AE 706

Assignment 5: Computation of Lid-Driven Cavity Flow Using Vorticity-Stream Function Formulation

Instructor
Prof. J.C Mandal
Department of Aerospace Engineering

Submitted by
ROLL NO: 24M0006
PRAJAPATI AJAYKUMAR NANDUBHAI
Department of Aerospace Engineering

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1 Problem Statement

The goal of this assignment is to numerically solve the steady-state lid-driven cavity (as shown in Figure 1) flow using the Vorticity-Stream Function formu lation. The flow is incompressible and two-dimensional. The velocity field is derived from the stream function, and the vorticity transport equation governs the evolution of vorticity in the domain

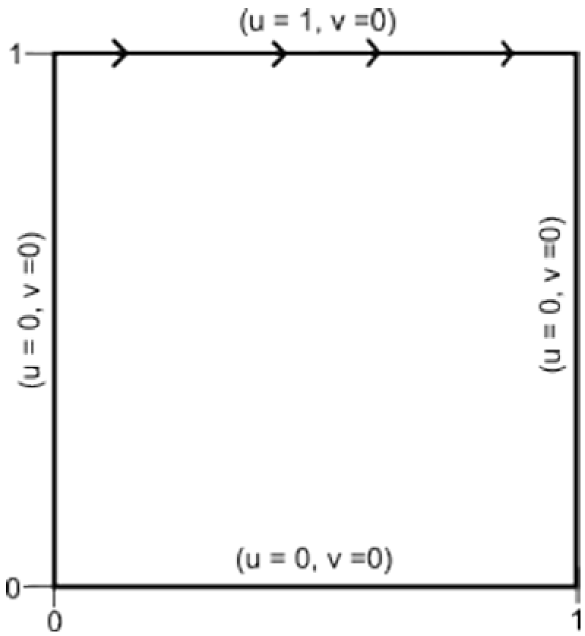

Figure 1: Lid driven cavity

The problem involves computing the flow field for a Reynolds number of 100, where the flow is governed by the Navier-Stokes equations expressed in terms of vorticity and stream function.

2 Governing Equations

The Navier-Stokes equations in terms of vorticity (ω) and stream function (ψ) are given as:

2.1 Vorticity Transport Equation

The vorticity transport equation is given by:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \tag{1}$$

where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{(vorticity)} \tag{2}$$

and

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{(velocity components)} \tag{3}$$

Here, ν is the kinematic viscosity.

2.2 Stream Function Equation (Poisson Equation)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} = -\omega \tag{4}$$

This equation ensures that the velocity field satisfies the incompressibility condition

3 Boundary Conditions

3.1 Lid (Top Wall, $y = 1\,\mathrm{m}$)

Driven by the lid:

$$\hat{u} = 1\,\mathrm{m/s}, \quad \hat{v} = 0\,\mathrm{m/s}$$

Imposed via vorticity:

$$\omega_{i,J} = -\frac{2(\psi_{i,J-1} - \psi_{i,J})}{\Delta y^2} - \frac{2\hat{u}_{i,J}}{\Delta y}$$

where an $I \times J$ grid is used; $i = 1, 2, \ldots, I$ and $j = 1, 2, \ldots, J$.

3.2 Bottom and Side Walls ($y = 0\,\mathrm{m}$, $x = 0\,\mathrm{m}$, $x = 1\,\mathrm{m}$)

No-slip condition:

$$\hat{u} = 0\,\mathrm{m/s}, \quad \hat{v} = 0\,\mathrm{m/s}$$

Imposed via vorticity:

$$\begin{aligned}
\text{Left wall} \quad & \omega_{1,j} = -\frac{2(\psi_{2,j} - \psi_{1,j})}{\Delta x^2} \\
\text{Right wall} \quad & \omega_{I,j} = -\frac{2(\psi_{I-1,j} - \psi_{I,j})}{\Delta x^2} \\
\text{Bottom wall} \quad & \omega_{i,1} = -\frac{2(\psi_{i,2} - \psi_{i,1})}{\Delta y^2}
\end{aligned}$$

4 Convergence Criteria

The residual $R_{i,j}$ at an internal grid point is the difference between the left-hand side (LHS) and the right-hand side (RHS) of the discretized streamfunction equation:

$$R_{i,j} = \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} + \omega_{i,j} \tag{5}$$

