Classification, reduction and stability of toric principal bundles

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(Joint work with Indranil Biswas, Arijit Dey, Bivas Khan and Mainak Poddar)

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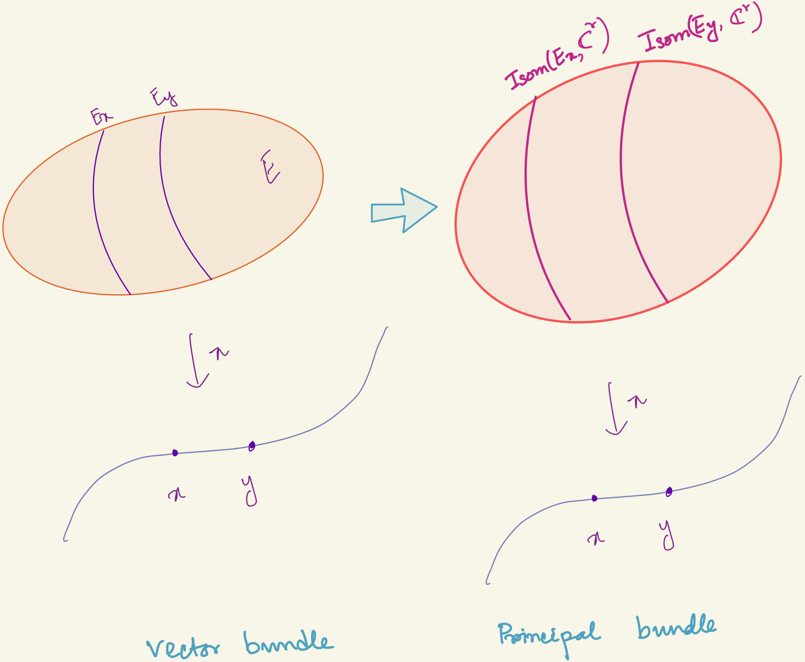
Ex
Ey
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Isom(Ex, C^r)
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π
x
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Vector bundle
Principal bundle

Principal bundles

- Let G denote a complex linear algebraic group.
- A principal G-bundle $\pi : \mathcal{E} \to X$ is a variety $\mathcal{E}$ with a right G-action, the action being free, such that π is G-equivariant, where X is being given the trivial G-action.

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- A principal G-bundle $\pi : \mathcal{E} \to X$ is a variety $\mathcal{E}$ with a right G-action, the action being free, such that π is G-equivariant, where X is being given the trivial G-action.
- Further, the bundle is assumed to be locally trivial in the étale topology. This means that, for every point $x \in X$, there exists a neighbourhood U and an étale morphism $U' \to U$ such that when $\mathcal{E}$ is pulled back to U', it is trivial as a G-bundle.

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Examples of Principal Bundles

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- Let $\mathcal{P} \to X$ be a principal G-bundle and let $f : Y \to X$ be any morphism. Then the pullback bundle $f^*(\mathcal{P}) := \mathcal{P} \times_X Y \to Y$ is a principal G-bundle over Y.

Toric Variety

Definition 1

A toric variety is a normal variety X such that

(1) *a torus $T \cong (\mathbb{C}^*)^n$ is a Zariski dense open subset of X, and*

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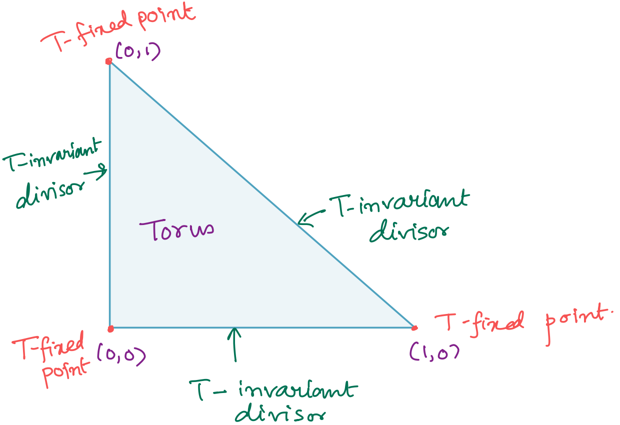
Example 2

- $(\mathbb{C}^*)^n$, $\mathbb{C}^n$, $\mathbb{P}^n$.
- *Projectivization of direct sum of line bundles on a toric variety.*
- *Blow up of a toric variety at an invariant subvariety.*



$$\Phi : (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^2$$

$$(t_1, t_2) \longmapsto [t_1^0 t_2^0 : t_1^1 t_2^0 : t_1^0 t_2^1] = [1 : t_1 : t_2]$$

Closure of the image of Φ is $\mathbb{P}^2$.

