Two generalizations of partial sum property, with two applications to weights of highest weight modules

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Notations for Kac–Moody algebras

1. $\mathfrak{g} =$ Kac–Moody Lie algebra over $\mathbb{C}$.
   Chevalley generators for $\mathfrak{g}$: $e_i, f_i, \alpha_i^\vee\ \forall\ 1 \leq i \leq n$.
   $\mathfrak{h} =$ Cartan subalgebra of $\mathfrak{g}$.
   $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ (triangular decomposition).
   $U(\mathfrak{g}) =$ universal enveloping algebra of $\mathfrak{g}$.
2. $\Delta =$ root system of $\mathfrak{g}$. Recall, $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\right)$.
3. $\mathcal{I}$ denotes the set of nodes in the Dynkin diagram for $\mathfrak{g}$.
   Ex: for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\mathcal{I} = \{1, 2\}$; for $\mathfrak{g} = \widehat{\mathfrak{sl}_3(\mathbb{C})}$, $\mathcal{I} = \{0, 1, 2\}$.
4. $\Pi = \{\alpha_i \mid i \in \mathcal{I}\}$ simple roots, $\Pi^\vee = \{\alpha_i^\vee \mid i \in \mathcal{I}\}$ simple co-roots
   ($\mathcal{I}$ is also an indexing set for simple roots and co-roots).
5. $\Delta^+ = \Delta \cap \mathbb{Z}_{\geq 0}\Pi$ positive roots.

Partial sum property for root systems

- ***Partial sum property*:** Every root in Δ is a sum of simple roots (or negative simple roots), whose partial sums are also roots.
  This is equivalent to saying given a positive root β, there exists a simple root α such that $\beta - \alpha$ is a root.

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Example: Let Δ be the root system of $\mathfrak{sl}_7(\mathbb{C})$, with a (fixed) base/set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_6\}$, with the Dynkin diagram:

We are indexing the nodes of the above diagram by $\{1, \ldots, 6\}$. Recall:

$$\Delta = \left\{ \pm \sum_{t=j}^{k} \alpha_t \;\middle|\; 1 \leq j \leq k \leq 6 \right\}.$$

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Choose $\beta = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. Observe:

$$\begin{aligned}\beta &= (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) + (\alpha_6)\\ &= (\alpha_2 + \alpha_3 + \alpha_4) + (\alpha_5) + (\alpha_6) = \cdots = (\alpha_2) + (\alpha_3) + (\alpha_4) + (\alpha_5) + (\alpha_6).\end{aligned}$$

Roots as sums of unit I-height roots, Example 1

Generalization 1 – parabolic: Let $I = \{2, 3, 5\}$ and define $\Pi_I := \{\alpha_2, \alpha_3, \alpha_5\}$.

Now every root that involves simple roots from Π_I in its expansion is a sum of *unit I-height roots*—roots which involve exactly one simple root from Π_I in their expansions.

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For example: Let $\beta = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$.

Observe:
$$\begin{aligned} \beta &= (\alpha_2 + \alpha_3 + \alpha_4) + (\alpha_5 + \alpha_6) \\ &= (\alpha_2) + (\alpha_3 + \alpha_4) + (\alpha_5 + \alpha_6). \end{aligned}$$

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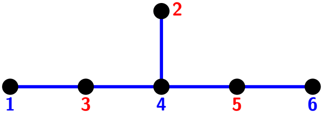
Question

Does this extension hold in all Kac–Moody types?

Roots as sums of unit I-height roots, Example 2

Let Δ be of type E_6, with the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_6\}$ and Dynkin diagram:



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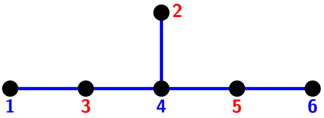

Choose $\beta = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \; = \; s_5\ s_4\ s_6\ s_5\ s_2\ s_4\ s_3\ (\alpha_1)$.

Observe:
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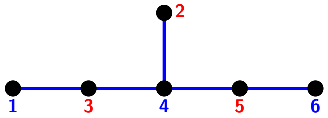

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The above two examples suggest

Given $\emptyset \neq S \subseteq \Pi$ and a positive root β involving some simple roots from S in its expansion, we can write β as an **ordered sum of roots**, each of which involves **only one simple root from** S.

