

Contents

1	Triangle	1
1.1	Angle Bisector	1

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad (1.1)$$

1.1. Angle Bisector

1.1.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$AE_3 = AF_3 = m, BD_3 = BF_3 = n, CD_3 = CE_3 = p. \quad (1.1.1.1)$$

Obtain m, n, p in terms of a, b, c obtained in Question 1.1.2.

Solution: From Question 1.1.2

$$a = \sqrt{97} \quad (1.1.1.2)$$

$$b = 5 \quad (1.1.1.3)$$

$$c = \sqrt{32} \quad (1.1.1.4)$$

From the given information,

$$a = m + n, \tag{1.1.1.5}$$

$$b = n + p, \tag{1.1.1.6}$$

$$c = m + p \tag{1.1.1.7}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.8}$$

$$\Rightarrow \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.9}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \quad (1.1.1.10)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \quad (1.1.1.11)$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \quad (1.1.1.12)$$

yielding

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right)^{-1} = \frac{1}{2} \left(\begin{array}{ccc} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{array}\right) \quad (1.1.1.13)$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{32}+5-\sqrt{97}}{2} \\ m &= \frac{a+c-b}{2} = \frac{\sqrt{97}+\sqrt{32}-5}{2} \\ n &= \frac{a+b-c}{2} = \frac{\sqrt{97}+5-\sqrt{32}}{2} \end{aligned} \quad (1.1.1.14)$$

on solving above equations we get

$$p = 0.4039982238 \quad (1.1.1.15)$$

$$m = 5.252856026 \quad (1.1.1.16)$$

$$n = 4.59600177 \quad (1.1.1.17)$$

1.1.2. Using section formula, find $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$.

Solution: Given

$$\mathbf{D}_3 = \frac{m\mathbf{C} + n\mathbf{B}}{m+n}, \mathbf{E}_3 = \frac{n\mathbf{A} + p\mathbf{C}}{n+p}, \mathbf{F}_3 = \frac{p\mathbf{B} + m\mathbf{A}}{p+m} \quad (1.1.2.1)$$

Here

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad (1.1.2.2)$$

$$p = 0.4039982238, m = 5.252856026, n = 4.59600177 \quad (1.1.2.3)$$

On substituting (1.1.2.2) and (1.1.2.3) in (??) We get

$$\mathbf{D}_3 = \frac{5.252856026 \begin{pmatrix} 1 \\ -5 \end{pmatrix} + 4.59600177 \begin{pmatrix} -6 \\ 1 \end{pmatrix}}{5.252856026 + 4.59600177} \quad (1.1.2.4)$$

$$\mathbf{E}_3 = \frac{4.59600177 \begin{pmatrix} -6 \\ -5 \end{pmatrix} + 0.4039982238 \begin{pmatrix} 1 \\ -5 \end{pmatrix}}{4.59600177 + 0.4039982238} \quad (1.1.2.5)$$

$$\mathbf{F}_3 = \frac{0.4039982238 \begin{pmatrix} -6 \\ 1 \end{pmatrix} + 5.252856026 \begin{pmatrix} -6 \\ -5 \end{pmatrix}}{0.4039982238 + 5.252856026} \quad (1.1.2.6)$$

On solving above equations We get

$$\mathbf{D}_3 = \begin{pmatrix} 0.1998795 \\ 1.86661311 \end{pmatrix} \quad (1.1.2.7)$$

$$\mathbf{E}_3 = \begin{pmatrix} 1.28566988 \\ 0.28566988 \end{pmatrix} \quad (1.1.2.8)$$

