AE 706

Numerical Solution of Flow through CD Nozzle using van Leer Flux Vector Splitting Method

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1 Problem Statement

Solve the quasi-one-dimensional Euler equations for the isentropic flow of a calorically perfect gas with specific heat ratio $\gamma = 1.4$ through a nozzle (as shown in Figure **??**) having an area distribution $A(x)$

$$A(x) = 1.0 + 2.0(x-1)^2, \quad 0 \le x \le 2 \tag{1}$$

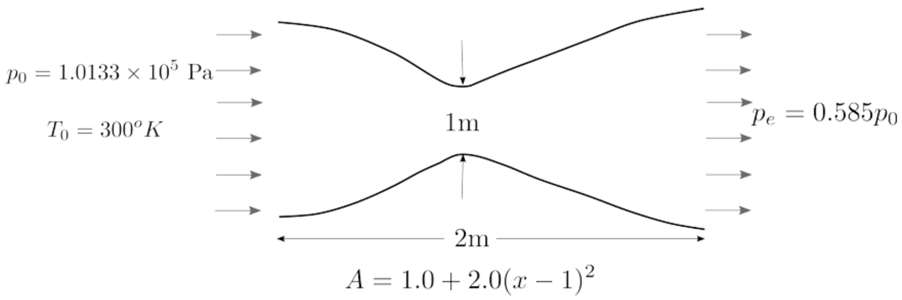


Figure 1: Flow through CD Nozzle

Solve the quasi-1D nozzle flow using van Leer Flux Vector Splitting method with:

- Exit to stagnation pressure ratio: $p_e/p_0 = 0.585$
- Reservoir temperature: $T_0 = 300$ K
- Reservoir pressure: $p_0 = 1.0133 \times 10^5$ Pa

Governing Equations

The quasi-1D Euler equations in conservative form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S} \tag{2}$$

where:

$$\mathbf{U} = \begin{bmatrix} \rho A \\ \rho u A \\ E A \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u A \\ (\rho u^2 + p)A \\ u(E+p)A \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ p\frac{\partial A}{\partial x} \\ 0 \end{bmatrix}$$

Numerical Scheme

Van Leer Flux Vector Splitting method:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_i^{+,n} - \mathbf{F}_{i-1}^{+,n}\right) - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1}^{-,n} - \mathbf{F}_i^{-,n}\right) + \Delta t \mathbf{S}_i \tag{3}$$

Source term discretization:

$$\mathbf{S}_i = \begin{bmatrix} 0 \\ p_i \frac{A_{i+1} - A_{i-1}}{2\Delta x} \\ 0 \end{bmatrix}$$

Van Leer Split Fluxes

- For $M \leq -1$:
$$\mathbf{F}^+_{VL} = 0, \quad \mathbf{F}^-_{VL} = \mathbf{F}$$

- For $-1 < M < 1$:
$$\mathbf{F}^+_{VL} = \frac{1}{4}\rho a (M+1)^2 A \begin{bmatrix} 1 \\ \frac{2a}{\gamma}\left(1 + \frac{\gamma-1}{2}M\right) \\ \frac{2a^2}{\gamma^2-1}\left(1 + \frac{\gamma-1}{2}M\right)^2 \end{bmatrix}, \quad \mathbf{F}^-_{VL} = \mathbf{F} - \mathbf{F}^+_{VL}$$

- For $M \geq 1$:
$$\mathbf{F}^+_{VL} = \mathbf{F}, \quad \mathbf{F}^-_{VL} = 0$$

where $a = \sqrt{\gamma R T}$ is the speed of sound ($\gamma = 1.4$ for air).