Chains of roots between comparable roots, Example

Let $\prec$ denote the usual partial order on $\mathfrak{h}^*$ defined by:

$$x, y \in \mathfrak{h}^* \text{ and } x \prec y \quad \text{if and only if} \quad y - x \in \mathbb{Z}_{\geq 0}\Pi.$$

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Generalization 2 – representations: Rather than going from 0 to a *root* through partial sums,
go between comparable *weights* of the adjoint/other modules.

Example: $\mathfrak{g} = \mathfrak{sl}_6(\mathbb{C})$, with fixed a Cartan subalgebra $\mathfrak{h}$ and simple roots Π.
Consider two roots $\alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$.

$$\text{Clearly} \quad \alpha_2 + \alpha_3 \prec \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5.$$

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$$\text{Clearly} \quad \alpha_2 + \alpha_3 \prec \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5.$$

Observe that each member in the following chain is a root, and also the difference between successive roots is a simple root:

$$\alpha_2 + \alpha_3 \prec \alpha_1 + \alpha_2 + \alpha_3 \prec \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \prec \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5.$$

Chains of weights between comparable weights, Example

Set $\lambda = \rho = \sum_{i=1}^{5} \varpi_i$ where $\varpi_1, \ldots, \varpi_5$ are the fundamental dominant weights in $\mathfrak{h}^*$ (corresponding to Π).
Let $L(\lambda)$ be the integrable (which is also simple) highest weight module over $\mathfrak{sl}_6(\mathbb{C})$ with highest weight $\lambda \in \mathfrak{h}^*$.

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Check: $\lambda - \alpha_2$ and $\lambda - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$ are weights of $L(\lambda)$, and

$$\lambda - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 \prec \lambda - \alpha_2.$$

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Observe that each member in the following chain is a weight of $L(\lambda)$:

$$\lambda - \alpha_2 \succ \lambda - \alpha_1 - \alpha_2 \succ \lambda - 2\alpha_1 - \alpha_2 \succ \lambda - 2\alpha_1 - \alpha_2 - \alpha_3$$
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The above two examples suggest

Given two comparable weights $\mu_0 \prec \mu$ of a module V over a semisimple Lie algebra, there exists a chain of weights of V between μ_0 and μ such that the successive differences of weights in the chain are simple roots.

The core of the talk

Summary + 'preview': Let $\mathfrak{g}$ be a Kac-Moody algebra (over $\mathbb{C}$). Broadly throughout the talk, we deal with two settings:
(a) the root systems, and
(b) the set of weights of arbitrary highest weight modules over Kac–Moody $\mathfrak{g}$.

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(a) the root systems, and
(b) the set of weights of arbitrary highest weight modules over Kac–Moody $\mathfrak{g}$.

In this talk, we present two generalizations of the partial sum property:

(G1) In structure theory, i.e., for the root system of $\mathfrak{g}$. We call this property as **Parabolic partial sum property**. We prove it to the best possible extent and the level of generality, i.e., for any Lie algebra graded over any free abelian semigroup, and moreover at the level of "*Lie words*".

(G2) In representation theory, i.e., for the weights of highest weight modules V over $\mathfrak{g}$. Given such a module V, this property is about the existence of a chain of weights between any two comparable weights (in the usual partial order on $\mathfrak{h}^*$) of V. We prove this property for all simple highest weight modules and many more.

The core of the talk

As an application of these two generalizations, we obtain notably the following two main results among many:

(A1) A minimal description for the set of weights of every simple highest weight $\mathfrak{g}$-module.

(A2) A Minkowski difference formula for the set of weights of every highest weight $\mathfrak{g}$-module with any highest weight in $\mathfrak{h}^*$. This extends the known formulas for the weights of simple highest weight $\mathfrak{g}$-modules due to Khare and Dhillon–Khare.

