Non-linear PDEs and positivity conditions in algebraic geometry

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Outline

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1. The Kähler cone and the Nakai criteria
2. Inverse Hessian equations
3. What next?

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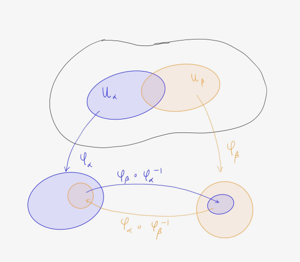
Complex manifolds

- Let M^n be a compact, complex manifold of dimension n without boundary.



- In the neighbourhood of each point we can choose *holomorphic coordinates* $(z^1, \cdots, z^n)$, where we put $z^i = x^i + \sqrt{-1}y^i$.
- (Almost complex structure) $J \in \mathrm{End}(TM)$ given by

$$J\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial y^i},\ J\Big(\frac{\partial}{\partial y^i}\Big) = -\frac{\partial}{\partial x^i}.$$

Definition independent of holomorphic coordinates $\iff$ CR equations.

Kähler manifolds

- (M, J, g) is called a Kähler manifold if
  1. (Hermitian condition) g is a Riemannian metric such that
  $$g(JX, JY) = g(X, Y).$$
  2. (Kähler condition) $\nabla J \equiv 0$ (or $\iff Hol(M, g) \subset GL(n, \mathbb{C}) \cap SO(2n)$.)
- So Kähler geometry $=$ (Complex Geometry)$\cap$ (Riemannian Geometry)
- (Kähler form) $\omega(X, Y) = g(JX, Y)$ is a real positive $(1,1)$ form. Locally
$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$
where $\{g_{i\bar{j}}\}$ is a hermitian, positive definite matrix of functions.
- The Kähler form determines g. Customary in Kähler geometry to consider the data (M, J, ω). We also sometimes call ω or g a Kähler metric.
- (Kähler condition) $\nabla J \equiv 0 \iff d\omega = 0$.
- Given a complex manifold (M, J) does it admit a Kähler metric? If so, how many of them?

Examples of Kähler manifolds

- **Riemann surfaces:** Let $n = 1$. Then there always exists a Kähler form. (Simply take the area form of any compatible Riemannian metric).
- **Complex flat space:** $\mathbb{C}^n$ with

$$\omega_{\mathbb{C}^n} := \frac{\sqrt{-1}}{2}\Big(dz^1 \wedge d\bar{z}^1 + \cdots + dz^n \wedge d\bar{z}^n\Big).$$

- **Complex projective space $\mathbb{P}^n$.** This is defined as $\mathbb{P}^n = \mathbb{C}^{n+1}/\mathbb{C}^*$, where $t \cdot (\xi^0, \cdots, \xi^n) = (t \cdot \xi^0, \cdots, t \cdot \xi^n)$. Then

$$\omega_{FS} := \sqrt{-1}\partial\overline{\partial}\log\Big(|\xi_0|^2 + \cdots + |\xi_n|^2\Big)$$

  is a Kähler metric on $\mathbb{P}^n$, called the *Fubini-Study* metric. All sub-manifolds $X \subset \mathbb{P}^n$ are also Kähler.
- **Complex Tori.** $\Lambda \subset \mathbb{C}^n$ a lattice. Then $\omega_{\mathbb{C}^n}$ induces a flat metric ω_Λ on $T_\Lambda := \mathbb{C}^n/\Lambda$. For a generic Λ, T_Λ is NOT projective if $n \geq 2$.
- Not all complex manifolds are Kähler. Eg - Hopf surfaces.

Kähler cone

- The Kähler form χ determines a cohomology class in $H^2_{dR}(M) = H^2(M, \mathbb{R})$ and $H^{1,1}_{\overline{\partial}}(M)$, and hence in

$$[\chi] \in H^{1,1}(M, \mathbb{R}) := H^{1,1}_{\overline{\partial}}(M) \cap H^2(M, \mathbb{R}).$$

.

