Functions

Functions vs. Relations

- A "relation" is just a relationship between sets of information.

- A “function” is a well-behaved relation, that is, given a starting point we know exactly where to go.
- A function is a special kind of relation where every input goes to exactly one output
  – It’s PREDICTABLE!

Definition

Given any sets *A*, *B*, a function *(or "mapping")* *f* from *A* *to* *B* (f:$A \rightarrow B$) is an assignment of **exactly one** element $\boldsymbol{f(x) \in B}$ to each element $\boldsymbol{x \in A}$.

Let *X* and *Y* be two nonempty sets.

A function from *X* into *Y* is a relation that associates with each element of *X* exactly one element of *Y*.

- Not every relation is a function(many to one allowed,but one to many not allowed,i.e., not a function.
- Every function is a relation.

So, in order to show that a relation f from A to B is a function, we must verify the following two conditions.

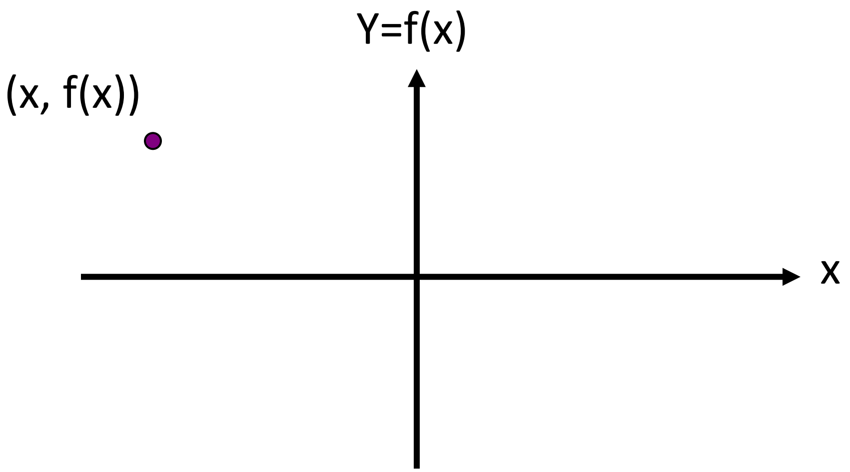
1. The domain of f is A, which means that every element of A has some image in B, and
2. an element of A cannot have more than one image in B.

Representations of Functions

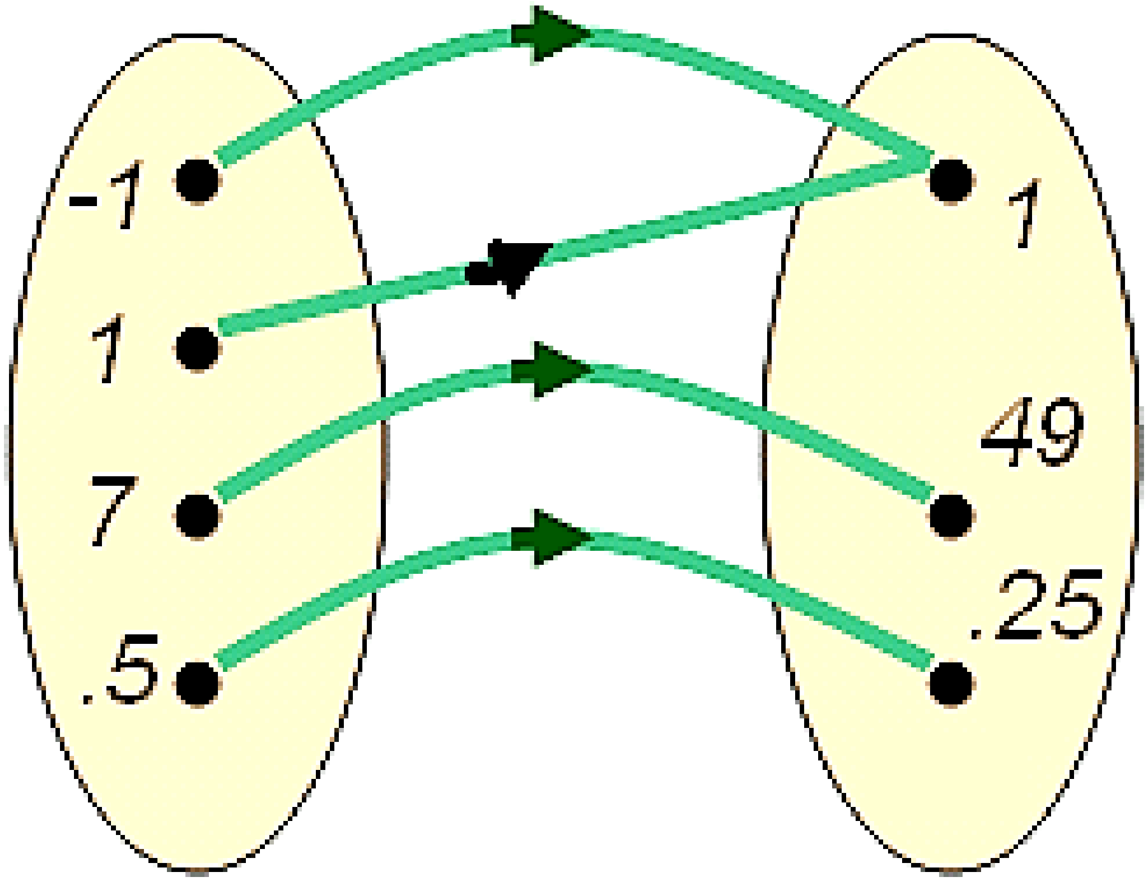
- Verbally
- Numerically, i.e. by a table
- Visually, i.e. by a graph/arrow diagram
- Algebraically, i.e. by an explicit formula

