On a theorem of Chernoff for quasi-analytic functions

Pritam Ganguly

Department of Mathematics
Indian Institute of Science
Bangalore-India

IISc–IISERP Joint Math 20-20 Symposium,
17th September, 2021.

Introduction: What is quasi-analyticity?

The key property of an analytic function is that it is completely determined by the values of the function and its derivatives at a single point.

Taylor series expansion of an analytic function on $\mathbb{R}$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

[1] E. Borel. *Leons sur les fonctions monognes uniformes d'une variable complexe. Collection de monographies sur la théorie des fonctions publiée sous la direction de M. E. Borel*. GauthierVillars, Paris, 1917

Introduction: What is quasi-analyticity?

The key property of an analytic function is that it is completely determined by the values of the function and its derivatives at a single point.

Taylor series expansion of an analytic function on $\mathbb{R}$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

- **Borel** [1] first observed that there is a larger class of smooth functions than that of analytic functions which has this magnificent property. He coined the term *quasi-analytic* for such class of functions.

[1] E. Borel. *Leons sur les fonctions monognes uniformes d'une variable complexe. Collection de monographies sur la théorie des fonctions publiée sous la direction de M. E. Borel.* GauthierVillars, Paris, 1917

Introduction: What is quasi-analyticity?

The key property of an analytic function is that it is completely determined by the values of the function and its derivatives at a single point.

Taylor series expansion of an analytic function on $\mathbb{R}$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

- **Borel** [1] first observed that there is a larger class of smooth functions than that of analytic functions which has this magnificent property. He coined the term *quasi-analytic* for such class of functions.

Definition

A subset $\mathcal{C}$ of the class of all smooth functions on (a, b) is said to be Quasi-analytic if for any $f \in \mathcal{C}$ and $x_0 \in (a, b)$, $\frac{d^n}{dx^n} f(x_0) = 0$ for all $n \in \mathbb{N}$ implies $f = 0$.

[1] E. Borel. *Leons sur les fonctions monognes uniformes d'une variable complexe. Collection de monographies sur la théorie des fonctions publiée sous la direction de M. E. Borel.* GauthierVillars, Paris, 1917

- Using the Taylor series expansion one can easily show that a function f on $I = (a, b)$ is analytic if and only if it satisfies the following growth condition:

$$\|\frac{d^n}{dx^n} f\|_{L^\infty(I)} \leq C n! A^n, \ \forall n$$

- Using the Taylor series expansion one can easily show that a function f on $I = (a, b)$ is analytic if and only if it satisfies the following growth condition:

$$\|\frac{d^n}{dx^n} f\|_{L^\infty(I)} \leq Cn!A^n, \ \forall n$$

- It is natural to ask whether relaxing growth condition on the derivatives generates quasi-analytic class.

- Using the Taylor series expansion one can easily show that a function f on $I = (a, b)$ is analytic if and only if it satisfies the following growth condition:

$$\|\frac{d^n}{dx^n} f\|_{L^\infty(I)} \leq C n! A^n, \ \forall n$$

- It is natural to ask whether relaxing growth condition on the derivatives generates quasi-analytic class.
- In 1912, **Hadamard** proposed the problem of finding sequence $\{M_n\}_n$ of positive numbers such that the class $C\{M_n\}$ of smooth functions on I satisfying $\|\frac{d^n}{dx^n} f\|_{L^\infty(I)} \leq A_f^n M_n$ for all $f \in C\{M_n\}$ is a quasi-analytic class.

- Using the Taylor series expansion one can easily show that a function f on $I = (a, b)$ is analytic if and only if it satisfies the following growth condition:

$$\|\frac{d^n}{dx^n} f\|_{L^\infty(I)} \leq Cn!A^n, \ \forall n$$

- It is natural to ask whether relaxing growth condition on the derivatives generates quasi-analytic class.
- In 1912, **Hadamard** proposed the problem of finding sequence $\{M_n\}_n$ of positive numbers such that the class $C\{M_n\}$ of smooth functions on I satisfying $\|\frac{d^n}{dx^n} f\|_{L^\infty(I)} \leq A_f^n M_n$ for all $f \in C\{M_n\}$ is a quasi-analytic class.

Theorem (Denjoy-Carleman)

$C\{M_n\}$ is a quasi-analytic class if and only if

$$\sum_{n=1}^{\infty} M_n{}^{-1/n} = \infty.$$

- **Examples:**

1. $M_n = n!$: This actually corresponds to the class of analytic functions.
2. $M_n = (\log n)^n, \ (\log n)^n(\log\log n)^n$ etc..

- **Examples:**

1. $M_n = n!$: This actually corresponds to the class of analytic functions.
2. $M_n = (\log n)^n, \ (\log n)^n (\log\log n)^n$ etc..

- In 1939, **S. Bochner** and **A.E. Taylor** proved a generalization of Denjoy-Carleman theorem on $\mathbb{R}^n$.

