Principal series component of Gelfand-Graev representation

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Preliminaries

- Let F denote a non-archimedean local field with its ring of integers $\mathcal{O}$ and the residue field $\mathbb{F}_q$, where $q = p^r$
- Let $\mathbb{G}$ be a connected reductive algebraic group defined over F and $G = \mathbb{G}(F)$ denote the group of F-rational points of $\mathbb{G}$.
- Fix a maximal F-split torus $\mathbb{S}$ in $\mathbb{G}$ and let $\mathbb{T} = Z_{\mathbb{G}}(\mathbb{S})$. Then $\mathbb{T}$ is the Levi factor of a minimal F-parabolic subgroup $\mathbb{B} = \mathbb{T}\mathbb{U}$ of $\mathbb{G}$ defined over F.
- If $\mathbb{H}$ is an algebraic group$/F$, denote $H = \mathbb{H}(F)$. eg. we denote $G, B, T, U, S\ldots$ for $\mathbb{G}(F), \mathbb{B}(F), \mathbb{T}(F), \mathbb{U}(F), \mathbb{S}(F)\ldots$ resp.
- Let $\widehat{U}$ be the space of all smooth characters $\psi : U \to \mathbb{C}^\times$ of U. Then, T acts on $\widehat{U}$ via

$$t \cdot \psi = \psi^t : x \mapsto \psi(txt^{-1}) \text{ for } t \in T \text{ and } \psi \in \widehat{U}.$$

Gelfand-Graev representations

- A smooth character $\psi : U \to \mathbb{C}^\times$ is called non-degenerate or, generic if its stabilizer in S lies in the center $Z(G)$ of G.
- The Gelfand-Graev representation $\boxed{c\text{-ind}_U^G(\psi)}$ of G (associated to ψ) is provided by the space of right G-smooth compactly supported modulo U functions $f : G \to \mathbb{C}$ satisfying: $f(ug) = \psi(u)f(g), \forall u \in U, g \in G$.
- Let M be a (B, T)-**standard** F-Levi subgroup of an F-parabolic $P = MN$ of G, i.e., $T \subset M$ and $B \subset P$. Then $B \cap M$ is a minimal parabolic subgroup of M with unipotent radical $U_M := U \cap M$. Then, $\psi_M := \psi|_{U_M}$ is a non-degenerate character of U_M. Let $P^- = MN^-$ be the P-opposite parabolic. Then Bushnell and Henniart showed

$$\boxed{c\text{-ind}_{U_M}^M(\psi_M) \cong \left(c\text{-ind}_U^G(\psi)\right)_{N^-}.} \tag{1}$$

Bernstein decomposition

- $\mathcal{R}(G)$: Category of smooth complex representations of G.
- Bernstein: decomposition of $\mathcal{R}(G)$ into indecomposable subcategories:

$$\mathcal{R}(G) = \prod_{\mathfrak{s} \in \mathcal{B}(G)} \mathcal{R}^{\mathfrak{s}}(G)$$

  Here $\mathcal{B}(G)$ is the set of inertial equivalence classes $\mathfrak{s} = [M, \sigma]_G$ of cuspidal pairs (M, σ), where M is an F-Levi subgroup, σ is a supercuspidal representation of M and equiv rel :$(M_1, \sigma_1) \sim (M_2, \sigma_2)$ if there exists $g \in G$ and a unramified character χ of G such that $M_2 = {}^g M_1$ and ${}^g \sigma_1 = \sigma_2 \otimes \chi$.
- The block $\mathcal{R}^{[M,\sigma]_G}(G)$ consists of those representations $\pi \in \mathcal{R}(G)$ whose each irreducible constituent appears in the parabolic induction of some supercuspidal representation in the equivalence class $[M, \sigma]_G$.

Hecke algera and Types

- Let (τ, V) be an irred repn of a compct open subgp J of G.
- The Hecke algebra $\mathcal{H}(G,\tau)$ is the space of compactly supported functions $f : G \to \mathrm{End}_{\mathbb{C}}(V^\vee)$ satisfying, $f(j_1 g j_2) = \tau^\vee(j_1) f(g) \tau^\vee(j_2)$ *for all* $j_1, j_2 \in J$ *and* $g \in G$.
- The standard convolution operation gives $\mathcal{H}(G,\tau)$ the structure of an associative $\mathbb{C}$-algebra with identity.
- $\mathcal{R}_\tau(G)$: the subcategory of $\mathcal{R}(G)$ whose objects are the representations $(\pi, \mathcal{V})$ of G generated by the τ-isotypic subspace $\mathcal{V}^\tau$ of $\mathcal{V}$. Then, there is a functor

$$\begin{aligned} \mathcal{M}_\tau : \mathcal{R}_\tau(G) &\to \mathcal{H}(G,\tau)\text{-Mod}, \\ \pi &\mapsto \mathrm{Hom}_J(\tau, \pi). \end{aligned}$$

- If $\mathcal{R}_\tau(G) = \mathcal{R}^{\mathfrak{s}}(G)$, the pair (J,τ) is called an $\mathfrak{s}$-type in G. In that case, the functor $\mathcal{M}_\tau$ gives an equivalence of categories.