A global residual norm is used to check for convergence. The most common choice is the **Root Mean Square (RMS) Residual**, defined as:

$$R_2 = \sqrt{\frac{1}{N}\sum_{i=2}^{I-1}\sum_{j=2}^{J-1} R_{i,j}^2}, \quad \text{where} \quad N = (I-2)\times(J-2) \tag{6}$$

Terminate the **Jacobi** or **Gauss-Seidel** iteration when $R_2 \leq 10^{-2}$, as this level of residual reduction is sufficient to obtain steady-state results.

4.1 Velocity Residuals (Time Iteration)

Terminate the **time iteration** when the **Root Mean Square (RMS) residuals** of the velocity components u and v fall below a threshold (i.e., 10^{-8}).

The RMS residual of a variable f (representing either u or v) is defined as:

$$\text{RMS}_f = \sqrt{\frac{1}{N}\sum_{i=2}^{I-1}\sum_{j=2}^{J-1}\left(f_{i,j}^{n+1} - f_{i,j}^{n}\right)^2} \tag{7}$$

Here, the superscript n indicates the time level.

5 Time Step Calculation

The time step Δt is determined based on stability conditions:

5.1 Convective Time Step

$$\Delta t_c = \sigma_c \frac{\Delta x \Delta y}{\|u_{\max}\|\Delta y + \|v_{\max}\|\Delta x} \tag{8}$$

where $\sigma_c = 0.4$ is the Courant number.

5.2 Diffusive Time Step

$$\Delta t_d = \sigma_d \frac{1}{2\nu} \cdot \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2} \tag{9}$$

where $\sigma_d = 0.6$ is the diffusion number.