- **Examples:**
  1. $M_n = n!$: This actually corresponds to the class of analytic functions.
  2. $M_n = (\log n)^n,\ (\log n)^n(\log\log n)^n$ etc..
- In 1939, **S. Bochner** and **A.E. Taylor** proved a generalization of Denjoy-Carleman theorem on $\mathbb{R}^n$.
  - **Notations:** Given a smooth function f on $\mathbb{R}^n$ denote

$$D_0 f(x) := |f(x)| \quad \text{and} \quad D_k f(x) = \left(\sum_{|\alpha|=k} \left| \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \right|^2 \right)^{\frac{1}{2}}$$

- **Examples:**
  1. $M_n = n!$: This actually corresponds to the class of analytic functions.
  2. $M_n = (\log n)^n, \ (\log n)^n(\log\log n)^n$ etc..
- In 1939, **S. Bochner** and **A.E. Taylor** proved a generalization of Denjoy-Carleman theorem on $\mathbb{R}^n$.
  - **Notations:** Given a smooth function f on $\mathbb{R}^n$ denote

$$D_0 f(x) := |f(x)| \ \text{ and } \ D_k f(x) = \left(\sum_{|\alpha|=k} \left| \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \right|^2 \right)^{\frac{1}{2}}$$

Theorem (Amer. J. Math, 1939)

Let f be a smooth function defined on a connected domain $\Omega \subset \mathbb{R}^n$ and x_0 be an interior point of Ω. Then the conditions

- $D_k f(x) \leq m_k$ *for all* $x \in \Omega$ *where* $\sum_{k=1}^{\infty} m_k^{-1/k} = \infty$,
- $D_k f(x_0) = 0$ *for all* $k \geq 0$

imply that f is identically zero on Ω.

Chernoff's theorem

Theorem (P.R. Chernoff, 1978)

Let f be a smooth function on $\mathbb{R}^n$. Let Δ be the standard Laplacian on $\mathbb{R}^n$. Assume that $\Delta^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and

$$\sum_{m=1}^{\infty} \|\Delta^m f\|_2^{-\frac{1}{2m}} = \infty.$$

If f and all its partial derivatives vanish at a point x_0, then f is identically zero.

Chernoff's theorem

Theorem (P.R. Chernoff, 1978)

Let f be a smooth function on $\mathbb{R}^n$. Let Δ be the standard Laplacian on $\mathbb{R}^n$. Assume that $\Delta^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and

$$\sum_{m=1}^{\infty} \|\Delta^m f\|_2^{-\frac{1}{2m}} = \infty.$$

If f and all its partial derivatives vanish at a point x_0, then f is identically zero.

1. **Questions:**
   - What happens if we replace the Laplacian by any other 2nd order operator?

Chernoff's theorem

Theorem (P.R. Chernoff, 1978)

Let f be a smooth function on $\mathbb{R}^n$. Let Δ be the standard Laplacian on $\mathbb{R}^n$. Assume that $\Delta^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and

$$\sum_{m=1}^{\infty} \|\Delta^m f\|_2^{-\frac{1}{2m}} = \infty.$$

If f and all its partial derivatives vanish at a point x_0, then f is identically zero.

1. **Questions:**
   - What happens if we replace the Laplacian by any other 2nd order operator?
   - What would be an analogue of this result for the Laplace-Beltrami operator on Riemannian symmetric spaces (of compact and non-compact type)?
   - Is this result true for the sublaplacian on the Heisenberg group?

Chernoff's theorem

Theorem (P.R. Chernoff, 1978)

Let f be a smooth function on $\mathbb{R}^n$. Let Δ be the standard Laplacian on $\mathbb{R}^n$. Assume that $\Delta^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and

$$\sum_{m=1}^{\infty} \|\Delta^m f\|_2^{-\frac{1}{2m}} = \infty.$$

If f and all its partial derivatives vanish at a point x_0, then f is identically zero.

1. **Questions:**
   - What happens if we replace the Laplacian by any other 2nd order operator?
   - What would be an analogue of this result for the Laplace-Beltrami operator on Riemannian symmetric spaces (of compact and non-compact type)?
   - Is this result true for the sublaplacian on the Heisenberg group?
   - Even if we replace the vanishing condition by a stronger one, can we get similar conclusion in the setting of symmetric spaces and Lie groups?

Riemannian symmetric spaces of noncompact type

- Let G be a connected, noncompact semisimple Lie group with finte center and K be a maximal compact subgroup. Let $X = G/K$ be the associated Riemannian symmetric space. This a noncompact Riemannian manifold.

Riemannian symmetric spaces of noncompact type

- Let G be a connected, noncompact semisimple Lie group with finte center and K be a maximal compact subgroup. Let $X = G/K$ be the associated Riemannian symmetric space. This a noncompact Riemannian manifold.
  •**Example:** $G = SL_2(\mathbb{R}),\ K = SO(2)$ and $X = SL_2(\mathbb{R})/SO(2) = H^2 = \{z \in \mathbb{C} : Im(z) > 0\}$ upper half space.

