2-2 Pachner move

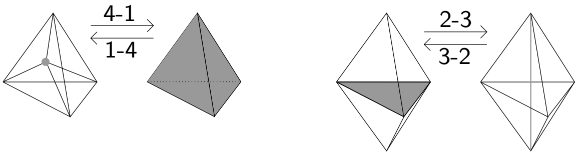
- Let K be a triangulation of an n-manifold M and let D be a disk subcomplex of K which is simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a Pachner move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

Pachner moves in dimension 3



- Let K be a triangulation of an n-manifold M and let D be a disk subcomplex of K which is simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a Pachner move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

Pachner moves in dimension 3

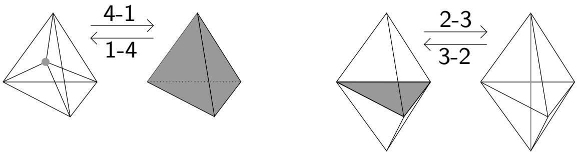


- (Pachner) Let K_1 and K_2 be PL-triangulations of an n-dimensional manifolds with p and q many n-simplexes and a common subdivision. Then K_1 is related to K_2 by a finite sequence of Pachner moves.

Pachner Moves

- Let K be a triangulation of an n-manifold M and let D be a disk subcomplex of K which is simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a Pachner move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

Pachner moves in dimension 3



- (Pachner) Let K_1 and K_2 be PL-triangulations of an n-dimensional manifolds with p and q many n-simplexes and a common subdivision. Then K_1 is related to K_2 by a finite sequence of Pachner moves.
- Let $f(n,p,q)$ be a bounding function on the length of this sequence. Solving the homeomorphism problem for PL n-manifolds is equivalent to obtaining such a bounding function $f(n,p,q)$.
  (Let K_M and K_N be triangulations of M and N with p and q many n-simplexes. Let $\mathcal{K} = \{K : d(K, K_M) < f(n,p,q)\}$. Then M is homeomorphic to N iff some $K \in \mathcal{K}$ is simplicially isomorphic to K_N.)

Bounding function on Pachner moves

- (Mijatovic) Let M be a closed orientable irreducible 3-manifold such that the closure of each component of the complement of the characteristic submanifold of M does not fiber over the circle. Then any two triangulations K_1 and K_2 of M with p and q many 3-simplexes are related by at most $f(p,q)$ Pachner moves where:

$$f(p,q) = \underbrace{2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{p}}}}}}}}_{2^{162p}} + \underbrace{2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{q}}}}}}}}_{2^{162q}}$$

- (Mijatovic) Let M be a closed orientable irreducible 3-manifold such that the closure of each component of the complement of the characteristic submanifold of M does not fiber over the circle. Then any two triangulations K_1 and K_2 of M with p and q many 3-simplexes are related by at most $f(p, q)$ Pachner moves where:

$$f(p,q) = \left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{p}}}}}}}\right\} 2^{162p} + \left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{q}}}}}}}\right\} 2^{162q}$$

Question

Is there a sharper bounding function $f(n, p, q)$ for the number of Pachner moves needed to relate geometric triangulations of constant curvature n-manifolds?

Theorem (K, Phanse)

Let M be closed spherical, Euclidean or hyperbolic n-manifold with geometric triangulations K_1 and K_2. Let K_1 and K_2 have p and q many n-simplexes respectively. Let Λ be an upper bound on the lengths of edges. When M is spherical, we require $\Lambda \leq \pi/2$. Let $inj(M)$ denote the injectivity radius of M.

When $n \leq 4$, then K_1 and K_2 are related by f many Pachner moves. In general, their 2^{n+1}-th barycentric subdivisions, $\beta^{2^{n+1}} K_1$ and $\beta^{2^{n+1}} K_2$ are related by f many Pachner moves.

$$f(n, p, q, \Lambda, inj(M)) = 2^{n+2}(n+1)!^{4+3m} pq(p+q)$$

where m is a non-negative integer greater than $\mu \ln(\Lambda/inj(M))$ and when $n > 4$ we also require $m \geq 2^{n+1}$.