Numerical Implementation

- Grid: $I_{max} = 101$ points
- Time step:
$$\Delta t = \nu \frac{\Delta x}{\lambda_{max}}, \quad \lambda_{max} = \max\left(|u_i| + a_i\right)$$
with CFL number $\nu \leq 1$ for stability.
- Initialization: Flow field initialized based on inlet conditions
- Boundary conditions:
  - Inlet: Stagnation conditions (T_0, p_0), velocity extrapolated
  - Outlet: Fixed pressure $p_e = 0.585 p_0$, density and velocity extrapolated

Required Results

1. Plot of p/p_0 versus x
2. Plot of Mach number versus x
3. Comparison with exact quasi-1D solution

2 Results

2.1 Comparison of numerical results with exact solution

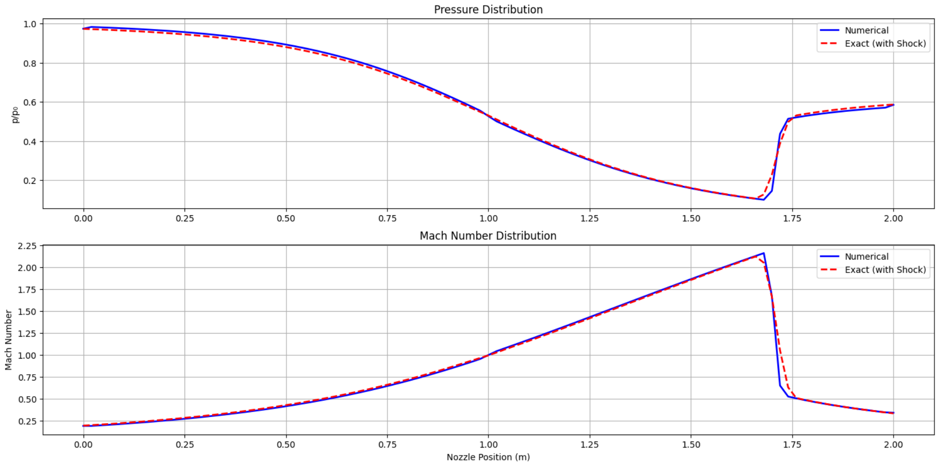


Figure 2: Comparison of numerical results with exact solution

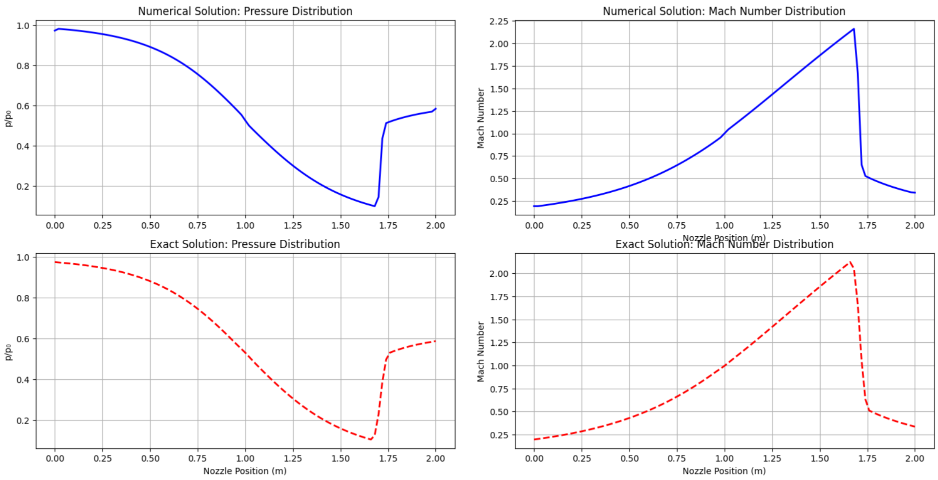


Figure 3: Individual graph of numerical results with exact solution

Table 1: Comparison of Numerical and Exact Solutions at Selected Locations

x (m)	$A(x)$	p/p_0 (num)	p/p_0 (exact)	M (num)	M (exact)
0.0000	3.0000	0.9740	0.9732	0.1945	0.1974
0.5000	1.5000	0.8929	0.8803	0.4180	0.4305
1.0000	1.0000	0.5273	0.5306	1.0002	0.9965
1.5000	1.5000	0.1588	0.1604	1.8632	1.8540
2.0000	3.0000	0.5850	0.5866	0.3443	0.3370

2.2 1. Pressure Distribution (p/p_0)

Pre-Shock Region ($x <\sim 1.5$ m):
The numerical solution (blue solid line) and the exact solution (red dashed line) show good agreement in the converging section ($x < 1.0$ m) and part of the diverging section ($x \approx 1.0$–1.5 m). Both exhibit a gradual decrease in pressure as the flow accelerates, consistent with isentropic expansion in a converging-diverging nozzle.
Minor deviations are observed, likely due to numerical diffusion or the finite grid resolution ($\Delta x = L/(I_{\max} - 1)$), which smooths the transition compared to the exact solution.