$$\mathbf{F}_3 = \begin{pmatrix} 0.59600178 \\ 0 \end{pmatrix} \quad (1.1.2.9)$$

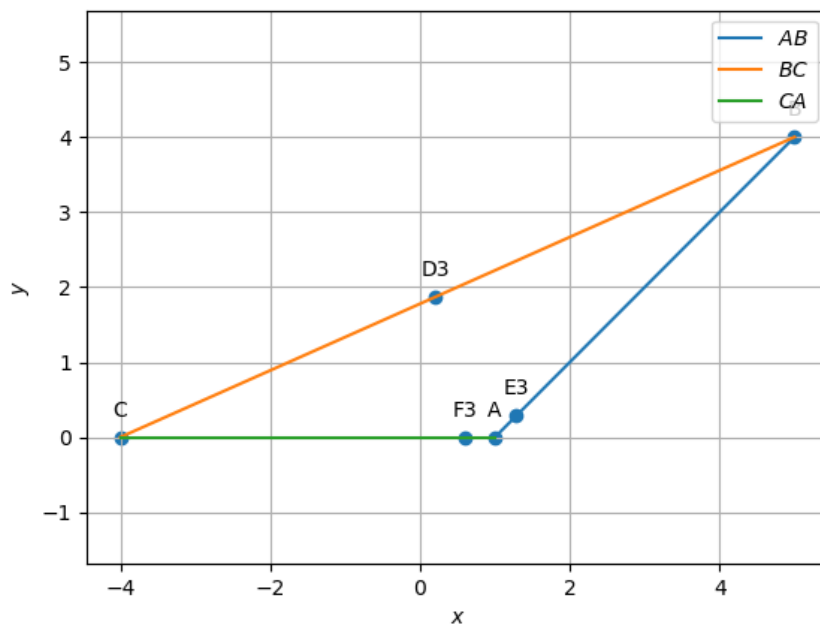


Figure 1.1: Points D_3, E_3, F_3

1.1.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the incentre and inradius of $\triangle ABC$.

Solution: Given

$$\mathbf{D}_3 = \begin{pmatrix} 0.1998795 \\ 1.86661311 \end{pmatrix} \quad (1.1.3.1)$$

$$\mathbf{E}_3 = \begin{pmatrix} 1.28566988 \\ 0.28566988 \end{pmatrix} \quad (1.1.3.2)$$

$$\mathbf{F}_3 = \begin{pmatrix} 0.59600178 \\ 0 \end{pmatrix} \quad (1.1.3.3)$$

(a) For circumcentre

Vector equation of $\mathbf{D} - \mathbf{E}$ is

$$(\mathbf{D}_3 - \mathbf{E}_3)^\top \left(\mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{E}_3}{2} \right) = 0 \quad (1.1.3.4)$$

$$(\mathbf{D}_3 - \mathbf{F}_3)^\top \left(\mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{F}_3}{2} \right) = 0 \quad (1.1.3.5)$$

on Substituting the values of $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$ and solving We get,

$$\begin{pmatrix} -1.08579038 & 1.58094323 \end{pmatrix} \mathbf{x} = 0.8894235909 \quad (1.1.3.6)$$

$$\begin{pmatrix} -0.39612228 & 1.86661311 \end{pmatrix} \mathbf{x} = 1.584489107 \quad (1.1.3.7)$$

Thus on solving (1.1.3.6) and (1.1.3.7) using gauss elimination

We get

$$\begin{pmatrix} -1.08579038 & 1.58094323 & 0.8894235909 \\ -0.39612228 & 1.86661311 & 1.584489107 \end{pmatrix} \quad (1.1.3.8)$$

$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0.59600178 \\ 0.97533799 \end{pmatrix} \quad (1.1.3.9)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 0.59600178 \\ 0.97533799 \end{pmatrix} \quad (1.1.3.10)$$

