AE 706

COMPUTATION FLUID DYNAMICS

Assignment-2: Grid generation using Transfinite interpolation (TFI) Method

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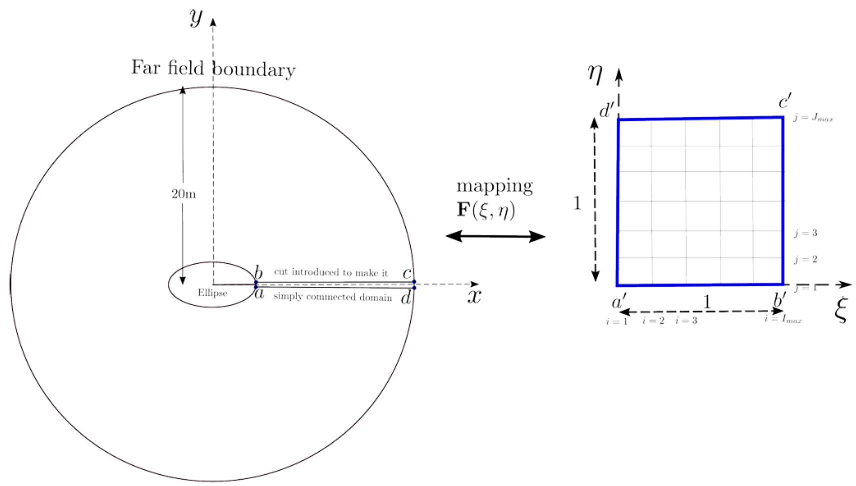
1 PROBLEM STATEMENT

Consider a physical domain between a ellipse (with semi-major axis a 1 m and semi-minor axis 0.5 m) and far field boundary located at 20m as shown in the Figure 1. You need to derive Transfinite interpolation mapping for generating grid in the physical domain. A (imaginary) cut is introduced so that physical domain becomes simply connected domain, which can then be mapped into unit square in the computational domain as shown in the Figure 1. Note that the co-ordinates of the points a and d are identical to the co-ordinates of b and c respectively.



(i) Physical Domain $\qquad$ (ii) Computational Domain

Figure 1: Grid mapping used for circular cylinder

Domain Specification and Discretization

Consider a ellipse (with semi-major axis a 1 m and semi-minor axis 0.5 m) over which incompressible irrotational flow is to be computed. The far field boundary is located at 20 m distance away as shown in the 1.

Generate a grid of the size 101 $\times$ 81, where there are 101 grid points in the ξ-diection and 81 grid points in the η-diection. There will be similarly 101 $\times$ 81 grids in the physical domain which can be obtained using Transfinite Interpolation method.

2 Transfinite Interpolation Mapping Derivation

Transfinite Interplation is a method used to convert physical domain into computation domain smoothly.It ensures that the generated grid conforms to boundary conditions while smoothly interpolating within the domain

2.1 Transfinite Interpolation Formula

The general transfinite interpolation function is defined as:

$$F(\xi, \eta) = P_\xi[F(\eta)] \oplus P_\eta[F(\xi)] = P_\xi[F(\eta)] + P_\eta[F(\xi)] - P_\xi P_\eta(F)$$

where $P_\xi[F(\eta)]$ and $P_\eta[F(\xi)]$ are projection operators that ensure interpolation along the coordinate lines.

2.2 Interpolation Along the Coordinate Lines

The projection operators for interpolation along the coordinate lines are given by:

$$P_\xi[F(\eta)] = \varphi_0(\xi)F(0, \eta) + \varphi_1(\xi)F(1, \eta) \tag{2}$$

$$P_\eta[F(\xi)] = \psi_0(\eta)F(\xi, 0) + \psi_1(\eta)F(\xi, 1) \tag{3}$$

where φ_0, φ_1 and ψ_0, ψ_1 are univariate interpolation functions satisfying:

$$\varphi_i(k) \equiv \delta_{ik} = \begin{cases} 1, & \text{for } i = k \\ 0, & \text{for } i \neq k \end{cases}$$

$$\psi_j(l) \equiv \delta_{jl} = \begin{cases} 1, & \text{for } j = l \\ 0, & \text{for } j \neq l \end{cases}$$

for $i, k = 0, 1$ and $j, l = 0, 1$.

These conditions ensure that the interpolation function correctly reproduces the boundary conditions at the edges of the computational domain

Product Projection $P_\xi P_\eta(\mathbf{F})$ is defined as

$$P_\xi P_\eta(\mathbf{F}) = \sum_{i=0}^{1}\sum_{j=0}^{1} \varphi_i(\xi)\psi_j(\eta)\mathbf{F}(\xi_i, \eta_j) \tag{4}$$

$$= \varphi_0(\xi)\psi_0(\eta)\mathbf{F}(0,0) + \varphi_0(\xi)\psi_1(\eta)\mathbf{F}(0,1)$$

$$+\varphi_1(\xi)\psi_0(\eta)\mathbf{F}(1,0) + \varphi_1(\xi)\psi_1(\eta)\mathbf{F}(1,1)$$

