Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \tag{1.1}$$

1.1. Vectors

1.2. Median

1.3. Altitude

1.4. Perpendicular Bisector

1.5. Angular Bisector

1.6. Matrix

The matrix of the veritices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{1.2}$$

$$= \begin{pmatrix} 1 & 5 & -4 \\ 0 & 4 & 0 \end{pmatrix} \tag{1.3}$$

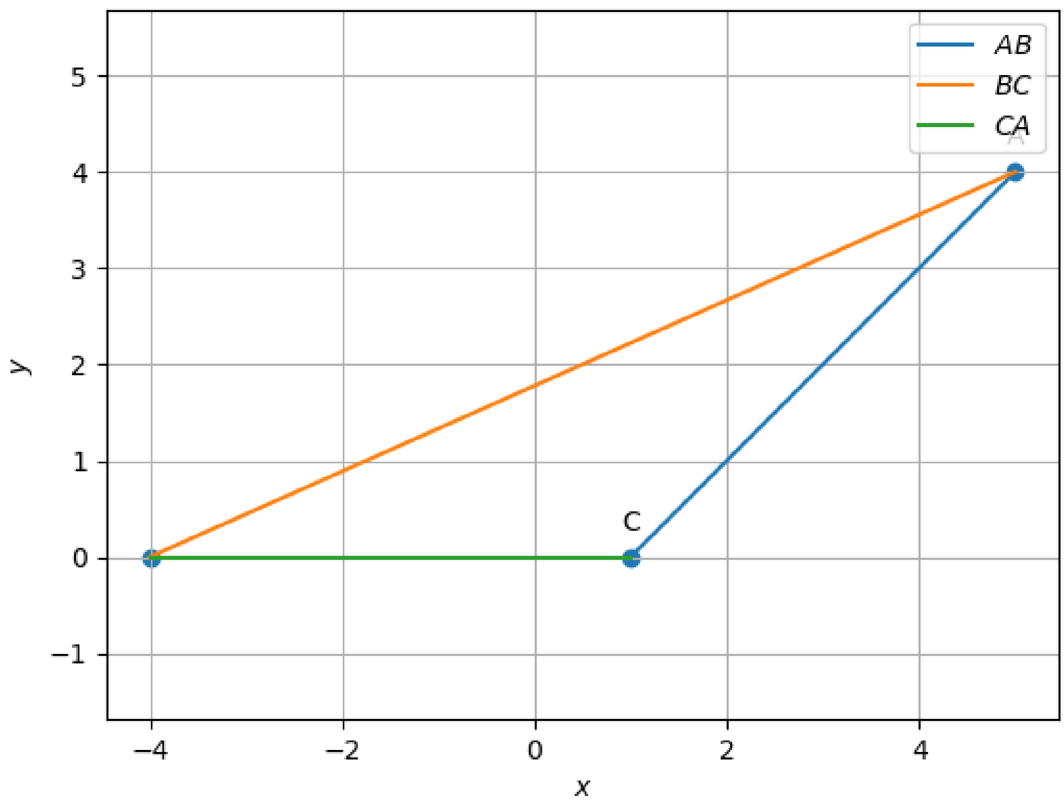

Figure 1.1: $\triangle$ ABC

1.6.1. Vectors

1.6.1.1. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.6.1.1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.6.1.1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.6.1.1.3}$$

$$= \begin{pmatrix} 1 & 5 & -4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.6.1.1.4}$$

Using Matrix multiplication

$$\mathbf{M} = \begin{pmatrix} -4 & 9 & -5 \\ -4 & 4 & 0 \end{pmatrix} \tag{1.6.1.1.5}$$

where the second matrix above is known as a circulant matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

1.6.1.2. Obtain the normal matrix of the sides of $\triangle ABC$

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.6.1.2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{1.6.1.2.2}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -4 & 9 & -5 \\ -4 & 4 & 0 \end{pmatrix} \tag{1.6.1.2.3}$$

Using matrix multiplication

$$\mathbf{N} = \begin{pmatrix} 4 & -4 & 0 \\ -4 & 9 & -5 \end{pmatrix} \tag{1.6.1.2.4}$$

1.6.1.3. Obtain a, b, c.

Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \tag{1.6.1.3.1}$$

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 4 & -4 \\ -4 & 9 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} 4 & -4 & 0 \\ -4 & 9 & -5 \end{pmatrix} \tag{1.6.1.3.2}$$

