Density of modular points in pseudo-deformation rings

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Introduction

Let p be an odd prime, $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$, $W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$ and $N \geq 1$ be an integer not divisible by p.
Denote the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside $Np\infty$ over $\mathbb{Q}$ by $G_{\mathbb{Q},Np}$.

Let $\bar{\rho} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, odd and semi-simple representation with Artin conductor N_0 and suppose $N_0 \mid N$.

Khare–Winterberger: $\bar{\rho}$ arises from a modular eigenform of level N. However, all lifts of $\bar{\rho}$ to finite extensions of $W(\mathbb{F})$ do not arise from modular forms (it can be seen using determinants).

Question

Does every lift of $\bar{\rho}$ to a finite extension of $W(\mathbb{F})$ arise from an arithmetic object?

A candidate for such an object would be p-adic modular forms.

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A natural approach to answer the question is to compare the 'universal deformation ring' $R_{\bar{\rho}}$ of $\bar{\rho}$ (which interpolates all the lifts of $\bar{\rho}$ to CNL $W(\mathbb{F})$-algebras) with the 'big' p-adic Hecke algebra acting on the space of modular forms of level N and all weights.

When $\bar{\rho}$ is absolutely irreducible, the universal deformation ring of $\bar{\rho}$ exists due to work of Mazur.
However, when $\bar{\rho}$ is reducible, the universal deformation ring $R_{\bar{\rho}}$ of $\bar{\rho}$ (in the sense of Mazur) does not exist.

So the appropriate object to consider here is the universal deformation ring of the pseudo-representation corresponding to $\bar{\rho}$ i.e. universal pseudo-deformation ring of $\bar{\rho}$ which we will denote by $R^{\text{pd}}_{\bar{\rho}}$.

Note that when $\bar{\rho}$ is absolutely irreducible, then universal pseudo-deformation ring of $\bar{\rho}$ is same as universal deformation ring of $\bar{\rho}$ i.e. $R_{\bar{\rho}} = R^{\text{pd}}_{\bar{\rho}}$.

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Definition

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Example: If $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(R)$ is a representation, then $(\mathrm{tr}(\rho), \det(\rho))$ is a 2-dimensional pseudo-representation of $G_{\mathbb{Q},Np}$. We call it the pseudo-representation corresponding to ρ.

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Big p-adic Hecke algebra

Let $M_k(N, W(\mathbb{F}))$ be the space of classical modular forms of weight k and level N with Fourier coefficients in $W(\mathbb{F})$.

Let $M(N, W(\mathbb{F})) = \sum_{k=0}^{\infty} M_k(N, W(\mathbb{F})) \subset W(\mathbb{F})[[q]]$.

Let $\mathbb{T}(N)$ be the $W(\mathbb{F})$-subalgebra of $\mathrm{End}_{W(\mathbb{F})}(M(N, W(\mathbb{F})))$ generated by the Hecke operators T_ℓ and S_ℓ for primes $\ell \nmid Np$. Here S_ℓ is the operator whose action on $M_k(N, W(\mathbb{F}))$ coincides with the action of the operator $\langle \ell \rangle \ell^{k-2}$.

Let $m_{\bar{\rho}}$ be the maximal ideal of $\mathbb{T}(N)$ corresponding to $\bar{\rho}$. Let $\mathbb{T}(N)_{\bar{\rho}}$ be the completion of $\mathbb{T}(N)$ at $m_{\bar{\rho}}$.

Note that $\mathbb{T}(N)_{\bar{\rho}}$ is a complete Noetherian local ring with residue field $\mathbb{F}$ and it is also reduced.

Theorems that compare $R^{\mathrm{pd}}_{\bar{\rho}}$ and $\mathbb{T}(N)_{\bar{\rho}}$ and assert that they are isomorphic are called 'big' $R = \mathbb{T}$ theorems.

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Theorems that compare $R^{\mathrm{pd}}_{\bar{\rho}}$ and $\mathbb{T}(N)_{\bar{\rho}}$ and assert that they are isomorphic are called 'big' $R = \mathbb{T}$ theorems.

Big p-adic Hecke algebra

Let $M_k(N, W(\mathbb{F}))$ be the space of classical modular forms of weight k and level N with Fourier coefficients in $W(\mathbb{F})$.
Let $M(N, W(\mathbb{F})) = \sum_{k=0}^{\infty} M_k(N, W(\mathbb{F})) \subset W(\mathbb{F})[[q]]$.

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Theorem (Gouvêa–Mazur)

- Every component of $\mathrm{Spec}(\mathbb{T}(N)_{\bar{\rho}})$ has Krull dimension at least 4.
- If $\bar{\rho}$ is absolutely irreducible, then there exists a surjective map $\phi : R_{\bar{\rho}} \to \mathbb{T}(N)_{\bar{\rho}}$ and it is an isomorphism when $\bar{\rho}$ is unobstructed (i.e. when $R_{\bar{\rho}} \simeq W(\mathbb{F})[[X, Y, Z]]$).

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If $\bar{\rho}$ is absolutely irreducible, then, under some mild hypotheses (Taylor–Wiles hypotheses), the surjective map $\phi : R_{\bar{\rho}} \to \mathbb{T}(N)_{\bar{\rho}}$ is an isomorphism.

