IISc-IISERP JOINT MATH 20-20 SYMPOSIUM

The Total Stiefel-Whitney class of an Orthogonal Representation of $\mathrm{SL}(n, q)$

Neha Malik
IISER Pune

(Joint work with Dr. Steven Spallone)

Stiefel-Whitney Classes

Let G be a finite group. To an orthogonal representation π of G, one can associate

$$w_i(\pi) \in H^i(G, \mathbb{Z}/2\mathbb{Z}) \text{ for } i = 0, 1, 2, \ldots$$

called the Stiefel-Whitney Classes (SWCs) of π.

Their sum

$$w(\pi) = w_0(\pi) + w_1(\pi) + w_2(\pi) + \ldots \in H^*(G, \mathbb{Z}/2\mathbb{Z})$$

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Properties of SWCs

1. $w_0(\pi)$ is the unit element $1 \in H^0(G, \mathbb{Z}/2\mathbb{Z})$.
2. $w_1 : \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} H^1(G, \mathbb{Z}/2\mathbb{Z})$.
3. $w_i(\pi) = 0$ for $i >$dim π.
4. Naturality. Given a group homomorphism $\varphi : G_1 \to G_2$ and an orthogonal representation π of G_2, we have
$$\varphi^*(w(\pi)) = w(\pi \circ \varphi),$$
where $\varphi^* : H^*(G_2, \mathbb{Z}/2\mathbb{Z}) \to H^*(G_1, \mathbb{Z}/2\mathbb{Z})$ is the map induced on cohomology.
5. Whitney Product Theorem. Let π_1 and π_2 be two orthogonal representations of G. Then,
$$w(\pi_1 \oplus \pi_2) = w(\pi_1) \cup w(\pi_2).$$

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Components of Orthogonal representations

Let π be an orthogonal representation of G. Then, it decomposes as

$$\pi \cong \underbrace{\bigoplus_i \sigma_i}_{\text{Irreducible orthogonal}} \oplus \underbrace{\bigoplus_j S(\rho_j)}_{\substack{\text{Orthogonally irreducible,} \\ \text{but not irreducible orthogonal}}}$$

We say ρ is orthogonally irreducible, if

- ρ is orthogonal,
- ρ can't be decomposed into a direct sum of orthogonal representations.

Example

Take ρ to be irreducible & non-orthogonal with dual $\rho^\vee$.
Then, $S(\rho) := \rho \oplus \rho^\vee$ is *orthogonally irreducible.*

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Detection by a subgroup

Question

Given a finite group G, find the total SWC of its orthogonal representations in terms of character values.

(N. Malik & S. Spallone) Special linear groups $\mathrm{SL}(n,q)$ for:

1. $n = 2$ with any q,
2. n, q both odd.

Detection by a subgroup (Important Tool)

For a finite group G and $H \xhookrightarrow{i} G$, we say that H detects the mod 2 cohomology of G, if the restriction map

$$H^*(G, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i^*} H^*(H, \mathbb{Z}/2\mathbb{Z})$$

is injective. In fact, $\mathrm{Im}(i^*) \subseteq H^*(H, \mathbb{Z}/2\mathbb{Z})^{N_G(H)}$.

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Detection by a subgroup

How do detection results help?

For an orthogonal representation π of G,

$$i^*(w(\pi)) = w(\pi|_H).$$

The detection results identify the total SWC of π with the SWC of its restriction to H.

Lemma ([1])

Let H be a Sylow 2-subgroup of G. Then, H detects mod 2 cohomology of G.

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$\mathrm{SL}(2,q)$ Detection for odd q

Let $G = \mathrm{SL}(2,q)$, q odd. Fix a, $b \in \mathbb{F}_q$ such that $a^2 + b^2 = -1$.
Then, Q_8 sits inside G via $\iota_{a,b} : Q_8 \to \mathrm{SL}(2,q)$ defined as,

$$i \mapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{ and } j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proposition (M., Sp.)

The cohomology ring $H^*(G, \mathbb{Z}/2\mathbb{Z})$ is detected by the quaternion group $\iota(Q_8)$. That is,

$$H^*(\mathrm{SL}(2,q), \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\iota^*} H^*(Q_8, \mathbb{Z}/2\mathbb{Z})^{N_G(Q_8)}$$

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Total SWC for $\mathrm{SL}(2, q)$, q odd

Let π be an orthogonal representation of G.

What is $\pi|_{Q_8}$??

Irreducible representations of Q_8 :

$$1, \chi_1, \chi_2, \chi_3 \quad \text{(1-dimensional orthogonal)}$$
$$\rho \quad \text{(2-dimensional symplectic)}$$

One can write

$$\pi|_{Q_8} = a_0 1 \oplus a_1 \chi_1 \oplus a_2 \chi_2 \oplus a_3 \chi_3 \oplus b S(\rho),$$

where $a_i, b \in \mathbb{Z}_{\geq 0}$.

What can we say about these coefficients a_i, b??

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Consider the outer automorphism of Q_8 defined by,

$$\begin{aligned} \theta : i &\mapsto j \\ j &\mapsto k \end{aligned}$$

Lemma (6.17, [2])

Let $Q_8 \subset G$. Then, there exists $T \in N_G(Q_8)$ with $T^3 = 1$ which acts by θ.

This induces the action:

$$\theta \circlearrowright \mathrm{Hom}(Q_8, \pm 1) = \langle \chi_1 \rangle \times \langle \chi_2 \rangle$$
$$\theta \cdot \chi(g) = \chi(\theta(g)) = \chi(TgT^{-1}).$$

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The θ- orbits of $\mathrm{Hom}(Q_8, \pm 1)$ are: $\{1\}$ and $\{\chi_1, \chi_2, \chi_3\}$.