Parabolic partial sum property

Given $\emptyset \neq I \subseteq \mathcal{I}$, and $\beta = \sum_{i \in \mathcal{I}} c_i \alpha_i \in \mathfrak{h}^*$, we define

$$\mathrm{ht}(\beta) = \sum_{i \in \mathcal{I}} c_i, \qquad \mathrm{ht}_I(\beta) := \sum_{i \in I} c_i \quad \text{the } I\text{-height of } \beta.$$

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Question (Khare)

Let $\emptyset \neq I \subseteq \mathcal{I}$, and $\beta \in \Delta^+$ with $\mathrm{ht}_I(\beta) > 0$. Do there exist positive roots:

$$\gamma_1, \dots, \gamma_n \in \Delta^+, \qquad n = \mathrm{ht}_I(\beta),$$

such that: (a) $\mathrm{ht}_I(\gamma_j) = 1 \ \forall \ \ 1 \leq j \leq n$, (b) $\sum_{j=1}^{n} \gamma_j = \beta$, and

(c) $\gamma_1 \prec \gamma_1 + \gamma_2 \prec \cdots \prec \gamma_1 + \cdots + \gamma_n$ are all roots?

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Answer (T.-)

Yes, for any Kac–Moody algebra $\mathfrak{g}$ (see next Theorem). We call this property for Kac–Moody root systems as Parabolic partial sum property. This property in fact holds **on the level of Lie words**, for any Lie algebra graded over an arbitrary free abelian semigroup:

Lifting the Parabolic partial sum property to Lie words

Theorem (T.-)

Let $\mathcal{G}$ be a Lie algebra graded over a free abelian semigroup $\mathbb{Z}_{\geq 0}\Pi$. Let $\mathcal{A} \subseteq \mathbb{Z}_{\geq 0}\Pi$ be the set of grades of $\mathcal{G}$, i.e., $\mathcal{G} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{G}_\alpha$. Fix $\emptyset \neq I \subseteq \mathcal{I}$ and $\beta \in \mathcal{A}$ with $\mathrm{ht}_I(\beta) > 0$. Then $\mathcal{G}_\beta$ is spanned by the Lie words:

$$\Big[x_{\gamma_1}, \big[x_{\gamma_2}, \cdots [x_{\gamma_{k-1}}, x_{\gamma_k}] \cdots \big]\Big], \quad \textit{where}$$

$$\gamma_i \in \mathcal{A}, \qquad \mathrm{ht}_I(\gamma_i) = 1, \qquad x_{\gamma_i} \in \mathcal{G}_{\gamma_i} \ \forall\ i, \quad \textit{and} \quad \gamma_1 + \cdots + \gamma_k = \beta.$$

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For Kac–Moody algebras, we have a stronger result:

Theorem (T.-)

Let $\mathfrak{g}$ be a Kac–Moody algebra and $\emptyset \neq I \subseteq \mathcal{I}$. Fix $x \in \bigoplus\limits_{\substack{\beta\in\Delta \\ \mathrm{ht}_I(\beta)>0}} \mathfrak{g}_\beta$. Then:

$$[x, f_\eta] = 0\ \ \forall\ \eta \in \Delta^-\ \textit{with}\ \mathrm{ht}_I(\eta) = -1,\ \textit{and}\ f_\eta \in \mathfrak{g}_\eta \quad \Longrightarrow \quad x = 0.$$

Notations for highest weight modules and integrability

Let $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall\ \alpha^\vee \in \Pi^\vee\}$ the integral weight lattice, and $P^+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \ \forall\ \alpha^\vee \in \Pi^\vee\}$ the cone of dominant integral weights, respectively.

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Let $\lambda \in \mathfrak{h}^*$.

1. $M(\lambda) =$ Verma module with highest weight λ.
2. $L(\lambda) =$ simple module " ——— " ——— ".
3. $M(\lambda) \twoheadrightarrow V =$ highest weight $\mathfrak{g}$-module V with highest weight λ.
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Integrability/ Integrable directions for $M(\lambda) \twoheadrightarrow V$:

1. $J_\lambda := \{j \in \mathcal{I} \mid \langle \lambda, \alpha_j^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$.
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Observation

$$J_\lambda \;=\; I_{L(\lambda)} \text{ the integrability of } L(\lambda).$$

Example: Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\mathcal{I} = \{1, 2\}$. Check that $I_{L(-\varpi_1)} = \{2\}$.

Definition of a parabolic Verma module

Let $\lambda \in \mathfrak{h}^*$ and $J \subseteq J_\lambda$.