- A class $\alpha \in H^{1,1}(M, \mathbb{R})$ is called Kähler, and we write it as $\alpha > 0$, if there exists a Kähler form $\omega \in \alpha$.
- The ***Kähler cone*** defined by

$$\mathcal{K} := \mathcal{K}_M := \{\alpha \in H^{1,1}(M, \mathbb{R}) \mid \alpha > 0\}$$

is an open, convex cone. Of course $\mathcal{K} \neq 0 \iff (M, J)$ admits a Kähler metric.
- $\mathcal{K}$ is a very interesting invariant of the Kähler manifold.
  - (Mirror symmetry) Degeneration of canonical Kähler metrics as class degenerates.
  - Singularity formation of Kähler Ricci flow.
- ($\sqrt{-1}\partial\overline{\partial}$-lemma) Let $\omega_1, \omega_2 \in \alpha \in H^{1,1}(M, \mathbb{R})$, and $\mathcal{K} \neq 0$. Then there exists $\varphi \in C^\infty(M, \mathbb{R})$ such that

$$\omega_2 = \omega_1 + \sqrt{-1}\partial\overline{\partial}\varphi.$$

Examples of the Kähler cone

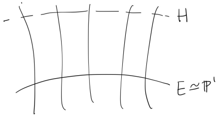
· Examples of Kähler cones

1) Riemann Surface X : $H^2(M,\mathbb{R}) \simeq \mathbb{R}$.

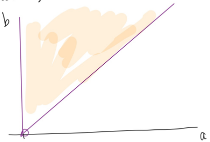
Then $\mathcal{K} = \mathbb{R}_+ \cong (0,\infty)$

2) $M = Bl_{[1:0:0]}\mathbb{P}^2 = \mathbb{P}^1$- bundle over $\mathbb{P}^1$

$H^{1,1}(M,\mathbb{R}) \simeq \mathbb{R}^2$
$= \{ b[H] - a[E] \mid a, b \in \mathbb{R}\}$.



(Seshadri criteria) $\Rightarrow \mathcal{K} = \{ b[H] - a[E] \mid b > a > 0\}$



A characterization of the Kähler cone

- If $V \subset M$ is a p-dimensional sub-variety and $\alpha = [\omega] \in \mathcal{K}$ for some Kähler form ω, then
$$\mathrm{Vol}(V,\alpha) := \int_V \alpha^p := \int_{V^{reg}} \omega^p > 0.$$
- Let $\alpha \in H^{1,1}(M,\mathbb{R})$.
    - (Nakai criteria) M projective and $\alpha = c_1(L)$, then $\alpha \in \mathcal{K} \iff \mathrm{Vol}(V,\alpha) > 0$ for all V.
    - (Demailly-Paun) (M,χ) Kähler, and for every irreducible analytic set $V \subset M$ of dimension p, and for every $k = 1, 2, \cdots, p$,
$$\int_V \alpha^k \wedge \chi^{p-k} > 0.$$
- Demailly-Paun proof uses Yau's solution to the Calabi conjecture i.e. solvability of Complex Monge Ampere equations:
$$\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi) = \text{given}.$$

The main steps in the proof of Demailly-Paun

Enough to show: If $\alpha + t[\chi] \in \mathcal{K}$ for all $t > 0$ and for all $V \subset M$,

$$\int_V \alpha^k \wedge \chi^{\dim V - k} > 0,$$

then $\alpha \in \mathcal{K}$.
Key Step: (Concentration of mass) Let $\tilde{\alpha}$ a class on $(\tilde{M}^m, \tilde{\chi})$ such that $\tilde{\alpha} + t\tilde{\chi} \in \mathcal{K}_{\tilde{M}}$ for all $t > 0$ and $\tilde{\alpha}^m > 0$. For any co-dimension p sub-variety Y, there exists a (p, p) current $\Theta \geq \beta_Y[Y]$ for some $\beta_Y > 0$.