(b) The circumradius is obtained from $r = \|\mathbf{I} - \mathbf{D}_3\|$

$$\mathbf{I} = \begin{pmatrix} 0.59600178 \\ 0.97533799 \end{pmatrix} \quad (1.1.3.11)$$

$$\mathbf{D}_3 = \begin{pmatrix} 0.1998795 \\ 1.86661311 \end{pmatrix} \quad (1.1.3.12)$$

$$\mathbf{I} - \mathbf{D}_3 = \begin{pmatrix} 0.39612228 \\ -0.89127512 \end{pmatrix} \quad (1.1.3.13)$$

$$r = \|\mathbf{I} - \mathbf{D}_3\| = \sqrt{(\mathbf{I} - \mathbf{D}_3)^\top (\mathbf{I} - \mathbf{D}_3)} \quad (1.1.3.14)$$

$$r = 0.9753379911888156 \quad (1.1.3.15)$$

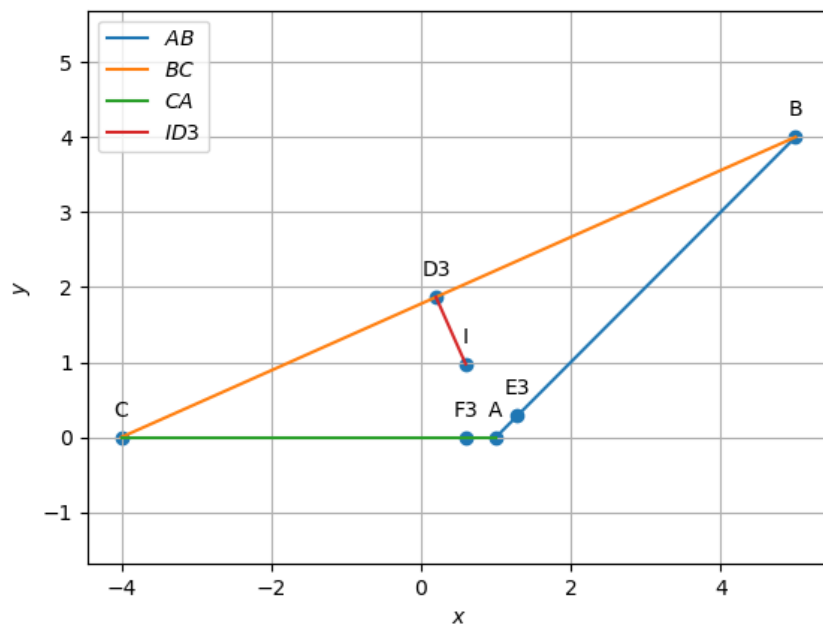


Figure 1.2: Incentre and Inradius of $\triangle ABC$

1.1.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the incircle of $\triangle ABC$.

Solution:

$$\mathbf{D}_3 = \begin{pmatrix} 0.1998795 \\ 1.86661311 \end{pmatrix} \quad (1.1.4.1)$$

$$\mathbf{E}_3 = \begin{pmatrix} 1.28566988 \\ 0.28566988 \end{pmatrix} \quad (1.1.4.2)$$

$$\mathbf{F}_3 = \begin{pmatrix} 0.59600178 \\ 0 \end{pmatrix} \quad (1.1.4.3)$$

$$\text{Incentre} \quad (1.1.4.4)$$

$$\mathbf{I} = \begin{pmatrix} 0.59600178 \\ 0.97533799 \end{pmatrix} \quad (1.1.4.5)$$

