Landau Singularities: Scope and Applications to QCD

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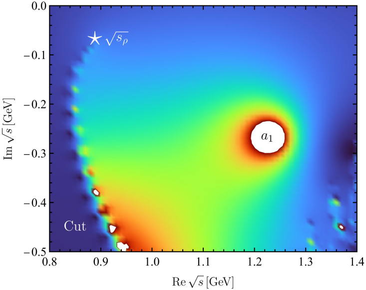
Presentation Outline

1. Introduction
2. Analytic Aspects
3. Infinite Volume Unitary Formalism
4. Summary & Conclusions

Introduction

- Hadronic resonances correspond to poles of the S-matrix.



Pole position of the $a_1(1260)$.[1]

[1] Figure from: 10.1103/PhysRevD.105.054020

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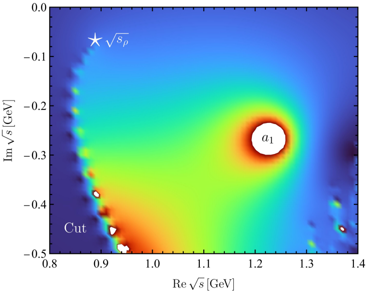
- Experimentally — peaks in invariant mass distributions.



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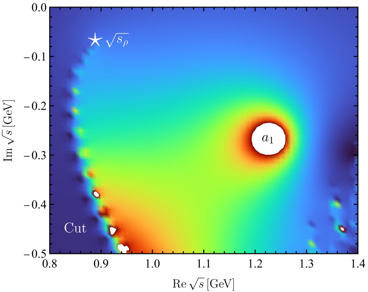
- Experimentally — peaks in invariant mass distributions.
- All observed peaks correspond to hadronic states? No.



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- Originally proposed by Landau; Recently used to explain a lot of observed peaks.

Singularities of Integrals

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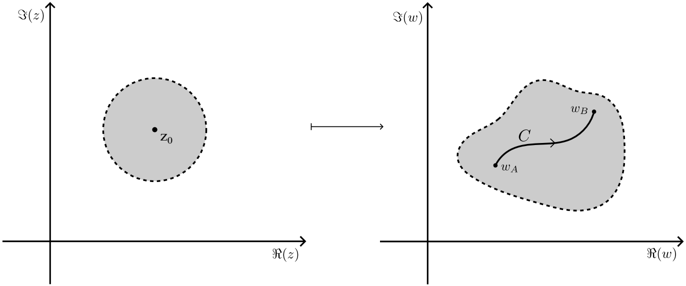
- What can one say about the singularities of f, given the knowledge of the singularities of g?

- Case: There are no singularities in the vicinity of the contour $\implies f$ is analytic.



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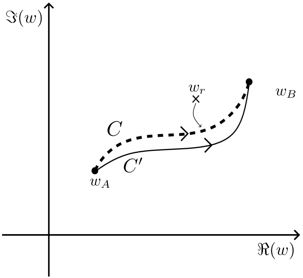

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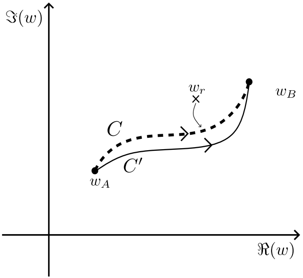

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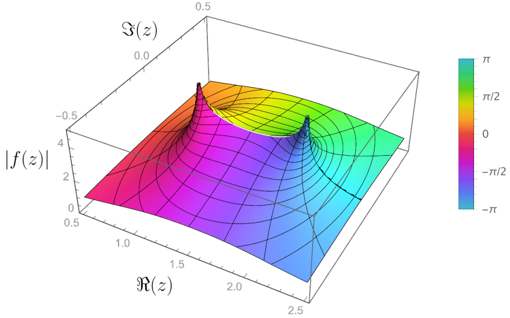