Using matrix multiplication

$$\mathbf{M} = \begin{pmatrix} 32 & -52 & 20 \\ -52 & 97 & -45 \\ -20 & 45 & -25 \end{pmatrix} \tag{1.6.1.3.3}$$

$$\mathbf{d} = \sqrt{\text{diag}\left(\begin{pmatrix} 32 & -52 & 20 \\ -52 & 97 & -45 \\ -20 & 45 & -25 \end{pmatrix}\right)} \tag{1.6.1.3.4}$$

$$= \begin{pmatrix} \sqrt{32} & \sqrt{97} & 5 \end{pmatrix} \tag{1.6.1.3.5}$$

1.6.1.4. Obtain the constant terms in the equations of the sides of the triangle.

Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag}\left\{\left(\mathbf{N}^\top \mathbf{P}\right)\right\} \tag{1.6.1.4.1}$$

$$\mathbf{N}^\top \mathbf{P} = \begin{pmatrix} 4 & -4 \\ -4 & 9 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} 1 & 5 & -4 \\ 0 & 4 & 0 \end{pmatrix} \tag{1.6.1.4.2}$$

$$\tag{1.6.1.4.3}$$

Using matrix multiplication

$$= \begin{pmatrix} 4 & 4 & -16 \\ -4 & 16 & 16 \\ 0 & 20 & 0 \end{pmatrix} \tag{1.6.1.4.4}$$

$$\mathbf{c} = \text{diag}\left(\begin{pmatrix} 4 & 4 & -16 \\ -4 & 16 & 16 \\ 0 & 20 & 0 \end{pmatrix}\right) \tag{1.6.1.4.5}$$

$$= \begin{pmatrix} 4 & 16 & 0 \end{pmatrix} \tag{1.6.1.4.6}$$

1.6.2. Median

1.6.2.1. Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.6.2.1.1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 5 & -4 \\ 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.6.2.1.2}$$

Using matrix multiplication

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-3}{2} & 3 \\ 2 & 0 & 2 \end{pmatrix} \tag{1.6.2.1.3}$$

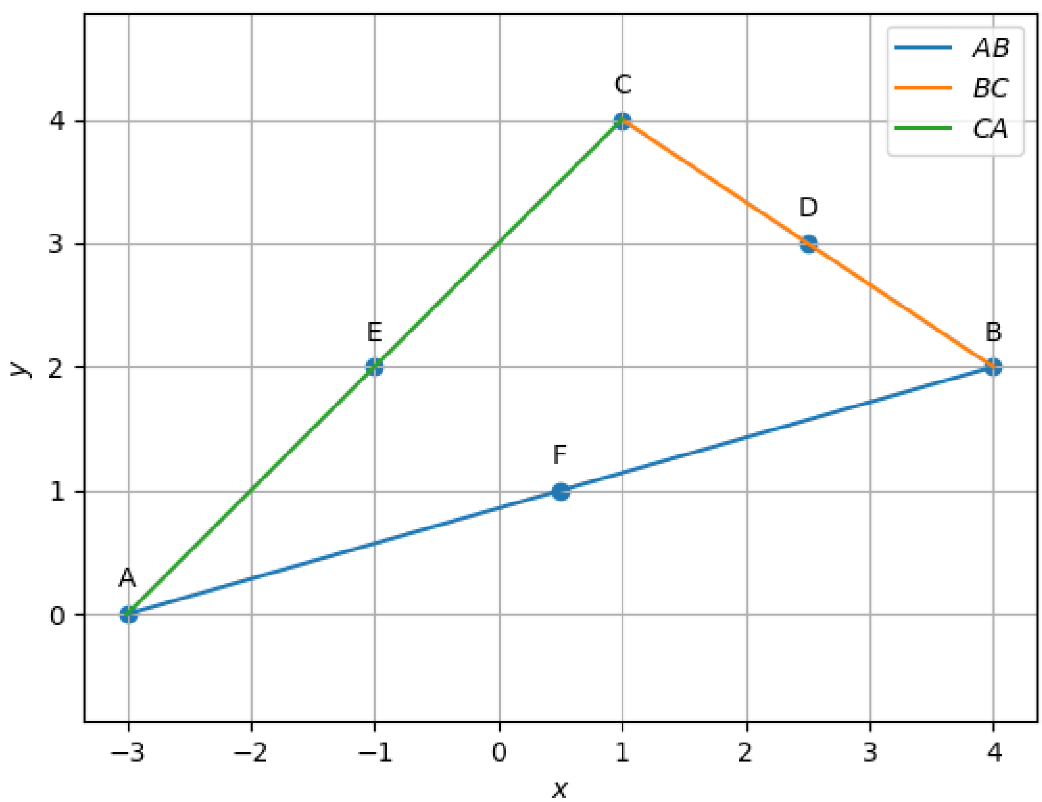

Figure 1.2: mid-points

1.6.2.2. Obtain the median direction matrix.

Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F} \end{pmatrix} \tag{1.6.2.2.1}$$

$$= \begin{pmatrix} \mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \tag{1.6.2.2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{1.6.2.2.3}$$

$$= \begin{pmatrix} 1 & 5 & -4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{1.6.2.2.4}$$