1. Let $\mathfrak{g}_J$ be the Kac–Moody subalgebra of $\mathfrak{g}$ generated by $\{e_j, f_j \mid j \in J\}$, $\mathfrak{l}_J = \mathfrak{g}_J + \mathfrak{h}$ be the Levi subalgebra corresponding to J, and $\mathfrak{p}_J := \mathfrak{l}_J + \mathfrak{n}^+ = \mathfrak{g}_J + \mathfrak{h} + \mathfrak{n}^+$ be the parabolic subalgebra corresponding to J.
2. Parabolic Verma module:

$$M(\lambda, J) := M(\lambda) \Big/ \left\langle f_j^{\langle \lambda, \alpha_j^\vee \rangle + 1} M(\lambda)_\lambda \;\middle|\; f_j \in \mathfrak{g}_{-\alpha_j} \ \forall j \in J \right\rangle.$$

3. Let $L_J(\lambda)$ denote the simple highest weight $\mathfrak{l}_J$-module (or $\mathfrak{p}_J$-module) with highest weight λ. Then $M(\lambda, J) \simeq U(\mathfrak{g}) \bigotimes_{U(\mathfrak{p}_J)} L_J(\lambda)$.

Weights of parabolic Verma modules

Fix $\emptyset \neq J \subseteq \mathcal{I}$.

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2. $\Delta_J := \Delta \cap \mathbb{Z}\Pi_J$.
3. $\Delta_J^+ := \Delta_J \cap \mathbb{Z}_{\geq 0}\Pi_J$. $\big($When $J = \mathcal{I}$: $\Pi_J = \Pi$, $\Delta_J = \Delta$, $\mathrm{ht}_J(.) = \mathrm{ht}(.)$.$\big)$

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$M(\lambda, J) \simeq U(\mathfrak{g}) \bigotimes_{U(\mathfrak{p})} L_J(\lambda)$ implies:

$$\mathrm{wt}\, M(\lambda, J) = \mathrm{wt}\, L_J(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+).$$

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Integrable slice decomposition (Khare [J.Alg.], Dhillon–Khare [Adv.Math.])

$$\mathrm{wt}\, M(\lambda, J) = \bigsqcup_{\xi \in \mathbb{Z}_{\geq 0}\Pi_{J^c}} \mathrm{wt}\, L_J(\lambda - \xi).$$

Formulas for weights of simple highest weight modules

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Let $\lambda \in \mathfrak{h}^*$ and $J \subseteq J_\lambda$. Then:

- *All highest weight modules of integrability J have the same weights*
  if and only if
  $|J_\lambda \setminus J| \leq 1$, or there is at least one edge between every two nodes in the Dynkin subdiagram on $J_\lambda \setminus J$.
- *In particular, by the above point:*

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Our interest: Obtain a **minimal description** for $\mathrm{wt} L(\lambda)$.

Minimal description for weights of simple modules

Let $J \subsetneq \mathcal{I}$, and set $I = \mathcal{I} \setminus J$.

Define $\quad \Delta_{I,1} := \{\beta \in \Delta^+ \mid \mathrm{ht}_I(\beta) = 1\} \;\subseteq\; \Delta^+ \setminus \Delta_J^+$

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Corollary [T.-]

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Observe in the above two weight formulas that $\Delta_{I,1}$ cannot be further reduced. This implies that the description we obtained is **minimal**. Also, we obtain the *unique minimal set of extremal rays* in $\mathrm{wt} M(\lambda, J)$.

Chain of weights between comparable weights

This section is due to the following question:

Question

Let $\mathfrak{g}$ be a Kac–Moody algebra, $\lambda \in \mathfrak{h}^*$, and $M(\lambda) \twoheadrightarrow V$. Suppose $\mu_0 \precneqq \mu \in \mathrm{wt}\, V$. Then does there exist a sequence of weights $\mu_i \in \mathrm{wt}\, V$, $1 \leq i \leq n = \mathrm{ht}(\mu - \mu_0)$, such that

$$\mu_0 \prec \cdots \prec \mu_i \prec \cdots \prec \mu_n = \mu \in \mathrm{wt}\, V \quad \text{and} \quad \mu_i - \mu_{i-1} \in \Pi \ \forall i?$$

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In finite type, this question was answered positively by Shrawan Kumar and Khare for all simple integrable highest weight modules and all simple highest weight modules among many others, respectively.