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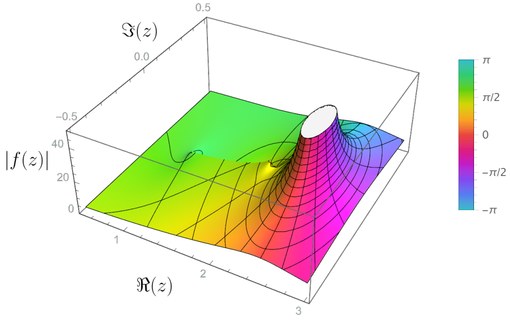

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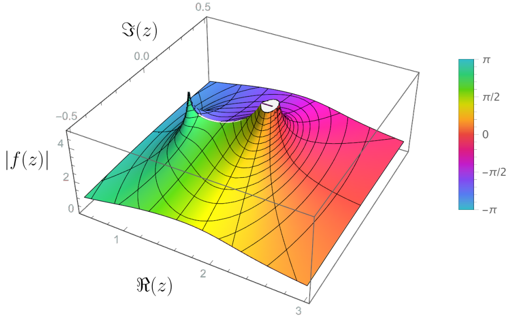

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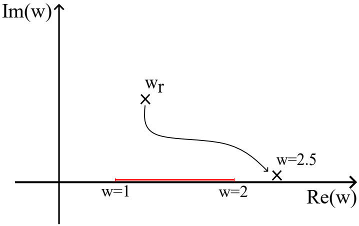

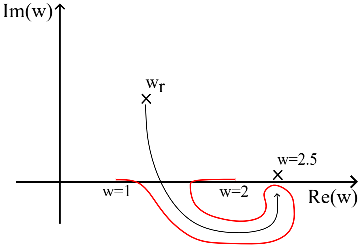

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- For some real parameters α_i, the Landau equations read,

$$\alpha_i(q_i^2 - m_i^2) = 0 \implies \alpha_i = 0 \text{ or } q_i^2 = m_i^2,$$
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- Third representation is interesting — one can use it study the nature of the singularities. Also, reveals the existence of a different class of singularities.

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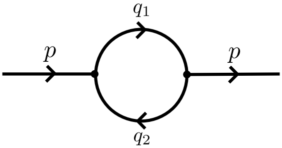
  2. For a graph with N internal propagators, $N - 1$ contractions results in a graph with trivial analyticity properties.
- Finally, as mentioned, we can also determine the nature of singularity, i.e., poles, algebraic/square root/logarithmic branch point.

The Bubble Graph

- First example of an application of the Landau equations: the bubble graph



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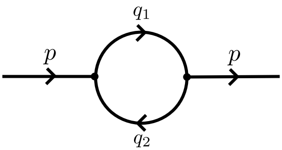

- The Landau equations for this graph read:

$$q_1^2 = m_1^2 \;\&\; q_2^2 = m_2^2,$$

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- To this end, dot the second equation with the loop momenta, and make use of conservation of 4-momentum:

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- The first singularity is exactly the normal threshold. Whereas, the second singularity is the anomalous threshold.
- One can determine the nature of the singularity as a square root branch point.

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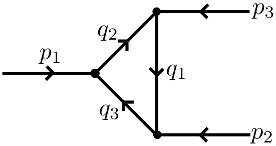
- Only the normal threshold presents itself as a cusp.
- How can one identify the actually relevant singularities?
- Something that was breifly mentioned before: we require the α's to be positive.

The Triangle Graph

- A relevant application of the Landau equations: the triangle graph.



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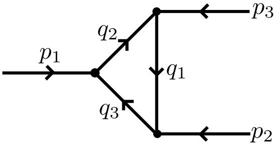

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- The above equation denotes a complex surface that can be understood for different cases.

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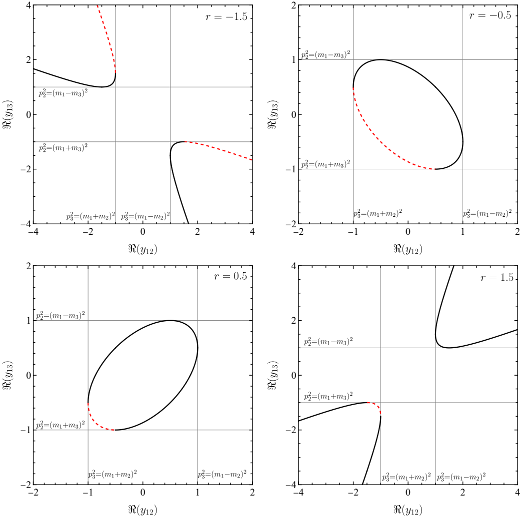
- We can attempt to gain a geometric understanding of the problem, with the imaginary parts set to 0:



