Volume and diameter of positively curved Kähler manifolds

Harish Seshadri

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Positively curved Riemannian manifolds

The general theme of comparison theorems in Riemannian geometry is the quantification of the intuitive idea that the more positively curved a space is, the "smaller" it is.

We recall some early results in this direction.

In the Riemannian setting, the "model" space with positive curvature in dimension n is the standard round n-sphere S^n of radius 1, for which $Ric = (n-1)g$ and $diam(S^n) = \pi$.

The Myers-Bonnet Theorem (1941): Let (M, g) be a complete Riemannian n-manifold with

$$Ric \geq (n-1)g.$$

Then

$$diam(M) \leq \pi.$$

In particular, M is compact.

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The standard proof of this result involves an application of the second variation formula for the length functional. The equality case is a well-known result of Cheng and involves a different approach:

Cheng's maximal diameter theorem (1975): Let (M, g) be a compact Riemannian n-manifold with

$$Ric \geq (n-1)g.$$

If

$$diam(M) = \pi,$$

then M is isometric to S^n.

Positively curved Riemannian manifolds

One has similar results for volume:

The Bishop volume comparison theorem (1963): Let (M, g) be a complete Riemannian n-manifold with

$$Ric \geq (n-1)g.$$

Then

$$vol(M) \leq Vol(S^n).$$

If equality holds, then M is isometric to S^n.

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In 1996, Colding proved an important “almost”-rigidity result which generalizes the equality case in Bishop's theorem. To state his result, we recall the notion of Gromov-Hausdorff distance between compact metric spaces.

Let (X, d_X), (Y, d_Y) be compact metric spaces. A map $f : X \to Y$ is said to be an ϵ-approximation if

(i) $|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| < \epsilon$ for all $x_1, x_2 \in X$ and

(ii) the ϵ-neighborhood of $f(X) \subset Y$ is equal to Y.

The Gromov-Hausdorff distance between X and Y is defined to be

$d_{GH}(X, Y) =$

$\inf\{\epsilon > 0 \ : \ \exists \ \epsilon\text{-approximations } f : X \to Y \ \text{ and } \ g : Y \to X\}$

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It can be checked that $d_{GH}(X, Y) = 0$ if and only if X and Y are isometric.

The G-H distance between Riemannian manifolds is defined to be the G-H distance between the corresponding metric spaces.

Theorem (Colding, 1996): Given $n \geq 2$ and $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon) > 0$ such that the following holds: If (M^n, g) is a compact Riemannian n-manifold with

$$Ric \geq (n-1)g \quad \text{and} \quad Vol(M) \geq Vol(S^n) - \delta,$$

then

$$d_{GH}(M, S^n) < \epsilon.$$

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In a separate paper, Colding also proved the converse: Given $n \geq 2$ and $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon) > 0$ such that $Ric \geq (n-1)g$ and $d_{GH}(M, S^n) < \delta$ imply $Vol(M) \geq Vol(S^n) - \epsilon$

Under the same hypotheses one also has the

Diffeomorphism stability theorem (Cheeger-Colding, 1997):
M is diffeomorphic to S^n.

Positively curved Kähler manifolds

The results in the Riemannian case only require the positivity of Ricci curvature. For Kähler manifolds, it turns out that the analogue of the Myers-Bonnet diameter theorem with a positive Ricci curvature assumption is not true.

First, we note that the "model" positively curved space in the Kähler setting is the complex projective space $\mathbb{C}P^n$ endowed with the Fubini-Study metric g_{FS} of constant homomorphic sectional curvature 2. We have

$$Ric = (n+1)g_{FS} \quad \text{and} \quad diam(\mathbb{C}P^n) = \frac{\pi}{\sqrt{2}}$$

Positively curved Kähler manifolds

Let M be the n-fold product $\mathbb{C}P1 \times ... \times \mathbb{C}P^1$ with the product metric, where the metric on $\mathbb{C}P^1$ is the round metric of curvature 1 scaled by $\frac{1}{n+1}$. Then $Ric = (n+1)g$ and

$$diam(M) = \sqrt{\frac{n}{n+1}}\pi,$$

which is larger than $\frac{\pi}{\sqrt{2}}$ if $n \geq 2$.