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- Each X_σ contains a unique T-fixed point denoted by x_σ. Ξ^* may be identified with the set of T-fixed points of X.
- We identify the open orbit O in X with T. Let $x_0 \in O$ correspond to $1_T \in T$.

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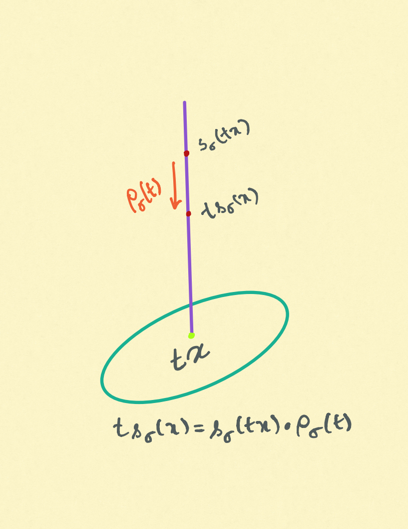
Goal: Study "toric principal bundles" over toric varieties.

Let G denote a complex linear algebraic group. Let X be a toric variety with dense torus T, then a principal G-bundle $\pi : \mathcal{E} \longrightarrow X$ is said to be a toric principal bundle if $\mathcal{E}$ is endowed with a lift of T-action on X.

Moreover, the T-action on $\mathcal{E}$ must commute with the G-action.

$$\begin{array}{ccc} T \curvearrowright & \mathcal{E} & \curvearrowleft G \\ & \Big\downarrow \pi & \\ T \curvearrowright & X & \end{array}$$

Equivariant trivialization over affine toric variety:



For any $\sigma \in \Xi^*$, set $\mathcal{E}_\sigma := \mathcal{E}|_{X_\sigma}$. A section $s_\sigma : X_\sigma \to \mathcal{E}_\sigma$ is called a distinguished section if

$$ts_\sigma(x) = s_\sigma(tx)\cdot\rho_\sigma(t),\ \forall x \in X_\sigma,\ \forall t \in T$$

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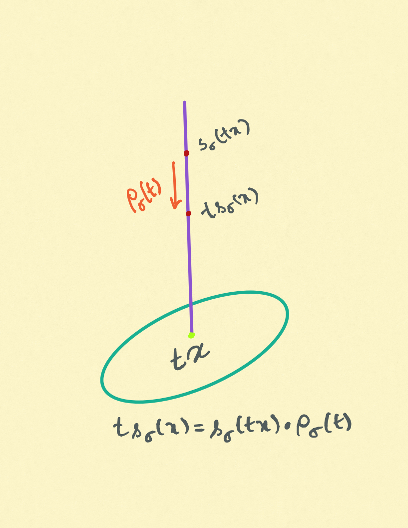

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where $\rho_\sigma : T \longrightarrow G$ is a homomorphism of algebraic groups.

$\mathcal{E}_\sigma$ is trivial and admits a distinguished section.