- i *When M is Euclidean, $\mu = n+1$*
- ii *When M is Spherical, $\mu = 2n+1$*
- iii *When M is Hyperbolic, $\mu = n\cosh^{n-1}(\Lambda) + 1$*

Corollary (K, Phanse)

Let M be closed spherical, Euclidean or hyperbolic n-manifold with geometric triangulations K_1 and K_2. Let K_1 and K_2 have p and q many n-simplexes respectively. Let Λ be an upper bound on lengths of edges. Let λ be a lower bound on lengths of edges. When M is spherical, we require $\Lambda \leq \pi/2$. Let Δ^n_λ denote the regular n-simplex with edges of length λ.

When $n \leq 4$, then K_1 and K_2 are related by f many Pachner moves. In general, their 2^{n+1}-th barycentric subdivisions, $\beta^{2^{n+1}} K_1$ and $\beta^{2^{n+1}} K_2$ are related by f many Pachner moves:

$$f(n, p, q, \Lambda, \lambda) = 2^{n+2}(n+1)!^{4+3m} pq(p+q)$$

where m is a non-negative integer greater than $\mu \ln(\Lambda \delta vol(S^n)/(\pi p\, vol(\Delta^n_\lambda))$ and when $n > 4$ we also require $m \geq 2^{n+1}$.

- i *When M is Euclidean, $\mu = n+1$, $\delta = p\Lambda$*
- ii *When M is Spherical, $\mu = 2n+1$, $\delta = \sin^{n-1}(p\Lambda)$*
- iii *When M is Hyperbolic, $\mu = n\cosh^{n-1}(\Lambda) + 1$, $\delta = \sinh^{n-1}(p\Lambda)$*

Corollary (K, Phanse)

Let M be closed hyperbolic 3-manifold with geometric triangulations K_1 and K_2. Let K_1 and K_2 have p and q many n-simplexes respectively. Let Λ be an upper bound on the lengths of edges. Let $t = p + q$.

Then K_1 and K_2 are related by f many Pachner moves:

$$f(t, \Lambda) = (1.07 \times 10^7) \cdot exp(83\, t\, exp(3\Lambda))$$

Corollary (K, Phanse)

Let M be closed hyperbolic 3-manifold with geometric triangulations K_1 and K_2. Let K_1 and K_2 have p and q many n-simplexes respectively. Let Λ be an upper bound on the lengths of edges. Let $t = p + q$.

Then K_1 and K_2 are related by f many Pachner moves:

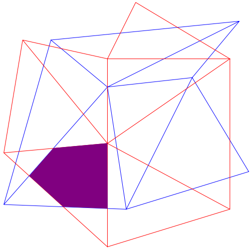
$$f(t, \Lambda) = (1.07 \times 10^7) \cdot exp(83\, t\, exp(3\Lambda))$$

Theorem (K, Raghunath)

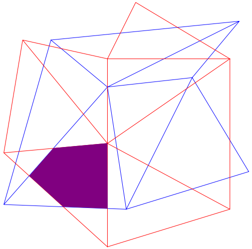
Let M be a complete orientable cusped hyperbolic 3-manifold. Let τ_1 and τ_2 be geometric ideal triangulations of M with at most p and q many tetrahedra respectively and all dihedral angles at least θ_0. Let $t = p + q$. Then the number of Pachner moves needed to relate τ_1 and τ_2 is less than

$$f(t, \theta_0) = (2.8 \times 10^{12}) \cdot \frac{t^{11/2}}{(\sin\theta_0)^{12t+27/2}}$$

- Let τ_1 and τ_2 be geometric triangulations of M. Then $\tau_1 \cap \tau_2$ is a common geometric polyhedral subdivision of τ_1 and τ_2.