$$\text{Radius} \quad (1.1.4.6)$$

$$\mathbf{r} = 0.9753379911888156 \quad (1.1.4.7)$$

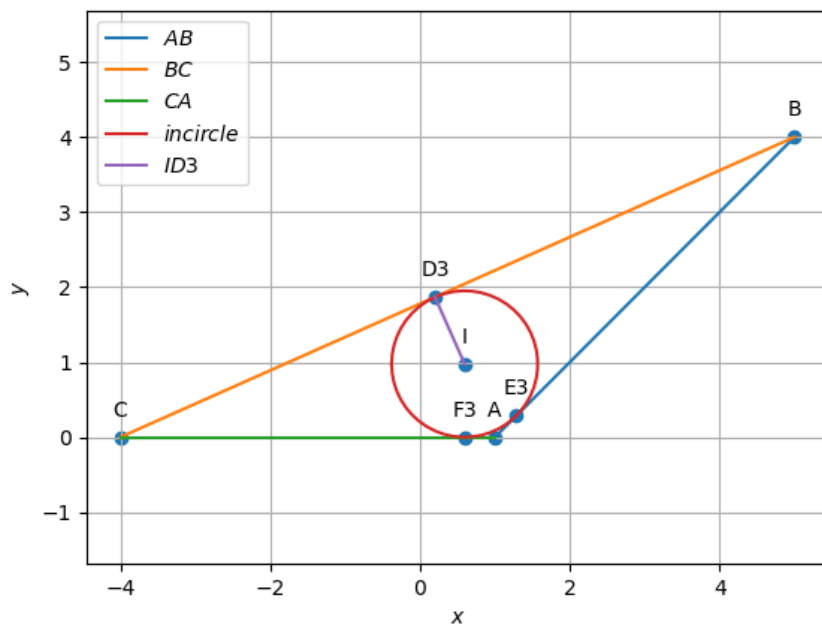


Figure 1.3: Incircle of $\triangle ABC$

1.1.5. Using (1.1.7) verify that

$$\angle BAI = \angle CAI. \quad (1.1.5.1)$$

AI is the bisector of $\angle A$.

Solution:

$$\cos \angle BAI \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{I} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{I} - \mathbf{A}\|} \quad (1.1.5.2)$$

$$\cos \angle CAI \triangleq \frac{(\mathbf{C} - \mathbf{A})^\top (\mathbf{I} - \mathbf{A})}{\|\mathbf{C} - \mathbf{A}\| \|\mathbf{I} - \mathbf{A}\|} \quad (1.1.5.3)$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and \mathbf{I} ,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.1.5.4)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -5 \\ 0 \end{pmatrix} \quad (1.1.5.5)$$

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} -0.40399822 \\ 0.97533799 \end{pmatrix} \quad (1.1.5.6)$$

also calculating the values of norms

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{32} \quad (1.1.5.7)$$

$$\|\mathbf{C} - \mathbf{A}\| = 5 \quad (1.1.5.8)$$

$$\|\mathbf{I} - \mathbf{A}\| = 0.8877326906 \quad (1.1.5.9)$$

$$(1.1.5.10)$$

(a) for $\angle BAI$:

On substituting the values in (1.1.5.2), We get

$$\cos \angle BAI \triangleq \frac{\begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} -0.40399822 \\ 0.97533799 \end{pmatrix}}{5.656854249 \times 0.8877326906} \quad (1.1.5.11)$$

$$(1.1.5.12)$$

On solving

$$\angle BAI = 62.92928^\circ \quad (1.1.5.13)$$

(b) for $\angle CAI$:

On substituting the values in (1.1.5.2) ,We get

$$\cos \angle CAI \triangleq \frac{\begin{pmatrix} -5 & 0 \end{pmatrix} \begin{pmatrix} -0.40399822 \\ 0.97533799 \end{pmatrix}}{5 \times 0.8877326906} \quad (1.1.5.14)$$

$$(1.1.5.15)$$

On solving

$$\angle CAI = 62.92928^\circ \quad (1.1.5.16)$$

Therefore $\angle BAI = \angle CAI$. and AI is the bisector of $\angle A$.

