The twisted supercritical deformed Hermitian-Yang-Mills equation on compact projective manifolds

Aashirwad Ballal

Indian Institute of Science

September 2021

Introduction

(M, χ) : a compact, connected Kähler manifold of complex dimension n.
$\alpha \in H^{1,1}(M, \mathbb{R})$

The dHYM equation:

$$\Im((\omega + \sqrt{-1}\chi)^n e^{\sqrt{-1}\hat{\theta}}) = 0 \tag{1}$$

where $\omega \in \alpha$ is also smooth and real.

If an $\omega_0 \in \alpha$ is fixed (which we assume from now on), then by the $\partial\bar{\partial}$-lemma, each $\omega \in \alpha$ can be written as $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ for some smooth real function ϕ (and conversely, for each such ϕ we get a member ω of α), so the dHYM equation is a non-linear 2nd order PDE for the function ϕ.

Introduction

Let $\lambda_i(x)$, $i = 1, \ldots, n$ denote the eigenvalues of $\chi^{-1}\omega$ at a point $x \in M$.

$$\Im((\omega + \sqrt{-1}\chi)^n e^{\sqrt{-1}\hat{\theta}})(x) = \prod_{i=1}^{n} \sqrt{1 + \lambda_i(x)^2} \cdot \sin(\hat{\theta} - \sum_{i=1}^{n} \theta_i(x))\chi^n$$

where $\theta_i(x) := \text{arccot}(\lambda_i(x))$ and $\text{arccot} : \mathbb{R} \to (0, \pi)$. So the dHYM equation can also be written as

$$\sum_{i=1}^{n} \text{arccot}(\lambda_i(x)) = \theta \tag{2}$$

for some $\theta \cong \hat{\theta} \mod \pi$. θ is called the phase angle.

Some necessary conditions

If a solution $\omega \in [\omega_0]$ to the dHYM equation exists, we must have

$$\int_M \Im((\omega + \sqrt{-1}\chi)^n e^{\sqrt{-1}\theta}) = \int_M \Im((\omega_0 + \sqrt{-1}\chi)^n e^{\sqrt{-1}\theta}) = 0 \quad (3)$$

And also

$$\sum_{i \in K} \theta_i(x) < \theta \quad (4)$$

for any $K \subsetneq \{1, \ldots, n\}$.

Supercritical Phase

If $\theta \in (0, \pi)$, the above inequalities can be rewritten in a more useful way.

Firstly, as $\sin(\theta) > 0$, equation (3) can be rewritten as

$$\int_M \Re(\omega_0 + \sqrt{-1}\chi)^n - \cot(\theta)\Im(\omega_0 + \sqrt{-1}\chi)^n = 0 \tag{5}$$

This shows that $\cot(\theta)$ is determined by the cohomology classes $[\omega_0], [\chi]$.

The inequalities (4) can be reformulated too. For example, as $\theta_i(x) < \theta$ and as cot is decreasing on $(0, \pi)$, we get $\lambda_i(x) > \cot(\theta)$ i.e.

$$\omega > \cot(\theta)\chi \tag{6}$$

Supercritical Phase

Similarly, taking $k := |K| = 1, \ldots, n-1$ in (4), these inequalities are equivalent to

$$\Re(\omega + \sqrt{-1}\chi)^k - \cot(\theta)\Im(\omega + \sqrt{-1}\chi)^k > 0 \tag{7}$$

For some small values of k, the inequalities are:

$$\begin{aligned} \omega - \cot(\theta)\chi &> 0 \\ \omega^2 - 2\cot(\theta)\omega\chi - \chi^2 &> 0 \\ \omega^3 - 3\cot(\theta)\omega^2\chi - 3\omega\chi^2 + \cot(\theta)\chi^3 &> 0 \end{aligned}$$

Cone and numerical conditions

- It was proved by Collins-Jacob-Yau that when $0 < \theta < \pi(1 - \frac{1}{n})$, the existence of an $\omega \in [\omega_0]$ satisfying the above inequalities is also sufficient for the existence of a solution to the dHYM equation.
- Such an ω is called a subsolution of the dHYM equation/said to satisfy the cone condition (from the results of Y. Yuan on the convexity of the level sets of the function $f(\lambda_1, \dots, \lambda_n) = \sum_i \operatorname{arccot}(\lambda_i)$, it follows that when the phase is supercritical the set of subsolutions is closed under convex linear combinations).

Cone and numerical conditions

Further, C-J-Y conjectured that the cone condition could be replaced by a numerical condition similar to the ones used in the numerical characterization of Kähler cones by Demailly and Paun. When M is projective, these numerical conditions reduce to

$$\int_V \Re(\omega_0 + \sqrt{-1}\chi)^k - \cot(\theta)\Im(\omega_0 + \sqrt{-1}\chi)^k > 0 \qquad (8)$$

(for $k = 1, \ldots, n-1$)

Some results

- G. Chen made significant progress towards proving the above conjecture by proving it (for all $\theta \in (0, \pi)$) under a slightly stronger hypothesis (uniform positivity).
- This was done by first producing a current T which satisfies the cone condition on M in some sense and then gluing its regularizations with a smooth form satisfying the cone condition in a neighborhood of a Lelong sublevel set of T.

