NCERT 12.10.5.14

1. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes,show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}, \vec{b} and \vec{c}$.

   Construction Steps

   Since $\vec{a}, \vec{b} and \vec{c}$ are mutually Perpendicular vectors, we have $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$. It is given that : $\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{c}\right|$ let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a}, \vec{b} and \vec{c}$ at angles $\theta_1, \theta_2 and \theta_3$ respectively.

   Then,we have:

$$\cos\theta_1 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|} \tag{1}$$

$$= \frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|} \tag{2}$$

$$= \frac{\left|\vec{a}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|} \tag{3}$$

$$\implies \frac{\left|\vec{a}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|} \tag{4}$$

$$\cos\theta_2 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|} \tag{5}$$

$$= \frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|} \tag{6}$$

$$= \frac{\left|\vec{b}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|} \tag{7}$$

$$\implies \frac{\left|\vec{b}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|} \tag{8}$$

$$\cos\theta_3 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|} \tag{9}$$

$$= \frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|} \tag{10}$$

$$= \frac{\left|\vec{c}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|} \tag{11}$$

$$\implies \frac{\left|\vec{c}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|} \tag{12}$$

$$\tag{13}$$

now , as

$$\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{c}\right|, \tag{14}$$

$$\cos\theta_1 = \cos\theta_2 = \cos\theta_3 \tag{15}$$

$$\therefore \theta_1 = \theta_2 = \theta_3 \tag{16}$$

Hence, the vector $\left(\vec{a}+\vec{b}+\vec{c}\right)$ is equally inclined to $\vec{a}, \vec{b} and \vec{c}$

let u assume that vector $\left(\vec{a}+\vec{b}+\vec{c}\right)$ as $\vec{p}$

$\vec{a}, \vec{b}, \vec{c}$ points are $(2,0,0),(0,2,0),(0,0,2)$ and $\left(\vec{a}+\vec{b}+\vec{c}\right)$ is $(2,2,2)$

let O be the origin and points are $(0,0,0)$
les us consider the $\angle POA$

$$\cos(POA) = \frac{(P-O)^\top (A-O)}{\|P-O\|\|A-O\|} \tag{17}$$

Finding angle POA

$$P-O = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \tag{18}$$

$$A-O = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

also calculating the values of norms

$$\|P-O\| = \sqrt{12} = 2\sqrt{3} \tag{20}$$

$$\|A-O\| = \sqrt{4} = 2 \tag{21}$$

$$\cos(POA) = \frac{(P-O)^\top (A-O)}{\|P-O\|\|A-O\|} \tag{22}$$

$$\implies \frac{\begin{pmatrix} 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}{4\sqrt{3}} \tag{23}$$

$$\implies \frac{1}{\sqrt{3}} \tag{24}$$

$$\angle POA = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \tag{25}$$

Similarly,$\angle POB, \angle POC$ are

Finding angle POB

$$P-O = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \tag{26}$$

$$B - O = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \tag{27}$$

also calculating the values of norms

$$\|P - O\| = \sqrt{12} = 2\sqrt{3} \tag{28}$$

$$\|B - O\| = \sqrt{4} = 2 \tag{29}$$

$$\cos(POB) = \frac{(-O)^\top (B - O)}{\|P - O\|\|B - O\|} \tag{30}$$

$$\implies \frac{\begin{pmatrix} 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}{4\sqrt{3}} \tag{31}$$

$$\implies \frac{1}{\sqrt{3}} \tag{32}$$

$$\angle POB = \cos^{-} 1 \left(\frac{1}{\sqrt{3}} \right) \tag{33}$$

Finding angle POC

$$P - O = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \tag{34}$$

$$C - O = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \tag{35}$$

also calculating the values of norms

$$\|P - O\| = \sqrt{12} = 2\sqrt{3} \tag{36}$$

$$\|C - O\| = \sqrt{4} = 2 \tag{37}$$

$$\cos(POA) = \frac{(P - O)^\top (C - O)}{\|P - O\|\|C - O\|} \tag{38}$$

$$\implies \frac{\begin{pmatrix} 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}{4\sqrt{3}} \tag{39}$$

$$\implies \frac{1}{\sqrt{3}} \tag{40}$$

$$\angle POC = \cos^{-} 1 \left(\frac{1}{\sqrt{3}} \right) \tag{41}$$

$\therefore \angle POA = \angle POB = \angle POC$
Hence here all the angles are equal so they are equally inclined to each other