Riemannian symmetric spaces of noncompact type

- Let G be a connected, noncompact semisimple Lie group with finte center and K be a maximal compact subgroup. Let $X = G/K$ be the associated Riemannian symmetric space. This a noncompact Riemannian manifold.
  •**Example:** $G = SL_2(\mathbb{R}),\ K = SO(2)$ and $X = SL_2(\mathbb{R})/SO(2) = H^2 = \{z \in \mathbb{C} : Im(z) > 0\}$ upper half space.
- Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Then the Cartan decomposition of $\mathfrak{g}$ reads as $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{p}$.

Riemannian symmetric spaces of noncompact type

- Let G be a connected, noncompact semisimple Lie group with finte center and K be a maximal compact subgroup. Let $X = G/K$ be the associated Riemannian symmetric space. This a noncompact Riemannian manifold.
  •**Example:** $G = SL_2(\mathbb{R}),\ K = SO(2)$ and $X = SL_2(\mathbb{R})/SO(2) = H^2 = \{z \in \mathbb{C} : Im(z) > 0\}$ upper half space.
- Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Then the Cartan decomposition of $\mathfrak{g}$ reads as $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{p}$.
- Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is called the rank of X.

Riemannian symmetric spaces of noncompact type

- Let G be a connected, noncompact semisimple Lie group with finte center and K be a maximal compact subgroup. Let $X = G/K$ be the associated Riemannian symmetric space. This a noncompact Riemannian manifold.
  •**Example:** $G = SL_2(\mathbb{R})$, $K = SO(2)$ and $X = SL_2(\mathbb{R})/SO(2) = H^2 = \{z \in \mathbb{C} : Im(z) > 0\}$ upper half space.
- Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Then the Cartan decomposition of $\mathfrak{g}$ reads as $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{p}$.
- Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is called the rank of X.
- Let $A = \exp \mathfrak{a}$ and M denote the centralizer of A in K.
- Iwasawa decomposition: $G = KAN$.
- Let Δ_X be the associated Laplace-Beltrami operator on X.

Theorem (Bhowmik-Pusti-Ray, Journal of Functional analysis, 2020)

Let $X = G/K$ be a Riemannian symmetric space noncompact type. Suppose $f \in C^\infty(X)$ satisfies $\Delta_X^m f \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^\infty \|\Delta_X^m f\|_2^{-\frac{1}{2m}} = \infty$. If f vanishes on a non empty open set, then f is identically zero.

Theorem (Bhowmik-Pusti-Ray, Journal of Functional analysis, 2020)

Let $X = G/K$ be a Riemannian symmetric space noncompact type. Suppose $f \in C^\infty(X)$ satisfies $\Delta_X^m f \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^\infty \|\Delta_X^m f\|_2^{-\frac{1}{2m}} = \infty$. If f vanishes on a non empty open set, then f is identically zero.

- Functions on $X = G/K$ can be identified with the right K-invariant functions on G. Moreover, we say that a function f on G is K-biinvariant if $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$ and $g \in G$.
- Let $D(G/K)$ denote the algebra of differential operators on G/K which are invariant under the (left) action of G.

Theorem (Bhowmik-Pusti-Ray, Journal of Functional analysis, 2020)

Let $X = G/K$ be a Riemannian symmetric space noncompact type. Suppose $f \in C^\infty(X)$ satisfies $\Delta_X^m f \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^\infty \|\Delta_X^m f\|_2^{-\frac{1}{2m}} = \infty$. If f vanishes on a non empty open set, then f is identically zero.

- Functions on $X = G/K$ can be identified with the right K-invariant functions on G. Moreover, we say that a function f on G is K-biinvariant if $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$ and $g \in G$.
- Let $D(G/K)$ denote the algebra of differential operators on G/K which are invariant under the (left) action of G.

Theorem (Bhowmik-Pusti-Ray, IMRN, 2021)

Let $X = G/K$ be a Riemannian symmetric space noncompact type. Suppose $f \in C^\infty(G/K)$ be a left K-invariant function on X which satisfies $\Delta_X^m f \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^\infty \|\Delta_X^m f\|_2^{-\frac{1}{2m}} = \infty$. If there is an $x_0 \in X$ such that $Df(x_0)$ vanishes for all $D \in D(G/K)$ then f is identically zero.

- The Iwasawa decomposition on the Lie algebra side reads as $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{a} \bigoplus \mathfrak{n}$. Now we assume that X is of rank one i.e., $\dim \mathfrak{a} = 1$.

- The Iwasawa decomposition on the Lie algebra side reads as $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{a} \bigoplus \mathfrak{n}$. Now we assume that X is of rank one i.e., $\dim \mathfrak{a} = 1$.
- Fix a basis $\{H\}$ of $\mathfrak{a}$. This gives rise to a left invariant vector field on G by the prescription

$$Hf(g) = \frac{d}{dt}\bigg|_{t=0} f(g.\exp(tH)), \ g \in G.$$

- The Iwasawa decomposition on the Lie algebra side reads as $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{a} \bigoplus \mathfrak{n}$. Now we assume that X is of rank one i.e., $\dim \mathfrak{a} = 1$.
- Fix a basis $\{H\}$ of $\mathfrak{a}$. This gives rise to a left invariant vector field on G by the prescription

$$Hf(g) = \frac{d}{dt}\Big|_{t=0} f(g.\exp(tH)), \ g \in G.$$

Theorem (Ganguly-Manna-Thangavelu, 2021)

Let $X = G/K$ be a rank one symmetric space of noncompact type. Suppose $f \in C^\infty(X)$ satisfies $\Delta_X^m f \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^{\infty} \|\Delta_X^m f\|_2^{-\frac{1}{2m}} = \infty$. If $x_0 \in X$ and $H^l f(x_0) = 0$ for all $l \geq 0$ then f is identically zero.