Shock Region ($x \approx 1.5$ m):
The numerical solution captures the shock as a smeared transition rather than a sharp discontinuity, which is characteristic of the Van Leer FVS scheme's upwind nature. This smearing is evident as a gradual pressure rise starting around $x = 1.5$ m, peaking near $x = 1.6$ m.
The exact solution shows a distinct jump at the shock location (around $x = 1.5$ m), followed by a rapid adjustment to the post-shock pressure. The numerical shock location aligns closely with the exact solution, indicating accurate prediction of the shock position based on the exit pressure ratio ($p_e/p_0 = 0.585$).

Post-Shock Region ($x > 1.5$ m):
After the shock, the numerical pressure rises to match the imposed exit pressure, but it exhibits oscillations or a slightly broader profile compared to the exact solution. This suggests some numerical dissipation or instability in the FVS scheme near the shock.
The exact solution stabilizes quickly to a constant pressure ratio (~ 0.6), consistent with the specified exit Mach number ($M = 0.337$) and boundary condition.

2.3 2. Mach Number Distribution

Pre-Shock Region ($x <\sim 1.5$ m):
Both solutions show a similar trend: the Mach number increases from a subsonic value (near 0.25) at the inlet, reaches unity at the throat ($x = 1.0$ m), and continues to rise into the supersonic regime in the diverging section. The numerical and exact curves are closely aligned, validating the isentropic acceleration modeled by both approaches.

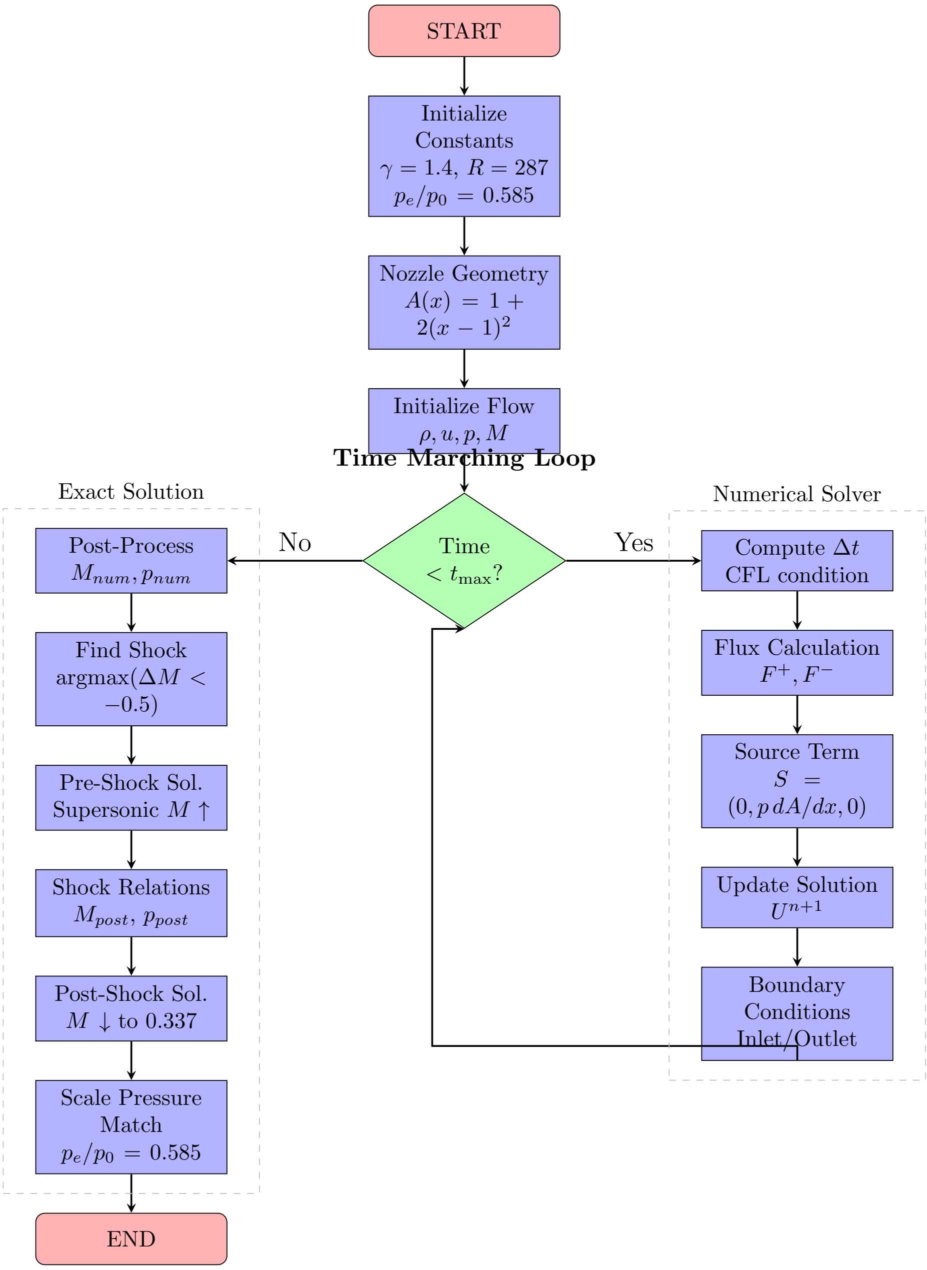
Shock Region ($x \approx 1.5$ m):
The numerical solution depicts the shock as a gradual drop in Mach number from a supersonic value (~ 1.8) to a subsonic value (~ 0.5), again due to the diffusive nature of the Van Leer FVS scheme. The transition is spread over a few grid points.
The exact solution shows a sharp decrease at the shock, dropping from approximately 1.8 to 0.5, reflecting the ideal normal shock relations. The shock location matches well between the two, confirming the numerical method's ability to predict the shock position.