Using matrix multiplication

$$\mathbf{M}_1 = \begin{pmatrix} \frac{1}{2} & \frac{13}{2} & -7 \\ -2 & 4 & -2 \end{pmatrix} \tag{1.6.2.2.5}$$

1.6.2.3. Obtain the median normal matrix.

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.6.2.3.1}$$

the normal matrix is obtained as

$$\mathbf{N}_1 = \mathbf{R}\mathbf{M}_1 \tag{1.6.2.3.2}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{13}{2} & -7 \\ -2 & 4 & -2 \end{pmatrix} \tag{1.6.2.3.3}$$

$$\mathbf{N}_1 = \begin{pmatrix} 2 & -4 & 2 \\ \frac{1}{2} & \frac{13}{2} & -7 \end{pmatrix} \tag{1.6.2.3.4}$$

1.6.2.4. Obtian the median equation constants.

$$\mathbf{c}_1 = \operatorname{diag}\left(\left(\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right)\right) \tag{1.6.2.4.1}$$

$$\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} 2 & \frac{1}{2} \\ -4 & \frac{13}{2} \\ 2 & -7 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{-3}{2} & 3 \\ 2 & 0 & 2 \end{pmatrix} \tag{1.6.2.4.2}$$

$$\tag{1.6.2.4.3}$$

Using matrix multiplication

$$= \begin{pmatrix} 2 & -3 & 7 \\ 11 & 6 & 1 \\ -13 & -3 & -8 \end{pmatrix} \tag{1.6.2.4.4}$$

$$\mathbf{c}_1 = \text{diag}\left(\begin{pmatrix} 2 & -3 & 7 \\ 11 & 6 & 1 \\ -13 & -3 & -8 \end{pmatrix}\right) \tag{1.6.2.4.5}$$

$$\mathbf{c}_1 = \begin{pmatrix} 2 & 6 & -8 \end{pmatrix} \tag{1.6.2.4.6}$$

1.6.2.5. Obtain the centroid by finding the intersection of the medians.

Solution:

$$\left(\mathbf{N}_1^\top \;\middle|\; \mathbf{c}^\top\right) = \left(\begin{array}{cc|c} 2 & \frac{1}{2} & 2 \\ -4 & \frac{13}{2} & 6 \\ 2 & -7 & -8 \end{array}\right) \tag{1.6.2.5.1}$$

Using Gauss-Elimination method:

$$\left(\begin{array}{cc|c} 2 & \frac{1}{2} & 2 \\ -4 & \frac{13}{2} & 6 \\ 2 & -7 & -8 \end{array}\right) \xleftrightarrow{R_1 \leftarrow R_1/2} \left(\begin{array}{cc|c} 1 & \frac{1}{4} & 1 \\ -4 & \frac{13}{2} & 6 \\ 2 & -7 & -8 \end{array}\right) \tag{1.6.2.5.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 4R_1} \left(\begin{array}{cc|c} 1 & \frac{1}{4} & 1 \\ 0 & \frac{15}{2} & 10 \\ 2 & -7 & -8 \end{array}\right) \tag{1.6.2.5.3}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 2R1} \left(\begin{array}{cc|c} 1 & \frac{1}{4} & 1 \\ 0 & \frac{15}{2} & 10 \\ 0 & \frac{-15}{2} & -10 \end{array}\right) \tag{1.6.2.5.4}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{2}{15} R_2} \left(\begin{array}{cc|c} 1 & \frac{1}{4} & 1 \\ 0 & 1 & \frac{4}{3} \\ 0 & \frac{-15}{2} & -10 \end{array}\right) \tag{1.6.2.5.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2/4} \left(\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{4}{3} \\ 0 & \frac{-15}{2} & -10 \end{array}\right) \tag{1.6.2.5.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 15R_2/2} \left(\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 \end{array}\right) \tag{1.6.2.5.7}$$

$$\text{Therefore } \mathbf{G} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \end{pmatrix} \tag{1.6.2.5.8}$$