- Let $\tilde{\chi}_t \in [\chi]$ sequence of (p, p) forms such that $\tilde{\chi}_t^p \sim [Y]$ if $t \sim 0$.
- Yau $\implies$ there exists $\omega_t \in \alpha + t[\chi]$ such that
$$\tilde{\omega}_t^m = c_t \tilde{\chi}_t^m.$$
- $\tilde{\alpha}_t^m > 0 \implies c_t > c_0 > 0$, and $\tilde{\omega}_t^p \to \Theta$.

To apply this:

- In projective case one can apply this to $\tilde{M} = M$ and $Y = H \cap M$.
- In general we have the Demailly-Paun diagonal trick: Apply this to $\tilde{M} = M \times M$, and $Y = \Delta \subset M \times M$.

Mabuchi functional and the cscK problem

A question of central interest in Kähler geometry is to contruct constant scalar curvature Kähler (cscK) metrics.

Conjecture

(Yau-Tian-Donaldson) Let $\alpha \in \mathcal{K}$. There exists a Kähler form $\omega \in \alpha$ whose scalar curvature $\mathbf{s}_\omega$ is constant if (and only if) the pair (M, α) is "stable".

- Let ω_0 be a reference metric in α, and let
$$\mathcal{H}_\alpha := \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$
- (Mabuchi energy, Chen) A metric ω_φ is cscK if and only if it is a smooth critical point of the functional
$$K(\varphi) = \int_M \log\Big(\frac{\omega_\varphi^n}{\omega_0^n}\Big)\frac{\omega_\varphi^n}{n!} + J_{-\mathrm{Ric}(\omega_0)}(\varphi),$$
where for any closed, real $(1,1)$ form χ, J_χ is defined by the variational formula
$$\delta J_\chi(\varphi) := \int_M \delta\varphi\Big(c_{n-1}\chi \wedge \frac{\omega_{\varphi^{n-1}}}{(n-1)!} - \frac{\omega_\varphi^n}{(n-1)!}\Big).$$

The J-equation

- From the previous slide:

$$\delta J_\chi(\varphi) := \int_M \delta\varphi\Big(c_{n-1}\chi\wedge\frac{\omega_{\varphi^{n-1}}}{(n-1)!} - \frac{\omega_\varphi^n}{(n-1)!}\Big).$$

- Clearly a metric ω_φ is a critical point if and only if it satisfies the so-called J-equation:

$$\omega_\varphi^n = c_{n-1}\chi\wedge\omega_\varphi^{n-1}.$$

- From now on we assume that $\chi > 0$. Then the above functional is "convex" on $\mathcal{H}_\alpha$. Moreover, if there is a solution to the J-equation, then J_χ is bounded below.
- In particular if M is general type i.e. $c_1(M) < 0$, then Yau $\implies$ there exists $\omega_0 \in \alpha$ such that $\chi := -\mathrm{Ric}(\omega_0) > 0$. Then if there is a solution to $\Lambda_\omega\chi = c_{n-1}$ for some $\omega \in \alpha$, then α has a cscK metric since J_χ will then be lower bounded, and hence K-energy will be proper.

Some necessary conditions

- Recall the J-equation:

$$\omega_\varphi^n = c_{n-1}\chi \wedge \omega_\varphi^{n-1}.$$

- A trivial necessary condition is obtained by integrating both sides, ie.

$$c_{n-1} = \frac{\int_M \omega_0^n}{\int_M \chi \wedge \omega_0^{n-1}}.$$

- If $n = 2$, then the completing squares, the equation is equivalent to

$$\left(\omega_0 - \frac{c_1}{2}\chi + \sqrt{-1}\partial\bar{\partial}\varphi\right)^2 = \chi^2.$$

- By Yau's solution to the Calabi conjecture a necessary and sufficient condition is that $[\omega_0] - \frac{c_1}{2}[\chi] > 0$, that is if we can choose a metric $\omega_0 \in [\omega_0]$ such that $\omega_0 - \frac{c_1}{2}\chi > 0$.
- (Demailly-Paun and Yau) $\implies$ For complex surfaces, solvability of J equation $\iff$ some numerical conditions hold: for all curves $C \subset M$,

$$\int_C \omega_0 > \frac{c_1}{2}\int_C \chi, \text{ and } \int_M (\omega_0 - \frac{c_1}{2}\chi)^2 > 0,$$