- Functions can be represented graphically in several ways:

- Variable x is called independent variable
- Variable y is called dependent variable
- For convenience, we use f(x) instead of y.
- The ordered pair in new notation becomes:
    - (x, y) = (x, f(x))



Arrow diagram representation of a function $f(x)=x^2$



Domain and Range

- Suppose, we are given a function from X into Y.
- Recall, for each element x in X there is exactly one corresponding element y=f(x) in Y.
- This element y=f(x) in Y we call the image of x.
- The domain of a function is the set X. That is a collection of all possible x-values.
- The range of a function is the set of all images as x varies throughout the domain.
- Y is the codomain.
- Domain- All inputs and Range- All outputs

If $f{:}A \to B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then:

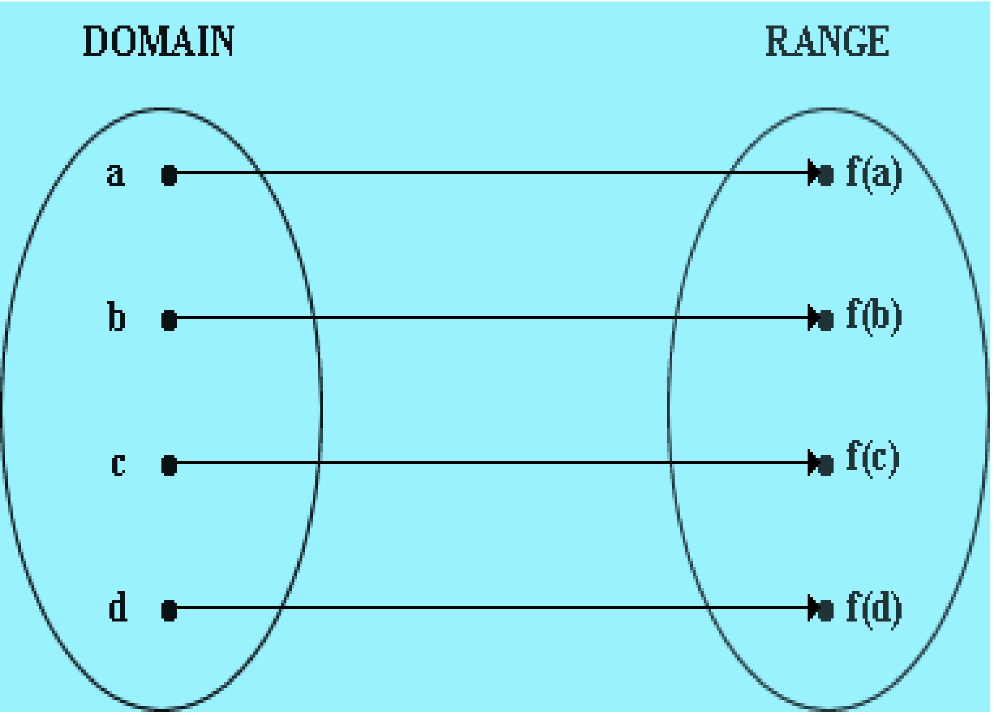
A is the *domain* of f.
B is the *codomain* of f.
b is the *image* of a under f.
a is a *pre-image* of b under f.

