

NCERT 12.10.5.14

1. If A, B, C are mutually perpendicular texttors of equal magnitudes, show that the $A + B + C$ is equally inclined to A, B and C .

Solution:

Suppose we have the following vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \quad (1)$$

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \quad (2)$$

$$\mathbf{v}_3 = \begin{pmatrix} -5 \\ -2 \\ -1 \end{pmatrix} \quad (3)$$

Step 1: Initialize

Set $\mathbf{u}_1 = \mathbf{v}_1$:

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$$

Step 2: Orthogonalization

For \mathbf{v}_2 :

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 \quad (4)$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(\mathbf{u}_1^\top \mathbf{v}_2 \right) \mathbf{u}_1 \quad (5)$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(-\frac{3}{2} \right) \mathbf{u}_1 \implies \begin{pmatrix} 1.5 \\ 1.5 \\ 2 \end{pmatrix} \quad (6)$$

For \mathbf{v}_3 :

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 \quad (7)$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \left(\mathbf{u}_2^\top \mathbf{v}_3 \right) \mathbf{u}_2 - \left(\mathbf{u}_1^\top \mathbf{v}_3 \right) \mathbf{u}_1 \quad (8)$$

$$\mathbf{u}_3 = \mathbf{v}_3 - (-2.121) \mathbf{u}_1 - (-4.28) \cdot \mathbf{u}_2 \quad (9)$$

$$\implies \begin{pmatrix} -1.302 \\ -1.302 \\ 1.93 \end{pmatrix} \quad (10)$$

Step 3: Normalization

Normalize each vector:

$$\mathbf{u}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \quad (11)$$

$$\mathbf{u}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \quad (12)$$

$$\mathbf{u}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \quad (13)$$

The final orthonormal basis is:

$$\mathbf{u}_1 = \begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix} \quad (14)$$

$$\mathbf{u}_2 = \begin{pmatrix} 0.514 \\ 0.514 \\ 0.685 \end{pmatrix} \quad (15)$$

$$\mathbf{u}_3 = \begin{pmatrix} -0.487 \\ -0.487 \\ -0.724 \end{pmatrix} \quad (16)$$

$$(17)$$

Step 4: QR Decoposition

we calculate Q by means of Gram-Schmidt process

Q is an orthogonal matrix

$$Q = \begin{pmatrix} 0.707 & 0.514 & -0.487 \\ -0.707 & 0.514 & -0.487 \\ 0 & 0.685 & -0.724 \end{pmatrix}$$

To verify it as a orthonormal matrix we have to check this property i.e, $Q^\top \cdot Q = I$

$$\Rightarrow Q^\top Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 5: Findings angles A,B,C and $A + B + C$

$$A = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \quad (18)$$

$$B = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \quad (19)$$

$$C = \begin{pmatrix} -5 \\ -2 \\ -1 \end{pmatrix} \quad (20)$$

$$A + B + C = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \quad (21)$$

$$(22)$$

Normalize each vector:

$$\|A\| = \sqrt{9} = 3 \quad (23)$$

$$\|B\| = \sqrt{13} \quad (24)$$

$$\|C\| = \sqrt{30} \quad (25)$$

$$\|A + B + C\| = \sqrt{9} = 3 \quad (26)$$

Finding angles:

$$\cos \theta_1 = \frac{\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \begin{pmatrix} -2 & -2 & 1 \end{pmatrix}}{\sqrt{9}\sqrt{9}} \quad (27)$$

$$\cos \theta_1 = \frac{1}{9} \quad (28)$$

$$\theta_1 = \cos^{-1} \left(\frac{1}{9} \right) \Rightarrow 84^\circ \quad (29)$$

$$\cos \theta_2 = \frac{\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} -2 & -2 & 1 \end{pmatrix}}{\sqrt{13}\sqrt{9}} \quad (30)$$

$$\cos \theta_2 = \frac{-4}{3\sqrt{13}} \quad (31)$$

$$\theta_2 = \cos^{-1} \left(\frac{-4}{3\sqrt{13}} \right) \Rightarrow 111^\circ \quad (32)$$

$$\cos \theta_3 = \frac{\begin{pmatrix} -5 \\ -2 \\ -1 \end{pmatrix} \begin{pmatrix} -2 & -2 & 1 \end{pmatrix}}{\sqrt{30}\sqrt{9}} \quad (33)$$

$$\cos \theta_3 = \frac{13}{\sqrt{30}\sqrt{9}} \quad (34)$$

$$\theta_3 = \cos^{-1} \left(\frac{13}{\sqrt{30}\sqrt{9}} \right) \Rightarrow 38^\circ \quad (35)$$

$$\therefore \theta_1 \neq \theta_2 \neq \theta_3$$

Hence we can say that $A + B + C$ is not equally inclined to A,B and C