Triangle Singularities: Scope and Applications to Effective Field Theories

Ajay S. Sakthivasan[1], M. Mai, A. Rusetsky, M. Döring
[1]HISKP, Universität Bonn

1. Motivation

Hadronic spectrum is rich, and provides a formidable playground for theoretical studies. Identifying hadronic resonances is challenging, both experimentally and theoretically. Experimentally, resonant states are identified as *bumps* in the invariant mass distributions of scattering experiments. At low energies, the non-perturbative nature of QCD poses a problem to meaningfully study hadronic resonant states. Instead, one can start with the minimal conditions for a consistent theory—unitarity, analyticity, crossing symmetry. This leads to the regime of S-matrix theory and effective field theories. Resonant states are identified as *poles* in the S-matrix theoretic matrix elements of the relevant scattering amplitudes.

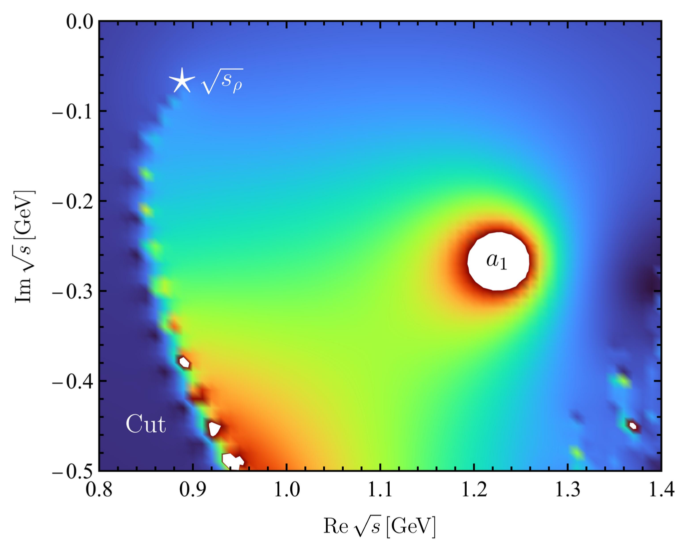

Figure 1: The $a_1(1260)$ pole [1].

One still needs to compare and match the resonant states observed experimentally with the resonant states obtained theoretically. **Do all observed peaks correspond to hadronic resonant states?** No! There are certain kinematical effects that don't depend on the underlying QCD dynamics, but still lead to observable peaks in the invariant mass distributions. One such example are the **Landau singularities**.

2. The Landau Equations

Contour integrals of entire functions on the complex plane are necessarily analytic. However, can one say something about integrals of functions that have singularities on the complex plane without actually evaluating the integrals? Owing to Cauchy's integral theorem, it turns out that the integrals are non-singular when the contour can be deformed. Such a deformation is not possible in specific cases: (1) The case when a singularity coincides with an endpoint of the contour, *endpoint singularities*; (2) The case when two or more singularities pinch the contour from opposite sides of the contour, preventing a deformation, *pinch singularities*.

The **Landau equations** [2] generalise to the case of a Feynman integral with l-loop momenta and N-internal propagators:

$$I = \int \prod_{i=1}^{l} d^4 k_i \, \frac{1}{\prod_{r=1}^{N} (q_r(k_i)^2 - m_r^2)}. \tag{1}$$

The Landau equations read:

$$\alpha_i(q_i^2 - m_i^2) = 0 \implies \alpha_i = 0 \text{ or } q_i^2 = m_i^2, \tag{2a}$$

$$\sum_{i \in \text{loop}} \alpha_i q_i^\mu(k_j) = 0. \tag{2b}$$

Eq. (2a) states that for the singularity, either the Feynman parameter vanishes or the corresponding internal propagator goes on mass shell. Eq. (2b) states that the components of the internal propagator momenta of a given loop are not linearly independent.

The Landau equations can be solved for specific cases. For example, solving the Landau equations for a one-loop diagram with two internal propagators, i.e., a bubble graph, one obtains two singularities: the two-body normal threshold and the anomalous threshold. One can also identify the nature of the singularity, which in this case is a square root branch point. This means that the singularity will present itself as a cusp in the relevant invariant mass distribution in the experiment. Solving the Landau equations for a one-loop diagram with three internal propagators, i.e., a triangle graph, is more interesting. In this case, if one fixes one of the external particles, one can obtain a range in the other two invariant masses in which a singularity can be found. Further, in this case, one can identify the nature of the singularity to be logarithmic. This is of significance, since a logarithmic branch point resembles a pole in experimental observations.