Theorem (Ganguly-Manna-Thangavelu, 2021)

Let $X = G/K$ be a rank one symmetric space of noncompact type. Suppose $f \in C^\infty(X)$ satisfies $\Delta_X^m f \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^\infty \|\Delta_X^m f\|_2^{-\frac{1}{2m}} = \infty$. If $x_0 \in X$ and $H^l f(x_0) = 0$ for all $l \geq 0$ then f is identically zero.

- The condition $H^l f(eK) = 0$ is the counterpart of $(\frac{d}{dr})^k f(r\omega)|_{r=0} = 0$ where $x = r\omega, r > 0, \omega \in \mathbb{S}^{n-1}$ is the polar decomposition of $x \in \mathbb{R}^n$. Indeed, as can be easily checked

$$\left(\frac{d}{dr}\right)^k f(r\omega) = \sum_{|\alpha|=k} \partial^\alpha f(r\omega)\,\omega^\alpha$$

and hence $(\frac{d}{dr})^k f(r\omega)|_{r=0} = 0$ for all k if and only if $\partial^\alpha f(0) = 0$ for all α.

Theorem (Ganguly-Manna-Thangavelu, 2021)

Let $X = G/K$ be a rank one symmetric space of noncompact type. Suppose $f \in C^\infty(X)$ satisfies $\Delta_X^m f \in L^2(X)$ for all $m \geq 0$ and $\sum_{m=1}^\infty \|\Delta_X^m f\|_2^{-\frac{1}{2m}} = \infty$. If $x_0 \in X$ and $H^l f(x_0) = 0$ for all $l \geq 0$ then f is identically zero.

- The condition $H^l f(eK) = 0$ is the counterpart of $(\frac{d}{dr})^k f(r\omega)|_{r=0} = 0$ where $x = r\omega, r > 0, \omega \in \mathbb{S}^{n-1}$ is the polar decomposition of $x \in \mathbb{R}^n$. Indeed, as can be easily checked

$$\left(\frac{d}{dr}\right)^k f(r\omega) = \sum_{|\alpha|=k} \partial^\alpha f(r\omega)\,\omega^\alpha$$

and hence $(\frac{d}{dr})^k f(r\omega)|_{r=0} = 0$ for all k if and only if $\partial^\alpha f(0) = 0$ for all α.

Open question

An exact analogue of Chernoff's theorem for Riemannian symmetric spaces of noncompact type without any restrictions on rank.

Compact symmetric spaces

- Let (G, K) be a compact symmetric pair and let $S = G/K$ be the associated symmetric space. We assume that X has rank one. Let $G = KAK$ be a Cartan decomposition of G where A is identified with $(0, R)$ for some $R > 0$.

[2] **H.-C. Wang, Two-point homogeneous spaces Ann. of Math. (2), 55 (1952), pp. 177-191.**

Compact symmetric spaces

- Let (G, K) be a compact symmetric pair and let $S = G/K$ be the associated symmetric space. We assume that X has rank one. Let $G = KAK$ be a Cartan decomposition of G where A is identified with $(0, R)$ for some $R > 0$.
- The rank one one compact symmetric spaces are completely classified by H.-C. Wang[2]

[2] **H.-C. Wang, Two-point homogeneous spaces Ann. of Math. (2), 55 (1952), pp. 177-191.**

Compact symmetric spaces

- Let (G, K) be a compact symmetric pair and let $S = G/K$ be the associated symmetric space. We assume that X has rank one. Let $G = KAK$ be a Cartan decomposition of G where A is identified with $(0, R)$ for some $R > 0$.
- The rank one one compact symmetric spaces are completely classified by H.-C. Wang[2]
- Following Wang any compact rank one symmetric space S is one of the following:
  1. the sphere $\mathbb{S}^q \subset \mathbb{R}^{q+1}$, $q \geq 1$;
  2. the real projective space $P_q(\mathbb{R})$, $q \geq 2$;
  3. the complex projective space $P_l(\mathbb{C})$, $l \geq 2$;
  4. the quaternionic projective space $P_l(\mathbb{H})$, $l \geq 2$;
  5. the Cauchy projective plane $P_2(\mathbb{C}ay)$.