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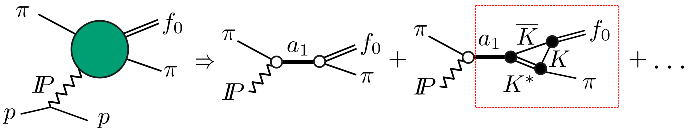

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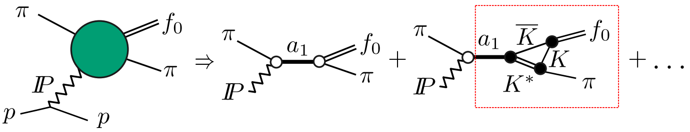

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Singularities of the $a_1(1420)$

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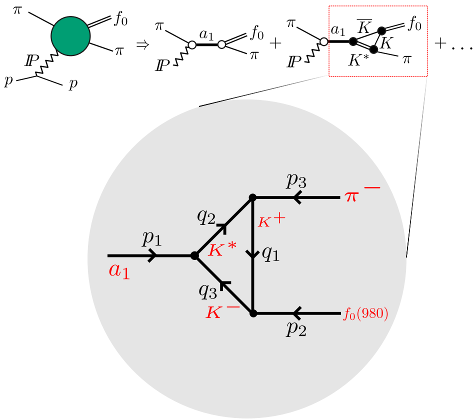

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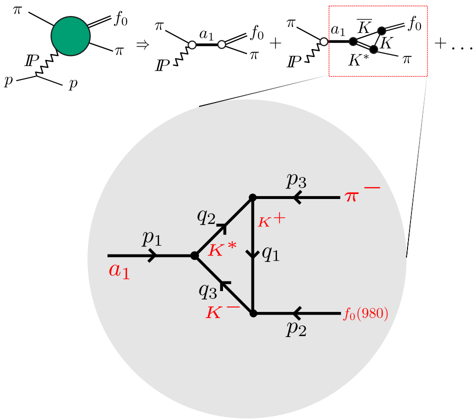

- Landau singularity for $p_3 = m_\pi$: $p_2/\mathrm{GeV} \equiv \sqrt{\sigma}/\mathrm{GeV} \in [0.986, 1.024]$ and $p_1/\mathrm{GeV} \equiv \sqrt{s}/\mathrm{GeV} \in [1.385, 1.436]$.

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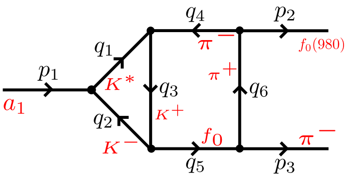

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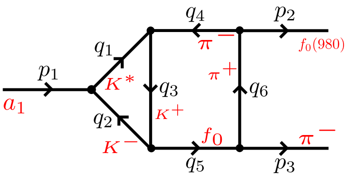

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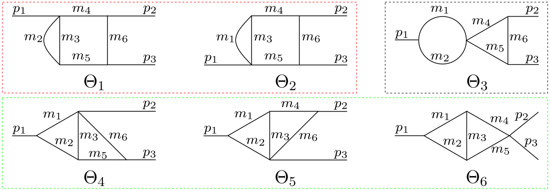

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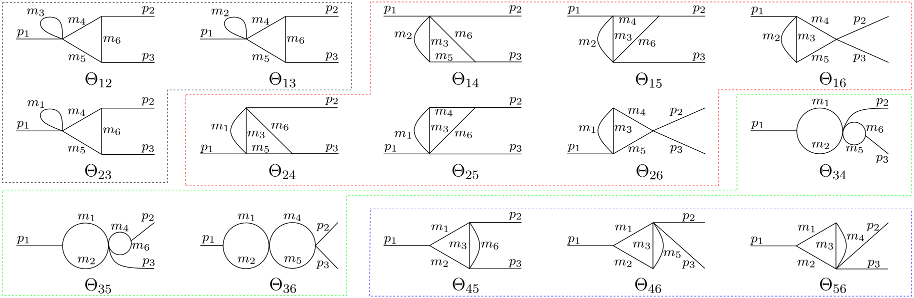