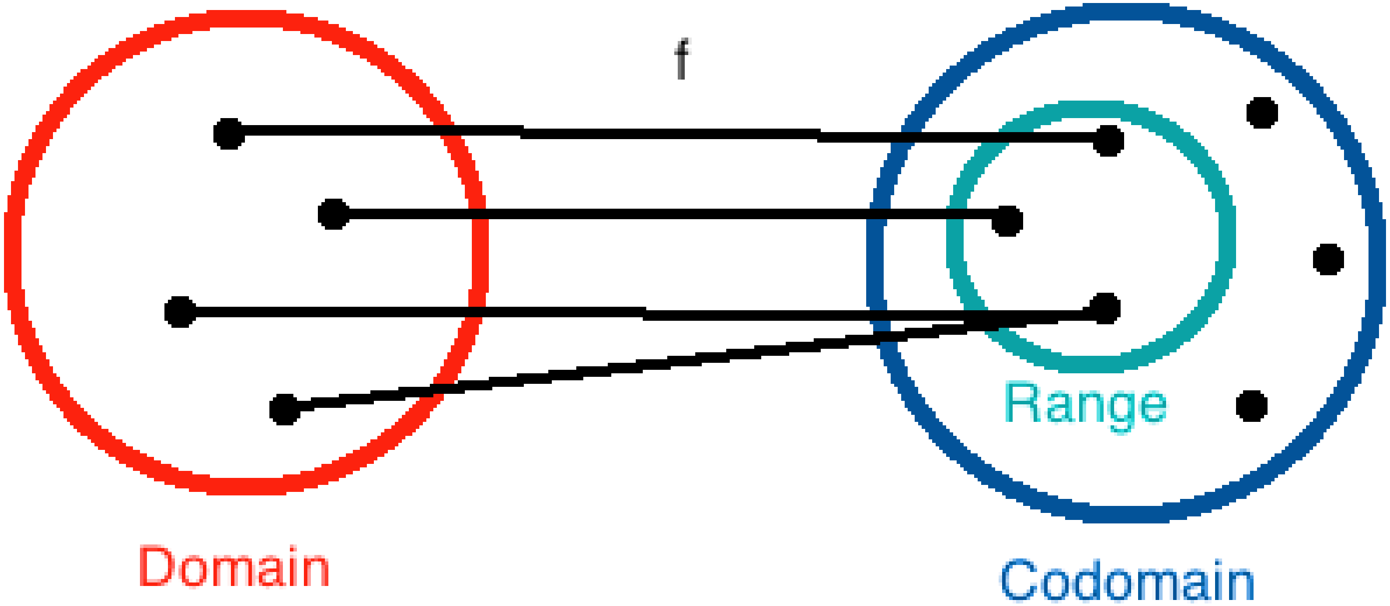
In general, b may have more than one pre-image

Range vs. Codomain - Example

Suppose that: *"f is a function mapping students in this class to the set of grades {A,B,C,D,E}."*
At this point, we know *f*'s co-domain is: ___________, and its range is __________ and it can be__________.

Suppose the grades turn out all A and B.
Then the range of *f* is ___________, but its co-domain is _______________________.

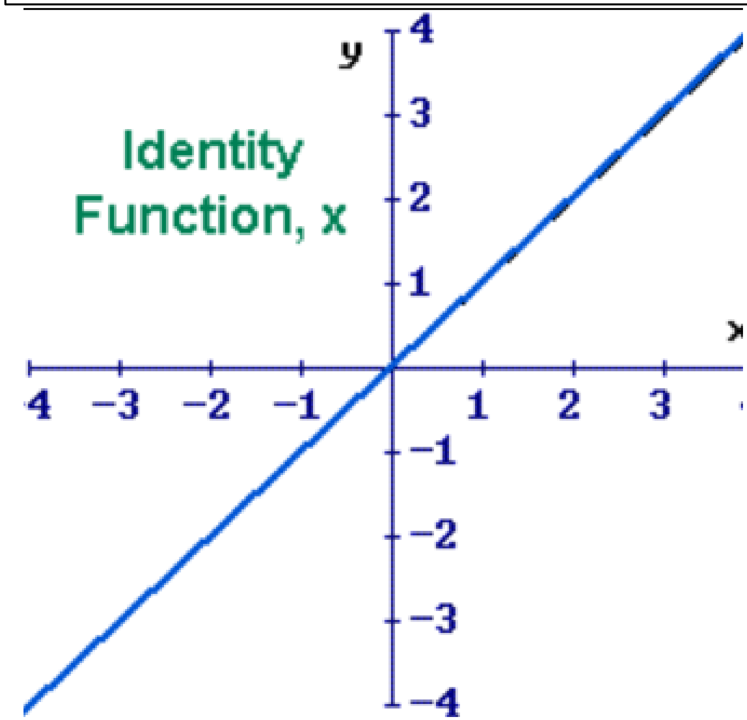

DOMAIN
RANGE
a
b
c
d
f(a)
f(b)
f(c)
f(d)