5.3 Final Time Step

$$\Delta t = \min(\Delta t_c, \Delta t_d) \tag{10}$$

6 Results

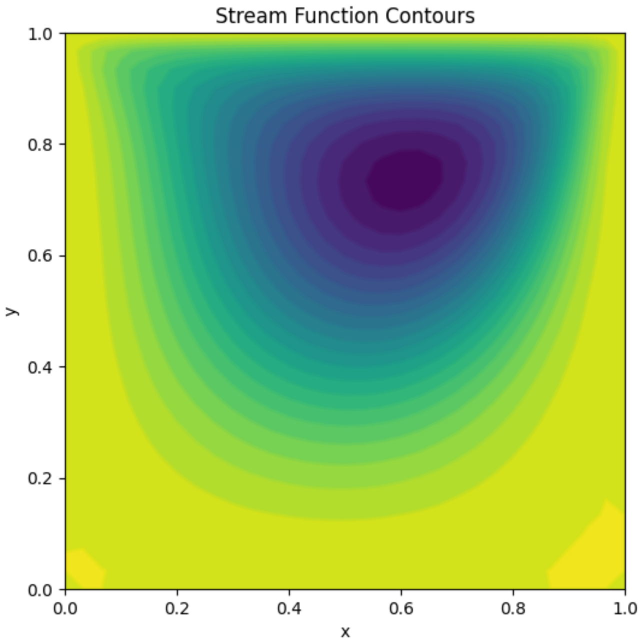

Figure 2: C Stream Function Contours

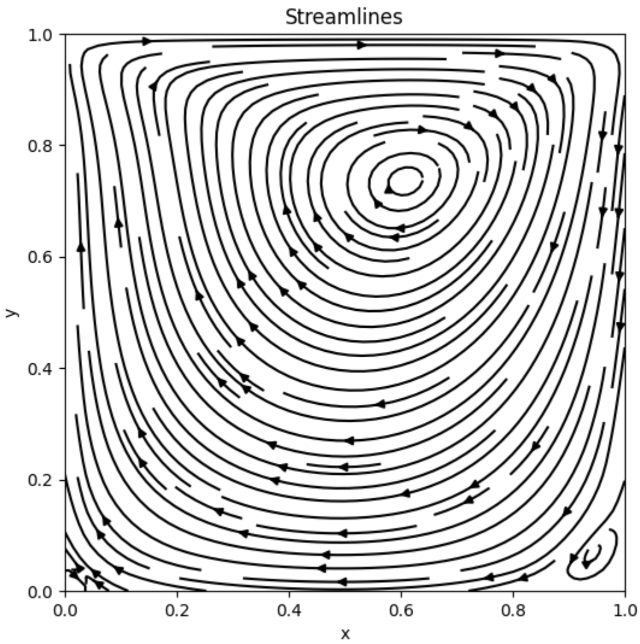

Figure 3: Streamlines of the flow

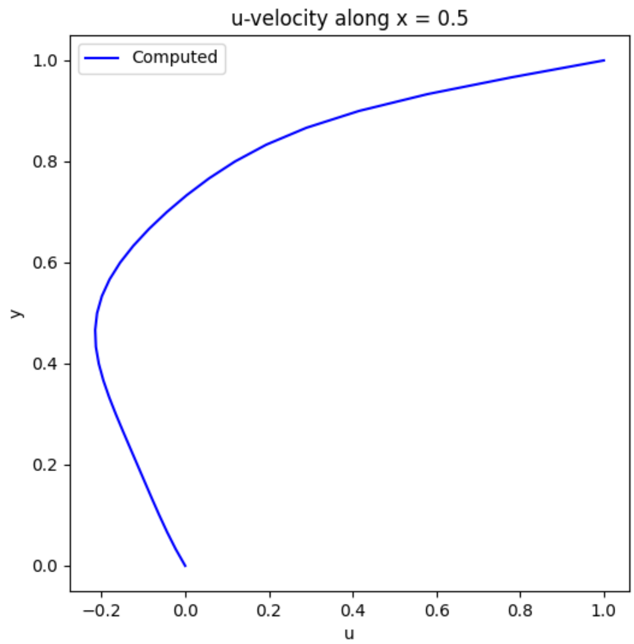

Figure 4: Distribution of the x-component of velocity vector (u) along the mid vertical line

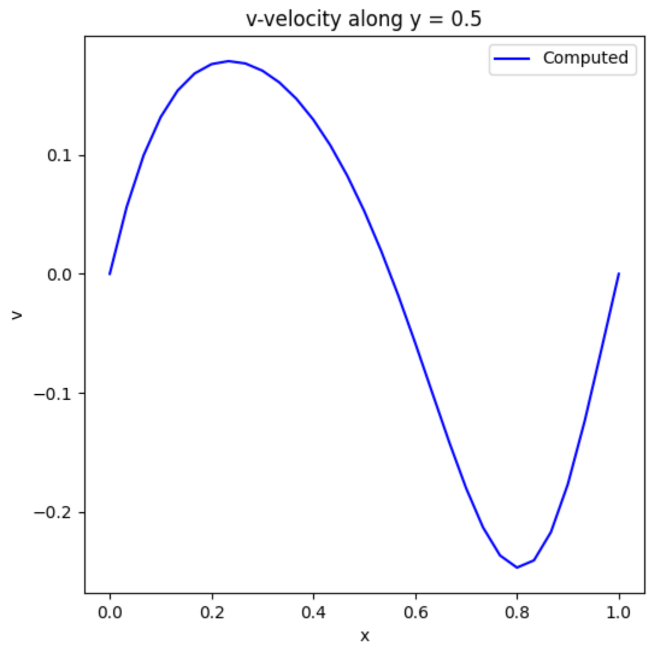

Figure 5: Distribution of the y-component of velocity vector (v) along the mid horizontal line

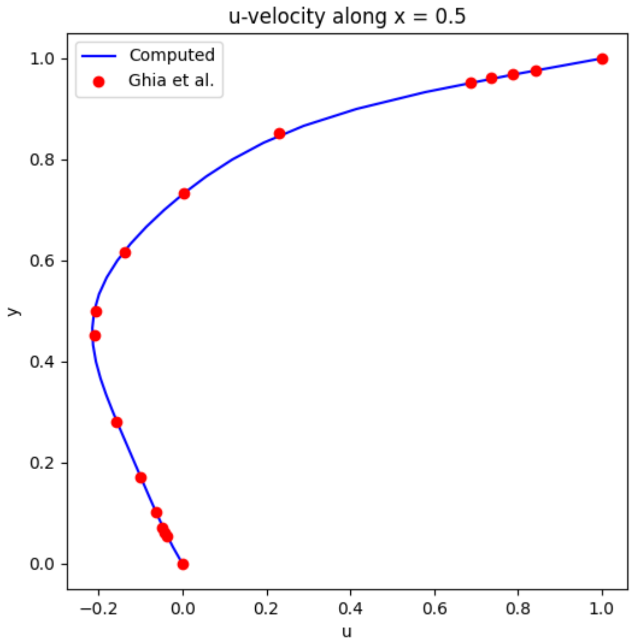

Figure 6: u-velocity along mid vertical line compared between CFD result and Ghia