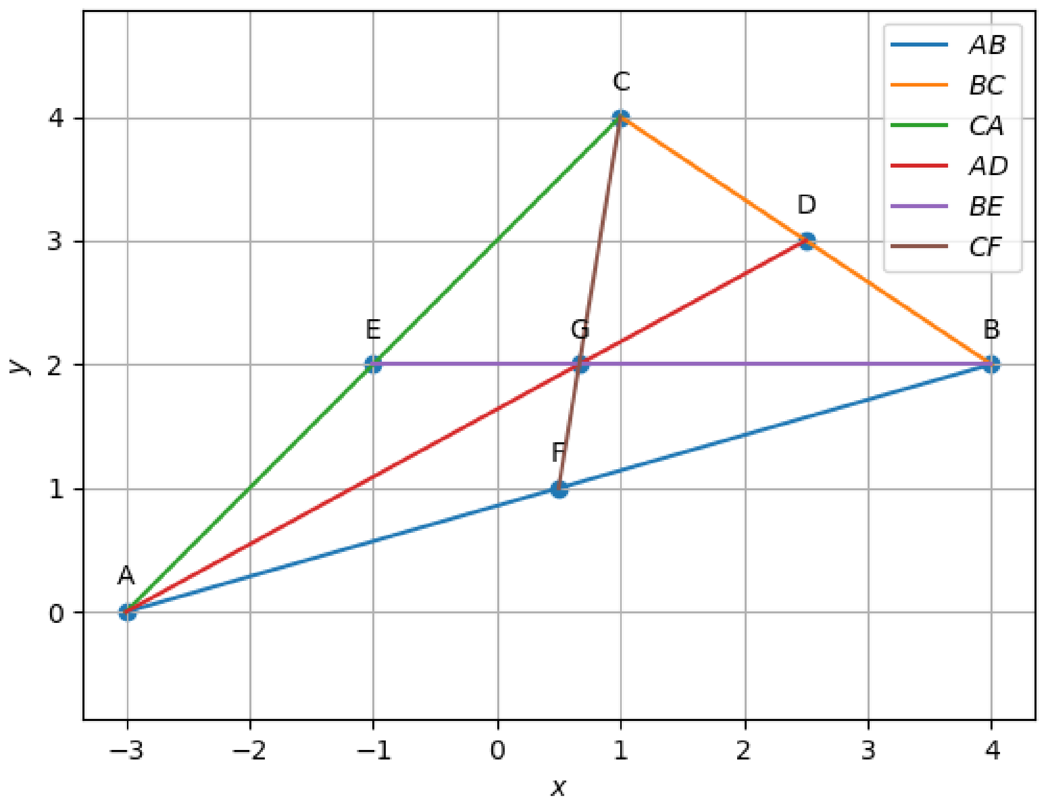

Figure 1.3: centroid of triangle ABC

1.6.3. Altitude

1.6.3.1. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B} \end{pmatrix} \tag{1.6.3.1.1}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.6.3.1.2}$$

$$= \begin{pmatrix} 1 & 5 & -4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.6.3.1.3}$$

Using Matrix multiplication

$$\mathbf{M}_2 = \begin{pmatrix} 9 & -5 & -4 \\ 4 & 0 & -4 \end{pmatrix} \tag{1.6.3.1.4}$$

1.6.3.2. Find the constants vector for the altitudes.

Solution: The desired vector is

$$\mathbf{c}_2 = \text{diag}\left\{ \left(\mathbf{M}^\top \mathbf{P} \right) \right\} \tag{1.6.3.2.1}$$

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} 9 & 4 \\ -5 & 0 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} 1 & 5 & -4 \\ 0 & 4 & 0 \end{pmatrix} \tag{1.6.3.2.2}$$

$$\tag{1.6.3.2.3}$$

Using matrix multiplication

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} 9 & 61 & -36 \\ -5 & -25 & 20 \\ -4 & -36 & 16 \end{pmatrix} \tag{1.6.3.2.4}$$

$$\mathbf{c}_2 = \text{diag}\left(\begin{pmatrix} -11 & 31 & -22 \\ -7 & -13 & -2 \\ 18 & -18 & 24 \end{pmatrix}\right) \tag{1.6.3.2.5}$$