Classification of distinguished sections

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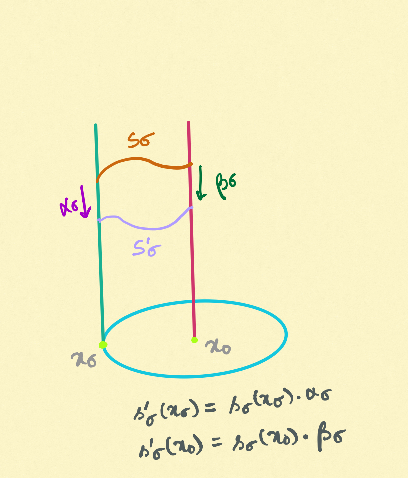
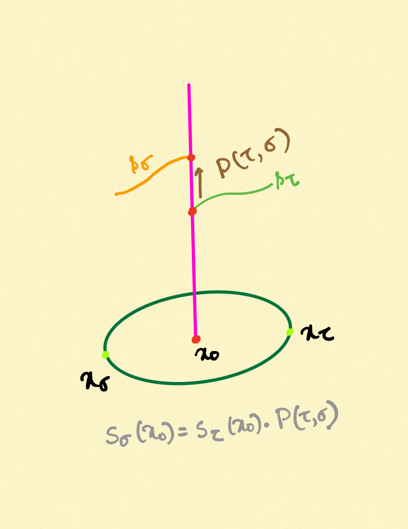
Suppose s_σ and s'_σ are two arbitrary distinguished sections for $\mathcal{E}_\sigma$ with homomorphisms ρ_σ, ρ'_σ respectively. Then,

1. $\rho'_\sigma(t) = \alpha_\sigma^{-1}\rho_\sigma(t)\,\alpha_\sigma$.
2. $\rho_\sigma(t)\beta_\sigma\alpha_\sigma^{-1}\rho_\sigma(t)^{-1}$ extends to a G-valued function over X_σ .





An admissible collection $\{\rho_\sigma, P(\tau, \sigma)\}$ consists of a collection of homomorphisms

$$\{\rho_\sigma : T \longrightarrow G \mid \sigma \in \Xi^*\}$$

and a collection of elements $\{P(\tau, \sigma) \in G \mid \tau, \sigma \in \Xi^*\}$.

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Theorem 1 (D, Khan, Biswas, Dey, Poddar (2021))

The isomorphism classes of T-equivariant principal G-bundles on X are in one-to-one correspondence with the "equivalence classes" of admissible collections $[\{\rho_\sigma, P(\tau,\sigma)\}]$.

Automorphisms of $\mathcal{E}$

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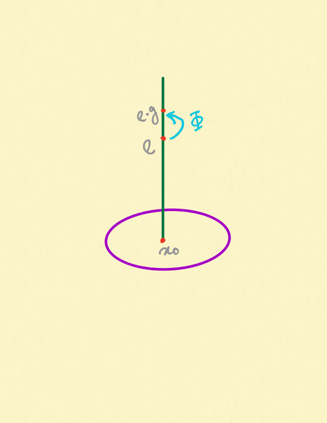

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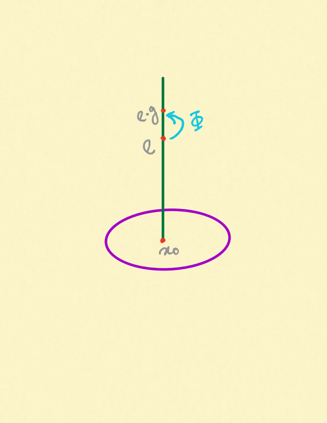

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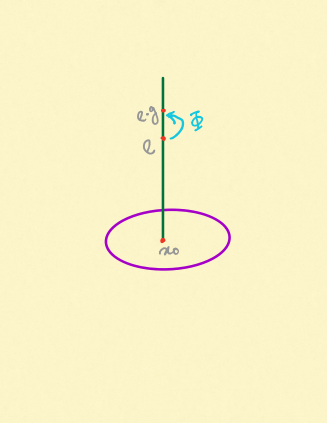

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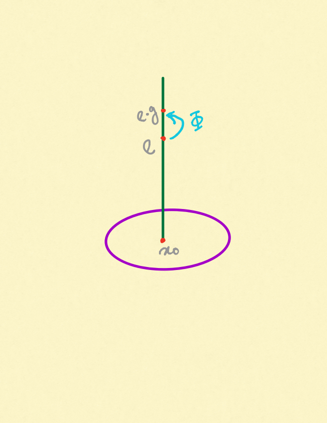

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- By T-equivariance, $\Phi|_{\mathcal{E}_O}$ is determined by $\Phi|_{\mathcal{E}_{x_0}}$.
- $\Phi|_{\mathcal{E}_{x_0}}$ is determined by $\Phi(e)$ using G-equivariance.