A more refined necessary condition

- More generally, if $0 < \lambda_1 \cdots < \lambda_n$ are the eigenvalues of $\chi^{-1}\omega_\varphi$, the equation is

$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} = \frac{n}{c_{n-1}}.$$

- So a necessary condition is that for any j,

$$\sum_{i \neq j} \frac{1}{\lambda_i} < \frac{n}{c_{n-1}} \iff n\omega_\varphi^{n-1} - c_{n-1}(n-1)\chi \wedge \omega_\varphi^{n-2} > 0.$$

- For an $(n-1, n-1)$ form Ψ, we say $\Psi > 0$ if $\Psi \wedge \sqrt{-1}\eta \wedge \overline{\eta}$ is a positive multiple of the volume form for all $(1,0)$ forms η.
- The above positivity or cone condition is almost as hard to check as solving the equation itself.

A numerical criteria, a la Demailly-Paun

Let (M, χ) be a Kähler manifold and ω_0 another Kähler metric and $c_{n-1} = \frac{\int_M \omega_0^n}{\int_M \chi \wedge \omega_0^{n-1}}$.

Theorem 2.1 (Song-Weinkove [5], 2009, Gao Chen in 2019 [1])

The following conditions are equivalent.

1. *There exists a Kähler metric $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ such that*
$$\begin{cases} \omega_\varphi^n = c_{n-1}\chi \wedge \omega_\varphi^{n-1} \\ n\omega_\varphi^{n-1} - c_{n-1}(n-1)\chi \wedge \omega_\varphi^{n-2} > 0. \end{cases} \tag{2.1}$$
2. *There exists a Kähler metric $\hat{\omega}_0 \in [\omega_0]$ satisfying the cone condition*
$$n\hat{\omega}_0^{n-1} - c_{n-1}(n-1)\chi \wedge \hat{\omega}_0^{n-2} > 0.$$
3. *There exists an $\varepsilon > 0$ such that for any p-dimensional sub-variety $V \subset X$,*
$$\int_V \left(n\omega_0^p - c_{n-1}p\chi \wedge \omega_0^{p-1}\right) > \varepsilon \int_V n\omega_0^p.$$

Remark

(1) $\iff$ (2) *by Song-Weinkove and* $\iff$ (3) *conjectured by Lejmi-Szekelyhidi and solved by Gao Chen.*

Generalized inverse Hessian equations

Let $c_1, \cdots, c_{n-1}$ be non-negative real numbers, such that at least one is positive. Suppose

$$\int_M \omega_\varphi^n = \int_M \sum_{k=1}^{n-1} c_k \chi^{n-k} \omega_\varphi^k.$$

Theorem (D.–Pingali [4], 2020, Conjecture of L-S)

Let M be a projective manifold, and χ, ω_0 be Kähler forms. TFAE:

1. *The generalised inverse Hessian equation has a solution $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfying*

$$\begin{cases} \omega_\varphi^n = \sum_{k=1}^{n-1} c_k \chi^{n-k} \omega_\varphi^k, \\ n\omega_\varphi^{n-1} - \sum_{k=1}^{n-1} c_k k \chi^{n-k} \omega_\varphi^{k-1} > 0. \end{cases} \tag{2.2}$$

2. *(Cone condition) There exists a Kähler metric $\hat{\omega}_0 \in [\omega_0]$ satisfying the cone condition, i.e.,*

$$n\hat{\omega}_0^{n-1} - \sum_{k=1}^{n-1} c_k k \chi^{n-k} \hat{\omega}_0^{k-1} > 0.$$

3. *(Numerical condition) For all subvarieties $V \subset M$ of co-dimension p, we have*

$$\int_V \left(\binom{n}{p} \omega_0^{n-p} - \sum_{k=p}^{n-1} c_k \binom{k}{p} \chi^{n-k} \wedge \omega_0^{k-p} \right) > 0.$$