$$\mathbf{c}_2 = \begin{pmatrix} 9 & -25 & 16 \end{pmatrix} \tag{1.6.3.2.6}$$

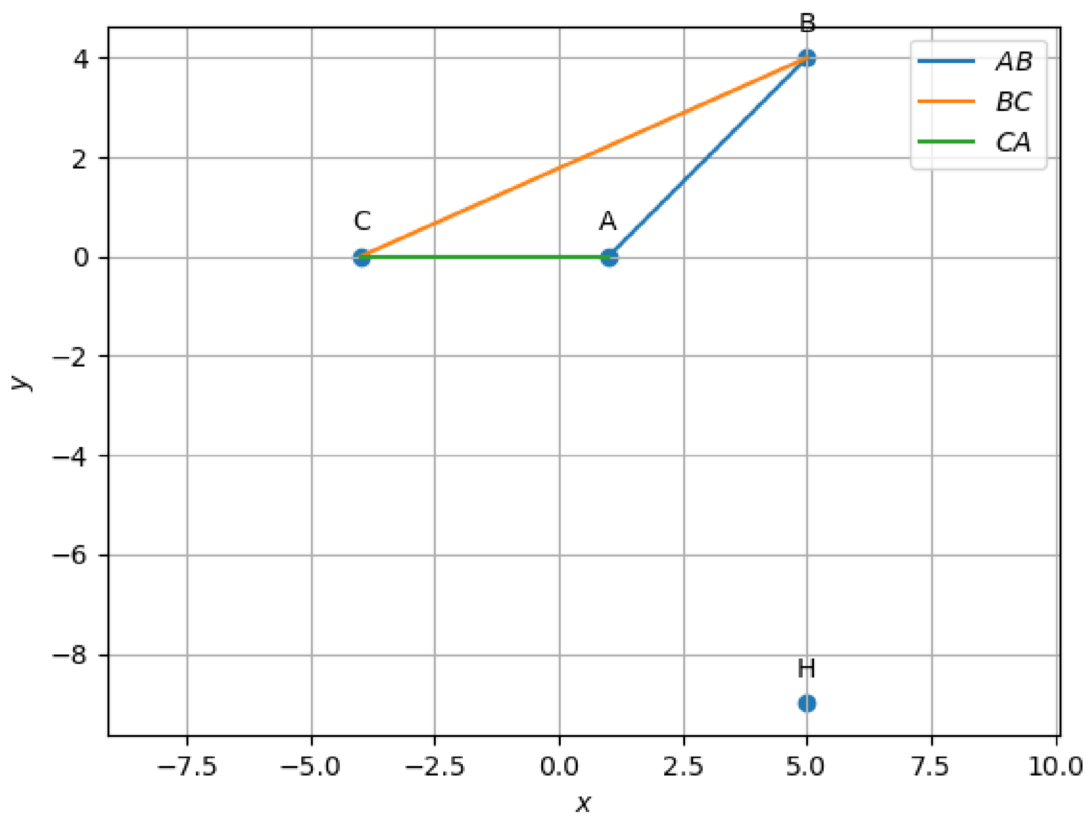

Figure 1.4: Ortho centre of △ ABC

1.6.4. Perpendicular Bisector

1.6.4.1. Find the normal matrix for the perpendicular bisectors

Solution: The normal matrix is $\mathbf{M}_2$

$$\mathbf{M}_2 = \begin{pmatrix} 9 & -5 & -4 \\ 4 & 0 & -4 \end{pmatrix} \tag{1.6.4.1.1}$$