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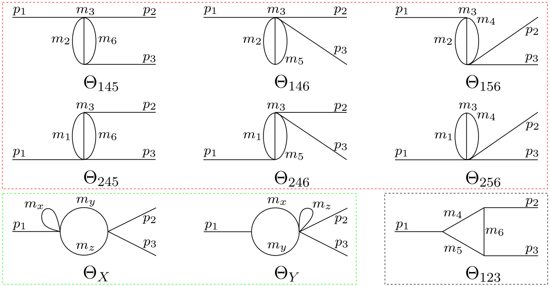

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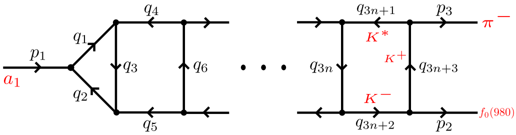
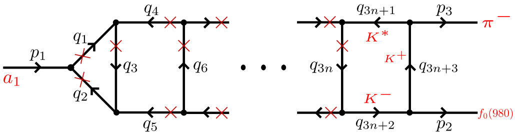
- The triangle singularity is present and no other singularities.





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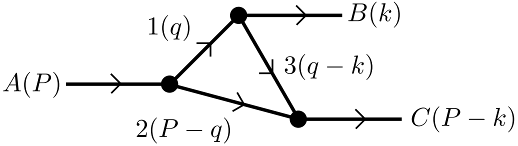

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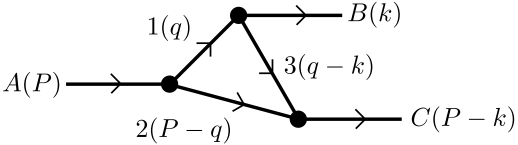
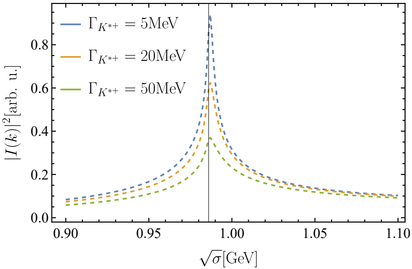
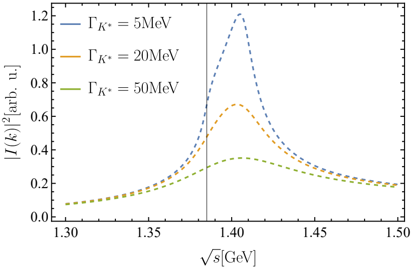


- Add an imaginary part to the masses to account for the instability (*width*) of the particles.



Amplitude squared vs. $\sqrt{\sigma}$, for $\sqrt{s} = 1.42$ GeV.



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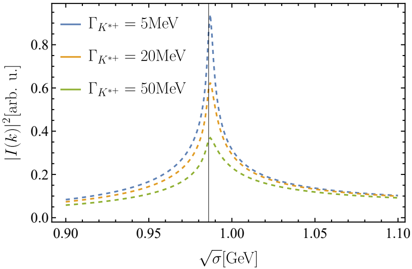

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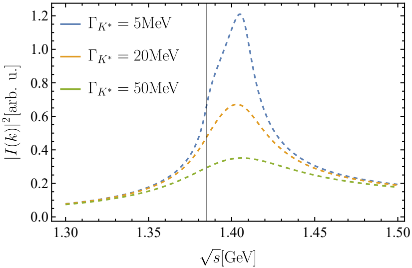

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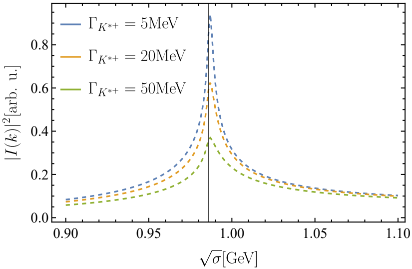