[2] **H.-C. Wang, Two-point homogeneous spaces Ann. of Math. (2), 55 (1952), pp. 177-191.**

Theorem (Ganguly-Thangavelu, Adv. Math, 2021)

Let $f \in C^\infty(G/K)$ be such that $\Delta_S^m f \in L^2(G/K)$ for all $m \geq 0$ and satisfies the Carleman condition $\sum_{m=1}^{\infty} \|\Delta_S^m f\|_2^{-1/(2m)} = \infty$. Then f cannot vanish on any open set unless it is identically zero.

Chernoff's theorem on Compact symmetric spaces

Theorem (Ganguly-Thangavelu, Adv. Math, 2021)

Let $f \in C^\infty(G/K)$ be such that $\Delta_S^m f \in L^2(G/K)$ for all $m \geq 0$ and satisfies the Carleman condition $\sum_{m=1}^\infty \|\Delta_S^m f\|_2^{-1/(2m)} = \infty$. Then f cannot vanish on any open set unless it is identically zero.

- Without any restriction on the rank, the following weaker result is known:

Theorem (Bhowmik-Pusti-Ray, IMRN, 2021)

Let $S = U/K$ be a Riemannian symmetric spaces of compact type and $f \in C^\infty(S)$ be K-invariant. Assume that $\sum_{m=1}^\infty \|\Delta_S^m f\|_2^{-1/(2m)} = \infty$. Then if $Df(eK) = 0$ for all $D \in D(S)$ then f is identically zero.

Polar coordinates on rank one compact symmetric spaces

- Each rank one compact symmetric space admits appropriate polar coordinate $(\theta, \xi) \in (0, \pi) \times \mathbb{S}^{ks}$, where $\mathbb{S}^{ks}$ is a unit sphere whose dimension depends on the associated symmetric space S.

Polar coordinates on rank one compact symmetric spaces

- Each rank one compact symmetric space admits appropriate polar coordinate $(\theta, \xi) \in (0, \pi) \times \mathbb{S}^{k_S}$, where $\mathbb{S}^{k_S}$ is a unit sphere whose dimension depends on the associated symmetric space S.
- In these coordinates, the Laplace-Beltrami operator turns out to be

$$-\Delta_S = \mathbb{L}_{\alpha,\beta} - \rho_S(\theta)\Delta_{\mathbb{S}^{k_S}}$$

Polar coordinates on rank one compact symmetric spaces

- Each rank one compact symmetric space admits appropriate polar coordinate $(\theta, \xi) \in (0, \pi) \times \mathbb{S}^{k_S}$, where $\mathbb{S}^{k_S}$ is a unit sphere whose dimension depends on the associated symmetric space S.
- In these coordinates, the Laplace-Beltrami operator turns out to be

$$-\Delta_S = \mathbb{L}_{\alpha,\beta} - \rho_S(\theta)\Delta_{\mathbb{S}^{k_S}}$$

- **Example:** Let $S = \mathbb{S}^q \subset \mathbb{R}^{q+1}$. Note that given $\xi \in \mathbb{S}^q$, we can write $\xi = (\cos\theta)e_1 + \xi_1^{'}(\sin\theta)e_2 + ... + \xi_q^{'}(\sin\theta)e_{q+1}$ for some $\theta \in (0, \pi)$ and $\xi^{'} = (\xi_1^{'}, ..., \xi_q^{'}) \in \mathbb{S}^{q-1}$ where $\{e_1, e_2, ..., e_{q+1}\}$ is the standard basis for $\mathbb{R}^{q+1}$.

Polar coordinates on rank one compact symmetric spaces

- Each rank one compact symmetric space admits appropriate polar coordinate $(\theta, \xi) \in (0, \pi) \times \mathbb{S}^{k_S}$, where $\mathbb{S}^{k_S}$ is a unit sphere whose dimension depends on the associated symmetric space S.
- In these coordinates, the Laplace-Beltrami operator turns out to be

$$-\Delta_S = \mathbb{L}_{\alpha,\beta} - \rho_S(\theta)\Delta_{\mathbb{S}^{k_S}}$$

- **Example:** Let $S = \mathbb{S}^q \subset \mathbb{R}^{q+1}$. Note that given $\xi \in \mathbb{S}^q$, we can write $\xi = (\cos\theta)e_1 + \xi_1^{'}(\sin\theta)e_2 + ... + \xi_q^{'}(\sin\theta)e_{q+1}$ for some $\theta \in (0, \pi)$ and $\xi^{'} = (\xi_1^{'}, ..., \xi_q^{'}) \in \mathbb{S}^{q-1}$ where $\{e_1, e_2, ..., e_{q+1}\}$ is the standard basis for $\mathbb{R}^{q+1}$.
- $\varphi : (0, \pi) \times \mathbb{S}^{q-1} \to \mathbb{S}^q$ defined by

$$\varphi(\theta, \xi') = (\cos\theta, \xi'_1 \sin\theta, \ldots, \xi'_q \sin\theta)$$

induces the polar coordinate system on $\mathbb{S}^q$

Polar coordinates on rank one compact symmetric spaces

- Each rank one compact symmetric space admits appropriate polar coordinate $(\theta, \xi) \in (0, \pi) \times \mathbb{S}^{k_S}$, where $\mathbb{S}^{k_S}$ is a unit sphere whose dimension depends on the associated symmetric space S.
- In these coordinates, the Laplace-Beltrami operator turns out to be

$$-\Delta_S = \mathbb{L}_{\alpha,\beta} - \rho_S(\theta)\Delta_{\mathbb{S}^{k_S}}$$