To produce this current, a "concentration of mass" technique was used. Here, one considers a subvariety Y of M and family of equations

$$\Re(\omega_t + \sqrt{-1}\chi)^n - \cot(\theta)\Im(\omega_t + \sqrt{-1}\chi)^n = f_t\chi^n \tag{9}$$

with $\omega_t \in [\omega_0]$ and $t \in (0, 1]$. As $t \to 0$, the functions f_t approach a logarithmic singularity along Y.

Some results

- One then shows the existence of a subsequence which converges weakly to a current T satisfying the cone condition and also having some mass concentrated on Y i.e. $T \geq \beta[Y]$ for some $\beta > 0$.
- The function f_t is called the twisting function. Hence, it is of interest to find conditions for the solvability of the twisted dHYM equation as well.
- Analogous to the result of C-J-Y, G. Chen showed that the existence of a subsolution suffices to prove the existence of a solution to the twisted dHYM, but for $\dim(M) > 3$ (the case $\dim(M) \leq 3$ was not needed for G. Chen's proof that uniform positivity $\implies$ the existence of a solution as the concentration of mass was done on the diagonal $\Delta \subset M \times M$).

Some results

- Recently, the uniform positivity assumption was removed by Chu-Lee-Takahashi for the non-twisted dHYM on Kähler manifolds by using the techniques of G. Chen and J. Song.
- For the twisted dHYM on projective manifolds, the uniformity assumption was removed by A. using the approach of G. Chen and Datar-Pingali.

The twisted dHYM

- The twisted dHYM is the equation

$$\Re(\omega + \sqrt{-1}\chi)^n - \cot(\theta)\Im(\omega + \sqrt{-1}\chi)^n = f\chi^n \qquad (10)$$

  where $\omega \in [\omega_0]$ as usual.
- The cone condition for the twisted dHYM is the same as that for the non-twisted dHYM: ω satisfies the cone condition if

$$\Re(\omega + \sqrt{-1}\chi)^k - \cot(\theta)\Im(\omega + \sqrt{-1}\chi)^k > 0 \qquad (11)$$

  for $k = \{1, \ldots, n\}$.
- As mentioned earlier, for $n > 3$, the implication cone condition $\implies$ solution was shown by G. Chen. For $n = 1, 2$ the proofs are simple, so we sketch an outline of the proof only for $n = 3$.

The twisted dHYM for n = 3

Let $\Omega = \omega - \cot(\theta)\chi$. The cone condition in terms of Ω is

$$\begin{aligned} \Omega &> 0 \\ \Omega^2 - \csc^2(\theta)\chi^2 &> 0 \end{aligned} \tag{12}$$

To prove the result, we use the method of continuity along the path

$$\Omega_{\phi_t}^3 = 3\csc^2(\theta)\chi^2\Omega_{\phi_t} + 2\csc^2(\theta)(tf + \cot(\theta) + d_t)\chi^3 \tag{13}$$

where $\Omega_{\phi_t} = \Omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_t$ and $t \in [0,1]$.

- For $t = 0$, existence of a solution follows by the result of G. Chen.
- Openness of the interval $S \subset [0,1]$ for which there exist solutions is a consequence of ellipticity.
- To prove that S is closed, a priori estimates must be obtained for the solutions of the family of equations (13)

A priori estimates

The C_0 estimates follow from the results of Z. Blocki and are standard. To prove the C_2 and higher estimates, we follow the approach of V. Pingali, who proved the result for $n = 3$ and $f = 0$ (non-twisted dHYM). The presence of a non-constant f necessitates some more delicate estimates in this case.

Twisted dHYM for general n

- To prove that (non-uniform) positivity $\implies$ the existence of a solution for the twisted dHYM equation on a projective manifold, we follow the approach of Datar-Pingali.
- Here, the idea is to select an very ample line bundle on M and let Y be a zero section of a hololmorphic section of this line bundle.
- Concentration of mass technique is then used to show that there exists a positive $(1,1)$-current $\Theta \geq 2\beta[Y], \Theta \in [\omega_0 - \cot(\theta)\chi]$ satisfying the cone condition with respect to χ.

Twisted dHYM for general n

- If χ_Y is a Kähler metric in the class $[Y]$, we can add the exact current $\beta\chi_Y - \beta[Y]$ to Θ to obtain a Kähler current $T \geq \beta[Y]$ satisfying the cone condition on $M \cap Y^c$.
- We then produce a smooth metric in a neighborhood of Y using induction, a degenerate concentration of mass and a few successive regularization arguments.
- This smooth metric is then glued together with the regularizations of T to get a smooth metric $\omega \in [\omega_0]$ satisfying the cone condition on M.

References

A. Ballal. "The supercritical deformed Hermitian Yang–Mills equation on compact projective manifolds". In: (2021). arXiv: 2108.00876

G. Chen. "The J-equation and the supercritical deformed Hermitian–Yang–Mills equation.". In: *Invent. Math.* 225 (2021), pp. 529–602. DOI: 10.1007/s00222-021-01035-3

V. Datar and V. Pingali. *A numerical criterion for generalized Monge–Ampère equations on projective manifolds*. 2020. arXiv: 2006.01530

T. Collins, A. Jacob, and S. T. Yau. "(1, 1) forms with specified Lagrangian phase: A priori estimates and algebraic obstructions.". In: *Camb. J. of Math.* 8 (2020), pp. 407–452

Thank You!