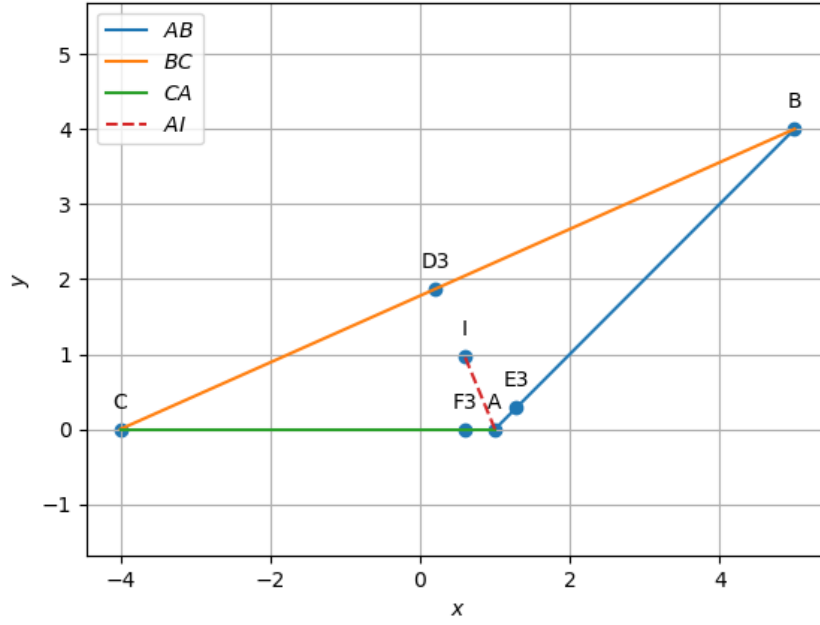


Figure 1.4: Angular bisector AI

1.1.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.

Solution:

(a) To prove BI is an angular bisector of $\angle B$

$$\cos \angle ABI \triangleq \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{I} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{I} - \mathbf{B}\|} \quad (1.1.6.1)$$

$$\cos \angle CBI \triangleq \frac{(\mathbf{C} - \mathbf{B})^\top (\mathbf{I} - \mathbf{B})}{\|\mathbf{C} - \mathbf{B}\| \|\mathbf{I} - \mathbf{B}\|} \quad (1.1.6.2)$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and \mathbf{I} ,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \quad (1.1.6.3)$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -9 \\ -4 \end{pmatrix} \quad (1.1.6.4)$$

$$\mathbf{I} - \mathbf{B} = \begin{pmatrix} -4.40399822 \\ -3.02466201 \end{pmatrix} \quad (1.1.6.5)$$

also calculating the values of norms

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{32} \quad (1.1.6.6)$$

$$\|\mathbf{C} - \mathbf{B}\| = \sqrt{97} \quad (1.1.6.7)$$

$$\|\mathbf{I} - \mathbf{B}\| = 5.342637981 \quad (1.1.6.8)$$

$$(1.1.6.9)$$

i. for $\angle ABI$:

On substituting the values in (1.1.6.1), We get

$$\cos \angle ABI \triangleq \frac{\begin{pmatrix} -4 & -4 \end{pmatrix} \begin{pmatrix} -4.40399822 \\ -3.02466201 \end{pmatrix}}{5.656854249 \times 5.342637981} \quad (1.1.6.10)$$

$$(1.1.6.11)$$

On solving

$$\angle ABI = 10.5194^\circ \quad (1.1.6.12)$$

ii. for $\angle CBI$:

On substituting the values in (1.1.6.1) ,We get

$$\cos \angle CBI \triangleq \frac{\begin{pmatrix} -9 & -4 \end{pmatrix} \begin{pmatrix} -4.40399822 \\ -3.02466201 \end{pmatrix}}{\sqrt{97} \times 5.342637981} \quad (1.1.6.13)$$

$$(1.1.6.14)$$

On solving

$$\angle CBI = 10.5194^\circ \quad (1.1.6.15)$$

Therefore $\angle ABI = \angle CBI$. and BI is the bisector of $\angle B$.