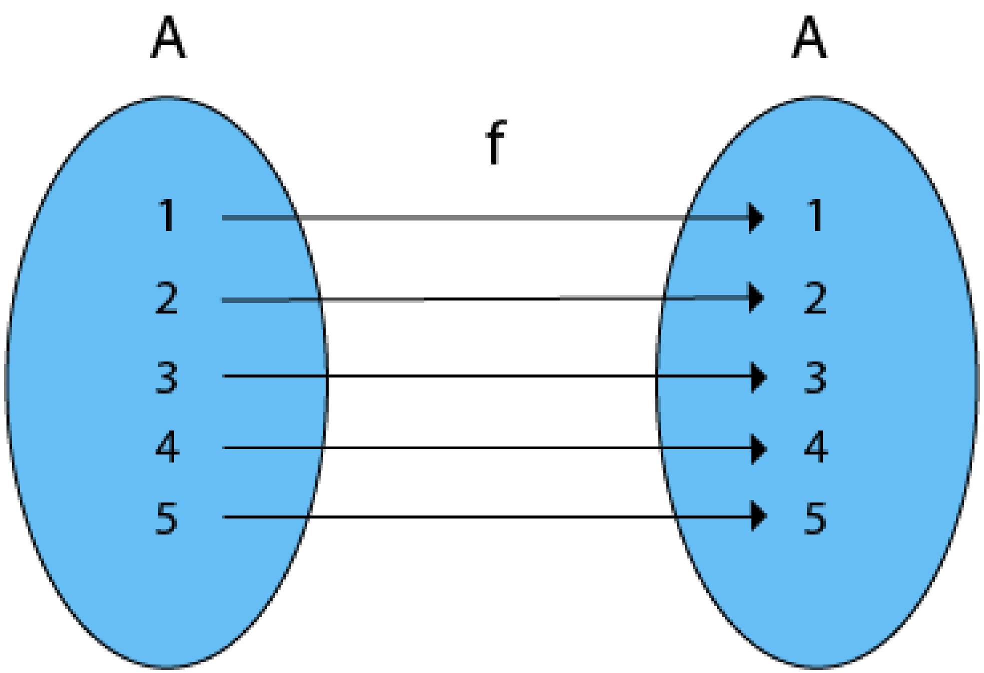

f
Range
Domain
Codomain

Special functions

Identity function

A function f: A→A is said to be the **identity function** if f(x)=x for all x ∈ A. this type of function is usually denoted by i_A