Note:

$$P_0P_0(\mathbf{F}) = \varphi_0(0)\psi_0(0)\mathbf{F}(0,0) = \mathbf{F}(0,0)$$

$$P_0P_1(\mathbf{F}) = \varphi_0(0)\psi_1(1)\mathbf{F}(0,1) = \mathbf{F}(0,1)$$

$$P_1P_0(\mathbf{F}) = \varphi_1(1)\psi_0(0)\mathbf{F}(1,0) = \mathbf{F}(1,0)$$

$$P_1P_1(\mathbf{F}) = \varphi_1(1)\psi_1(1)\mathbf{F}(1,1) = \mathbf{F}(1,1)$$

Finally, the key equation:

$$\boxed{\mathbf{F}(\xi,\eta) = P_\xi[\mathbf{F}(\eta)] \oplus P_\eta[\mathbf{F}(\xi)] = P_\xi[\mathbf{F}(\eta)] + P_\eta[\mathbf{F}(\xi)] - P_\xi P_\eta(\mathbf{F})} \tag{5}$$

2.3 Physical Domain

2.3.1 Inner Boundary (Ellipse)

At $r = 1$ (normalized), the equation of the inner boundary is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a = 1, \quad b = 0.5 \tag{6}$$

Parametrically, this can be written as:

$$x = a\cos\theta = \cos\theta, \quad y = b\sin\theta = 0.5\sin\theta \tag{7}$$

where $\theta \in [0, 2\pi]$.

2.3.2 Outer Boundary

The outer boundary is located at 20 meters. Since the shape is not specified, we assume a circular far-field boundary with radius 20 meters:

$$x = 20\cos\theta, \quad y = 20\sin\theta \tag{8}$$

2.3.3 The Cut

The cut makes the domain simply connected. Assume it is along the positive x-axis, from $(1,0)$ on the ellipse to $(20,0)$ on the far-field boundary. This split defines points:

- a and b on the ellipse.
- c and d on the far-field boundary.

These points have identical coordinates across the cut.

2.4 Computational Domain

The computational domain is a unit square:

$$\xi \in [0,1], \quad \eta \in [0,1] \tag{9}$$

where:

- ξ represents the angular direction (like θ).
- η represents the radial direction (from the inner to the outer boundary).

2.4.1 Boundary Correspondence

- $\eta = 0$: Inner ellipse.
- $\eta = 1$: Outer far-field boundary.
- $\xi = 0$ and $\xi = 1$: Along the cut ($\theta = 0$ or $\theta = 2\pi$), with ξ mapped from 0 to 1 over $\theta = 0$ to 2π.

2.5 Transfinite mapping

2.5.1 linear curves finding

$$F(\xi, 0) = \begin{bmatrix} \cos(2\pi\xi) \\ 0.5\sin(2\pi\xi) \end{bmatrix} \tag{10}$$

$$F(\xi, 1) = \begin{bmatrix} 20\cos(2\pi\xi) \\ 20\sin(2\pi\xi) \end{bmatrix} \tag{11}$$

$$F(0, \eta) = \begin{bmatrix} (1-\eta)\cdot 1 + \eta \cdot 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 19\eta \\ 0 \end{bmatrix} \tag{12}$$

$$F(1, \eta) = \begin{bmatrix} (1-\eta)\cdot 1 + \eta \cdot 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 19\eta \\ 0 \end{bmatrix} \tag{13}$$

2.5.2 Corner points finding

$$F(0, 0) = \begin{bmatrix} (1-0)\cdot 1 + 0 \cdot 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{14}$$

$$F(0, 1) = \begin{bmatrix} (1-1)\cdot 1 + 1 \cdot 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix} \tag{15}$$

$$F(1, 0) = \begin{bmatrix} (1-0)\cdot 1 + 0 \cdot 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{16}$$

$$F(1, 1) = \begin{bmatrix} (1-1)\cdot 1 + 1 \cdot 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix} \tag{17}$$

substituting all these terms in equation 5 we,

$$F(\xi,\eta) = (1-\xi)\begin{bmatrix}1+19\eta\\0\end{bmatrix} + \xi\begin{bmatrix}1+19\eta\\0\end{bmatrix} + (1-\eta)\begin{bmatrix}\cos(\xi 2\pi)\\0.5\sin(\xi 2\pi)\end{bmatrix} + \eta\begin{bmatrix}20\cos(\xi 2\pi)\\20\sin(\xi 2\pi)\end{bmatrix}$$

$$-\left\{(1-\xi)(1-\eta)\begin{bmatrix}1\\0\end{bmatrix} + (1-\xi)\eta\begin{bmatrix}20\\0\end{bmatrix} + \xi(1-\eta)\begin{bmatrix}1\\0\end{bmatrix} + \xi\eta\begin{bmatrix}20\\0\end{bmatrix}\right\}$$

After simplifying the therm we get

$$x(\xi,\eta) = (1+19\eta)\cos(2\pi\xi) \tag{18}$$

$$y(\xi,\eta) = [0.5(1-\eta)+20\eta]\sin(2\pi\xi) \tag{19}$$

3 Results

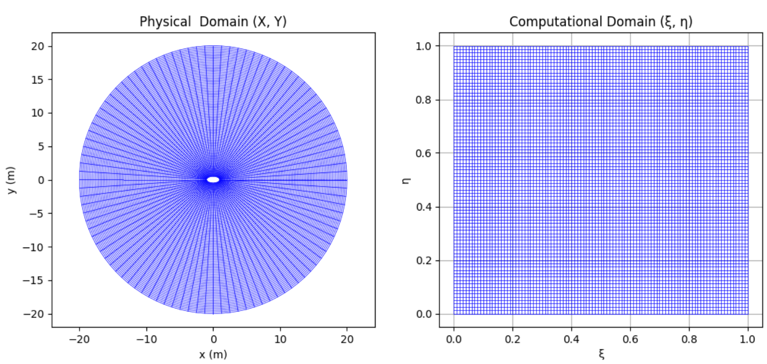


Figure 2: Grid plots in computational and physical domain (ξ, η).