Bimodal Wilson System in $L^2(\mathbb{R})$

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IISC–IISERP JOINT MATH 20-20 SYMPOSIUM

September 17, 2021

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  is an ONB for $L^2(\mathbb{R})$, where $\chi_{[0,1)}$ is the characteristic function of $[0,1)$.
- $\mathcal{G}(\chi, 1, 1)$ is the simplest example of Gabor systems.

- More generally, given $\alpha, \beta > 0$ and $\phi \in L^2(\mathbb{R})$, the set

$$\mathcal{G}(\phi, \alpha, \beta) = \{\phi_{j,m}(\cdot) := \phi(\cdot - \beta j)e^{2\pi i \alpha m \cdot} : j, m \in \mathbb{Z}\} \tag{0.1}$$

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- $\mathcal{G}(\phi, \alpha, \beta)$ is called a Gabor frame if there exist $0 < A \leq B$ such that for every $f \in L^2(\mathbb{R})$ we have

$$A\|f\|^2 \leq \sum_{j,m \in \mathbb{Z}} |\langle f, \phi_{j,m}\rangle|^2 \leq B\|f\|^2. \tag{0.2}$$

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- If in addition, $A = B = 1$ we call the system a Parseval (Gabor) frame.

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- Balian–Low theorem: If $\mathcal{G}(\phi, \alpha, 1/\alpha)$ is an ONB, then, the window ϕ must be poorly localized in time or frequency that is

$$\int_{\mathbb{R}} |x|^2 |\phi(x)|^2 \, dx = \infty \quad or \quad \int_{\mathbb{R}} |\xi|^2 \, |\hat{\phi}(\xi)|^2 \, d\xi = \infty$$

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- Thus BLT imposes strict limits on Gabor systems that form an ONB.
- Question: How tight Gabor frame $\mathcal{G}(\phi, 1, \frac{1}{\alpha})$ of redundancy α can be transformed into ONB for $L^2(\mathbb{R})$ with well localized generator ϕ?

Bimodal Wilson System

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- More precisely, given a Gabor system $\mathcal{G}(\phi,\alpha,\beta)$, the associated (bimodal) Wilson system $\mathcal{W}(\phi,\alpha,\beta)$ is

$$\mathcal{W}(\phi,\alpha,\beta)=\{\psi_{j,m} : j\in\mathbb{Z}, m\in\mathbb{N}_0\} \tag{0.3}$$

where

$$\psi_{j,m}(x)=\begin{cases}\sqrt{2\beta}\phi_{2j,0}(x)=\sqrt{2\beta}\phi(x-2\beta j); & \text{if } j\in\mathbb{Z}, m=0,\\ \sqrt{\beta}\left[e^{-2\pi i\beta j\alpha m}\phi_{j,m}(x)+(-1)^{j+m}e^{2\pi i\beta j\alpha m}\phi_{j,-m}(x)\right]; & \\ & \text{if } (j,m)\in\mathbb{Z}\times\mathbb{N}.\end{cases} \tag{0.4}$$

Theorem (Daubechies-Jaffard-Journé, SIAM J. Math. Anal. - 1991)

Let $\phi \in L^2(\mathbb{R})$ *be such that* $\hat{\phi}(\xi) = \overline{\hat{\phi}(\xi)}$ *and* $\|\phi\|_2 = 1$. *Then the Gabor system* $\mathcal{G}(\phi, 1, 1/2)$ *is a tight frame for* $L^2(\mathbb{R})$ *if, and only if, the Wilson system* $\mathcal{W}(\phi, 1, 1/2)$ *is an orthonormal basis for* $L^2(\mathbb{R})$. *Furthermore, one can choose* $\phi \in C^\infty(\mathbb{R})$ *with compact support.*

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- The underlying theme in all these results is a one-to-one association of a tight Gabor frame of redundancy $(\alpha\beta)^{-1} = 2$ with a bimodal Wilson basis.
- K. Gröchenig Question: How tight Gabor frame $\mathcal{G}(\phi, 1, \frac{1}{3})$ can be transformed into ONB by taking suitable linear combination?