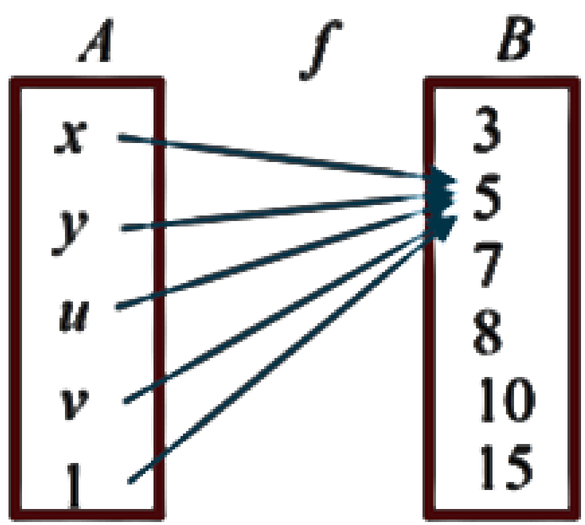


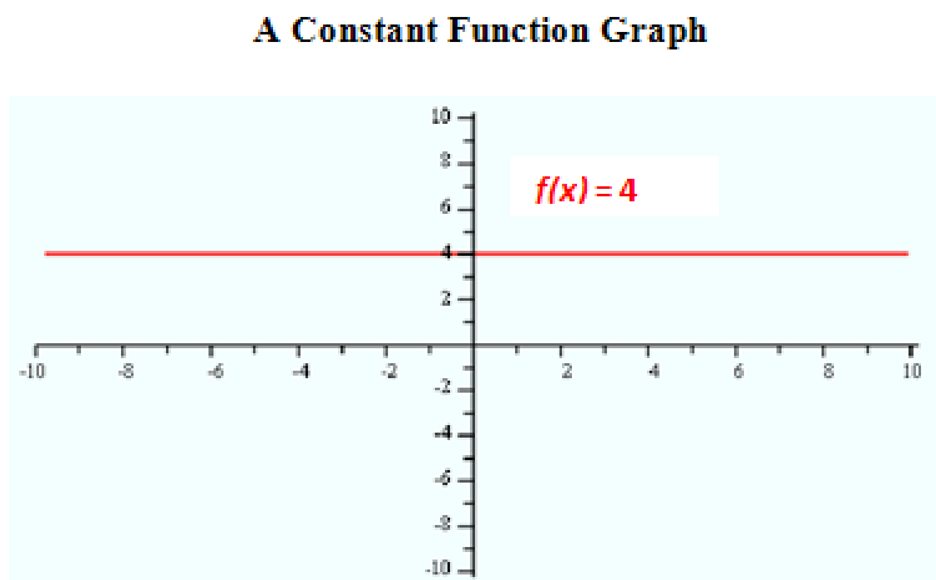


Constant function

A function f: A→B is said to be the **constant function** If there exists b ∈ B such that f(x)=b for all x ∈ A. That is, all element of A are mapped to only one element of B(constant).

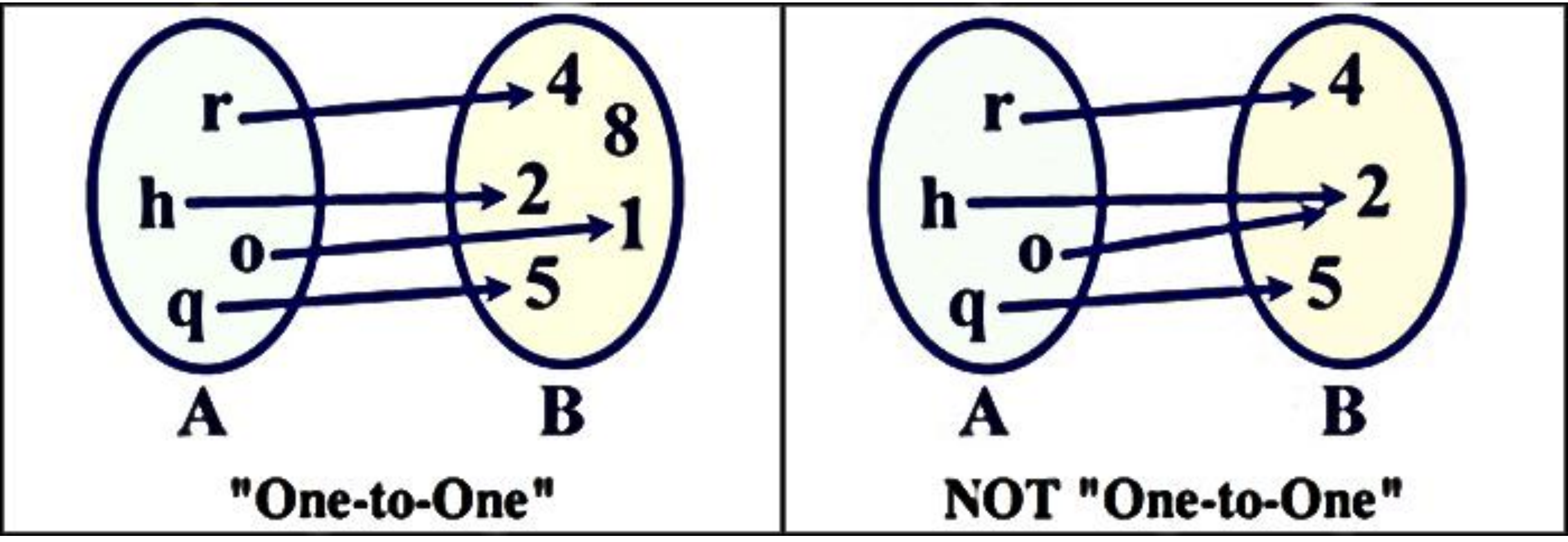
Under a constant function, every element of domain set goes to some fixed element in the codomain.





