NCERT 12.10.5.14

1. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are mutually perpendicular vectors of equal magnitudes,show that the $\mathbf{A}+\mathbf{B}+\mathbf{C}$ is equally inclined to $\mathbf{A}, \mathbf{B}$and$\mathbf{C}$.

   Solution:

   Suppose we have the following vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

   Step 1: Initialize

   Set $\mathbf{u}_1 = \mathbf{v}_1$:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

   Step 2: Orthogonalization

For $\mathbf{v}_2$:

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 \tag{1}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(\mathbf{u}_1^\top \mathbf{v}_2 \right) \mathbf{u}_1 \tag{2}$$

$$\mathbf{u}_2 \implies \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{3}$$

For $\mathbf{v}_3$:

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 \tag{4}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \left(\mathbf{u}_2^\top \mathbf{v}_3 \right) \mathbf{u}_2 - \left(\mathbf{u}_1^\top \mathbf{v}_3 \right) \mathbf{u}_1 \tag{5}$$

$$\mathbf{u}_3 \implies \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \tag{6}$$

Step 3: Normalization

Normalize each vector:

$$\mathbf{u}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \tag{7}$$

$$\mathbf{u}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \tag{8}$$

$$\mathbf{u}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \tag{9}$$

The final orthonormal basis is:

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\implies \begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix} \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix} \mathbf{u}_3 = \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$\implies \begin{pmatrix} -0.408 \\ -0.408 \\ 0.816 \end{pmatrix}$$

Step 4: QR Decoposition

we calculate Q by means of Gram–Schmidt process

Q is an orthogonal matrix

$$Q = \begin{pmatrix} 0.577 & 0.707 & -0.408 \\ 0.577 & -0.707 & -0.408 \\ 0.577 & 0 & 0.816 \end{pmatrix}$$

To verify it as a orthonormal matrix we have to check this property i.e, $Q^\top.Q = I$

$$\implies Q^\top Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 5: Findings angles between $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \text{and} \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$

$$\mathbf{u}_1 = \begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix} \tag{10}$$

$$\mathbf{u}_2 = \begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix} \tag{11}$$

$$\mathbf{u}_3 = \begin{pmatrix} -0.408 \\ -0.408 \\ 0.816 \end{pmatrix} \tag{12}$$

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \implies \mathbf{y} = \begin{pmatrix} 0.876 \\ -0.538 \\ 1.393 \end{pmatrix} \tag{13}$$

Normalize each vector:

$$\|\mathbf{u}_1\| = 0.9987 \tag{14}$$

$$\|\mathbf{u}_2\| = 0.9998 \tag{15}$$

$$\|\mathbf{u}_3\| = 0.9987 \tag{16}$$

$$\|\mathbf{y}\| = 2.9972 \tag{17}$$

Finding angles:

$$\cos\theta_1 = \frac{\begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix} \begin{pmatrix} 0.876 & -0.538 & 1.393 \end{pmatrix}}{(0.9987)\,(2.9972)} \tag{18}$$

$$\cos\theta_1 = \frac{1}{3} \tag{19}$$

$$\theta_1 = \cos^{-1}\left(\frac{1}{3}\right) \implies 70^\circ \tag{20}$$

$$\cos\theta_1 = \frac{\begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix} \begin{pmatrix} 0.876 & -0.538 & 1.393 \end{pmatrix}}{(0.99967)\,(2.9972)} \tag{21}$$

$$\cos\theta_1 = \frac{1}{3} \tag{22}$$

$$\theta_1 = \cos^{-1}\left(\frac{1}{3}\right) \implies 70^\circ \tag{23}$$

$$\cos\theta_1 = \frac{\begin{pmatrix} -0.408 \\ -0.408 \\ 0.816 \end{pmatrix} \begin{pmatrix} 0.876 & -0.538 & 1.393 \end{pmatrix}}{(0.9987)\,(2.9972)} \tag{24}$$

$$\cos\theta_1 = \frac{1}{3} \tag{25}$$

$$\theta_1 = \cos^{-1}\left(\frac{1}{3}\right) \implies 70^\circ \tag{26}$$

$$\therefore \theta_1 = \theta_2 = \theta_3$$

Hence we can say that $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$ is equally inclined to $\mathbf{u}_1, \mathbf{u}_2 \text{and} \mathbf{u}_3$