NCERT 12.10.5.14

1. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes,show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}and\vec{c}$.

   Construction Steps

   Since $\vec{a},\vec{b}and\vec{c}$ are mutually Perpendicular vectors, we have $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$. It is given that : $\left|\vec{a}\right|=\left|\vec{b}\right|=\left|\vec{c}\right|$ let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b}and\vec{c}$ at angles $\theta_1,\theta_2and\theta_3$ respectively.

   Then,we have:

$$\cos\theta_1 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|} \tag{1}$$

$$= \frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|} \tag{2}$$

$$= \frac{\left|\vec{a}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|} \tag{3}$$

$$\implies \frac{\left|\vec{a}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|} \tag{4}$$

$$\cos\theta_2 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|} \tag{5}$$

$$= \frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|} \tag{6}$$

$$= \frac{\left|\vec{b}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|} \tag{7}$$

$$\implies \frac{\left|\vec{b}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|} \tag{8}$$

$$\cos\theta_3 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|} \tag{9}$$

$$= \frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|} \tag{10}$$

$$= \frac{\left|\vec{c}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|} \tag{11}$$

$$\implies \frac{\left|\vec{c}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|} \tag{12}$$

$$\tag{13}$$

now , as

$$\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{c}\right|, \tag{14}$$

$$\cos\theta_1 = \cos\theta_2 = \cos\theta_3 \tag{15}$$

$$\therefore \theta_1 = \theta_2 = \theta_3 \tag{16}$$

Hence, the vector $\left(\vec{a}+\vec{b}+\vec{c}\right)$ is equally inclined to $\vec{a},\vec{b}and\vec{c}$