Theorem (Bhimani-Okoudjou, JMAA-2020)

Let $\beta \in (0, 1/2)$. There exists $\phi \in S(\mathbb{R})$ with $\hat{\phi} \in C_c^\infty(\mathbb{R})$ such that the Gabor system $\mathcal{G}(\phi, 1, \beta)$ is a tight frame for $L^2(\mathbb{R})$ with frame bound β^{-1} if and only if the Wilson system $\mathcal{W}(\phi, 1, \beta)$ is a Parseval frame for $L^2(\mathbb{R})$.

Characterization for Wilson bases in L^2

Theorem

Let $\alpha, \beta > 0$, and $\{\psi_{j,m}\}_{j\in\mathbb{Z}, m\in\mathbb{N}_0}$ is defined by (0.4). The following statements are equivalent:

(a) $\mathcal{W}(\phi, \alpha, \beta) = \{\psi_{j,m}\}_{j\in\mathbb{Z}, m\in\mathbb{N}_0}$ *is a Parseval frame for* $L^2(\mathbb{R})$.

(b) $\Phi_k(\xi) = \delta_{k,0}\, a.e.$, *and* $\Delta_k(\xi) = 0\, a.e.$ *for each* $k \in \mathbb{Z}$, *where*

$$\begin{cases} \Phi_k(\xi) = \sum_{m\in\mathbb{Z}} \hat{\phi}(\xi - \alpha m)\overline{\hat{\phi}(\xi + \beta^{-1}k - \alpha m)}, \\ \Delta_k(\xi) = \sum_{m\in\mathbb{Z}} (-1)^m \hat{\phi}(\xi + \alpha m)\overline{\hat{\phi}(\xi + \beta^{-1}(k + 1/2) - \alpha m)}. \end{cases}$$

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- To show: $\|f\|_{L^2}^2 = \sum_{m\in\mathbb{N}_0} \sum_{j\in\mathbb{Z}} |\langle f, \psi_{j,m}\rangle|^2 := \mathcal{I}(f)$.

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- It is enough to prove above formula in dense class

$$\mathcal{D} = \left\{ f \in L^2 : \hat{f} \in L^\infty \text{ and support of } \hat{f} \text{ is a compact subset of } \mathbb{R}\setminus\{0\} \right\}$$

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- Develop decomposition formula $\mathcal{I}(f) = \mathcal{I}_0(f) + \mathcal{I}_1(f)$ (and invoke hypothesis)

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- **Proposition**: Let $\alpha, \beta > 0$ and $\phi \in L^2(\mathbb{R})$. For any $f \in \mathcal{D}$ we have the following decomposition

$$\mathcal{I}(f) = \mathcal{I}_0(f) + \mathcal{I}_1(f).$$

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Sketch Proof: (b) implies (a)

$$\widehat{\phi_{j,m}}(\xi) = e^{-2\pi i\beta j(\xi-\alpha m)}\hat{\phi}(\xi-\alpha m), \ (\xi \in \mathbb{R}). \tag{0.5}$$

$$\widehat{\psi_{j,m}}(\xi) = \begin{cases} \sqrt{2\beta}e^{-4\pi i\beta j\xi}\hat{\phi}(\xi); \quad \text{if } \ j \in \mathbb{Z}, m=0, \\ \sqrt{\beta}\left[e^{-2\pi i\beta j\xi}\hat{\phi}(\xi-\alpha m) + (-1)^{j+m}e^{-2\pi i\beta j\xi}\hat{\phi}(\xi+\alpha m)\right]; \\ \qquad \text{if } \ (j,m) \in \mathbb{Z}\times\mathbb{N}. \end{cases}$$

$$\sum_{m\in\mathbb{N}_0}\sum_{j\in\mathbb{Z}}|\langle f,\psi_{j,m}\rangle|^2 = \int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}\hat{f}(\xi+\beta^{-1}k)\overline{\hat{f}(\xi)}\Phi_k(\xi)d\xi + \mathcal{I}_1(f)$$

- technical part: long computations...

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- Hypothesis: $\mathcal{W}(\phi, \alpha, \beta) = \{\psi_{j,m}\}_{j \in \mathbb{Z}, m \in \mathbb{N}_0}$ is a Parseval frame.

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$0=\int_{\mathbb{R}}\sum_{0\neq k\in\mathbb{Z}}\hat{f}(\xi+\beta^{-1}k)\overline{\hat{f}(\xi)}\Phi_k(\xi)d\xi+\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}\overline{\hat{f}(\xi)}\hat{f}(\xi+\beta^{-1}(k+1/2))\Delta_k(\xi)d\xi.$

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- We claim that $\Phi_k = 0$ a.e. for all $0 \neq k \in \mathbb{Z}$ and $\Delta_k = 0$ a.e. for all $k \in \mathbb{Z}$.

