Nodal sets of Gaussian Laplace eigenfunctions

Lakshmi Priya

Department of Mathematics
Indian Institute of Science, Bangalore

Introduction

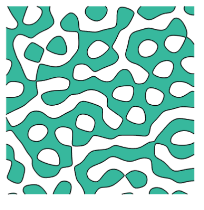
Nodal sets

For M be a manifold and $f : M \to \mathbb{R}$, define the following:

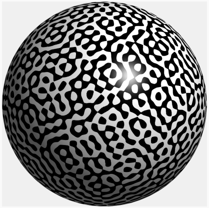
$$\textbf{Nodal set of } f : \mathcal{Z}(f) := f^{-1}\{0\},$$

$$\textbf{Nodal component of } f \text{ is a connected component of } \mathcal{Z}(f),$$

$$\textbf{Nodal domain of } f \text{ is a connected component of } M \setminus \mathcal{Z}(f).$$



$\Delta f + f = 0$ on $\mathbb{R}^2$



$\Delta f + \lambda f = 0$ on $\mathbb{S}^2$

A band limited function on $\mathbb{S}^2$

The above pictures are taken from Prof. Dmitry Belyaev's webpage.

Laplace eigenfunctions on smooth manifolds

Let (M, g) be a smooth Riemannian manifold and Δ_g be the **Laplace–Beltrami operator** or **Laplacian**, then in local coordinates

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_i (g^{ij} \sqrt{|g|} \partial_j f) \longleftrightarrow \Delta = \partial_1^2 + \cdots + \partial_n^2.$$

$\lambda \in \mathbb{R}$ is an **eigenvalue** of Δ_g if $\exists f : M \to \mathbb{R}$ s.t. $f \not\equiv 0$ and $-\Delta_g f = \lambda f$.

f is called an **eigenfunction** corresponding to the eigenvalue λ.

If M is closed, then the eigenvalues of Δ_g are non-negative and can be enumerated as $0 \leq \lambda_1 < \lambda_2 < \cdots$ with $\lambda_n \nearrow \infty$ and each λ_j has finite multiplicity.

There is an ONB for $L^2(M, d\mathrm{Vol}_g)$ consisting of eigenfunctions of Δ_g.

Ex: $M = \mathbb{S}^1$ w/ eigenvalues n^2, $n \in \mathbb{N}$ and eigenfunctions $\sin n\theta, \cos n\theta$.

Regularity of Δ eigenfunctions

Harmonifying Δ eigenfunctions: For f satisfying $\Delta f + \lambda f = 0$, the function $F : M \times \mathbb{R} \to \mathbb{R}$ defined by $F(z,t) := f(z)e^{-\sqrt{\lambda}t}$ is harmonic on $M \times \mathbb{R}$.

At the wavelength scale ($\sim 1/\sqrt{\lambda}$), f resembles a harmonic function.

Estimates at wavelength scale for Δ *eigenfunctions on* $\mathbb{R}^2$, $\mathbb{S}^2$ *and* $\mathbb{T}^2$: For every $p \in M$, $n \in \mathbb{N}_0$ and $r > 0$, $\exists\ C_{r,n} > 0$ s.t. $\forall f, \lambda$ satisfying $\Delta f + \lambda f = 0$,

$$|\nabla^n f(p)|^2 \leq C_{r,n} \lambda^{n+1} \int_{B(p, \frac{r}{\sqrt{\lambda}})} f(x)^2 dV(x).$$

Regularity of the nodal sets of Δ eigenfunctions

Let M be a smooth closed Riemannian manifold and let $\Delta\phi + \lambda\phi = 0$.

- *Courant's nodal domain theorem:* A bound for # of nodal domains is

$$\#\text{ nodal domains of } \phi \ \le \sum_{\lambda' \le \lambda} \text{multiplicity of } \Delta \text{ eigenvalue } \lambda'.$$

- *Yau's conjecture* (resolved for analytic g by Donnelly & Fefferman): Asymptotics for nodal volume

$$c_M\sqrt{\lambda} \le \text{Volume of } \mathcal{Z}(\phi) \le C_M\sqrt{\lambda}.$$

For a smooth Riemannian manifold M of dim. n, if $\Delta\phi + \lambda\phi = 0$

- *Faber–Krahn*: Every nodal domain of ϕ is *large* enough

$$\text{Volume of every nodal domain of } \phi \gtrsim 1/\lambda^{n/2}.$$

- The nodal set forms a $c/\sqrt{\lambda}$ net on M.