Results of Kumar and Khare in finite type

Proposition (S. Kumar, 2016)

Let $\lambda \in P^+$ (dominant weight) and $\mu_0 \precneqq \mu \in \mathrm{wt} L(\lambda)$. Then there exists a sequence of weights $\mu_i \in \mathrm{wt} L(\lambda)$ such that

$$\mu_0 \prec \cdots \prec \mu_i \prec \cdots \prec \mu_n = \mu \quad \text{and} \quad \mu_i - \mu_{i-1} \in \Pi \ \forall i.$$

Theorem (Khare, J.Alg. 2016)

Let $\mathfrak{g}$ be semisimple, $\lambda \in \mathfrak{h}^*$, and $M(\lambda) \twoheadrightarrow V$. Suppose $\mu_0 \precneqq \mu \in \mathrm{wt} V$ and one of the following hold:

1. $|J_\lambda \setminus I_V| \leq 1$ *(e.g. $V = L(\lambda)$ is simple); or*
2. $V = M(\lambda, J)$ *for some $J \subseteq J_\lambda$.*

Then we get a sequence of weights with the same properties as above.

Extension to saturated subsets in the semisimple case

Recall, $U \subseteq P$ is a **saturated subset** if for every $x \in U$ and $\alpha \in \Pi$, $x - t\alpha \in U$ for all $t = 0, \ldots, \langle x, \alpha^\vee \rangle$.

Observe that every saturated subset is W-invariant.
Example: $\mathrm{wt}\, V$ for any integrable (not necessarily highest weight) module V.

Proposition (T.-)

Let $\mathfrak{g}$ be semisimple, and $U \subseteq P$ be a saturated subset. Fix $\mu_0 \precneqq \mu \in U$. Then there exists a sequence $\mu_i \in U$, $1 \leq i \leq n = \mathrm{ht}(\mu - \mu_0)$, such that

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Corollaries

- This reproves the above result of Kumar, as $\mathrm{wt} L(\lambda)$ for $\lambda \in P^+$ is saturated, and moreover extends it to all integrable modules.
- Let $U \subseteq P$ have the above "chain property". Then, U is invariant under W if and only if U is saturated.

Extension to Kac–Moody setting

We extend the above results for over Kac–Moody algebras in two ways:
(1) for the root system Δ,
(2) for sets of weights of arbitrary submodules of parabolic Verma modules (including all simple highest weight modules).

Theorem (T.-)

Let $\mathfrak{g}$ be a Kac–Moody algebra and V be a $\mathfrak{g}$-module. Suppose

- $V = \mathfrak{g}$ *(adjoint representation) or*
- $\lambda \in \mathfrak{h}^*$, $J \subseteq J_\lambda$, *and V is a submodule of $M(\lambda, J)$,*

and $\mu_0 \precneqq \mu \in \mathrm{wt}\, V$. Then there exists a sequence of weights $\mu_i \in \mathrm{wt}\, V$, $1 \leq i \leq n = \mathrm{ht}(\mu - \mu_0)$, such that

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Question

Can we extend this result to more highest weight $\mathfrak{g}$-modules?

A Minkowski difference formula for weights

As an important application of the above two generalizations of the partial sum property, we obtain a Minkowski difference formula for $\mathrm{wt}\, V$, for any highest weight module V over any Kac–Moody algebra $\mathfrak{g}$.
For $J \subseteq \mathcal{I}$, we define $\mathrm{wt}_J V := \mathrm{wt}\, V \cap (\lambda - \mathbb{Z}_{\geq 0}\Pi_J)$.

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More strongly, we have for arbitrary $J \subseteq \mathcal{I}$:

(2) $$\mathrm{wt}_J V - \mathbb{Z}_{\geq 0}\Pi_{J^c} \subset \mathrm{wt}\, V \quad \Longleftrightarrow \quad \mathrm{wt}\, V = \mathrm{wt}_J V - \mathbb{Z}_{\geq 0}\Delta_{J^c, 1}.$$

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Similarity of the formulas in (1) with those for $\mathrm{wt}\, L(\lambda)$

It is a trivial check that $\mathrm{wt}_{J_\lambda} L(\lambda) = \mathrm{wt}\, L_{J_\lambda}(\lambda)$.

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We prove the above implication by our analysis of the subsets $\Delta_{I,1}$ of Δ.

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Significance of formula (1)

Studying the weights of $M(\lambda) \twoheadrightarrow V$ is reduced to studying for $\lambda \in P^+$.

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