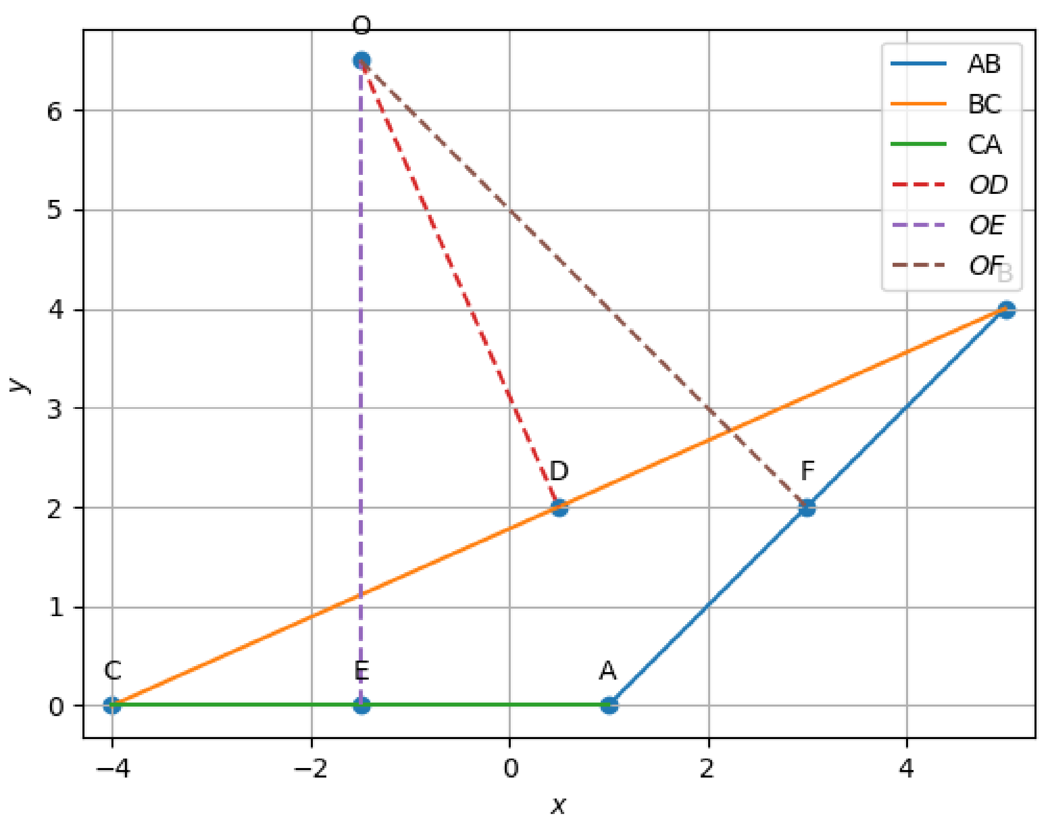

Figure 1.5: plot of perpendicular bisectors

1.6.4.2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right\} \tag{1.6.4.2.1}$$

Solution:

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right\} \tag{1.6.4.2.2}$$

$$\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ -5 & 0 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{-3}{2} & 3 \\ 2 & 0 & 2 \end{pmatrix} \tag{1.6.4.2.3}$$

$$\tag{1.6.4.2.4}$$

Using matrix multiplication

$$\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} \frac{25}{2} & \frac{-27}{2} & 35 \\ \frac{-5}{2} & \frac{15}{2} & -15 \\ -10 & 6 & -20 \end{pmatrix} \tag{1.6.4.2.5}$$

$$\mathbf{c}_3 = \text{diag}\left(\begin{pmatrix} \frac{25}{2} & \frac{-27}{2} & 35 \\ \frac{-5}{2} & \frac{15}{2} & -15 \\ -10 & 6 & -20 \end{pmatrix}\right) \tag{1.6.4.2.6}$$

$$\mathbf{c}_3 = \begin{pmatrix} \frac{25}{2} & \frac{15}{2} & -20 \end{pmatrix} \tag{1.6.4.2.7}$$