$0 = \int_{\mathbb{R}} \sum_{0 \neq k \in \mathbb{Z}} \hat{g}(\xi + \beta^{-1}k)\overline{\hat{f}(\xi)}\Phi_k(\xi)d\xi + \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \overline{\hat{f}(\xi)}\hat{g}(\xi + \beta^{-1}(k + 1/2))\Delta_k(\xi)d\xi$ for all $f, g \in \mathcal{D}$

- Let us fix $k_0 \neq 0$ and $0 \neq \xi_0 \neq \xi_0 + \beta^{-1}k_0$ and $\Phi_{k_0} \in L^1_{loc}(\mathbb{R})$.

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$$\lim_{\delta\to 0} \frac{1}{\mu(B_\delta(\xi_0))} \int_{\mathbb{R}} \Phi_{k_0}(\xi)d\xi = \Phi_{k_0}(\xi_0). \tag{0.6}$$

sketch proof: (a) implies (b)

- We consider $\delta > 0$ sufficiently small so that both $B_\delta(\xi_0)$ and $B_\delta(\xi_0 + k_0)$ lie within $\mathbb{R} \setminus \{0\}$.
- Let f_δ and g_δ in $\mathcal{D}$ be functions such that

$$\hat{f}_\delta(\xi) = \frac{1}{\sqrt{\mu(B_\delta(\xi_0))}} \chi_{B_\delta(\xi_0)}(\xi),$$

and

$$\hat{g}_\delta(\xi) = \frac{1}{\sqrt{\mu(B_\delta(\xi_0))}} \chi_{B_\delta(\xi_0 + \beta^{-1} k_0)}(\xi).$$

- Note that $\hat{g}_\delta(\xi) = \hat{f}_\delta(\xi - \beta^{-1} k_0)$ and

$$\overline{\hat{f}_\delta(\xi)} \hat{g}_\delta(\xi + \beta^{-1} k_0) = \frac{1}{\mu(B_\delta(\xi_0))} \chi_{B_\delta(\xi_0)}(\xi).$$

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$$
\begin{aligned}
0 \;&=\; \frac{1}{\mu(B_\delta(\xi_0))}\int_{B_\delta(\xi_0)} \Phi_{k_0}(\xi)d\xi + \int_{\mathbb{R}} \sum_{k\neq 0,k_0} \hat{g}_\delta(\xi+\beta^{-1}k)\overline{\hat{f}_\delta(\xi)}\Phi_k(\xi)d\xi \\
&\quad + \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \overline{\hat{f}_\delta(\xi)}\hat{g}(\xi+\beta^{-1}(k+1/2))\Delta_k(\xi)d\xi \\
&=\; \frac{1}{\mu(B_\delta(\xi_0))}\int_{B_\delta(\xi_0)} \Phi_{k_0}(\xi)d\xi + J_\delta + P_\delta.
\end{aligned}
$$

- To establish that $\Phi_{k_0}(\xi_0)=0$, it suffices to prove that

$$\lim_{\delta\to 0} J_\delta = \lim_{\delta\to 0} P_\delta = 0.$$

Characterization for Wilson bases in L^2

- Suppose $\{e_j : j = 1, 2, ...\} \subset L^2$ is a Parseval frame. If $\|e_j\|_{L^2} = 1$ for all j, then $\{e_j : j = 1, 2, ...\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

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Corollary

Let $\alpha, \beta > 0$, and $\{\psi_{j,m}\}_{j\in\mathbb{Z}, m\in\mathbb{N}_0}$ is defined by (0.4). Suppose that one of the statements (a) or (b) in Theorem 3 hold (hence all of them hold), then $\{\psi_{j,m}\}_{j\in\mathbb{Z}, m\in\mathbb{N}_0}$ is an ONB for $L^2(\mathbb{R})$ if and only if

$$\begin{cases} \|\phi\|_{L^2} = \frac{1}{\sqrt{2\beta}}, \\ \Re\langle X_{j,m}, Y_{j,m}\rangle = 0. \end{cases}$$

where $X_{j,m} = e^{-2\pi i\beta j\alpha m}\phi_{j,m}, Y_{j,m} = (-1)^{j+m}e^{2\pi i\beta j\alpha m}\phi_{j,-m}$.