Motivation

Although nodal sets of Δ eigenfunctions are *regular*, there is quite a bit of *variability* in their behaviour and it is interesting to understand their *typical* behaviour. The following are some properties of the nodal set of Δ eigenfunctions which have been studied:

- Nodal domain/component count in compact subsets of the manifold:
  - Total count.
  - Specialized count: Count of nodal components/domains with a specified topology, volume, boundary volume etc.
- Length (or more generally volume in higher dim.) of the nodal set.

Hence we *randomize* and study the above mentioned random quantities for the nodal set of random functions.

Q: Expectation, higher moments, LLN, CLT, concentration?

Motivation

Although nodal sets of Δ eigenfunctions are *regular*, there is quite a bit of *variability* in their behaviour and it is interesting to understand their *typical* behaviour. The following are some properties of the nodal set of Δ eigenfunctions which have been studied:

- Nodal domain/component count in compact subsets of the manifold:
  - Total count.
  - Specialized count: Count of nodal components/domains with a specified topology, volume, boundary volume etc.
- Length (or more generally volume in higher dim.) of the nodal set.

Hence we *randomize* and study the above mentioned random quantities for the nodal set of random functions.

Q: Expectation, higher moments, LLN, CLT, concentration?

Conjectures about Δ eigenfunctions

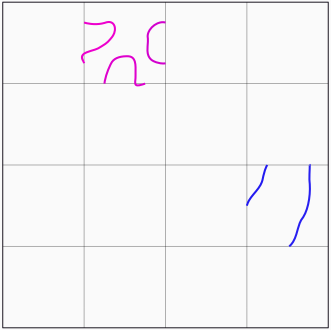
- Random wave conjecture: Random plane wave (RPW) is a universal object which *models* Δ eigenfns. corresponding to high eigenvalues on any manifold whose geodesic flow is ergodic (these include manifolds with negative curvature).

- Semi-classical eigenfunction hypothesis: If $\{\phi_n : n \geq 1\}$ is an ONB s.t. $\Delta\phi_n + \lambda_n\phi_n = 0$ where λ_n's are non-decreasing, then there is a density one subsequence of $\{\phi_n : n \geq 1\}$, call it $\{\phi_{n_i}\}$, whose L^2 mass equidistributes at scales slightly larger than wavelength scale,

$$\lim_{i\to\infty} \sup_{x\in M, r\geq r_{n_i}} \left| \frac{\int_{B(x,\frac{r}{\sqrt{\lambda_{n_i}}})} \phi_{n_i}^2}{\mathrm{Vol}(B(x, \frac{r}{\sqrt{\lambda_{n_i}}}))} - 1 \right| = 0,$$

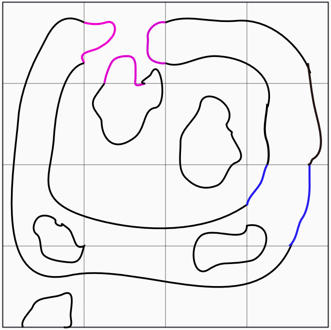
whenever $\lim_{i\to\infty} r_{n_i} = \infty$.

Local vs. Non-local vs. Semi-local



A part of the nodal set of $f : \mathbb{R}^2 \to \mathbb{R}$.

Local vs. Non-local vs. Semi-local



Nodal set of $f : \mathbb{R}^2 \to \mathbb{R}$.