It turns out that the correct diameter bound holds under a stronger notion of positivity of curvature.

Positively curved Kähler manifolds

We say that M has *positive holomorphic bisectional curvature* bounded below by $K > 0$ if

$$R(X, Y, X, Y) + R(X, JY, X, JY) \geq K(1 + \langle X, Y\rangle^2 + \langle X, JY\rangle^2)$$

for all unit tangent vectors X, Y. Here $J : TM \to TM$ denotes the almost-complex structure.

This condition, denoted by

$$B \geq K,$$

is stronger than positive Ricci curvature and, in fact, implies that

$$Ric \geq K(n+1)g.$$

Positively curved Kähler manifolds

Theorem (Tam-Yu, 2005): Let M be a compact kähler manifold with $B \geq 1$. Then

$$diam(M) \leq diam(\mathbb{C}P^n).$$

The proof is based on a comparison theorem for the Laplacian of distance functions. The same proof can be used to recover the Myers-Bonnet theorem for Riemannian manifolds.

Recently, the analogue of Cheng's maximal diameter theorem was established:

Theorem (V. Datar-HS, 2021) *Let M be a compact Kähler manifold with $B \geq 1$. If*

$$diam(M) = diam(\mathbb{C}P^n),$$

then M is isometric to $\mathbb{C}P^n$.

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The analogue of Bishop's theorem about volume was also proved recently:

Theorem (Zhang, 2019): Let M be a compact Kähler manifold with

$$Ric \geq (n+1)g.$$

Then

$$Vol(M) \leq Vol(\mathbb{C}P^n).$$

If equality holds, then M is holomorphically isometric to $\mathbb{C}P^n$.

In an appendix to the paper of Zhang, Liu proved a holomorphic analogue of the Cheeger-Colding diffeomorphism stability theorem:

Theorem (Liu, 2019): There exists $\epsilon = \epsilon(n) > 0$ with the following property: If M^n is a compact Kähler manifold with

$$Ric \geq (n+1)g \quad \text{and} \quad Vol(M) \geq Vol(\mathbb{C}P^n) - \epsilon,$$

then M is biholomorphic to $\mathbb{C}P^n$.

Positively curved Kähler manifolds

Subsequently, the analogue of Colding's Gromov-Hausdorff closeness theorem was proved:

Theorem (Datar-HS-Song, 2020): Given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n)$ with the following property: If M^n is a compact Kähler manifold with

$$Ric \geq (n+1)g \quad \text{and} \quad Vol(M) \geq Vol(\mathbb{C}P^n) - \epsilon,$$

then

$$d_{GH}(M, \mathbb{C}P^n) < \epsilon.$$

Remarks on proofs:

Diameter rigidity: The positive bisectional curvature assumption has the following strong implication:

Theorem (Proof of Frankel's Conjecture due to Siu-Yau, Mori, 1980): Any compact Kähler n-manifold with positive bisectional curvature is biholomorphic to $\mathbb{C}P^n$.

So we can assume that the underlying manifold in our theorem is $\mathbb{C}P^n$.

Remarks on proofs:

The main idea is to establish a monotonicity formula for a function arising from Lelong numbers of positive currents on $\mathbb{C}P^n$.

This step only requires a positive current and, in particular, the positive bisectional curvature hypothesis is not used.

Let T be a closed non-negative $(1,1)$-current on $\mathbb{C}P^n$, and $q \in \mathbb{C}P^n$.
Lemma: Then

$$\Theta(T, q, r) := \frac{1}{(2\pi)^{n-1} \sin^{2n-2}(r/\sqrt{2})} \int_{B_{\mathbb{C}P^n}(q,r)} T \wedge \omega_{FS}^{n-1}$$

is increasing in r. Here $B_{FS}(q, r)$ is the ball of radius r with respect to g_{FS} and ω_{FS} is the Kähler form of g_{FS}.

Remarks on proofs:

Moreover, we also have that

$$\lim_{r \to 0^+} \Theta(T, q, r) = \nu(T, q),$$

where $\nu(T, q)$ is the Lelong number of T at q.