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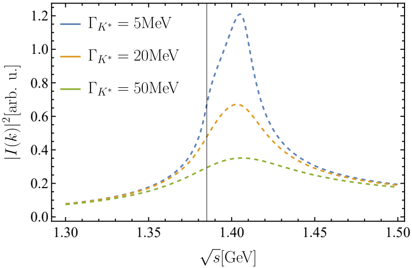

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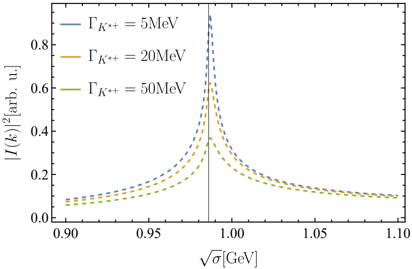

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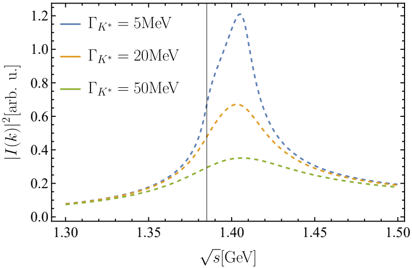

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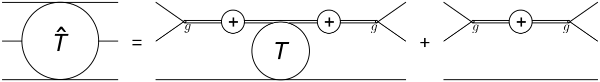

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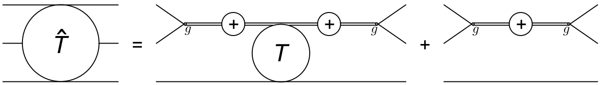

- It can be seen that the connected part contains the exchange dynamics, including the triangle diagram.

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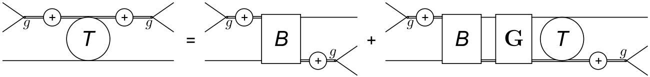
- One considers the Bethe-Salpeter ansatz, $T = B + BGT$, to determine T.[5]



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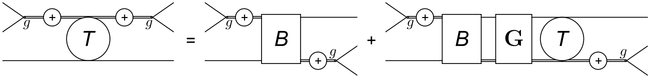

- The equation to be solved,

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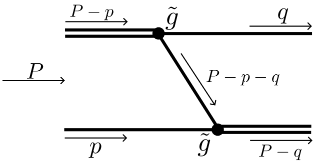

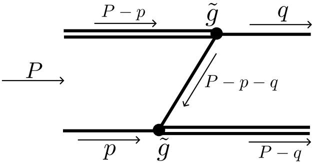

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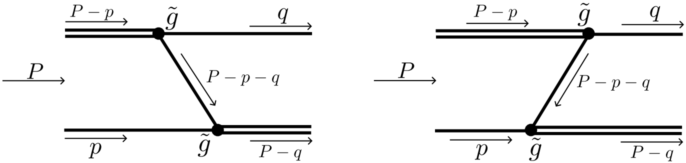

- The first diagram corresponds to a (time-)forward propagating exchange particle. A diagram of the second type can be added to the above equation to make it Lorentz covariant.

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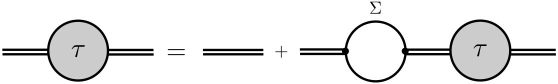

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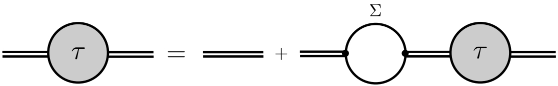

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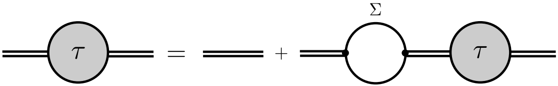

- A fine point: in principle, one also takes 3-body contact interactions into account. This amounts to replacing B with $B + C$.
- In practice, C is suitably parametrised and experimental input is used.