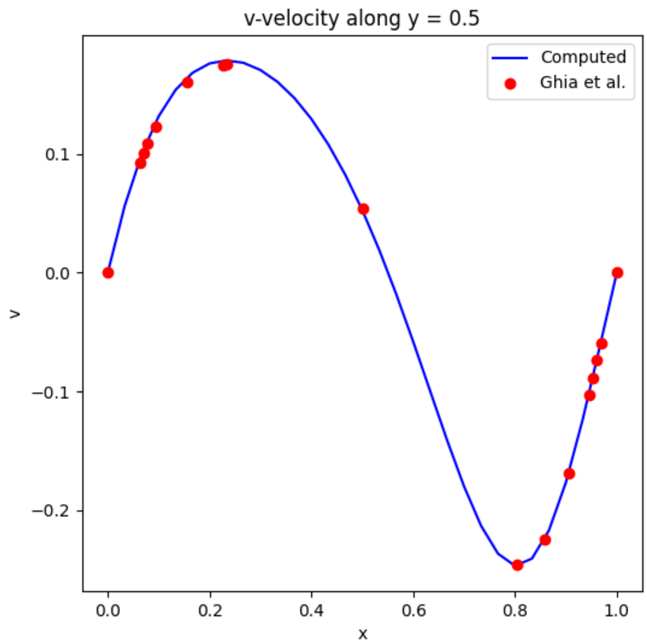

Figure 7: v-velocity along mid horizontal line compared between CFD result and Ghia

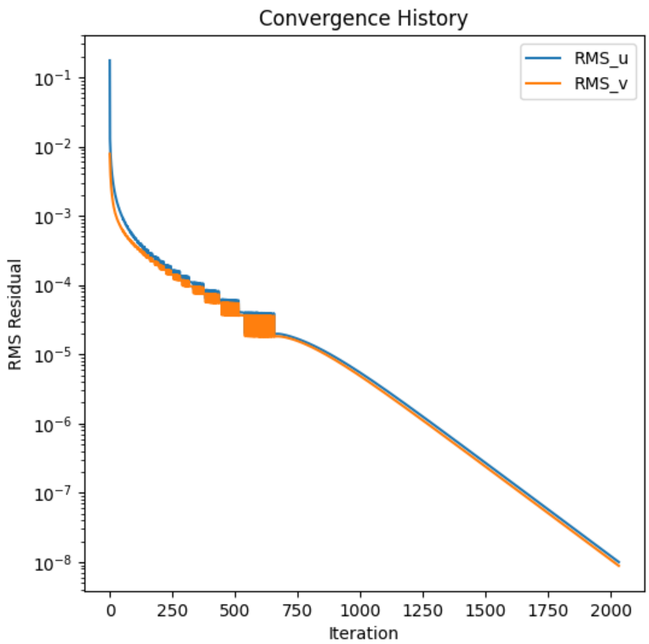

Figure 8: Convergence history for (RMS)u and (RMS)v with iterations

7 Conclusion

The numerical simulation effectively captured the key characteristics of lid-driven cavity flow at a Reynolds number of 100. The results demonstrate strong agreement with established benchmark data, thereby validating the implementation. The vorticity–stream function formulation proved to be a robust and efficient approach for solving this two-dimensional incompressible flow problem.