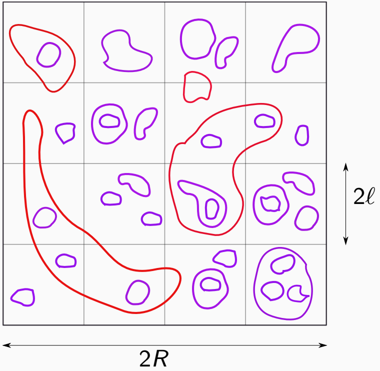
- Local Nodal length in $[0, T]^2 = \sum$ Nodal length in the tiling boxes.
- Non-local $\#$ Nod. comp. in $[0, T]^2 \neq \sum \#$ Nod. comp. in the tiling boxes.
- Semi-local $\#$ Nod. comp. in $[0, T]^2 \approx \sum \#$ Nod. comp. in the tiling boxes, if the tiling boxes are *large* enough.

Counting nodal components: A simple example

$F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth SGP. What can we say about $\mathbb{E}[N_R(F)]/R^2$?

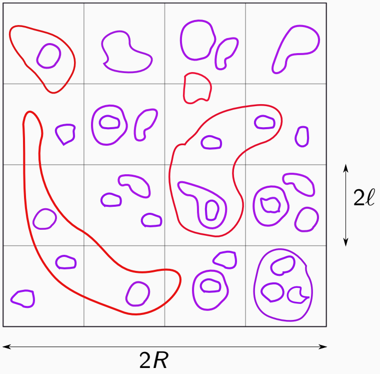
Counting nodal components: A simple example

$F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth SGP. What can we say about $\mathbb{E}[N_R(F)]/R^2$?



Counting nodal components: A simple example

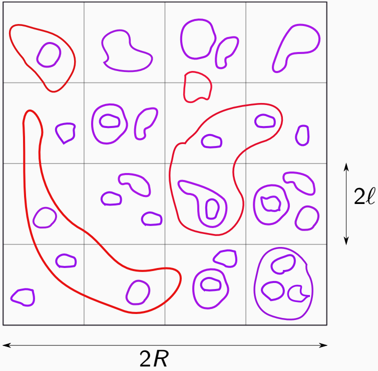
$F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth SGP. What can we say about $\mathbb{E}[N_R(F)]/R^2$?



$$\sum \#\mathrm{NC} \le N_R = \sum \#\mathrm{NC} + \#\ \mathrm{NC},$$

Counting nodal components: A simple example

$F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth SGP. What can we say about $\mathbb{E}[N_R(F)]/R^2$?



$$\sum \#\mathrm{NC} \leq N_R = \sum \#\mathrm{NC} + \#\ \mathrm{NC},$$

$$\frac{R^2}{\ell^2}\ \mathbb{E}[N_\ell] \leq \mathbb{E}[N_R] = \frac{R^2}{\ell^2}\ \mathbb{E}[N_\ell] + \mathbb{E}[\#\mathrm{NC}],$$

$$\frac{\mathbb{E}[N_\ell]}{\ell^2} \leq \frac{\mathbb{E}[N_R]}{R^2} \leq \frac{\mathbb{E}[N_\ell]}{\ell^2} + \frac{\mathbb{E}[\#\mathrm{NC}]}{R^2},$$

$$\frac{\mathbb{E}[N_\ell]}{\ell^2} \leq \frac{\mathbb{E}[N_R]}{R^2} \leq \frac{\mathbb{E}[N_\ell]}{\ell^2} + \frac{1}{\ell}.$$

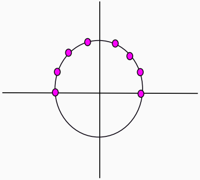
This proves that $\mathbb{E}[N_R]/R^2$ is Cauchy and hence converges to a constant $c \geq 0$.