Proposition

Let $\phi \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$. The Gabor system $\mathcal{G}(\phi, \alpha, \beta)$ is a tight frame for $L^2(\mathbb{R})$ with frame bound β^{-1} if and only if ϕ satisfies $\sum_{m\in\mathbb{Z}} \hat{\phi}(\xi - \alpha m)\overline{\hat{\phi}(\xi + \beta^{-1}k - \alpha m)} = \delta_{k,0}$ a.e. for each $k \in \mathbb{Z}$.

From tight Gabor frames to Parseval Wilson frames

Theorem

Let $\phi \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$. The following two statements are equivalent.

(a) *The Gabor system $\mathcal{G}(\phi, \alpha, \beta)$ is a tight frame for $L^2(\mathbb{R})$ with frame bound β^{-1}, and $\Delta_k = 0$ a.e. for all $k \in \mathbb{Z}$.*

(b) *The Wilson system $\mathcal{W}(\phi, \alpha, \beta)$ is a Parseval frame for $L^2(\mathbb{R})$.*

Corollary

Let $\phi \in L^2(\mathbb{R})$ *and* $\alpha, \beta > 0$. *Let* $X_{j,m}$ *and* $Y_{j,m}$ *be defined by*

$$\begin{cases} X_{j,m} = e^{-2\pi i\beta j\alpha m}\phi_{j,m}, \\ Y_{j,m} = (-1)^{j+m}e^{2\pi i\beta j\alpha m}\phi_{j,-m}. \end{cases}$$

Suppose that the Gabor system $\mathcal{G}(\phi, \alpha, \beta)$ *is a tight frame for* $L^2(\mathbb{R})$ *with frame bound* β^{-1}, *and* $\Delta_k = 0$ *a.e. for all* $k \in \mathbb{Z}$. *Then, the Wilson system* $\mathcal{W}(\phi, \alpha, \beta)$ *is an orthonormal basis for* $L^2(\mathbb{R})$ *if and only if*

$$\begin{cases} \|\phi\|_{L^2} = \frac{1}{\sqrt{2\beta}} \\ \Re\langle X_{j,m}, Y_{j,m}\rangle = 0 \end{cases}$$

for all $(j, m) \in \mathbb{Z} \times \mathbb{N}$.

Examples of generator of Wilson systems

- We wish to find rapidly decaying C^{∞} function ϕ satisfying the hypothesis of previous theorems.

Examples of generator of Wilson systems

- We wish to find rapidly decaying C^∞ function ϕ satisfying the hypothesis of previous theorems.
- we seek a function $\phi \in L^2(\mathbb{R})$ which satisfies

$$\Phi_k(\xi) = \sum_{m\in\mathbb{Z}} \hat{\phi}(\xi - \alpha m)\overline{\hat{\phi}(\xi + \beta^{-1}k - \alpha m)} = \delta_{k,0} \; a.e \text{ for each } k \in \mathbb{Z},$$

$$\Delta_k(\xi) = \sum_{m\in\mathbb{Z}} (-1)^m \hat{\phi}(\xi+\alpha m)\overline{\hat{\phi}(\xi + \beta^{-1}(k+1/2) - \alpha m)} = 0 \; a.e \text{ for each } k$$

- Let $\beta \in (0, 1/2)$, and $\alpha = 1$.
- we choose a function $\hat{\phi} : \mathbb{R} \to \mathbb{C}$ supported in $B_\gamma(0) = \{\xi \in \mathbb{R} : |\xi| \leq \gamma\}$, where $\gamma = \frac{1}{4\beta} - \epsilon$ for $\epsilon > 0$ suitable small enough so that $1 < 2\gamma$, that is, $1 < \frac{1}{2\beta} - 2\epsilon$.