(b) To prove CI is an angular bisector of $\angle C$

$$\cos \angle BCI \triangleq \frac{(\mathbf{B} - \mathbf{C})^\top (\mathbf{I} - \mathbf{C})}{\|\mathbf{B} - \mathbf{C}\| \|\mathbf{I} - \mathbf{C}\|} \quad (1.1.6.16)$$

$$\cos \angle ACI \triangleq \frac{(\mathbf{A} - \mathbf{C})^\top (\mathbf{I} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{I} - \mathbf{C}\|} \quad (1.1.6.17)$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and \mathbf{I} ,

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \quad (1.1.6.18)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (1.1.6.19)$$

$$\mathbf{I} - \mathbf{C} = \begin{pmatrix} 4.5960178 \\ 0.97533799 \end{pmatrix} \quad (1.1.6.20)$$

also calculating the values of norms

$$\|\mathbf{B} - \mathbf{C}\| = \sqrt{97} \quad (1.1.6.21)$$

$$\|\mathbf{A} - \mathbf{C}\| = 5 \quad (1.1.6.22)$$

$$\|\mathbf{I} - \mathbf{C}\| = 4.698368207 \quad (1.1.6.23)$$

$$(1.1.6.24)$$

i. for $\angle BCI$:

On substituting the values in (1.1.6.16), We get

$$\cos \angle BCI \triangleq \frac{\begin{pmatrix} 9 & 4 \end{pmatrix} \begin{pmatrix} 4.5960178 \\ 0.97533799 \end{pmatrix}}{\sqrt{97} \times 4.698368207} \quad (1.1.6.25)$$

$$(1.1.6.26)$$

On solving

$$\angle BCI = 11.98120388^\circ \quad (1.1.6.27)$$

ii. for $\angle ACI$:

On substituting the values in (1.1.6.16) ,We get

$$\cos \angle ACI \triangleq \frac{\begin{pmatrix} 5 & 0 \end{pmatrix} \begin{pmatrix} 4.5960178 \\ 0.97533799 \end{pmatrix}}{5 \times 4.698368207} \quad (1.1.6.28)$$

$$(1.1.6.29)$$

On solving

$$\angle ACI = 11.98120388^\circ \quad (1.1.6.30)$$

Therefore $\angle BCI = \angle ACI$. and CI is the bisector of $\angle C$.

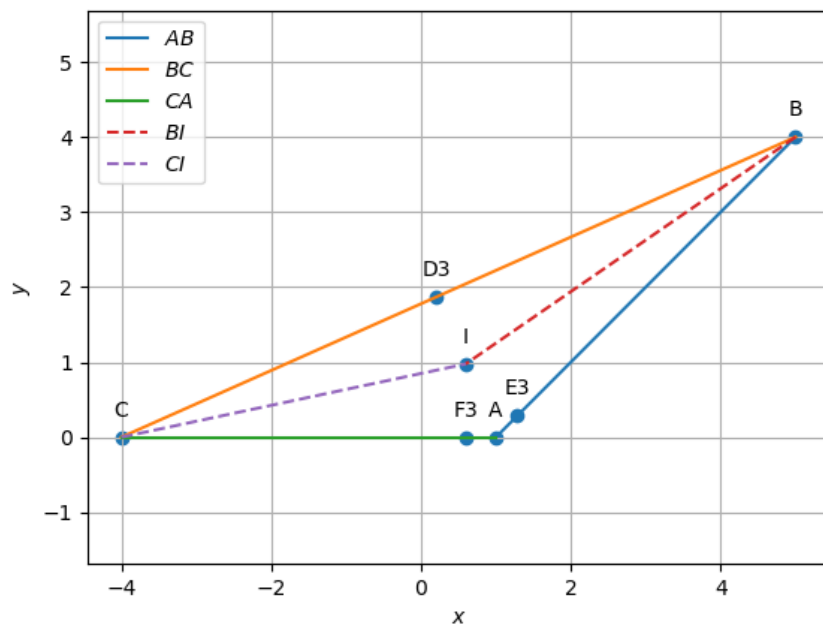


Figure 1.5: Angular bisectors BI, CI