The definition of $\nu(T, q)$ is as follows: Since T is closed, the $\partial\bar{\partial}$-Lemma implies that we can write $T = \partial\bar{\partial}\varphi$ in a neighbourhood of q.

One then defines

$$\nu(T, q) := \lim_{r \to 0^+} \frac{\sup_{B_{\mathbb{C}^n}(0,r)} \varphi(z)}{\log r},$$

where z is a holomorphic coordinate in a neighbourhood of q such that $z(q) = 0$.

Remarks on proofs:

Now, we assume that g is a Kähler metric on $\mathbb{C}P^n$ with $B \geq 1$.

Let $p \in \mathbb{C}P^n$ and let d_p be the corresponding distance function. Based on computations of Tam-Yu, Lott proved that if

$$\psi = \log \cos^2(\frac{d_p}{\sqrt{2}}),$$

then the current

$$T = \omega + \sqrt{-1}\partial\bar{\partial}\psi,$$

which is *a priori* defined on the complement of the cut-locus of g with respect to p, extends to a non-negative current on $\mathbb{C}P^n$.

Remarks on proofs:

By the maximal diameter hypothesis, there are $p, q \in \mathbb{C}P^n$ such that $d_p(q) = \pi/\sqrt{2}$.

Claim:

$$\nu(T, q) \geq 2\pi.$$

If $R\pi/\sqrt{2}$, the monotonicity lemma then implies that

$$\Theta(T, q, R) \geq \lim_{r \to 0^+} \Theta(T, q, r) = \nu(T, q) \geq 2\pi.$$

Remarks on proofs:

To evaluate $\Theta(T, q, R)$, we observe that $H^{(}\mathbb{C}P^n, \mathbb{R}) = \mathbb{R}$. Hence the cohomology classes of the Kähler forms ω and ω_{FS} are related by

$$[\omega] = c[\omega_{FS}]$$

for some $c > 0$.

Hence

$$\Theta(T, q, R) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{C}P^n} T \wedge \omega_{FS}^{n-1} = 2\pi c.$$

Therefore

$$c \geq 1.$$

Remarks on proofs:

On the other hand, the bisectional curvature lower bound gives

$$Ric(\omega) \geq (n+1)\omega.$$

Since $[Ric(\omega)] = [Ric(\omega_{FS})] = (n+1)[\omega_{FS}]$, we get $c \leq 1$.

Hence $c = 1$ and $[Ric(\omega)] = (n+1)[\omega]$. Again using $Ric(\omega) \geq (n+1)\omega$ and the $\partial\bar{\partial}$-Lemma, one concludes that $Ric(\omega) = (n+1)\omega$. By the uniqueness of Kähler-Einstein metrics on $\mathbb{C}P^n$, one cocludes that g is isometric to g_{FS}.

Remarks on proofs:

The main step in proving the above theorem is the following result. In what follows ω will denote the Kähler form of a Kähler metric.

Lemma: Let (M^n, g_{KE}) be a Kähler-Einstein manifold with $Ric = g_{KE}$. Suppose that $\delta_i \to 0$ and g_i is a sequence of Kähler metrics on M such that
(i) $\omega_i \in c_1(M)$ and
(ii) $Ric(g_i) \geq (1 - \delta_i) g_i$.
Then

$$(M, g_i) \xrightarrow{d_{GH}} (M, g_{KE}).$$

Remarks on proofs:

Given this lemma, the theorem follows: By Liu's result, M is biholomorphic to $\mathbb{C}P^n$ and hence $H^2(M, \mathbb{R}) = \mathbb{R}$. This implies that, after rescaling, condition (i) of the lemma is satisfied by any Kähler metric on M. If one rescales so that $Vol(g_i) = Vol(\mathbb{C}P^n)$, then (i) holds. With this normalization, one gets condition (ii) as well. Since the rescaling constants will be close to 1, one can see that the conclusion of the theorem follows.

The proof of the lemma uses some recent nontrivial results arising from the work of Tian and Chen-Donaldson-Sun about Kähler-Einstein metrics and stability.