Back to the $a_1(1420)$

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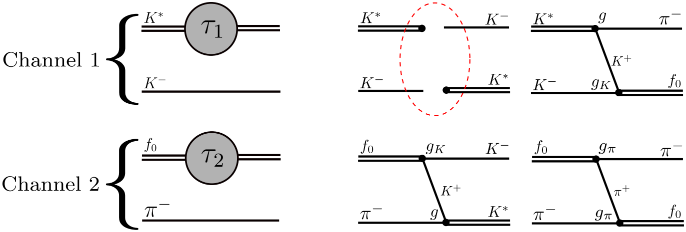
- We work with a toy model: all particles and the source are scalar; isobars and spectators are always in relative s-wave; a little sloppy with isospin invariance. Concentrate only on understanding the analytic structure within this framework.
- We have two possible isobars: K^* and f_0 *implies* need to carry out a coupled channel analysis.



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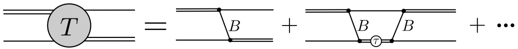
- For f_0 we also make use of g_K/g_p ratio from BaBaR collaboration.

- Further, the solution of the Bethe-Salpeter equation can be given in form of the Born series.



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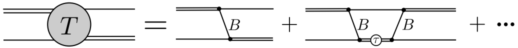

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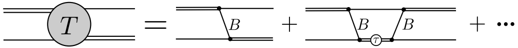

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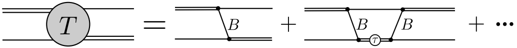

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- The integral in consideration is the Fredholm equation of the second kind.
- To note, the discrete version of the integral can be cast into a matrix equation, and then inverted to obtain the connected amplitude, given that the matrix is non-singular, i.e.,

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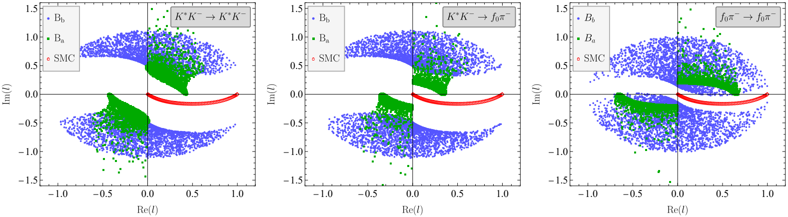

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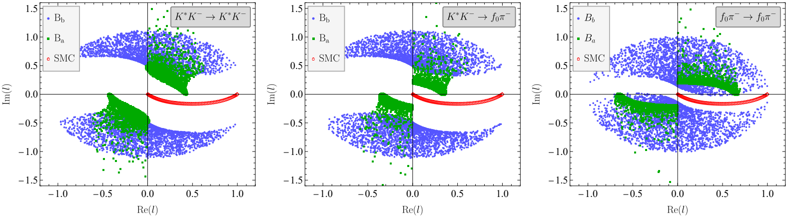

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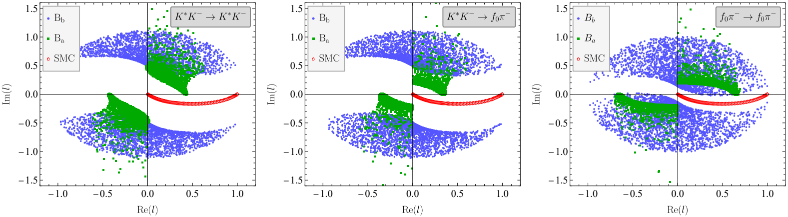

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- But, we need the amplitudes for $q \in \mathbb{R} \rightarrow$ can be done in different ways!

- Step 2.1: **Rational Analytic Continuation**: We use a continued fraction to obtain the isobar-spectator amplitude for $q \in \mathbb{R}$.

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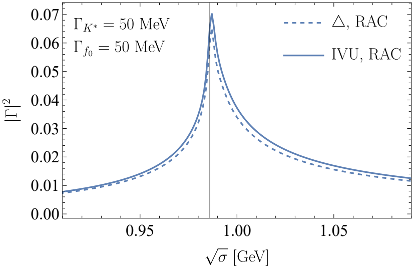
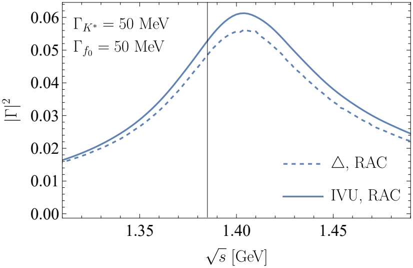
- Why not Padé? Padé was unable to reproduce sharper features like cusps.