Allen has extended the results of Böckle to prove big $R = \mathbb{T}$ theorems in the context of polarized Galois representations for CM fields and also for Hilbert modular forms under some hypotheses.
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Aim and setup

Aim

To explore big $R = \mathbb{T}$ theorems for reducible $\bar{\rho}$.

Setup

Suppose $\bar{\rho} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$ is odd and reducible such that $\bar{\rho} = \bar{\chi}_1 \oplus \bar{\chi}_2$ for some continuous characters $\bar{\chi}_1, \bar{\chi}_2 : G_{\mathbb{Q},Np} \to \mathbb{F}^\times$. Let $\bar{\chi} = \bar{\chi}_1\bar{\chi}_2^{-1}$. Let N_0 be the Artin conductor of $\bar{\rho}$. Suppose $N_0 \mid N$.

Note that $\bar{\rho}$ defines a maximal ideal $m_{\bar{\rho}}$ of $\mathbb{T}(N)$ and let $\mathbb{T}(N)_{\bar{\rho}}$ be the completion of $\mathbb{T}(N)$ at $m_{\bar{\rho}}$.

Let $R^{\mathrm{pd}}_{\bar{\rho}}$ be the universal deformation ring of the pseudo-representation $(\mathrm{tr}(\bar{\rho}), \det(\bar{\rho})) : G_{\mathbb{Q},Np} \to \mathbb{F}$ in the category $\mathcal{C}$ of complete Noetherian local rings with residue field $\mathbb{F}$. So it interpolates all pseudo-representations which take values in objects of $\mathcal{C}$ and lift $(\mathrm{tr}(\bar{\rho}), \det(\bar{\rho}))$. The existence of $R^{\mathrm{pd}}_{\bar{\rho}}$ is due to Chenevier.

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Main result

Lemma

There exists a pseudo-representation $(\tau_N, \delta_N) : G_{\mathbb{Q},Np} \to \mathbb{T}(N)_{\bar{\rho}}$ lifting $(\mathrm{tr}(\bar{\rho}), \det(\bar{\rho}))$ such that the morphism $\Phi : R^{\mathrm{pd}}_{\bar{\rho}} \to \mathbb{T}(N)_{\bar{\rho}}$ induced by it is surjective.

Theorem (D.)

Suppose we are in the setup above. Suppose $\bar{\chi}|_{G_{\mathbb{Q}_p}} \neq 1, \omega_p^{-1}|_{G_{\mathbb{Q}_p}}$, $\dim_{\mathbb{F}}(H^1(G_{\mathbb{Q},Np}, \bar{\chi})) = 1$ and $p \nmid \phi(N)$. Then:

- The morphism $\Phi : R^{\mathrm{pd}}_{\bar{\rho}} \to \mathbb{T}(N)_{\bar{\rho}}$ induces an isomorphism $(R^{\mathrm{pd}}_{\bar{\rho}})^{\mathrm{red}} \simeq \mathbb{T}(N)_{\bar{\rho}}$ of local complete intersection rings of Krull dimension 4.
- Moreover, if $1 \leq \dim_{\mathbb{F}}(H^1(G_{\mathbb{Q},Np}, \bar{\chi}^{-1})) \leq 3$, then the map $\Phi : R^{\mathrm{pd}}_{\bar{\rho}} \to \mathbb{T}(N)_{\bar{\rho}}$ is an isomorphism of local complete intersection rings of Krull dimension 4.

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Level raising of modular forms

Theorem (D.)

Let p be a regular prime and k_0 be an even integer such that $2 < k_0 < p-1$. Let $\ell_1, \cdots, \ell_r$ be primes such that

- $p \nmid \ell_i^2 - 1$,
- Either $p \mid \ell_i^{k_0} - 1$ for all $1 \leq i \leq r$ or $p \mid \ell_i^{k_0-2} - 1$ for all $1 \leq i \leq r$.

If k is an integer such that $k \equiv k_0 \pmod{p-1}$, then there exists a newform f of level $\Gamma_0(\ell_1 \cdots \ell_r)$ and weight k such that ρ_f lifts $1 \oplus \omega_p^{k_0-1}$.

For $r = 1$, the theorem is known, under less restrictive hypotheses, by work of Billerey–Menares. Their method is different from ours.

For $r > 1$, such results have been obtained by Yoo and Wake–Wang-Erickson in the case of $\bar{\rho} = 1 \oplus \omega_p$ using different methods.

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For $r = 1$, the theorem is known, under less restrictive hypotheses, by work of Billerey–Menares. Their method is different from ours.

For $r > 1$, such results have been obtained by Yoo and Wake–Wang-Erickson in the case of $\bar{\rho} = 1 \oplus \omega_p$ using different methods.

Level raising of modular forms

Theorem (D.)

Let p be a regular prime and k_0 be an even integer such that $2 < k_0 < p - 1$. Let $\ell_1, \cdots, \ell_r$ be primes such that

- $p \nmid \ell_i^2 - 1$,
- Either $p \mid \ell_i^{k_0} - 1$ for all $1 \leq i \leq r$ or $p \mid \ell_i^{k_0-2} - 1$ for all $1 \leq i \leq r$.

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Thank You!