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- $$\hat{\phi}(\xi)\overline{\hat{\phi}(\xi + \beta^{-1}k)} = 0, \quad 0 \neq k \in \mathbb{Z},$$
  $$\hat{\phi}(\xi)\overline{\hat{\phi}(\xi + \beta^{-1}(k + 1/2))} = 0, \quad k \in \mathbb{Z}$$

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- $$\hat{\phi}(\xi)\overline{\hat{\phi}(\xi + \beta^{-1}k)} = 0, \quad 0 \neq k \in \mathbb{Z},$$
  $$\hat{\phi}(\xi)\overline{\hat{\phi}(\xi + \beta^{-1}(k + 1/2))} = 0, \quad k \in \mathbb{Z}$$
- $\Phi_k = 0$ a.e. for all $0 \neq k \in \mathbb{Z}$ and $\Psi_k = 0$ a.e. for all $k \in \mathbb{Z}$.

- Next, we wish to show that $\Phi_0(\xi) = \sum_{m\in\mathbb{Z}} |\hat{\phi}(\xi - m)|^2 = 1$ a.e..
- Since this sum is periodic in ξ with period 1, we only needs to check what happen for $0 \leq \xi \leq 1$.
- To this end, consider smooth function $G : \mathbb{R} \to [0, 1]$ satisfying the following properties:

$$G(x) = \begin{cases} 0 & \text{if } \; x \leq -\gamma + 1, \\ 1 & \text{if } \; x \geq \gamma. \end{cases}$$

-
- We define the function $\hat{\phi} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{\phi}(\xi) = \begin{cases} \sin\left[\frac{\pi}{2} G(\xi + 1)\right] & \text{if } \xi \leq 0, \\ \cos\left(\frac{\pi}{2} G(\xi)\right) & \text{if } \xi \geq 0. \end{cases}$$

- Since $\hat{\phi} \in C_c^\infty(\mathbb{R})$, we have $\phi \in \mathcal{S}(\mathbb{R})$.

Theorem (Bhimani-Okoudjou, JMAA-2020)

Let $\beta \in (0, 1/2)$. There exists $\phi \in S(\mathbb{R})$ with $\hat{\phi} \in C_c^{\infty}(\mathbb{R})$ such that the Gabor system $\mathcal{G}(\phi, 1, \beta)$ is a tight frame for $L^2(\mathbb{R})$ with frame bound β^{-1} if and only if the Wilson system $\mathcal{W}(\phi, 1, \beta)$ is a Parseval frame for $L^2(\mathbb{R})$.

Given $\beta > 0$, we define the Zak transform of $f \in \mathcal{S}(\mathbb{R})$ by

$$Z_\beta f(x,\xi) = \frac{1}{\sqrt{\beta}} \sum_{k\in\mathbb{Z}} f(\beta^{-1}(\xi - k))e^{2\pi ikx}. \tag{0.7}$$

- The two-variable function $F = Z_\beta f$ is periodic in the first variable and "semi-periodic" in the second variable:

$$Z_\beta f(x+1,\xi) = Z_\beta f(x,\xi), \quad Z_\beta f(x,\xi \pm 1) = e^{\pm 2\pi ix} Z_\beta f(x,\xi). \tag{0.8}$$

- The set of all functions F of two variables satisfying the periodicity conditions (0.8) can be equipped with the norm

$$\|F\|^2 = \int_0^1 \int_0^1 |F(x,\xi)|^2 dxd\xi. \tag{0.9}$$

Theorem

Let $\hat{\phi}$ be real functions such that $|\hat{\phi}(\xi)| \lesssim (1+|\xi|)^{-1-\epsilon}$ and $\beta = 1/(2n)$ where n is any odd natural number. Then the following are equivalent:

1. *The Gabor system $\mathcal{G}(\phi, 1, \beta)$ is a tight frame for $L^2(\mathbb{R})$ with frame bound β^{-1}.*
2. *The Wilson system $\mathcal{W}(\phi, 1, \beta)$ is a Parseval frame for $L^2(\mathbb{R})$.*
3. *The Zak transform $Z_\beta \hat{\phi}$ of $\hat{\phi}$ satisfies*

$$\sum_{r=0}^{\beta^{-1}-1} \left| Z_\beta \hat{\phi}\,(x, \xi - \beta r) \right|^2 = \frac{1}{\beta}$$

   for all most all $x, \xi \in [0,1]$.