- At $1 + \infty$-loop level,





Amplitude squared vs. $\sqrt{\sigma}$, for $\sqrt{s} = 1.42$ GeV.

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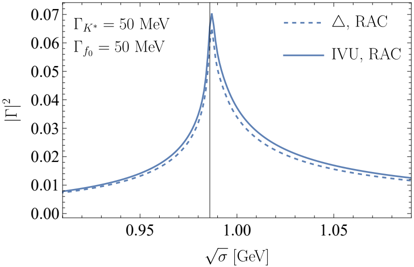

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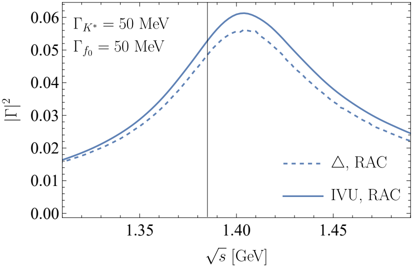

Amplitude squared vs. $\sqrt{s}$, for $\sqrt{\sigma} = 0.99$ GeV.

- We see the Landau singularity in both the plots.

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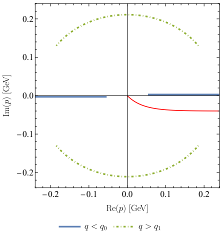
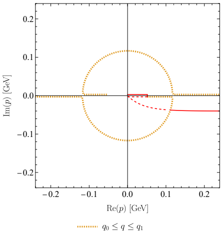
- To expound, instead of a continued fraction, use the knowledge of the location of the branch cuts to analytically continue the amplitude for $q \in \mathbb{R}$.[7]



Branch cuts of B for q outside the triangle singularity region.



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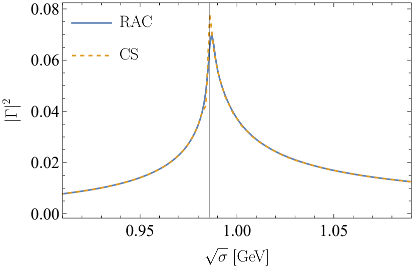
[7] For a comprehensive review: *The Quantum Mechanical Three-Body Problem*, Schmid & Ziegelmann.

- Matches with the other method.



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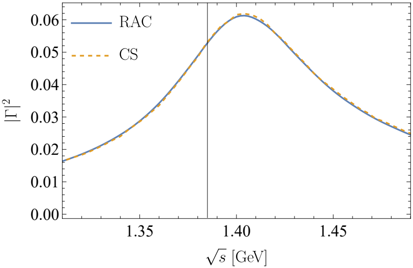

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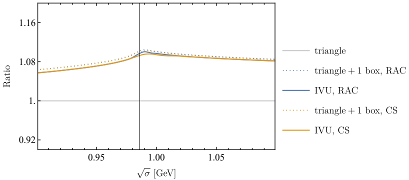
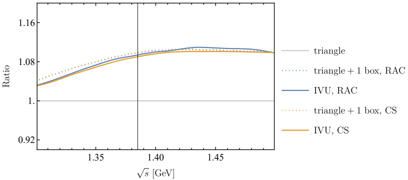


Ratios of amplitudes to triangle amplitudes vs. $\sqrt{\sigma}$, for $\sqrt{s} = 1.42$ GeV.



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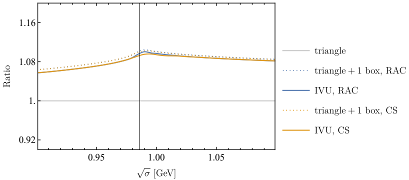

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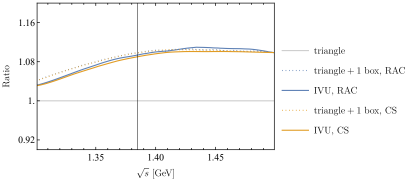

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- The amplitudes at triangle $+1$ box are about 1.1x the triangle diagram amplitudes.

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[8] 10.1140/epja/i2017-12440-1

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Thanks for listening!