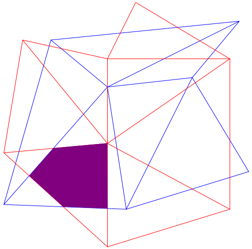


- Let τ_1 and τ_2 be geometric triangulations of M. Then $\tau_1 \cap \tau_2$ is a common geometric polyhedral subdivision of τ_1 and τ_2.



- *Bounded Common Subdivision:* Find a bound on the number of polytopes in $\tau_1 \cap \tau_2$ (Use thick-thin decomposition for hyperbolic cusped manifolds).

- Let τ_1 and τ_2 be geometric triangulations of M. Then $\tau_1 \cap \tau_2$ is a common geometric polyhedral subdivision of τ_1 and τ_2.

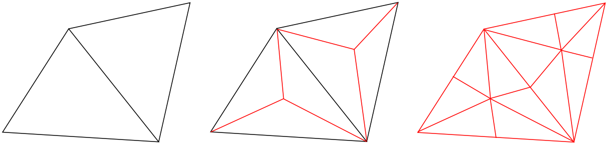


- *Bounded Common Subdivision:* Find a bound on the number of polytopes in $\tau_1 \cap \tau_2$ (Use thick-thin decomposition for hyperbolic cusped manifolds).
- *Pachner sequence to common subdivision:* Find a bounded sequence of Pachner moves using the shelling of a derived subdivisions of triangulated polytopes:

$$\tau_1 \sim \beta\tau_1 \sim \beta(\tau_1 \cap \tau_2) \sim \beta\tau_2 \sim \tau_2$$

- To begin with, our aim is to bound the number of Pachner moves to go from a triangulation K of M to its subdivision αK.

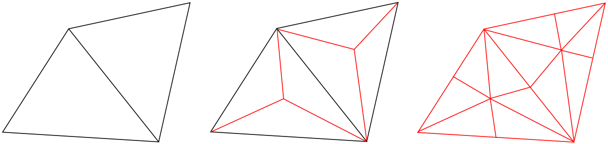
- To begin with, our aim is to bound the number of Pachner moves to go from a triangulation K of M to its subdivision αK.
- Partial Barycentric subdivisions of K:



$\beta_2 K = K$ $\beta_1 K$ $\beta_0 K = \beta K$

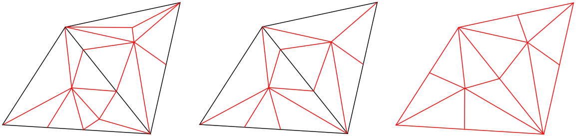
Partial barycentric subdivisions

- To begin with, our aim is to bound the number of Pachner moves to go from a triangulation K of M to its subdivision αK.
- Partial Barycentric subdivisions of K:



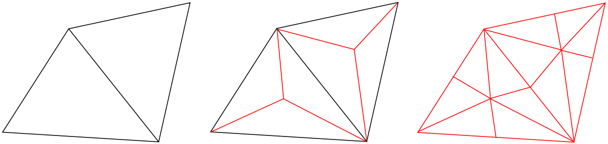
$\beta_2 K = K$ $\quad$ $\beta_1 K$ $\quad$ $\beta_0 K = \beta K$

- Partial Barycentric subdivision of K relative to αK:



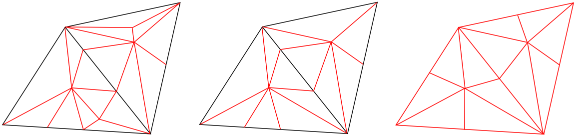
$\beta_2^\alpha K = \alpha K$ $\quad$ $\beta_1^\alpha K$ $\quad$ $\beta_0^\alpha K = \beta K$

- To begin with, our aim is to bound the number of Pachner moves to go from a triangulation K of M to its subdivision αK.
- Partial Barycentric subdivisions of K:



$\beta_2 K = K$ $\quad$ $\beta_1 K$ $\quad$ $\beta_0 K = \beta K$

- Partial Barycentric subdivision of K relative to αK:

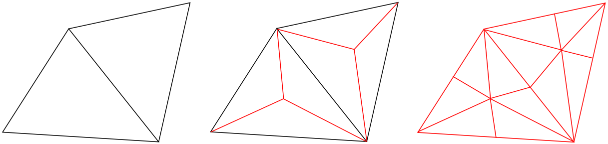


$\beta_2^\alpha K = \alpha K$ $\quad$ $\beta_1^\alpha K$ $\quad$ $\beta_0^\alpha K = \beta K$

- $\alpha K = \beta_n^\alpha K,\ \beta_{n-1}^\alpha K,\ \beta_{n-2}^\alpha K,\ \ldots,\ \beta_1^\alpha K,\ \beta_0^\alpha K = \beta K.$

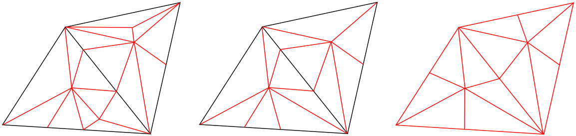
Partial barycentric subdivisions

- To begin with, our aim is to bound the number of Pachner moves to go from a triangulation K of M to its subdivision αK.
- Partial Barycentric subdivisions of K:



$\beta_2 K = K$ $\qquad$ $\beta_1 K$ $\qquad$ $\beta_0 K = \beta K$

- Partial Barycentric subdivision of K relative to αK:



$\beta_2^\alpha K = \alpha K$ $\qquad$ $\beta_1^\alpha K$ $\qquad$ $\beta_0^\alpha K = \beta K$

- $\alpha K = \beta_n^\alpha K,\ \beta_{n-1}^\alpha K,\ \beta_{n-2}^\alpha K,\ \dots,\ \beta_1^\alpha K,\ \beta_0^\alpha K = \beta K.$
- For A an r-simplex, let $S(A) = \alpha A \star link(A, \beta_r K)$. So, $\alpha K \sim \beta K \Leftarrow \beta_r^\alpha K \sim \beta_{r-1}^\alpha K \Leftarrow S(A) \sim C(\partial S(A))$ for all r-simplexes A.

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.
- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, \ldots, \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.
- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, ..., \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

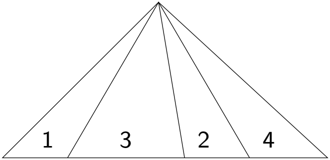


Figure: Not a shelling sequence

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.

- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, ..., \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

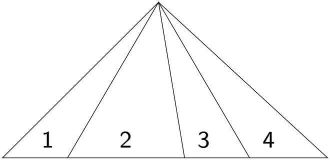


Figure: A shelling sequence

Shellability $\Rightarrow$ Starrability

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.

- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, \ldots, \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

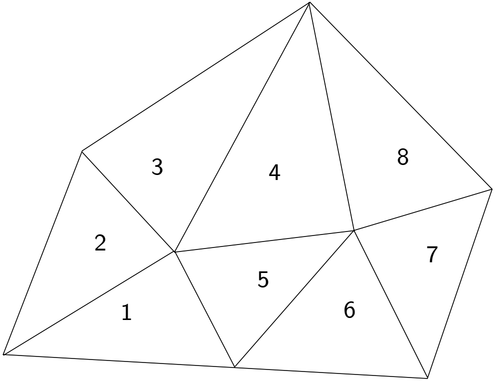


Figure: Shelling sequence on triangulated polytope K

Shellability $\Rightarrow$ Starrability

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.

- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, \ldots, \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

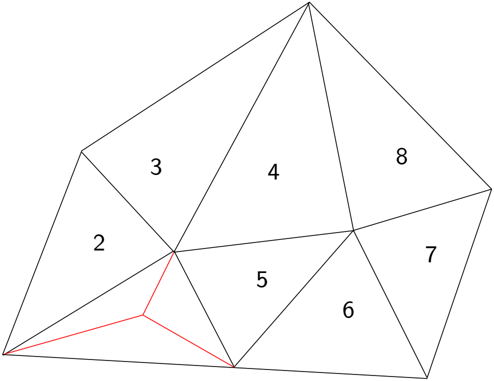


Figure: Perform a 1-3 move

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.
- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, ..., \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

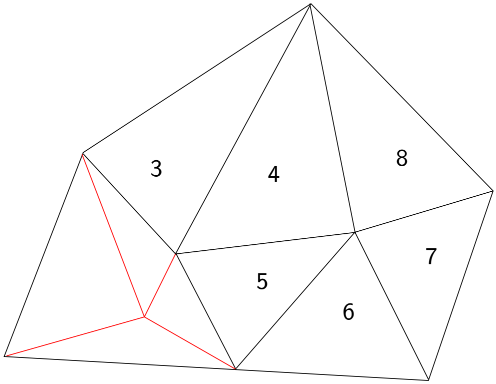


Figure: Perform a 2-2 move

Shellability $\Rightarrow$ Starrability

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.

- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, ..., \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

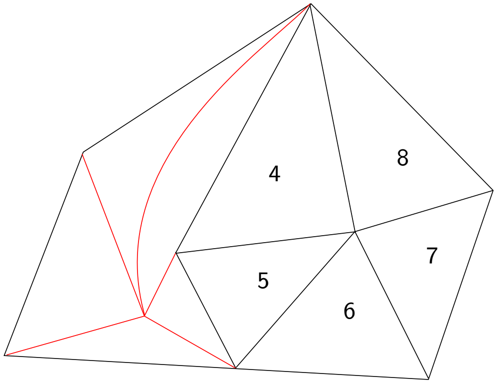


Figure: Perform a 2-2 move

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.

- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, \ldots, \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

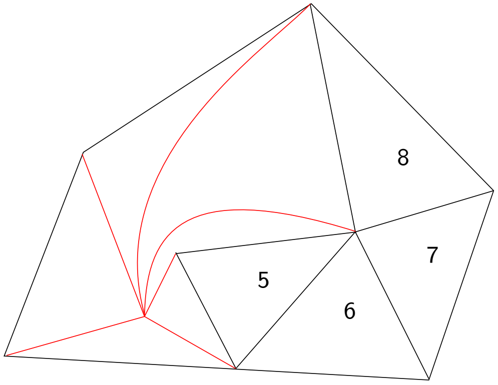


Figure: Perform a 2-2 move

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.
- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, \ldots, \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

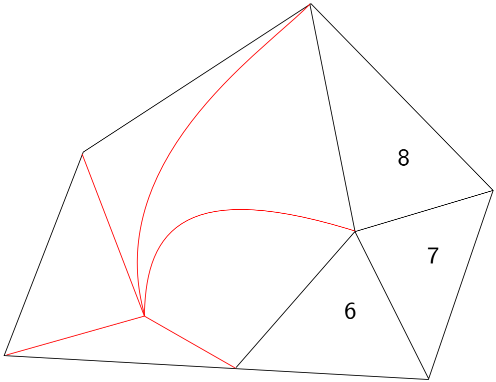


Figure: Perform a 3-1 move

Shellability $\Rightarrow$ Starrability

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.

- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, \ldots, \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

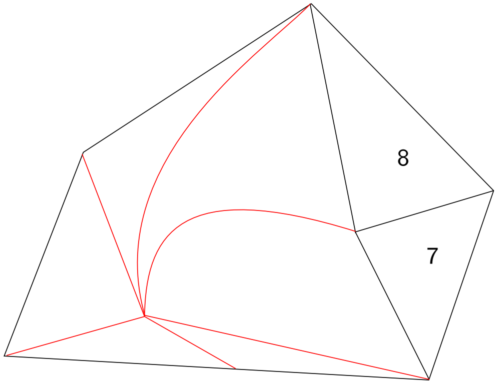


Figure: Perform a 2-2 move

Shellability $\Rightarrow$ Starrability

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.

- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, \ldots, \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

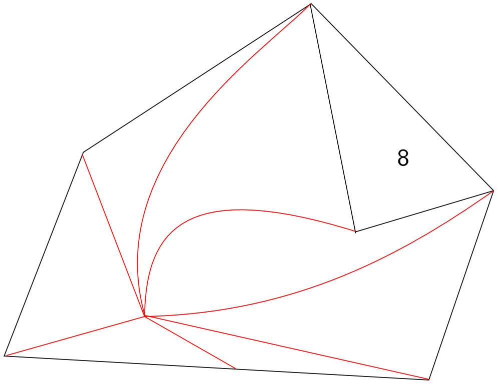


Figure: Perform a 2-2 move

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.
- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, \ldots, \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.

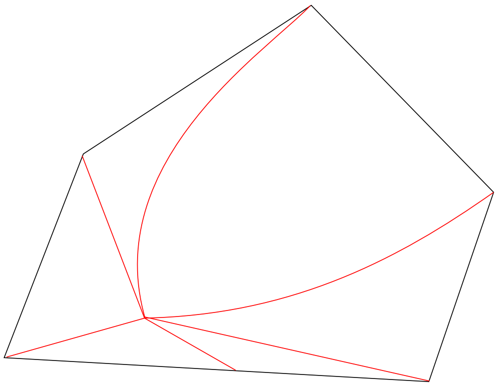


Figure: Finally perform a 3-1 move to get $C(\partial K)$ from K in 8 Pachner moves.

Shellability $\Rightarrow$ Starrability

- Aim: $S(A) \sim C(\partial S(A))$ by a controlled number of Pachner moves.
- We say that a triangulated n ball K is shellable if there is an enumeration of its n-simplexes $\Delta_1, ..., \Delta_m$ such that $\Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j)$ is an $n-1$ dimensional disk subcomplex of $\partial \Delta_i$.
- (Adiprasito - Benedetti) If K is any (Euclidean) subdivision of a convex (Euclidean) polytope then $\beta^2 K$ is shellable.
- Links of all simplexes in $\beta^m K$ are shellable for $m = 2^{n+1}$.
- And so after taking suitably many barycentric subdivisions, $S(A) = \alpha A \star link(A, \beta_r(K))$ is the join of shellable complexes, and is therefore shellable $\Rightarrow S(A) \sim C(\partial S(A))$ by as many Pachner moves as n-simplexes in $S(A)$.
- Hence we can go from αK to K by a controlled number of Pachner moves through the various $\beta_r^\alpha K$.

- Given triangulations K_1 and K_2 of M, we take a common geometric subdivision $\beta(K_1 \cap K_2)$.

- Given triangulations K_1 and K_2 of M, we take a common geometric subdivision $\beta(K_1 \cap K_2)$.

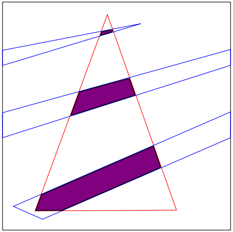


Figure: Disconnected intersection of simplexes of two linear triangulations of a torus

- Given triangulations K_1 and K_2 of M, we take a common geometric subdivision $\beta(K_1 \cap K_2)$.

Theorem

Let $\beta^m\Delta$ be the m-th geometric barycentric subdivision of an n-simplex Δ with new vertices added at the centroid of simplexes. Let Λ be an upper bound on the length of edges of Δ. Then the diameter of simplexes of $\beta^m\Delta$ is at most $\kappa^m\Lambda$ where

$$\kappa = \begin{cases} \frac{n}{n+1} & \text{for } M \text{ Euclidean} \\ \frac{2n}{2n+1} & \text{for } M \text{ spherical} \\ \frac{n\cosh^{n-1}(\Lambda)}{n\cosh^{n-1}(\Lambda)+1} & \text{for } M \text{ hyperbolic} \end{cases}$$

- Given triangulations K_1 and K_2 of M, we take a common geometric subdivision $\beta(K_1 \cap K_2)$.