References

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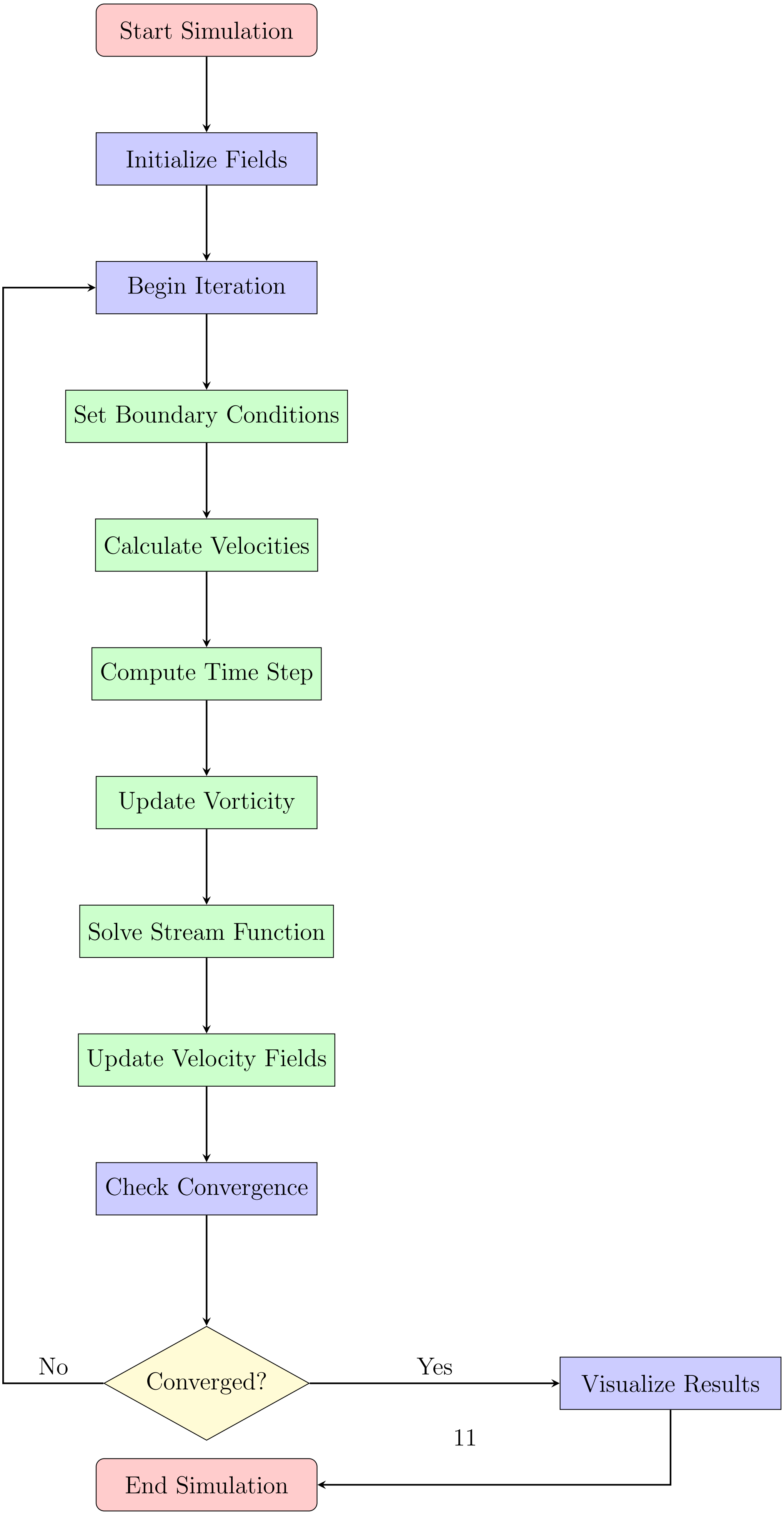
[2] Hoffmann, K. A., & Chiang, S. T., *Computational Fluid Dynamics for Engineers*, Vol. I, 4th ed., Engineering Education Systems, 2000.

[3] Pletcher, R. H., Tannehill, J. C., & Anderson, D. A., *Computational Fluid Dynamics and Heat Transfer*, 3rd ed., Taylor & Francis, 2011.

8 Flowchart



Start Simulation
Initialize Fields
Begin Iteration
Set Boundary Conditions
Calculate Velocities
Compute Time Step
Update Vorticity
Solve Stream Function
Update Velocity Fields
Check Convergence
Converged?
No
Yes
Visualize Results
End Simulation