3. Implementation

In the last decade, there has been a plethora of observations which were explained to be triangle singularities. One such example is the $a_1(1420)$ [3]. A peak was observed in the $f_0\pi^-$ p-wave final state of the proton-pion collision in COMPASS experiment, approximately at $\sqrt{s} = 1.42$ GeV. This was tentatively considered a dynamically generated resonance, and in particular, an excited state of the $a_1(1260)$. However, this particular peak was very narrow and lied very close to the ground state. This prompted other explanations, and now is widely accepted to be a triangle singularity. The analyses carried out were rather ad hoc, and in our work [4] we aim to mainly **study the effects of final-state interactions on the Landau singularity**, while also making use of **a framework that consistently describes the unstable particles present in the system**. To this end, we implement the **Infinite Volume Unitary** framework [5] to a toy model.

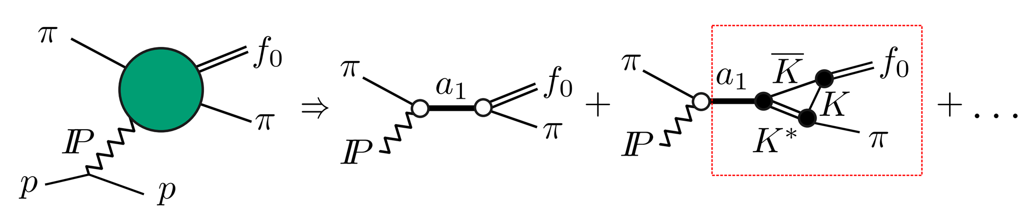

Figure 2: The $a_1(1420)$ as a triangle singularity [3].

As a first step, one needs to identify all the singularities associated with the different cases—one or more Feynman parameters vanishing—corresponding to the full rescattering diagram. The integral can be solved by discretisation and inversion, if the kernel is non-singular. This is ensured through a contour deformation. Still, numerical implementation is rather involved. Various methods to handle the ill-behaviour of the amplitude around the singularity were used [6].

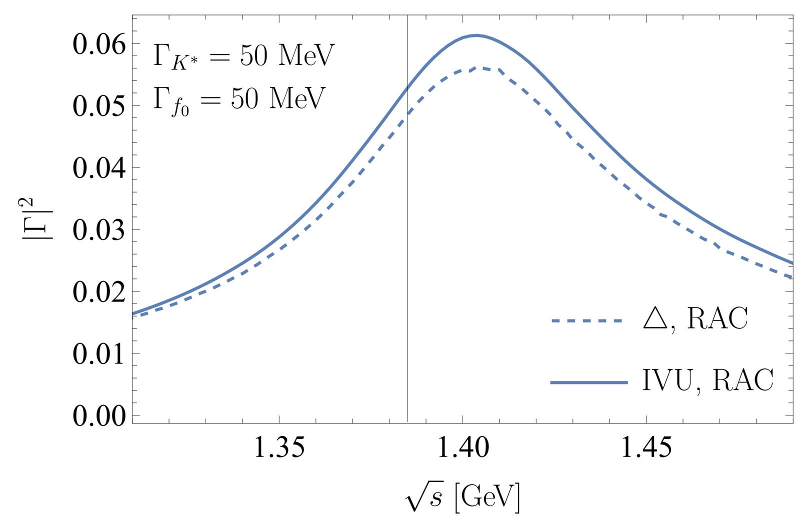

Figure 3: The amplitudes in the relevant invariant masses.

4. Conclusions

The Landau singularities are well-known for half a century now, and due to the rich nature of hadronic spectrum, they offer a field-agnostic test of different approaches. In this work, we studied the effects of final-state interactions on the triangle singularity for the so-called $a_1(1420)$. A three-body unitary framework was employed for this purpose. It was shown that only the original triangle singularity is present at all orders of the rescattering, which presents itself as a logarithmic singularity. Further, for our toy model, it was shown that the overall additional contribution is about 10% of the original amplitude, with the second rescattering contributing the most. From here, we currently aim to implement this to a realistic system, and also study Landau singularities on the lattice.

References

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