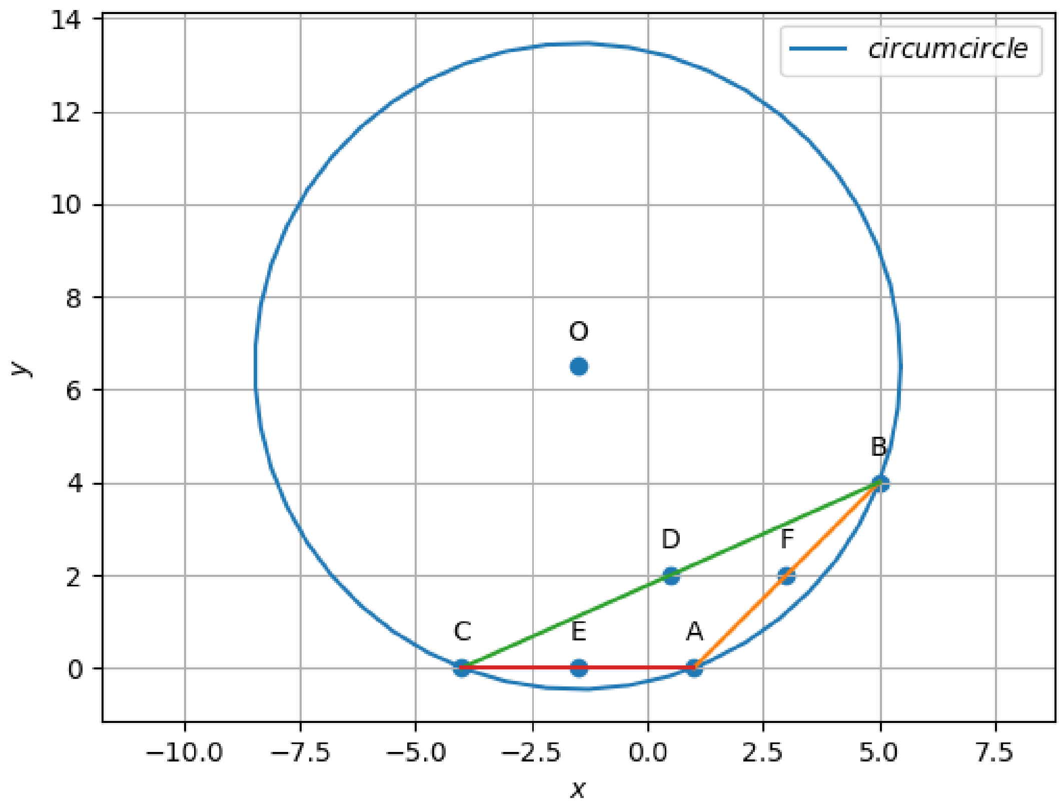

Figure 1.6: circumcentre and circumcircle of $\triangle$ ABC

1.6.5. Angle Bisector

1.6.5.1. Find the points of contact.

Solution: The points of contact are given by

$$\begin{pmatrix} \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} & \frac{m\mathbf{C}+n\mathbf{B}}{m+n} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \frac{n}{b} & \frac{m}{c} & 0 \\ 0 & \frac{p}{c} & \frac{n}{a} \\ \frac{p}{b} & 0 & \frac{m}{a} \end{pmatrix} \tag{1.6.5.1.1}$$

$$\begin{pmatrix} \mathbf{p} & \mathbf{m} & \mathbf{n} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.6.5.1.2}$$

$$= \frac{1}{2} \begin{pmatrix} \sqrt{97} & \sqrt{25} & \sqrt{32} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.6.5.1.3}$$

$$= \frac{1}{2} \begin{pmatrix} 9.848857802 & 5 & 5.65684549 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.6.5.1.4}$$

Using matrix multiplication

$$\begin{pmatrix} \mathbf{p} & \mathbf{m} & \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0.403993844 & 5.252851646 & 4.596006156 \end{pmatrix} \tag{1.6.5.1.5}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \frac{n}{b} & \frac{m}{c} & 0 \\ 0 & \frac{p}{c} & \frac{n}{a} \\ \frac{p}{b} & 0 & \frac{m}{a} \end{pmatrix} = \begin{pmatrix} 1 & 5 & -4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} \frac{4.596006156}{\sqrt{25}} & \frac{5.252851646}{\sqrt{32}} & 0 \\ 0 & \frac{0.403993844}{\sqrt{32}} & \frac{4.596006156}{\sqrt{97}} \\ \frac{0.403993844}{\sqrt{25}} & 0 & \frac{5.252851646}{\sqrt{97}} \end{pmatrix} \tag{1.6.5.1.6}$$

Using matrix multiplication We get the points of contact

$$= \begin{pmatrix} 0.596006156 & 1.28566722899 & 0.1998835027956 \\ 0 & 0.28566722899833 & 1.8666148901313 \end{pmatrix} \tag{1.6.5.1.7}$$