Post-Shock Region ($x > 1.5$ m):
Post-shock, the numerical Mach number decreases gradually toward the exit value ($\sim$ 0.337), with some minor fluctuations, possibly due to boundary condition enforcement or numerical artifacts.
The exact solution smoothly approaches the specified exit Mach number (0.337), demonstrating a more stable subsonic deceleration. The numerical solution's slight deviation suggests that the boundary condition implementation or time-stepping may introduce small errors.

3 Conclusion

The Van Leer FVS numerical solution provides a reasonable approximation of the exact solution for the nozzle flow with a shock, accurately predicting the shock location and overall flow trends. However, the scheme's inherent diffusion leads to smeared shocks and minor post-shock oscillations, differing from the sharp transitions and stability of the exact solution. With refinements in grid resolution and shock-capturing techniques, the numerical results could be brought closer to the exact solution, enhancing its utility for engineering applications.

4 Flowchart


START
Initialize Constants
$\gamma = 1.4,\ R = 287$
$p_e/p_0 = 0.585$
Nozzle Geometry
$A(x) = 1 + 2(x-1)^2$
Initialize Flow
ρ, u, p, M
Time Marching Loop
Time $< t_{\max}$?
No
Yes
Exact Solution
Post-Process
M_{num}, p_{num}
Find Shock
argmax($\Delta M < -0.5$)
Pre-Shock Sol.
Supersonic $M \uparrow$
Shock Relations
M_{post}, p_{post}
Post-Shock Sol.
$M \downarrow$ to 0.337
Scale Pressure
Match
$p_e/p_0 = 0.585$
END
Numerical Solver
Compute Δt
CFL condition
Flux Calculation
F^+, F^-
Source Term
$S = (0, p\,dA/dx, 0)$
Update Solution
U^{n+1}
Boundary Conditions
Inlet/Outlet

5 Reference

1. Charles Hirsch, *Numerical Computation of Internal and External Flows*, Volume II, Wiley, 1990. (Section 20.2.3, pp. 420–425)