- **Example:** Let $S = \mathbb{S}^q \subset \mathbb{R}^{q+1}$. Note that given $\xi \in \mathbb{S}^q$, we can write $\xi = (\cos\theta)e_1 + \xi_1'(\sin\theta)e_2 + ... + \xi_q'(\sin\theta)e_{q+1}$ for some $\theta \in (0, \pi)$ and $\xi' = (\xi_1', ..., \xi_q') \in \mathbb{S}^{q-1}$ where $\{e_1, e_2, ..., e_{q+1}\}$ is the standard basis for $\mathbb{R}^{q+1}$.
- $\varphi : (0, \pi) \times \mathbb{S}^{q-1} \to \mathbb{S}^q$ defined by

$$\varphi(\theta, \xi') = (\cos\theta, \xi_1' \sin\theta, \ldots, \xi_q' \sin\theta)$$

  induces the polar coordinate system on $\mathbb{S}^q$
- $-\Delta_{\mathbb{S}^q} = -\frac{\partial^2}{\partial\theta^2} - (q-1)\cot\theta\frac{\partial}{\partial\theta} - \sin^{-2}\theta\Delta_{\mathbb{S}^{q-1}}$

• Vanishing condition:

• Euclidean space: $\mathbb{R}^n$
Polar form: $(0,\infty)\times\mathbb{S}^{n-1}$
• $\partial^\alpha f(0)=0$ for all $\alpha\in\mathbb{N}^n$ is equivalent to

$$\frac{d^k}{dr^k}\Big|_{r=0} f(r\omega)=0 \text{ for all } \omega\in\mathbb{S}^{n-1}$$

and $k\geq 0$.

• Compact symmetric space: S
Polar form: $(0,\pi)\times\mathbb{S}^{ks}$

• **Vanishing condition:**

• **Euclidean space:**$\mathbb{R}^n$
Polar form: $(0, \infty) \times \mathbb{S}^{n-1}$
• $\partial^\alpha f(0) = 0$ for all $\alpha \in \mathbb{N}^n$ is equivalent to

$$\frac{d^k}{dr^k}\Big|_{r=0} f(r\omega) = 0 \text{ for all } \omega \in \mathbb{S}^{n-1}$$

and $k \geq 0$.

• **Compact symmetric space:** S
Polar form: $(0, \pi) \times \mathbb{S}^{ks}$
• Every function f on S corresponds to a function F on $(0, \pi) \times \mathbb{S}^{ks}$. So the natural vanishing condition would be

$$\frac{d^k}{d\theta^k}\Big|_{\theta=0} F(\theta, \xi) = 0, \ \forall \xi \in \mathbb{S}^{ks}, \ k \geq 0.$$

• Vanishing condition:

• **Euclidean space:**$\mathbb{R}^n$
Polar form: $(0,\infty)\times\mathbb{S}^{n-1}$
• $\partial^\alpha f(0)=0$ for all $\alpha\in\mathbb{N}^n$ is equivalent to

$$\frac{d^k}{dr^k}\Big|_{r=0} f(r\omega)=0 \text{ for all } \omega\in\mathbb{S}^{n-1}$$

and $k\geq 0$.

• **Compact symmetric space:** S
Polar form: $(0,\pi)\times\mathbb{S}^{ks}$
• Every function f on S corresponds to a function F on $(0,\pi)\times\mathbb{S}^{ks}$. So the natural vanishing condition would be

$$\frac{d^k}{d\theta^k}\Big|_{\theta=0} F(\theta,\xi)=0,\ \forall\xi\in\mathbb{S}^{ks},\ k\geq 0.$$

Theorem (Ganguly-Manna-Thangavelu, 2021)

Let S be a rank one Riemannian symmetric space of compact type. Suppose $f\in C^\infty(S)$ satisfies $\Delta_S^m f\in L^2(S)$ for all $m\geq 0$ and $\sum_{m=1}^\infty \|\Delta_S^m f\|_2^{-\frac{1}{2m}}=\infty$. If the function F on $(0,\pi)\times\mathbb{S}^{ks}$ associated to f on S satisfies $\frac{\partial^m}{\partial\theta^m}\big|_{\theta=0}F(\theta,\xi)=0$ for all $m\geq 0$, then f is identically zero.

Heisenberg group and related operators

- Consider the Heisenberg group $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t).(w, s) := \left(z + w, t + s + \frac{1}{2}\,\mathrm{Im}(z.\bar{w})\right).$$

This is a step two Nilpotent Lie group where the Lebesgue measure *dzdt* plays the role of Haar measure.