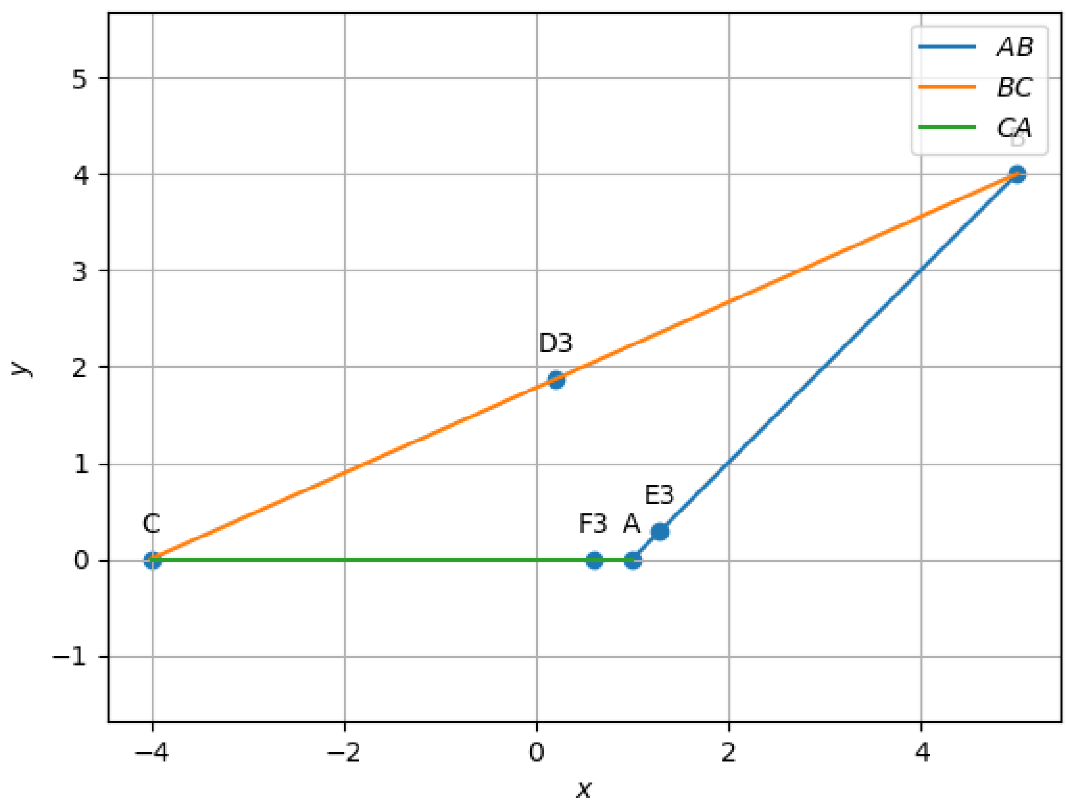

Figure 1.7: Contact points of incircle of *triangle* ABC

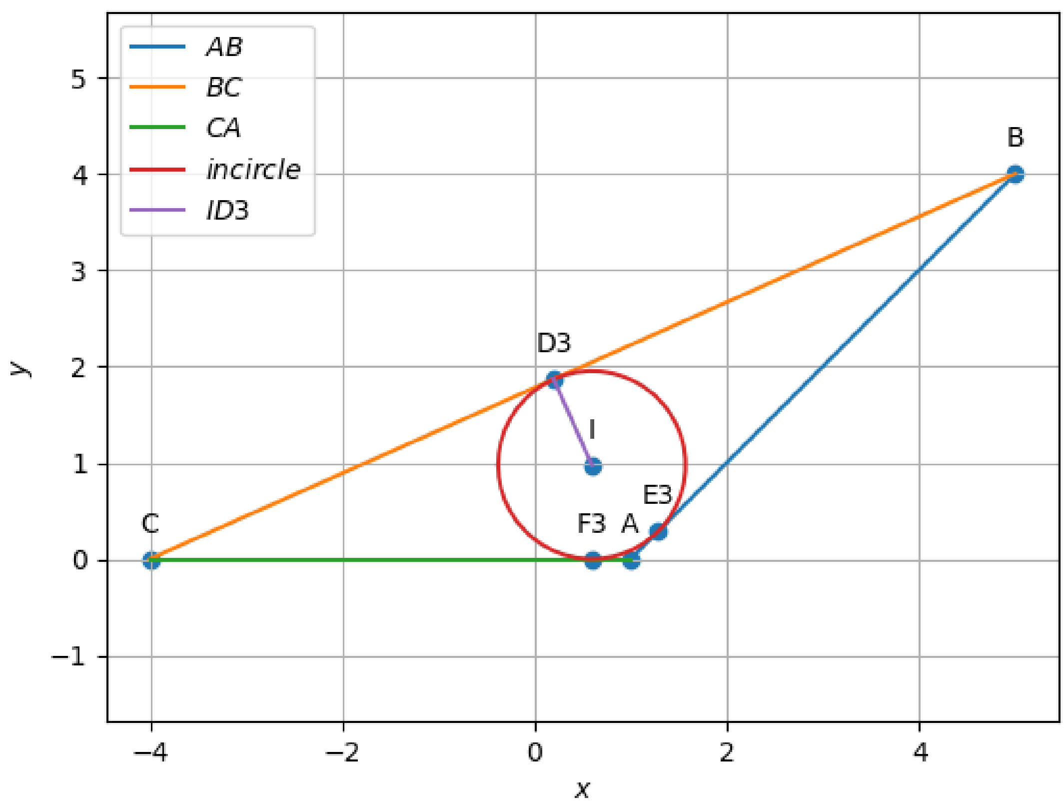

Figure 1.8: Incircle and Incentre of △ ABC