Consider the map $\xi : \mathsf{Aut}_T(\mathcal{E}) \to G$, uniquely determined by the relation

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Theorem 2 (D, Khan, Biswas, Dey, Poddar (2021))

$Aut_T(\mathcal{E})$, the group of T-equivariant automorphisms of $\mathcal{E}$, is given by

$$Aut_T(\mathcal{E}) = \{g \in G \mid \textit{for all } \sigma \in \Xi^*,\ \rho_\sigma(t) g \rho_\sigma(t)^{-1} \textit{ extends to } X_\sigma \ \}.$$

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In general, we have the inclusions

$$Z(G) \subseteq \mathrm{Aut}_T(\mathcal{E}) \subseteq G.$$

For $X = \mathbb{P}^m \times \mathbb{P}^n$ and $G = GL(m+n, \mathbb{C})$, we have $Z(G) \neq \mathrm{Aut}_T(\mathcal{E})$, where $\mathcal{E}$ is the tangent frame bundle of X.

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Let $\mathcal{E}_G$ be a toric principal G-bundle over X. $\mathcal{E}_G$ is said to admit an equivariant reduction of structure group to $H \leq G$ if there exists a toric principal H-bundle $\mathcal{E}_H$ such that the toric principal G- bundle $\mathcal{E}_H \times_H G$ is equivariantly isomorphic to $\mathcal{E}_G$.

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Theorem 3 (D, Khan, Biswas, Dey, Poddar (2021))

$\mathcal{E}_G$ has an equivariant reduction of structure group to a Levi subgroup H of G if and only if

$$Z^0(H) \subseteq Aut_T(\mathcal{E}_G).$$

Equivariant splitting of a toric principal bundle

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Theorem 4 (D, Khan, Biswas, Dey, Poddar (2021))

Let $\phi : G \to G'$ be an injective homomorphism of reductive algebraic groups. Let $\mathcal{E}_{G'} := \mathcal{E}_G \times_G G'$, where $\mathcal{E}_G$ is a toric principal G-bundle on X. Suppose that $\mathcal{E}_{G'}$ is equivariantly split, then $\mathcal{E}_G$ itself splits equivariantly.

Principal bundle analogue of Kaneyama's result

Theorem 5 (D, Khan, Biswas, Dey, Poddar (2021))

Let G be a reductive subgroup of $\mathrm{GL}(r, \mathbb{C})$. Any toric principal G-bundle on $\mathbb{P}^n$ splits equivariantly if $r < n$.

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This theorem has an alternative proof using certain results of Biswas and Parameswaran.

Why study (semi)stability

Building blocks

↓

Harder-Narsimhan filtration

↓

Jordan-Hölder filtration

Moduli Spaces

↓

arise in connection with classification problems

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Definition 3 (μ-stability or Mumford-Takemoto stability)

$\mathcal{E}$ is said to be **(semi)stable** *if for any coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}$, we have $\mu(\mathcal{F})(\leq) < \mu(\mathcal{E})$.*

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- *Line bundles are stable.*

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Equivariant stability of toric vector bundles

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An equivariant vector bundle $\mathcal{E}$ on a toric variety X is said to be **equivariantly (semi)stable** *with respect to an equivariant ample line bundle H if for any equivariant coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \text{rank } \mathcal{F} < \text{rank } \mathcal{E}$, we have $\mu(\mathcal{F})(\leq) < \mu(\mathcal{E})$.*

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Theorem 6 (Kool (2011), Biswas, Dey, Genc, and Poddar (2018))

Let $\mathcal{E}$ be an equivariant vector bundle on a nonsingular projective toric variety X. Then $\mathcal{E}$ is (semi)stable if and only if it is equivariantly (semi)stable.

Stability of toric principal bundles

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Theorem 7 (D, Khan, Biswas, Dey, Poddar (2021))

Let $\mathcal{E}_G$ be a T-equivariant principal G-bundle on a projective toric variety X. Then $\mathcal{E}_G$ is stable if and only if $\mathcal{E}_G$ is equivariantly stable.


Thank You