Heisenberg group and related operators

- Consider the Heisenberg group $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t).(w, s) := \left(z + w, t + s + \frac{1}{2}\operatorname{Im}(z.\bar{w})\right).$$

This is a step two Nilpotent Lie group where the Lebesgue measure *dzdt* plays the role of Haar measure.

- The Heisenberg Lie algebra, $\mathfrak{h}_n$ consists of left invariant vector fields on $\mathbb{H}^n$. A basis for $\mathfrak{h}_n$ is provided by the $2n+1$ vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \;\; Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \;\; j = 1, 2, \ldots, n, \text{ and } T = \frac{\partial}{\partial t}.$$

Heisenberg group and related operators

- Consider the Heisenberg group $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t).(w, s) := \left(z + w, t + s + \frac{1}{2}\operatorname{Im}(z.\bar{w})\right).$$

  This is a step two Nilpotent Lie group where the Lebesgue measure *dzdt* plays the role of Haar measure.

- The Heisenberg Lie algebra, $\mathfrak{h}_n$ consists of left invariant vector fields on $\mathbb{H}^n$. A basis for $\mathfrak{h}_n$ is provided by the $2n+1$ vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t},\ \ Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t},\ j = 1, 2, ..., n,\ \text{and}\ T = \frac{\partial}{\partial t}.$$

- The **sublaplacian** on $\mathbb{H}^n$ is defined by $\mathcal{L} := -\sum_{j=1}^{\infty}(X_j^2 + Y_j^2)$ which is given explicitly by

$$\mathcal{L} = -\Delta_{\mathbb{C}^n} - \frac{1}{4}|z|^2\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial t}$$

  where $\Delta_{\mathbb{C}^n}$ stands for the Laplacian on $\mathbb{C}^n$.

Chernoff's theorem on the Heisenberg group

Theorem (Bagchi-Ganguly-Sarkar-Thangavelu, 2020)

Let f be a smooth function on $\mathbb{H}^n$ satisfying $f(z,t) = f_0(|(z,t)|)$ where $|(z,t)| = (|z|^4 + t^2)^{1/4}$ is the Koranyi norm on $\mathbb{H}^n$. Assume that $\mathcal{L}^m f \in L^2(\mathbb{H}^n)$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \|\mathcal{L}^m f\|_2^{-\frac{1}{2m}} = \infty$. If f and all its partial derivatives vanish at 0, then f is identically zero.

Chernoff's theorem on the Heisenberg group

Theorem (Bagchi-Ganguly-Sarkar-Thangavelu, 2020)

Let f be a smooth function on $\mathbb{H}^n$ satisfying $f(z,t) = f_0(|(z,t)|)$ where $|(z,t)| = (|z|^4 + t^2)^{1/4}$ is the Koranyi norm on $\mathbb{H}^n$. Assume that $\mathcal{L}^m f \in L^2(\mathbb{H}^n)$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \|\mathcal{L}^m f\|_2^{-\frac{1}{2m}} = \infty$. If f and all its partial derivatives vanish at 0, then f is identically zero.

- **Full Laplacian:**

$$\Delta_{\mathbb{H}} := -\sum_{j=1}^{n} (X_j^2 + Y_j^2) - T^2.$$

Chernoff's theorem on the Heisenberg group

Theorem (Bagchi-Ganguly-Sarkar-Thangavelu, 2020)

Let f be a smooth function on $\mathbb{H}^n$ satisfying $f(z,t) = f_0(|(z,t)|)$ where $|(z,t)| = (|z|^4 + t^2)^{1/4}$ is the Koranyi norm on $\mathbb{H}^n$. Assume that $\mathcal{L}^m f \in L^2(\mathbb{H}^n)$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \|\mathcal{L}^m f\|_2^{-\frac{1}{2m}} = \infty$. If f and all its partial derivatives vanish at 0, then f is identically zero.

- **Full Laplacian:**

$$\Delta_{\mathbb{H}} := -\sum_{j=1}^{n} (X_j^2 + Y_j^2) - T^2.$$

Theorem (Ganguly-Thangavelu, 2021)

Let f be a smooth function on $\mathbb{H}^n$ such that $\Delta_{\mathbb{H}}^m f \in L^2(\mathbb{H}^n)$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \|\Delta_{\mathbb{H}}^m f\|_2^{-\frac{1}{2m}} = \infty$. If f vanishes on a nonempty open set, then f is identically zero.

- Two very close relatives of the sublaplacian are the Hermite and special Hermite operators. and $\mathcal{L}$ are connected via the relations:

$$\mathcal{L}(f(z)e^{it}) = e^{it}Lf(z), \text{ and } W(Lf) = W(f)H$$

where

$$L = -\Delta_{\mathbb{C}^n} + \frac{1}{4}|z|^2 + i\sum_{j=1}^{n}(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}), \; H = -\Delta + |x|^2.$$

- Two very close relatives of the sublaplacian are the Hermite and special Hermite operators. and $\mathcal{L}$ are connected via the relations:

$$\mathcal{L}(f(z)e^{it}) = e^{it}Lf(z), \text{ and } W(Lf) = W(f)H$$

where

$$L = -\Delta_{\mathbb{C}^n} + \frac{1}{4}|z|^2 + i\sum_{j=1}^{n}(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}), \quad H = -\Delta + |x|^2.$$