Some remarks

- In particular we obtain a stronger version of the theorem of Gao Chen when M is projective.
- Our theorem also solves a well known conjecture of Lejmi-Szekelyhidi (albeit in the projective case) on solvability of the equation

$$\omega^n = c_k \omega^k \wedge \chi^{n-k}.$$

  Note that $k = n - 1$ is the J equation.
- We also obtain an equivariant version. In particular, for toric manifolds, one needs to only check the numerical criteria on torus invariant sub-varieties. For J-equation, this recovers results of Collins-Szekelyhidi.
- In [6], Jian Song removes the assumption of uniform positivity in the numerical condition for the J-equation for *all* Kähler manifolds.

Impressionistic outline of the proof

- (PDE result) (1) $\iff$ (2). In fact to prove a version of Mass concentration we need to solve the PDE with an additional term of $f\chi^n$ with f allowed to be slightly negative.
- Enough to assume that $\alpha + t[\chi]$ admits a solution to the PDE for all $t > 0$. Show that it also admits a solution at $t = 0$.
- (Mass concentration). If Y is an ample divisor and $0 < \beta, \delta << 1$, then there exists a current $\Theta \in (1-\delta)\alpha$ such that $\Theta \geq \beta[Y]$ and Θ satisfies the cone condition on $M \setminus Y$.
- (Siu) For $c > 0$, $E_c(\Theta) = \{x \in M \mid \nu(\Theta, x) > c\}$ is analytic.
- (Induction) There is a solution to the equation on Z in the class $\alpha\Big|_Z$ if Z is smooth, and so there is a ω_U in a neighbourhood of Z satisfying cone condition. Glue this with a regularization of Θ (need $c << 1$) to obtain a metric satisfying the cone condition on M. Then use the PDE result.
- If Z is not smooth, then take a resolution of singularities. The numerical criteria comes into play here.

1 The Kähler cone and the Nakai criteria

2 Inverse Hessian equations

3 What next?

1. One can try to prove the D-Pingali theorem for non-projective Kähler manifolds. The problem is that one needs to solve an appropriate PDE on $M \times M$. In our case, it is not clear what this PDE should be. This is mainly a linear algebra and PDE problem, and should be surmountable.
2. Convergence of the J-flow:
$$\begin{cases} \frac{\partial\varphi}{\partial t} = 1 - c_{n-1}\frac{\omega_\varphi^{n-1}\wedge\chi}{\omega_\varphi^n} \\ \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0. \end{cases}.$$
Long time existence is well known. Convergence is known when a solution to the J equation exists. In general convergence to singular solutions is known for some examples with a lot of symmetry such as $\mathrm{Bl}_p\mathbb{P}^n$. What about toric manifolds? Some sort of Harder-Narasimhan filtration?
3. Solve the J equation directly for any pair of classes α, β. Allow singularities.

Select references

- G. Chen. On J-equation., to appear in Invent. Math., arXiv : 1905.10222, .
- Collins, T. and Tosatti, V. “Kähler currents and null loci”, Invent. Math., n.0 3. (2015): 1167–1198
- Datar, V. and Pingali, V *A numerical criterion for generalised Monge-Ampere equations on projective manifolds*, to appear in Geom and Func. Anal. (GAFA), arXiv:2006.01530.
- Demailly, J-P and Paun, M. “Numerical characterization of the Kähler cone of a compact Kähler manifold”. Ann. Math. (2004) : 1247-1274.
- J. Song, and B. Weinkove. “The degenerate J-flow and the Mabuchi energy on minimal surfaces of general type.” Univ. Iagellonicae Acta Math. 50 : 89-106.
- J. Song “Nakai-Moishezon criterions for complex Hessian equations”, arXiv:2012.07956.
- G. Székelyhidi. “Fully non-linear elliptic equations on compact Hermitian manifolds.” J. Differential Geom. 109(2)(2018) : 337-378.

Thank You for your attention!