Theorem

Let ϕ be a real-valued function such that $\hat{\phi}$ and ϕ have exponential decay. Suppose that $\alpha = 1$ and $\beta^{-1} \in \mathbb{N}$. Then

$$\sum_{m\in\mathbb{Z}} \hat{\phi}(\xi - m)\hat{\phi}(\xi + \beta^{-1}k - m) = \delta_{k,0} \quad a.e.\ for\ each\ k \in \mathbb{Z}$$

if and only if the Zak transform $Z_\beta\hat{\phi}$ of $\hat{\phi}$ satisfies

$$\sum_{r=0}^{\beta^{-1}-1} \left| Z_\beta \hat{\phi}\,(x, \xi - \beta r) \right|^2 = \frac{1}{\beta} \tag{0.10}$$

for all most all $x, \xi \in [0, 1]$.

- We start with a real-valued function g with exponential decay,

$$\begin{cases} |g(x)| \leq Ce^{-\lambda|x|}, & x \in \mathbb{R}, \lambda > 0, \\ |\hat{g}(\xi)| \leq Ce^{-\mu|\xi|}, & \xi \in \mathbb{R}, \mu > 0. \end{cases} \tag{0.11}$$

- We start with a real-valued function g with exponential decay,

$$\begin{cases} |g(x)| \leq Ce^{-\lambda|x|}, & x \in \mathbb{R}, \lambda > 0, \\ |\hat{g}(\xi)| \leq Ce^{-\mu|\xi|}, & \xi \in \mathbb{R}, \mu > 0. \end{cases} \tag{0.11}$$

- $G := Z_\beta g$ is a well-defined continuous and bounded function. Furthermore, since g is real-valued we have, for $x, \xi \in \mathbb{R}$,

$$G(-x, \xi) = \overline{G(x, \xi)}. \tag{0.12}$$

- We start with a real-valued function g with exponential decay,

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- $G := Z_\beta g$ is a well-defined continuous and bounded function. Furthermore, since g is real-valued we have, for $x, \xi \in \mathbb{R}$,

$$G(-x, \xi) = \overline{G(x, \xi)}. \tag{0.12}$$

- Assume further that

$$\inf_{x,\xi \in [0,1]} \sum_{r=0}^{\beta^{-1}-1} |G(x, \xi - \beta r))|^2 > 0. \tag{0.13}$$

We then define

$$\hat{\phi} = Z_{\beta}^{-1}\Psi, \tag{0.14}$$

where

$$\Psi(x,\xi) = \frac{1}{\sqrt{\beta}} \frac{G(x,\xi)}{\left(\sum_{r=0}^{\beta^{-1}-1} \left|G\left(x,\xi-\beta r\right)\right|^{2}\right)^{1/2}}, \tag{0.15}$$

and

$$Z_{\beta}^{-1}\Psi(\xi) = \sqrt{\beta}\int_{0}^{1} \Psi(x,\beta\xi)dx.$$

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and

$$Z_{\beta}^{-1}\Psi(\xi) = \sqrt{\beta}\int_0^1 \Psi(x,\beta\xi)dx.$$

Theorem

The function $\hat{\phi}$, defined by (0.14), is real-valued and satisfies (0.10). Furthermore, ϕ and $\hat{\phi}$ have exponential decay.

Theorem (Bhimani-Okoudjou, JMAA-2020)

Let $\beta \in (0, 1/2)$. There exists $\phi \in S(\mathbb{R})$ with $\hat{\phi} \in C_c^\infty(\mathbb{R})$ such that the Gabor system $\mathcal{G}(\phi, 1, \beta)$ is a tight frame for $L^2(\mathbb{R})$ with frame bound β^{-1} if and only if the Wilson system $\mathcal{W}(\phi, 1, \beta)$ is a Parseval frame for $L^2(\mathbb{R})$.

Theorem (Bhimani-Okoudjou, JMAA-2021)

Let $3 \leq \beta^{-1} \in \mathbb{N}$. There exists no function $\phi \in L^2(\mathbb{R})$ with either $\hat{\phi}$ compactly supported, or ϕ and $\hat{\phi}$ having exponential decay, such that the Wilson system $\mathcal{W}(\phi, 1, \beta)$ is an ONB for $L^2(\mathbb{R})$.

Open Question

Given $\mathcal{G}(\phi, \alpha, \beta)$ of redundancy β^{-1}, can we construct Wilson system $\mathcal{W}(\phi, \alpha, \beta)$ (may be suitable linear combination of β^{-1} elements from $\mathcal{G}(\phi, \alpha, \beta)$) forms an ONB for $L^2(\mathbb{R})$?

THANK YOU