Theorem

Let $\beta^m\Delta$ be the m-th geometric barycentric subdivision of an n-simplex Δ with new vertices added at the centroid of simplexes. Let Λ be an upper bound on the length of edges of Δ. Then the diameter of simplexes of $\beta^m\Delta$ is at most $\kappa^m\Lambda$ where

$$\kappa = \begin{cases} \frac{n}{n+1} & \text{for } M \text{ Euclidean} \\ \frac{2n}{2n+1} & \text{for } M \text{ spherical} \\ \frac{n\cosh^{n-1}(\Lambda)}{n\cosh^{n-1}(\Lambda)+1} & \text{for } M \text{ hyperbolic} \end{cases}$$

- So we take m large enough such that

$$\kappa^m\Lambda < 2 \cdot \text{Convexity radius of } M = inj(M)$$

Then all simplexes of $\beta^m K_1$ and $\beta^m K_2$ are strongly convex, and therefore intersect at most once.

- Let K_1 and K_2 be given simplicial triangulations of M.

Outline of Proof for compact case

- Let K_1 and K_2 be given simplicial triangulations of M.
- If $n > 4$ then take 2^{n+1} barycentric subdivisions of K_i so that link of each simplex is shellable. (Using Adiprasito-Benedetti)

Outline of Proof for compact case

- Let K_1 and K_2 be given simplicial triangulations of M.
- If $n > 4$ then take 2^{n+1} barycentric subdivisions of K_i so that link of each simplex is shellable. (Using Adiprasito-Benedetti)
- Take sufficiently many barycentric subdivisions such that each simplex lies in a strongly convex ball of M.

Outline of Proof for compact case

- Let K_1 and K_2 be given simplicial triangulations of M.
- If $n > 4$ then take 2^{n+1} barycentric subdivisions of K_i so that link of each simplex is shellable. (Using Adiprasito-Benedetti)
- Take sufficiently many barycentric subdivisions such that each simplex lies in a strongly convex ball of M.
- Calculate the number of n-simplexes in the common subdivision $\alpha K = \beta(K_1 \cap K_2)$.

- Let K_1 and K_2 be given simplicial triangulations of M.
- If $n > 4$ then take 2^{n+1} barycentric subdivisions of K_i so that link of each simplex is shellable. (Using Adiprasito-Benedetti)
- Take sufficiently many barycentric subdivisions such that each simplex lies in a strongly convex ball of M.
- Calculate the number of n-simplexes in the common subdivision $\alpha K = \beta(K_1 \cap K_2)$.
- Calculate the number of n-simplexes in $S(A) = \alpha A \star link(A, \beta_r K)$ for each r-simplex A. This gives the number of Pachner moves needed to go from $S(A)$ to $C(\partial S(A))$ and therefore from $\beta_r^\alpha K$ to $\beta_{r-1}^\alpha K$. Summing this up from $r = 1...n$, gives a bound on number of Pachner moves from αK to K_1 and similarly αK to K_2 . Adding these gives the required bound on number of moves from K_1 to K_2.

Related Publications

- *Bounds on Pachner moves and systoles of cusped 3-manifolds*, Tejas Kalelkar and Sriram Raghunath, Accepted in Journal of Algebraic & Geometric Topology, arXiv:2007.02781
- *An upper bound on Pachner moves relating geometric triangulations*, Tejas Kalelkar and Advait Phanse, Journal of Discrete and Computational Geometry, Volume 66, 2021, Number 3, 809830
- *Geometric bistellar moves relate geometric triangulations*, Tejas Kalelkar and Advait Phanse, Topology and its Applications, Volume 285, 2020, 107390107397