G-cover

- Let (K_M, ρ_M) be a $[M, \sigma]_M$-type in M. Let (K, ρ) be a pair consisting of a compact open subgroup K of G and an irreducible representation ρ of K. The pair (K, ρ) is called the G-cover of (K_M, ρ_M) if for any opposite pair of F-parabolic subgroups $P = MN$ and $P^- = MN^-$, the following properties hold:
  (1) $K = (K \cap N)(K \cap M)(K \cap N^-)$.
  (2) $K_M = K \cap M, \rho|_{K_M} = \rho_M$ and $K \cap N, K \cap N^- \subset \ker(\rho)$.
  (3) For any smooth representation Υ of G; the natural projection Υ to the Jacquet module Υ_N induces an injection on Υ^ρ i.e., $\Upsilon^\rho \to (\Upsilon_N)^{\rho_M}$ is injective.
- In that case, (K, ρ) is an $[M, \sigma]_G$-type in G.
- Then for any F-parabolic subgroup $P' = MN'$ and for any smooth representation Υ of G, there is an isomorphism of $\mathcal{H}(M, \rho_M)$-modules(via $\mathcal{M}_\rho$):

$$\boxed{(\Upsilon_{N'})^{\rho_M} \cong \Upsilon^\rho.} \tag{2}$$

Bernstein components of GGR:

- Through Bernstein decomposition, the G.G.R. $c\text{-}\mathrm{ind}_U^G(\psi)$ decomposed into the direct sum of certain representations $c\text{-}\mathrm{ind}_U^G(\psi)_{\mathfrak{s}} \in \mathcal{R}^{\mathfrak{s}}(G)$ for $\mathfrak{s} \in \mathcal{B}(G)$.
- Bushnell and Henniart[1, Theorem 4.2]: for each $\mathfrak{s} \in \mathcal{B}(G)$ $\boxed{c\text{-}\mathrm{ind}_U^G(\psi)_{\mathfrak{s}} \text{ is finitely generated over } G}$.
- Focus: $\boxed{c\text{-}\mathrm{ind}_U^G(\psi)_{\mathfrak{s}} \text{ where, } \mathfrak{s} = [T, \chi]_G}$, where T is a maximal F-torus of G and $\chi \in \widehat{T}$.
- Let (K, ρ) be a $[T, \chi]_G$-type in G (Exists!) and $\mathcal{H}(G, \rho) =$ rel. Hecke alg. Then, $c\text{-}\mathrm{ind}_U^G(\psi)^{\rho}$ generates $c\text{-}\mathrm{ind}_U^G(\psi)_{\mathfrak{s}}$.
- We show $\boxed{(c\text{-}\mathrm{ind}_U^G(\psi))^{\rho} \text{ is a cyclic } \mathcal{H}(G, \rho)\text{-module.}}$
- Moreover, $\mathbb{T}$ split, $\boxed{(c\text{-}\mathrm{ind}_U^G(\psi))^{\rho} \cong \mathcal{H}(G, \rho) \otimes_{\mathcal{H}_{W_\chi}} \mathrm{sgn}.}$
- This generalize the main result of Chan and Savin in [3], who treated the case of $\chi = 1$ for $\mathbb{T}$ split.

Theorem 1

Theorem (Mishra,....)

There is an isomorphism $(c\text{-ind}_U^G(\psi))^\rho \cong \mathcal{H}(T,\rho_T)$ of $\mathcal{H}(T,\rho_T)$-modules. Consequently, $(c\text{-ind}_U^G(\psi))^\rho$ is a cyclic $\mathcal{H}(G,\rho)$-module.

Proof.

Put $M = T$ in Eq.(1) $\boxed{c\text{-ind}_{U_M}^M(\psi_M) \cong \left(c\text{-ind}_U^G(\psi)\right)_{N^-}.}$ Then $U_M = 1$, and get an isomorphism of T-representations $(c\text{-ind}_U^G(\psi))_{U^-} \cong c\text{-ind}_1^T(\mathbb{C}) \cong C_c^\infty(T)$. Consequently, $(c\text{-ind}_U^G(\psi))_{U^-}^{\rho_T} \cong C_c^\infty(T)^{\rho_T} \cong \mathcal{H}(T,\rho_T)$ as $\mathcal{H}(T,\rho_T)$-mods.
Now Eq.(2) $\boxed{\Upsilon^\rho \cong (\Upsilon_{N^-})^{\rho_M}.}$ $\Rightarrow$ as $\mathcal{H}(T,\rho_T)$-modules,

$$(c\text{-ind}_U^G(\psi))^\rho \cong (c\text{-ind}_U^G(\psi))_{U^-}^{\rho_T} \cong \mathcal{H}(T,\rho_T)$$

The result follows.