One-to-One Functions

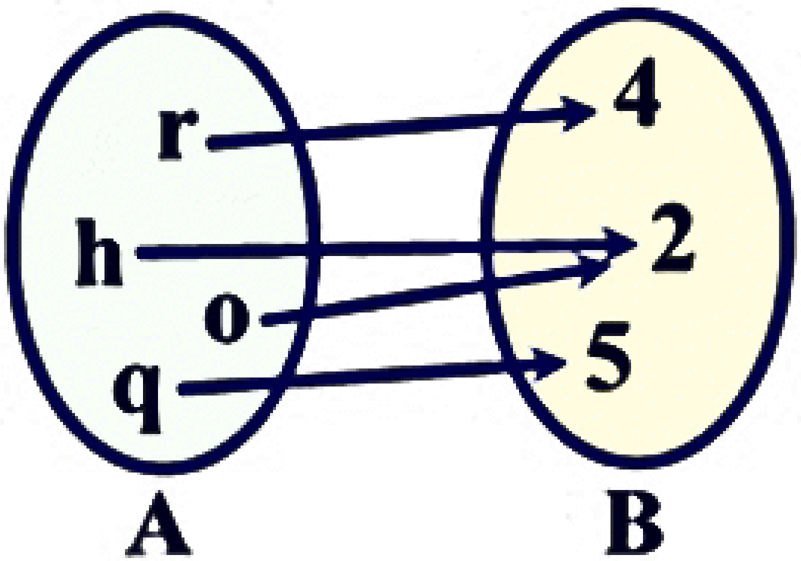
- A function is *one-to-one (1-1)*, or *injective*, or *an* *injection*, iff every element of its range has **only one** pre-image.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.



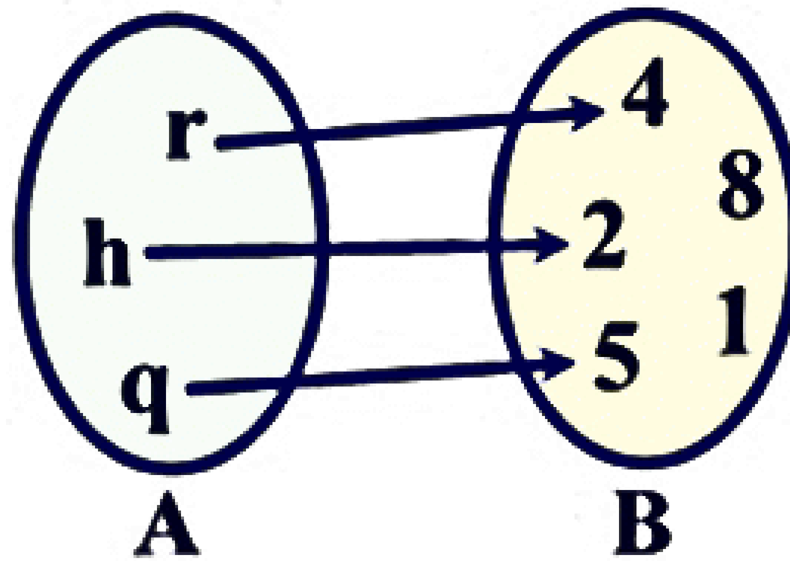
Onto function(surjective/surjection)

- A **function** f: A -> B is called an **onto function** if the range of function f is entire set B.
- In other words, if each b ∈ B there exists at least one a ∈ A such that f(a) = b, then f is an onto **function**.
- An **onto function** is also called surjective **function**.