$\because\ \mathbb{E}[\#\mathrm{NC}] \leq \mathbb{E}[\#\text{zeros of } F \text{ restricted to the hor.\& vert.lines}] = C\frac{R}{\ell} \cdot 2R.$

Random functions & their nodal sets

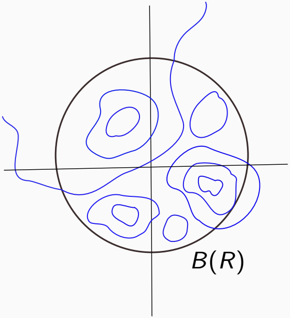
Random plane wave (RPW)

Consider n angularly equidistributed points $\{a_1, a_2, \dots, a_n\}$ on the half circle $[0, \pi] \subset \mathbb{S}^1$ and consider the following Gaussian process, w/ $\xi_j, \eta_j \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$:



$$F_n(z) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \xi_j \cos(z \cdot a_j) + \eta_j \sin(z \cdot a_j).$$

RPW F_{RPW} is the *limit* of F_n as $n \to \infty$. A.s., F_n and F_{RPW} satisfy $\Delta f + f = 0$ on $\mathbb{R}^2$.



Formally, the RPW is the centered stationary Gaussian process on $\mathbb{R}^2$ whose spectral measure is the uniform measure on $\mathbb{S}^1$.

Quantities of interest:

$N_R := \#\{\text{Nodal comp. of } F \text{ contained in } B(R)\}$,

$\mathcal{L}_R := \text{length}(\{F = 0\} \cap B(R))$.

Questions

$X_n \overset{\text{i.i.d.}}{\sim} \nu$, $\mathbb{E}[X_1] = \mu$ & $\text{Var}(X_1) = \sigma^2$	Random plane wave	
$S_n := \sum_{k=1}^n X_k$	Nodal length $\mathcal{L}_R$	Nodal domain count N_R
Law of large numbers $\frac{S_n - n\mu}{n} \overset{\text{a.s.}}{\to} 0$	$\frac{\mathcal{L}_R}{R^2} \xrightarrow{\text{a.s.}, L^1} C$	$\frac{N_R}{R^2} \xrightarrow{\text{a.s.}, L^1} c$
Central limit theorem $\frac{S_n - n\mu}{\sigma\sqrt{n}} \overset{\text{d}}{\to} \mathcal{N}(0,1)$	$\text{Var}(\mathcal{L}_R) \sim R^2 \log R$ $\frac{\mathcal{L}_R - \mathbb{E}[\mathcal{L}_R]}{\sqrt{\text{Var}(\mathcal{L}_R)}} \overset{d}{\to} \mathcal{N}(0,1)$	×
Large deviation/exp. conct. (w/ extra assumptions on ν) $\mathbb{P}(\lvert S_n - n\mu\rvert \geq \epsilon n) \approx e^{-I(\epsilon)n}$	×	$\mathbb{P}\left(\lvert N_R - cR^2\rvert \geq \epsilon R^2\right) \lesssim e^{-c_\epsilon R}$

Nodal component count of the RPW

Since F_{RPW} is an ergodic process, Wiener's ergodic theorem implies the following LLN type result.

Theorem (Nazarov & Sodin, 2016)

There is $c_{NS} > 0$ s.t.

$$\frac{N_R(F_{RPW})}{R^2} \xrightarrow{a.s., L^1} c_{NS} \text{ as } R \to \infty.$$

Nodal component count of the RPW

Since F_{RPW} is an ergodic process, Wiener's ergodic theorem implies the following LLN type result.

Theorem (Nazarov & Sodin, 2016)

There is $c_{NS} > 0$ s.t.

$$\frac{N_R(F_{RPW})}{R^2} \xrightarrow{a.s., L^1} c_{NS} \text{ as } R \to \infty.$$

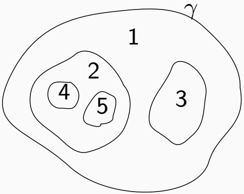
Following result/proof is heavily inspired from Nazarov–Sodin's result/proof of concentration for RSH. Main tool: Gaussian concentration result.

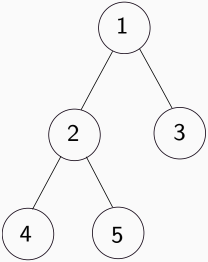
Theorem (LP)

Given $\epsilon > 0$, $\exists\ c_\epsilon, \widetilde{c}_\epsilon, C_\epsilon, \widetilde{C}_\epsilon > 0$ such that

$$\widetilde{C}_\epsilon \exp\left(-\widetilde{c}_\epsilon R\right) \le \mathbb{P}\left(\left|\frac{N_R(F_{RPW})}{R^2} - c_{NS}\right| \ge \epsilon\right) \le C_\epsilon \exp\left(-c_\epsilon R\right).$$

Nesting configurations





To every nodal component (here γ), we associate a finite rooted tree (called its *tree end*) with the nodes $\leftrightarrow$ nodal domains contained in the interior of γ and an edge between two nodes if the corresponding nodal domains share a boundary.