Theorem (Ganguly-Thangavelu, Adv. Math., 2021)

Let $f \in C^\infty(\mathbb{C}^n)$ (resp. $f \in C^\infty(\mathbb{R}^n)$) be such that $L^m f \in L^2(\mathbb{C}^n)$ (resp. $H^m f \in L^2(\mathbb{R}^n)$) for all $m \geq 0$ and satisfies the Carleman condition $\sum_{m=1}^{\infty} \|L^m f\|_2^{-1/(2m)} = \infty$. (resp. $\sum_{m=1}^{\infty} \|H^m f\|_2^{-1/(2m)} = \infty$.)Then f cannot vanish on any open set unless it is identically zero.

Open questions

- Analogue of Chernoff's theorem for the sublaplacian with the stronger vanishing condition (i.e.,f vanishes on any open set)

Open questions

- Analogue of Chernoff's theorem for the sublaplacian with the stronger vanishing condition (i.e., f vanishes on any open set)
- Exact analogue of Chernoff's theorem for $\mathcal{L}$ or $\Delta_{\mathbb{H}}$ with a weaker vanishing condition:

$$X^{\alpha} Y^{\beta} T^{m} f(z,t) = 0, \ \forall \alpha, \beta \in \mathbb{N}^{n}, m \in \mathbb{N}$$

where $X = (X_1, X_2, ..., X_n)$ and X^{α} stands for $X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n}$.

Open questions

- Analogue of Chernoff's theorem for the sublaplacian with the stronger vanishing condition (i.e.,f vanishes on any open set)
- Exact analogue of Chernoff's theorem for $\mathcal{L}$ or $\Delta_{\mathbb{H}}$ with a weaker vanishing condition:

$$X^\alpha Y^\beta T^m f(z,t) = 0, \ \forall \alpha, \beta \in \mathbb{N}^n, m \in \mathbb{N}$$

  where $X = (X_1, X_2, ..., X_n)$ and X^α stands for $X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n}$.
- Exact analogue of Chernoff's theorem for H and L with vanishing condition: $\partial^\alpha f(x_0) = 0$ for all $\alpha \in \mathbb{N}^n$.

Open questions

- Analogue of Chernoff's theorem for the sublaplacian with the stronger vanishing condition (i.e.,f vanishes on any open set)
- Exact analogue of Chernoff's theorem for $\mathcal{L}$ or $\Delta_{\mathbb{H}}$ with a weaker vanishing condition:

$$X^\alpha Y^\beta T^m f(z,t) = 0, \ \forall \alpha, \beta \in \mathbb{N}^n, m \in \mathbb{N}$$

where $X = (X_1, X_2, ..., X_n)$ and X^α stands for $X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n}$.
- Exact analogue of Chernoff's theorem for H and L with vanishing condition: $\partial^\alpha f(x_0) = 0$ for all $\alpha \in \mathbb{N}^n$.
  • Recently for $x_0 = 0$, we have proved the result. But $x_0 \neq 0$ case is still OPEN!.

Open questions

- Analogue of Chernoff's theorem for the sublaplacian with the stronger vanishing condition (i.e.,f vanishes on any open set)
- Exact analogue of Chernoff's theorem for $\mathcal{L}$ or $\Delta_{\mathbb{H}}$ with a weaker vanishing condition:

$$X^\alpha Y^\beta T^m f(z,t) = 0, \ \forall \alpha, \beta \in \mathbb{N}^n, m \in \mathbb{N}$$

  where $X = (X_1, X_2, ..., X_n)$ and X^α stands for $X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n}$.
- Exact analogue of Chernoff's theorem for H and L with vanishing condition: $\partial^\alpha f(x_0) = 0$ for all $\alpha \in \mathbb{N}^n$.
  • Recently for $x_0 = 0$, we have proved the result. But $x_0 \neq 0$ case is still OPEN!.

Application

- Chernoff type theorems can be used to prove sharp decay for the spectral projections (associated to the operator) of functions which are allowed to vanish on an open set.

References

- P. R. Chernoff, Quasi-analytic vectors and quasi-analytic functions. *Bull. Amer. Math. Soc.* **81** (1975), 637-646.
- S. Bagchi, P. Ganguly, J. Sarkar and S. Thangavelu, On theorems of Chernoff and Ingham on the Heisenberg group, arXiv:2009.14230 (2020).
- P. Ganguly and S. Thangavelu, Theorems of Chernoff and Ingham for certain eigenfunction expansions, *Adv. Math*, Volume 386 (2021). https://doi.org/10.1016/j.aim.2021.107815
- P. Ganguly, R. Manna and S. Thangavelu, On a theorem of Chernoff on rank one Riemannian symmetric spaces, arXiv:2103.13888 (2021)
- P. Ganguly and S. Thangavelu, Analogues of theorems of Chernoff and Ingham on the Heisenberg group, arXiv:2106.02704 (2021)

THANK YOU!