Roche's(Principal series) Hecke algebra [6]

- Assume that $\mathbb{G}$ is split. so that $\mathbb{B} = \mathbb{T}\mathbb{U}$ is now an F-Borel subgroup of $\mathbb{G}$ containing the maximal F-split torus $\mathbb{T} = \mathbb{S}$.
- The pair $(\mathbb{B}, \mathbb{T})$ determines a based root datum $\Psi = (X^*, \Phi, \Pi, X_*, \Phi^\vee, \Pi^\vee)$. Here X^* (resp. X_*) is the character (resp. co-character) lattice of $\mathbb{T}$ and Π (resp. $\Pi^\vee$) is a basis (resp. dual basis) for the set of roots $\Phi = \Phi(\mathbb{G}, \mathbb{T})$ (resp.$\Phi^\vee$) of $\mathbb{T}$ in $\mathbb{G}$.
- Let $W = N_{\mathbb{G}}(\mathbb{T})(F)/\mathbb{T}(F)$ be the Weyl group of $\mathbb{G}$ and $\widetilde{W} = N_{\mathbb{G}}(\mathbb{T})(F)/\mathbb{T}(\mathcal{O})$ be the Iwahori-Weyl group of $\mathbb{G}$. Then $\widetilde{W} = X_* \rtimes W$.
- Let $\chi^\#$ be a character of T and put $\chi = \chi^\#|_{\mathbb{T}(\mathcal{O})}$. Then $(\mathbb{T}(\mathcal{O}), \chi)$ is a $[T, \chi^\#]_T$-type in T. Roche constructed a G-cover (K, ρ) of$(\mathbb{T}(\mathcal{O}), \chi)$. Then (K, ρ) is a $[T, \chi^\#]_G$-type in G.

- Let $\widetilde{W}_\chi = X_* \rtimes W_\chi$ be the subgroup of $\widetilde{W}$ which fixes χ. Consider the Hecke algebra $\mathcal{H}(G,\rho)$ associated to the pair (K,ρ). Roche gave generators $\{\mathcal{T}_w \mid w \in \widetilde{W}_\chi\}$ of the algebra $\mathcal{H}(G,\rho)$ and their relations.
- For $w \in \widetilde{W}_\chi$, define $\mathcal{T}_w$ to be the unique element of $\mathcal{H}(G,\rho)$ supp on Kn_wK and $\mathcal{T}_w(n_w) = q^{\frac{-\mathfrak{l}(w)}{2}} \tilde{\chi}(n_w)^{-1}$.
- $\mathcal{H}_{W_\chi}$ = subalg of $\mathcal{H}(G,\rho)$ generated by $\{\mathcal{T}_w \mid w \in W_\chi\}$.
- Bernstein presentation:$\mathcal{H}(G,\rho) \cong \mathcal{H}_{W_\chi} \otimes_{\mathbb{C}} \mathcal{H}(T,\rho_T)$.
- sgn : 1-dime repn of $\mathcal{H}_{W_\chi}$ in which $\mathcal{T}_w$ acts by $(-1)^{\mathfrak{l}'(w)}$.

Remark

If χ has positive depth, we assume all the assumptions taken by Roche i.e., F has characteristic 0 and some restrictions on p.

Principal series components

- Let x hyperspecial point in the Bruhat-Tits building which gives $\mathbb{G}$ the structure of a Chevalley group. The hyperspecial subgroup $\mathbb{G}(F)_{x,0}$ at x is $\mathbb{G}(\mathcal{O})$ with pro-unipotent radical $\mathbb{G}(F)_{x,0+}$. Then $\mathbb{G}(F)_{x,0}/\mathbb{G}(F)_{x,0+} \cong \mathbb{G}(\mathbb{F}_q)$.
- We say that ψ is of generic depth-zero at x if $\psi|_{\mathbb{U}(F)\cap\mathbb{G}(F)_{x,0}}$ factors through a generic character ψ_q of $\mathbb{U}(\mathbb{F}_q) \cong \mathbb{U}(F) \cap \mathbb{G}(F)_{x,0}/\mathbb{U}(F) \cap \mathbb{G}(F)_{x,0+}$.

Theorem (Mishra,....)

If $\chi = 1$, then assume that the T-orbit of ψ contains a character of generic depth zero at x. If $\chi \neq 1$, then assume that the center of $\mathbb{G}$ is connected. Then as $\mathcal{H}(G,\rho)$-module

$$\boxed{(c\text{-ind}_U^G(\psi))^\rho \cong \mathcal{H}(G,\rho) \otimes_{\mathcal{H}_{W_\chi}} \mathrm{sgn}.}$$

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Thank you.