Since $\pi|_{Q_8}$ is θ- invariant, it will have the form

$$\pi|_{Q_8} \cong a_0 1 \oplus a_1(\chi_1 \oplus \chi_2 \oplus \chi_3) \oplus b(S(\rho)).$$

By Whitney product theorem for SWCs, we have

$$\begin{aligned} w(\pi|_{Q_8}) &= \underbrace{w(1)^{a_0}}_{1} \cup \underbrace{w(\chi_1 \oplus \chi_2 \oplus \chi_3)^{a_1}}_{1} \cup w(S(\rho))^b \\ &= w(S(\rho))^b \\ &= (1+e)^b, \text{ where } e(\neq 0) \in H^4(Q_8, \mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

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Let $e \in H^4(G, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ be the unique non-trivial element.

Theorem (M., Sp.)

Let π be an orthogonal representation of G. Then the total SWC of π is

$$w(\pi) = (1 + e)^{r_\pi}$$

where $r_\pi = \frac{1}{8}(\chi_\pi(1) - \chi_\pi(-1))$.

Corollary

Let $G = \mathrm{SL}(2, q)$ with q odd. Let π be an irreducible orthogonal representation of G. Then, its associated total SWC $w(\pi) = 1$.

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Gow's formula & Proof to Corollary

For π irreducible, its Frobenius-Schur indicator $\varepsilon(\pi)$ is given as

$$\varepsilon(\pi) = \frac{1}{|G|}\sum_{g\in G}\chi_\pi(g^2) = \begin{cases} 1 & \pi \text{ orthogonal} \\ 0 & \pi \text{ not self-dual} \\ -1 & \pi \text{ symplectic} \end{cases}.$$

Let π be an irreducible orthogonal representation of G, with central character ω_π.
Now, $\mathrm{SL}(2, q)$-representations satisfy

$$\begin{aligned} \omega_\pi(-1) &= \varepsilon(\pi) \quad \text{" Gow's formula [3]"} \\ &= 1 \quad \text{for } \pi \text{ irreducible orthogonal.} \end{aligned}$$

Since $\chi_\pi(-1) = \omega_\pi(-1)\chi_\pi(1) = \chi_\pi(1)$, we have

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Detection result for $\mathrm{SL}(n, q)$ for n, q odd

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Denote the diagonal subgroup of $\mathrm{SL}(n, q)$ by A. It is a direct product of cyclic groups

$$A \cong C_{q-1} \times \ldots \times C_{q-1} (n-1 \text{ times}).$$

Lemma (M.,Sp.)

Let n, q be odd. Then, A detects the mod 2 cohomology of $\mathrm{SL}(n, q)$. In fact,

$$H^*(G, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^*(A, \mathbb{Z}/2\mathbb{Z})^W,$$

where $W \cong S_n$ is the Weyl group of G.

When $q \equiv 3 \pmod 4$, let $A_2 = \mathrm{Syl}_2(A) = C_2^{n-1}$.
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The group $A_2 = C_2 \times C_2 \subset \mathrm{SL}(3,q)$ viewed as,

$$(a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1^{-1}a_2^{-1} \end{pmatrix} = t$$

detects mod 2 cohomology of G.

Notation

1. Denote the character group of A_2 by $\hat{A}_2$.
2. The linear characters of A_2 are:

$$\mathrm{sgn}_{ij} = \mathrm{sgn}^i \otimes \mathrm{sgn}^j \quad \text{where } i, j \in \{0, 1\}.$$

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The Weyl action on $\hat{A}_2$

The Weyl group $W \cong S_3$ of $\mathrm{SL}(3, q)$ acts on $\hat{A}_2$ as,

$$\begin{aligned} S_3 \times \hat{A}_2 &\to \hat{A}_2 \\ (g, \chi) &\mapsto {}^g\chi \end{aligned}$$

where ${}^g\chi : t \mapsto \chi(gtg^{-1})$.

Lemma

All non-trivial linear characters of A_2 are W-conjugate.

Let π be an orthogonal representation of G. Then,

$$\pi|_{A_2} = m_0(\mathrm{sgn}_{00}) \bigoplus m_1(\mathrm{sgn}_{10} \oplus \mathrm{sgn}_{01} \oplus \mathrm{sgn}_{11})$$

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Total SWC for $\mathrm{SL}(3, q)$, $q \equiv 3 \pmod 4$

By Whitney Product theorem for SWCs,

$$w(\pi) = \left(w(\mathrm{sgn}_{10}) \cup w(\mathrm{sgn}_{01}) \cup w(\mathrm{sgn}_{11})\right)^{m_1}.$$

Theorem (M.,Sp.)

Let π be an orthogonal representation of G. Define

$$m_1 = \frac{1}{4}(\chi_\pi(1) - \chi_\pi(g_0)),$$

where $g_0 = diag(-1, -1, 1) \in G$. Then, the total SWC of π is

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$\mathrm{SL}(2, q)$ Detection when $q = 2^r$

Let $G = \mathrm{SL}(2, q)$ for even q. Consider its subgroup

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{F}_q \right\}.$$

Here, N is a 2-Sylow subgroup of G.

The restriction map induced by the inclusion $j : N \hookrightarrow G$ is an injection.

$$j^* : H^*(G, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^*(N, \mathbb{Z}/2\mathbb{Z}).$$

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$$w(\pi) = \Big(\prod_{v \in H^1(N, \mathbb{Z}/2\mathbb{Z})} (1+v) \Big)^m.$$

where $\prod_{v \in H^1(N, \mathbb{Z}/2\mathbb{Z})} (1+v) = 1 + \sum_{i=0}^{r-1} c_{r,i}$ *is in terms of Dickson invariants.*

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Thank you