➢ In short, for a function to be **onto** every output must have an input

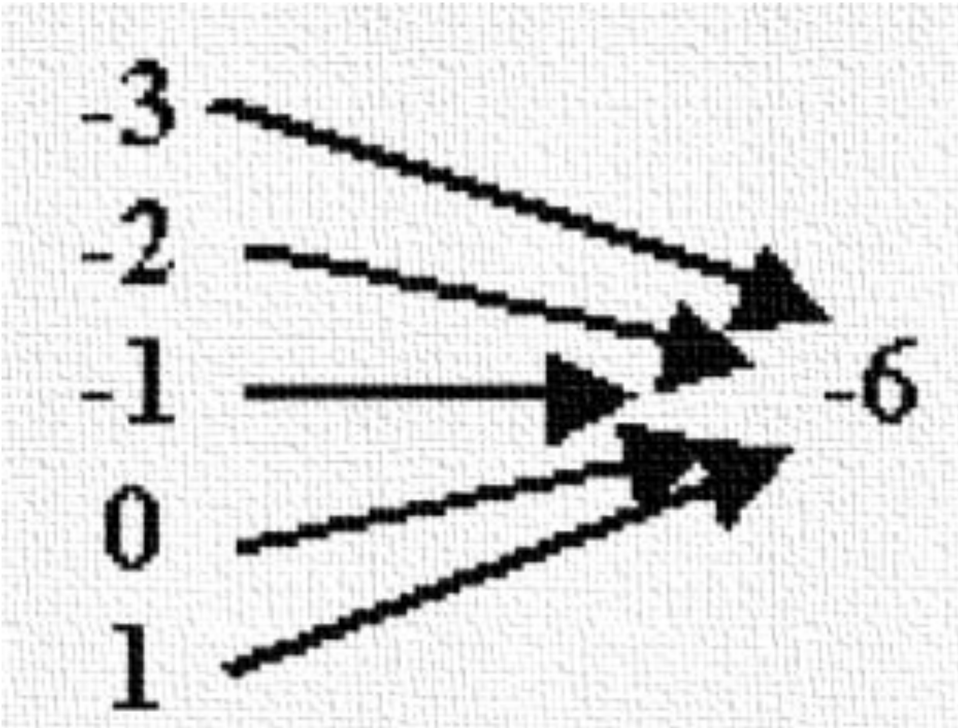


"Onto"



NOT "Onto"

Your task!!



- Domain?
- Range?
- Function?
- One to one?
- Onto?

Bijective/bijection function

A function f is *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff **it is both one-to-one and onto**.

DEFINITION 5.1.5 ▸ Let A and B be sets and $f : A \to B$. Then

(i) f is called **one-one** (or **injective** or **injection**) if for all $a_1, a_2 \in A$,

$$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$$

(i.e., images of distinct elements of the domain are distinct).

(ii) f is called **onto** B (or **surjective** or **surjection**) if for every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$, i.e.,

$$\mathrm{Im}(f) = B.$$

(iii) f is called **one-to-one correspondence** (or **bijective** or **bijection**) if f is both one-one and onto.

Composition of functions

DEFINITION 5.1.6 ▶ Let $f : A \to B$ and $g : B \to C$ be functions. The **composition** of f and g, written $g \circ f$, is the function from A to C defined as

$$(g \circ f)(a) = g(f(a)), \quad \text{for all } a \in A.$$

We sometimes write the composition, $g \circ f$, of the function f and g as gf.

Let $f : A \to B$ and $g : B \to C$ be functions. Pictorially, $g \circ f$ is described in Figure 5.13.

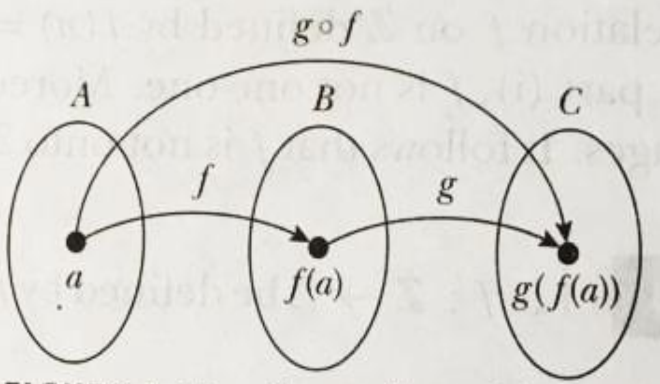


FIGURE 5.13 Composition of the functions f and g

Your task!!

- Given $f(x) = 5x + 8$ and $g(x) = 2$, find $f(g(x))$.

- Given $f(x) = x^2 + x$ and $g(x) = x - 4$, find $f(g(x))$ and $g(f(x))$.

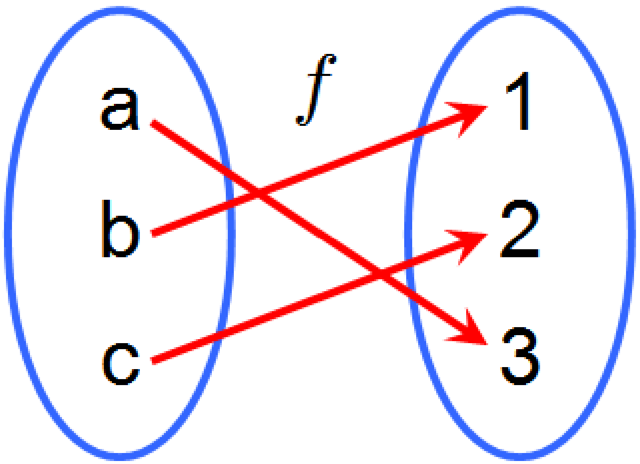
- Given $f(x) = 2x + 5$ and $g(x) = 8 + x$, find $f(g(-2))$.

