Central limit theorems in number theory and graph theory

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- A "vertical" variant of the Sato-Tate distribution.
- General principles of asymptotic distribution of families in compact intervals of $\mathbb{R}$.
- Thematic similarities between the eigenvalues of Hecke operators and eigenvalues of families of regular graphs.

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- Let $\{P_n(t)\}$, $P_n(t)$ is a degree n polynomial. For all $n \geq 0$,

$$\lim_{V\to\infty} \frac{1}{|A_V|} \sum_{\lambda\in A_V} P_n(\lambda) = \int_\Omega P_n(t)\mu(t)dt.$$

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- We denote $Z_n(x) = X_n(x) - X_{n-2}(x),\ n \geq 2.$

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- For $n \geq 1$, let $T_n : S_k(N) \to S_k(N)$ denote the n-th Hecke operator.

- By the Ramanujan-Deligne bound, for a prime p such that $(p, N) = 1$, the eigenvalues of T_p lie in the interval

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- Any Hecke newform $f(z) \in \mathcal{F}_k(N)$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n,$$

  where $a_f(1) = 1$ and

$$\frac{T_n(f(z))}{n^{\frac{k-1}{2}}} = a_f(n) f(z),\ n \geq 1.$$

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$$\lim_{\substack{k+N \to \infty \\ p \nmid N}} \frac{N_I(p,N,k)}{s_k(N)} = \int_I \nu_p(t)\,dt,$$

where

$$\nu_p(t) = \frac{p+1}{\pi} \frac{\sqrt{1-\frac{t^2}{4}}}{(p^{1/2}+p^{-1/2})^2 - t^2}.$$

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- $\sigma_p^2 = \frac{1}{s_k(N)} \sum_{f \in \mathcal{F}_k(N)} (Y_p(f) - \mu_p)^2$.
- Is there a central limit-type theorem that predicts the distribution of

$$\frac{\sum_{p \leq x}(Y_p - \mu_p)}{\sqrt{\sum_{p \leq x} \sigma_p^2}}.$$

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$$\sum_{p \leq x} \mu_p := \sum_{p \leq x} \frac{1}{s_k(N)} \sum_{f \in \mathcal{F}_k(N)} Y_p(f) \sim \pi(x)\mu(I).$$

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Theorem (Ctd.)

For any integer $r \geq 0$,

$$\lim_{x \to \infty} \frac{1}{s_k(1)} \sum_{f \in \mathcal{F}_k(1)} \left(\frac{\sum_{p \leq x} \chi_I(a_f(p)) - \pi(x)\mu(I)}{\sqrt{\pi(x)[\mu(I) - \mu(I)^2]}} \right)^r$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-t^2/2} dt.$$

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That is, $\frac{\sum_{p\leq x}(Y_p - \mu_p)}{\sqrt{\sum_{p\leq x} \sigma_p^2}}$ converges to the normal distribution as $x \to \infty$. Our motivation was a result of Nagoshi, which showed that

$$\frac{\sum_{p\leq x} a_f(p)}{\sqrt{\pi(x)}}$$

converges to the normal distribution as $x \to \infty$ (as $\log k / \log x \to \infty$).

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Let $G(t) = \sum_{m \geq 0} U(m) X_{2m}(t)$, $|U(m)| \ll e^{-\alpha m^{\omega}}$. *Define* $Y_p : \mathcal{F}_k(1) \to \mathbb{R}$ *given by*

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$$V_G = \int_0^1 G(t)^2 \mu(t)dt - \left(\int_0^1 G(t)\mu(t)dt \right)^2.$$

Theorem (Contd.)

- *For any integer* $r \geq 0$,

$$\frac{1}{s_k(1)} \sum_{f \in \mathcal{F}_k(1)} \left(\frac{\sum_{p \leq x} Y_p(f) - \pi(x) \int_0^1 G(t)\mu(t)dt}{\sqrt{\pi(x) V_G}} \right)^r$$

$$\sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-t^2/2} dt.$$

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- Let us choose a sequence $\{X_m\}$ of graphs such that $X_m \in \mathcal{G}(m, d)$. How are the families $A(X_m)$ distributed as $m \to \infty$?

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Note: if $d = p + 1$, we recover $\nu_p(t)$.

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- Moreover, for an interval $I \subset [-2, 2]$,

$$\mathbb{E}\left[B_{\chi_I}\right] \sim m \int_I \mu(t).$$

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Then,

$$V[B_f] := \mathbb{E}\left[(B_f - \mathbb{E}[B_f])^2\right] \sim m^{\frac{1}{2}} \sum_{k=3}^{\infty} k a_k^2 \text{ as } m \to \infty.$$

Theorem (Ctd.)

For every $r \geq 0$,

$$\lim_{m\to\infty} \frac{1}{|\mathcal{G}(m,d(m))|} \sum_{Y\in\mathcal{G}(m,d(m))} \left(\frac{\left(B_f(Y) - m\int_{-2}^{2} f(t)\mu(t)dt\right)}{\sqrt{V[B_f]}} \right)^r$$

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$$= \begin{cases} 0 & \text{if } r \text{ is odd} \\ \frac{r!}{(r/2)!2^{r/2}} & \text{if } r \text{ if even.} \end{cases}$$

Can we obtain asymptotics $\mathbb{E}[B_f]$, $V[B_f]$ and a central limit theorem under "looser" growth conditions for d in terms of m(possibly for a restricted class of test functions)?

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- One takes a real-valued, even test function $g \in C^\infty(\mathbb{R})$.

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- Suppose the Fourier transform $\widehat{g}$ has compact support, without loss of generality, in $[-1, 1]$.
- $G_L(\theta)$ is a periodic function with Fourier expansion

$$G_L(\theta) = \frac{1}{L} \sum_{|m| \leq L} \widehat{g}\left(\frac{m}{L}\right) e(m\theta), \; e(x) = e^{2\pi i x}.$$

- In fact,
  $H_L(t) := G_L(\theta) = \sum_{0 \leq m \leq L} U(m) X_{2m}(t), \; t = 2\cos\pi\theta,$ where

$$U(m) = \frac{1}{L}\left(\widehat{g}\left(\frac{m}{L}\right) - \widehat{g}\left(\frac{m+1}{L}\right)\right).$$

Theorem (Podder, S)

Let $L \geq 1$. *Define* $B_{H_L} : \mathcal{G}(m, d) \to \mathbb{R}$ *by*

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We should look at the asymptotics of the average

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Can we find suitable growth conditions for d under which the above $\sim V_{H_L}$ and

$$\mathbb{E}\left[\left(\frac{\left(B_{H_L}(Y) - m\int_{-2}^{2} H_L(t)\mu(t)dt\right)}{\sqrt{V_{H_L}}}\right)^r\right]$$

converges to Gaussian moments as $m\to\infty$?

Trace formula

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Proposition

Let $Y \in \mathcal{G}(m,d)$, $d \geq 3$. Let $A(Y)$ denote the family of eigenvalues of $\mathcal{T}'$ (with respect to Y). For a positive integer n, let $f_n(Y)$ denote the number of closed non-backtracking walks (CNBW)'s of length n in Y and let $C_n(Y)$ denote the number of circuits of length n in Y.

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