NCERT 12.10.5.14

1. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are mutually perpendicular vectors of equal magnitudes,show that the vector $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is equally inclined to $\mathbf{A}, \mathbf{B} and \mathbf{C}$.

   Construction Steps

   Since $\mathbf{A}, \mathbf{B} and \mathbf{C}$ are mutually Perpendicular vectors, we have $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} = 0$.
   It is given that :

   $$\left|\mathbf{A}\right| = \left|\mathbf{B}\right| = \left|\mathbf{C}\right|$$

   let vector $\mathbf{A}+\mathbf{B}+\mathbf{C}$ be inclined to $\mathbf{A}, \mathbf{B} and \mathbf{C}$ at angles $\theta_1, \theta_2 and \theta_3$ respectively.

   Then,we have:

   $$\|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^\top \mathbf{B} - \mathbf{A}} \tag{1}$$

   let u assume that vector $(\mathbf{A} + \mathbf{B} + \mathbf{C})$ as $\mathbf{P}$

   $\mathbf{A}, \mathbf{B}, \mathbf{C}$ points are $(2, 0), (2, 0), (2, 0)$ in three different axis and $(\mathbf{A} + \mathbf{B} + \mathbf{C})$ is $(2, 2)$

   let $\mathbf{O}$ be the origin and points are $(0, 0)$
   let us consider the $\angle\mathbf{POA}$

   $$\cos(\theta_1) = \frac{(\mathbf{P} - \mathbf{O})^\top (\mathbf{A} - \mathbf{O})}{\|\mathbf{P} - \mathbf{O}\| \|\mathbf{A} - \mathbf{O}\|} \tag{2}$$

   Finding angle POA

$$\mathbf{P}-\mathbf{O}=\begin{pmatrix}2\\2\end{pmatrix} \tag{3}$$

$$\mathbf{A}-\mathbf{O}=\begin{pmatrix}2\\0\end{pmatrix} \tag{4}$$

we know that

$$\begin{aligned}\|\mathbf{P}-\mathbf{O}\| &\triangleq \sqrt{(\mathbf{P}-\mathbf{O})^\top \mathbf{P}-\mathbf{O}}\\ \|\mathbf{A}-\mathbf{O}\| &\triangleq \sqrt{(\mathbf{A}-\mathbf{O})^\top \mathbf{A}-\mathbf{O}}\end{aligned} \tag{5}$$

$$\|\mathbf{P}-\mathbf{O}\| = \sqrt{8} = 2\sqrt{2} \tag{6}$$

$$\|\mathbf{A}-\mathbf{O}\| = \sqrt{4} = 2 \tag{7}$$

$$\cos(\theta_1) = \frac{(\mathbf{P}-\mathbf{O})^\top(\mathbf{A}-\mathbf{O})}{\|\mathbf{P}-\mathbf{O}\|\|\mathbf{A}-\mathbf{O}\|} \tag{8}$$

$$\implies \frac{\begin{pmatrix}2 & 2\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}}{4\sqrt{2}} \tag{9}$$

$$\implies \frac{1}{\sqrt{2}} \tag{10}$$

$$\angle POA = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \tag{11}$$

let us consider the $\angle\mathbf{POB}$

$$\cos(\theta_2) = \frac{(\mathbf{P}-\mathbf{O})^\top(\mathbf{B}-\mathbf{O})}{\|\mathbf{P}-\mathbf{O}\|\|\mathbf{B}-\mathbf{O}\|} \tag{12}$$

Finding angle **POB**

$$\mathbf{P}-\mathbf{O}=\begin{pmatrix}2\\2\end{pmatrix} \tag{13}$$

$$\mathbf{B}-\mathbf{O}=\begin{pmatrix}2\\0\end{pmatrix} \tag{14}$$

we know that

$$\begin{aligned}\|\mathbf{P}-\mathbf{O}\| &\triangleq \sqrt{(\mathbf{P}-\mathbf{O})^\top \mathbf{P}-\mathbf{O}}\\ \|\mathbf{B}-\mathbf{O}\| &\triangleq \sqrt{(\mathbf{B}-\mathbf{O})^\top \mathbf{B}-\mathbf{O}}\end{aligned} \tag{15}$$

$$\|\mathbf{P}-\mathbf{O}\| = \sqrt{8} = 2\sqrt{2} \tag{16}$$

$$\|\mathbf{B}-\mathbf{O}\| = \sqrt{4} = 2 \tag{17}$$

$$\cos(\theta_2) = \frac{(\mathbf{P}-\mathbf{O})^\top(\mathbf{B}-\mathbf{O})}{\|\mathbf{P}-\mathbf{O}\|\|\mathbf{B}-\mathbf{O}\|} \tag{18}$$

$$\implies \frac{\begin{pmatrix}2 & 2\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}}{4\sqrt{2}} \tag{19}$$

$$\implies \frac{1}{\sqrt{2}} \tag{20}$$

$$\angle\mathbf{POB} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \tag{21}$$

let us consider the $\angle\mathbf{POC}$

$$\cos(\theta_3) = \frac{(\mathbf{P}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})}{\|\mathbf{P}-\mathbf{O}\|\left\|\mathbf{C}-\mathbf{O}\right\|} \tag{22}$$

Finding angle **POC**

$$\mathbf{P} - \mathbf{O} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{23}$$

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{24}$$

we know that

$$\begin{aligned} \|\mathbf{P} - \mathbf{O}\| &\triangleq \sqrt{(\mathbf{P} - \mathbf{O})^\top \mathbf{P} - \mathbf{O}} \\ \left\|\mathbf{C} - \mathbf{O}\right\| &\triangleq \sqrt{(\mathbf{C} - \mathbf{O})^\top \mathbf{C} - \mathbf{O}} \end{aligned} \tag{25}$$

$$\|\mathbf{P} - \mathbf{O}\| = \sqrt{8} = 2\sqrt{2} \tag{26}$$

$$\left\|\mathbf{C} - \mathbf{O}\right\| = \sqrt{4} = 2 \tag{27}$$

$$\cos(\theta_3) = \frac{(\mathbf{P} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O})}{\|\mathbf{P} - \mathbf{O}\| \left\|\mathbf{C} - \mathbf{O}\right\|} \tag{28}$$

$$\implies \frac{\begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}}{4\sqrt{2}} \tag{29}$$

$$\implies \frac{1}{\sqrt{2}} \tag{30}$$

$$\angle\mathbf{POC} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \tag{31}$$

$\therefore \angle\mathbf{POA} = \angle\mathbf{POB} = \angle\mathbf{POC}$
Hence here all the angles are equal so they are equally inclined to each other