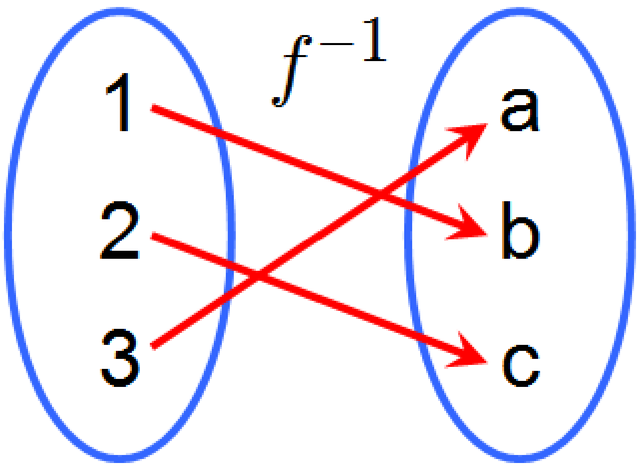
Inverse of a function

- Quick review of inverse of a relation.
- {(2, 3), (5, 0), (-2, 4), (3, 3)}
- Domain & Range = ?
- Inverse = ?
- D = {2, 5, -2, 3}
- R = {3, 0, 4}

Inverse of a function

- {(2, 3), (5, 0), (-2, 4), (3, 3)}
- Inverse = switch the x and y(reverse the order of ordered pairs in a relation, (domain and range)
- Inverse = {(3, 2), (0, 5), (4, -2), (3, 3)}





Right and left Inverse of a function

DEFINITION 5.2.1 ▶ Let $f : A \to B$ be a function from the set A into the set B.

(i) f is called **left invertible** if there exists $g : B \to A$ such that $g \circ f = i_A$. Moreover, if such a function g exists, then g is called a **left inverse** of f.

(ii) f is called **right invertible** if there exists $h : B \to A$ such that $f \circ h = i_B$. Moreover, if such a function h exists, then h is called a **right inverse** of f.

(i) g(f(a))=a right inverse

(ii) f(g(b))=b left inverse

Floor and Ceiling functions

Floor function: for any real number x, the floor of x, written $\lfloor X \rfloor$,is the greatest integer less than or equal to x.

Ceiling function: for any real number x, the ceiling of x, written $\lceil X \rceil$,is the greatest integer less than or equal to x.

Floor and Ceiling functions

$\lfloor X \rfloor$ Floor function: the largest integer $\leq$ X

$\lfloor 2.7 \rfloor = 2$ $\lfloor -2.7 \rfloor = -3$ $\lfloor 2 \rfloor = 2$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq$ X

$\lceil 2.3 \rceil = 3$ $\lceil -2.3 \rceil = -2$ $\lceil 2 \rceil = 2$

Note: floor and ceiling functions are neither one-one nor onto.

Cardinality of a Set

Definition: The number of elements in a set.

Let A be a set.

- If $A = \varnothing$ (the empty set), then the cardinality of A is 0.

- b. If A has exactly n elements, n a natural number, then the cardinality of A is n. The set A is a *finite set.*

- c. Otherwise, A is an *infinite set.*

Notation

The cardinality of a set A is denoted by $| A |$.

- If $A = \varnothing$, then $| A | = 0$.
- If A has exactly n elements, then $| A | = n$.
- c. If A is an infinite set, then $| A | = \infty$.

Cardinality of a Set

Example	Cardinality
$A = \{ 5 \}$	$\|A\| = 1$
$B = \{ 7,2 \}$	$\|B\| = 2$
$C = \{ 1,3,4 \}$	$\|C\| = 3$
$D = \{ 9,1,5,8 \}$	$\|D\| = 4$
$E = \{ 5,5,5,5,5 \}$	$\|E\| = 1$

Note: Let A and B be sets. Then, $|A| = |B|$ if and only if there is a one-to-one correspondence between the elements of A and the elements of B.

Example:

$$A = \{1, 2, 3, 4, 5\}, \quad B = \{a, e, i, o, u\}$$

$$1 \to a,\ 2 \to e,\ 3 \to i,\ 4 \to o,\ 5 \to u;$$

$$|A| = |B| = 5$$