This tree contains information about the *nesting* of the nodal components in the interior of γ.

Connectivity of $\gamma := 1 +$ degree of the root of its tree end.

Let $\mathcal{T}$ be the collection of finite rooted trees and let $T \in \mathcal{T}$. For $f : \mathbb{R}^2 \to \mathbb{R}$, define

$$N_R(f, T) := \#\text{nodal comp. of } f \text{ in } B(R) \text{ whose tree end is } T.$$

Theorem (Sarnak & Wigman, 2019)

If ν has no atoms, $\exists\ \mu$ a probability measure on $\mathcal{T}$ w/ supp $(\mu) = \mathcal{T}$ s.t. for every $T \in \mathcal{T}$,

$$\frac{N_R(F_{RPW}, T)}{R^2} \xrightarrow{a.s., L^1} c_{NS} \cdot \mu(T), \text{ as } R \to \infty.$$

Theorem (LP)

In the setting of the above thm, $\exists\ c_\epsilon, C_\epsilon > 0$ s.t. $\forall \epsilon > 0$

$$\mathbb{P}\left(\left|\frac{N_R(F_{RPW}, T)}{R^2} - c_{NS} \cdot \mu(T)\right| \geq \epsilon\right) \leq C_\epsilon \exp\left(-c_\epsilon R\right).$$

Random spherical harmonics (RSH) on $\mathbb{S}^2$

- Eigenvalues of Δ: $n(n+1),\ n \in \mathbb{N}$.

Eigenspace $\mathcal{H}_n$: $(2n+1)$ dimensional space of degree n spherical harmonics (restriction to $\mathbb{S}^2$ of deg n hom. harmonic polynomials in $\mathbb{R}^3$).

- RSH of degree n: Let $\{\phi_j : 0 \leq j \leq 2n\}$ be any ONB for $\mathcal{H}_n$, then

$$F_n := \frac{1}{\sqrt{2n+1}}(\xi_0\phi_0 + \cdots + \xi_{2n}\phi_{2n}),\ \xi_j \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1).$$

- Covariance kernel: $K(z,w) = \mathbb{E}[F_n(z)F_n(w)] = P_n(\cos\Theta(z,w))$, where P_n is the degree n Legendre polynomial s.t. $P_n(1) = 1$.

- If $N(\cdot) := \#$nodal component count, then for every $f \in \mathcal{H}_n$,

Courant's ND thm. $\Rightarrow 1 \leq N(f) \leq n^2$.

Nodal component count for RSH

Theorem (Nazarov & Sodin, 2009)

There exists $c_{NS} > 0$ satisfying: given $\epsilon > 0$, $\exists\ c, C_\epsilon > 0$ such that

$$\mathbb{P}\left(\left|\frac{N(F_n)}{n^2} - c_{NS}\right| > \epsilon\right) \leq C_\epsilon e^{-c\epsilon^{\mathbf{15}} n}.$$

For $p \in \mathbb{S}^2$ and $\lambda_n = \sqrt{n(n+1)}$, we study nodal count at wavelength scale.
Let $N(F_n, r_n) := \#$nodal components of F_n in $B(p, \frac{r_n}{\sqrt{\lambda_n}})$.

Theorem (LP)

There are constants $c, C > 0$ such that $\forall \epsilon > 0$ and sequence $r_n \to \infty$

$$\mathbb{P}\left(\left|\frac{N(F_n, r_n)}{n^2 \mathrm{Vol}\ B(p, r_n/\sqrt{\lambda_n})} - c_{NS}\right| > \epsilon\right) \leq C\ e^{-c\epsilon^{\mathbf{16}} r_n}.$$

Thank You!