NCERT 12.10.5.14

1. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are mutually perpendicular vectors of equal magnitudes,show that the $\mathbf{A}+\mathbf{B}+\mathbf{C}$ is equally inclined to $\mathbf{A}, \mathbf{B}\text{and}\mathbf{C}$.

   Solution:

   Suppose we have the following vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} -5 \\ -2 \\ -1 \end{pmatrix}$$

   Step 1: Initialize

   Set $\mathbf{u}_1 = \mathbf{v}_1$:

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$$

   Step 2: Orthogonalization

For $\mathbf{v}_2$:

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 \tag{1}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(\mathbf{u}_1^\top \mathbf{v}_2 \right) \mathbf{u}_1 \tag{2}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(-\frac{3}{2} \right) \mathbf{u}_1 \implies \begin{pmatrix} 1.5 \\ 1.5 \\ 2 \end{pmatrix} \tag{3}$$

For $\mathbf{v}_3$:

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 \tag{4}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \left(\mathbf{u}_2^\top \mathbf{v}_3 \right) \mathbf{u}_2 - \left(\mathbf{u}_1^\top \mathbf{v}_3 \right) \mathbf{u}_1 \tag{5}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - (-2.121) \, \mathbf{u}_1 - (-4.28) \cdot \mathbf{u}_2 \tag{6}$$

$$\implies \begin{pmatrix} -1.302 \\ -1.302 \\ 1.93 \end{pmatrix} \tag{7}$$

Step 3: Normalization

Normalize each vector:

$$\mathbf{u}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \tag{8}$$

$$\mathbf{u}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \tag{9}$$

$$\mathbf{u}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \tag{10}$$

The final orthonormal basis is:

$$\mathbf{u}_1 = \begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} 0.514 \\ 0.514 \\ 0.685 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} -0.487 \\ -0.487 \\ -0.724 \end{pmatrix}$$

Step 4: QR Decoposition

we calculate Q by means of Gram–Schmidt process

Q is an orthogonal matrix

$$Q = \begin{pmatrix} 0.707 & 0.514 & -0.487 \\ -0.707 & 0.514 & -0.487 \\ 0 & 0.685 & -0.724 \end{pmatrix}$$

To verify it as a orthonormal matrix we have to check this property i.e, $Q^\top.Q = I$

$$\implies Q^\top Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 5: Findings angles $\mathbf{A}, \mathbf{B}, \mathbf{C} \text{and} \mathbf{A}+\mathbf{B}+\mathbf{C}$

$$\mathbf{A} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \tag{11}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \tag{12}$$

$$\mathbf{C} = \begin{pmatrix} -5 \\ -2 \\ -1 \end{pmatrix} \tag{13}$$

$$\mathbf{A}+\mathbf{B}+\mathbf{C} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \tag{14}$$

Normalize each vector:

$$\|\mathbf{A}\| = \sqrt{9} = 3 \tag{15}$$

$$\|\mathbf{B}\| = \sqrt{13} \tag{16}$$

$$\|\mathbf{C}\| = \sqrt{30} \tag{17}$$

$$\|\mathbf{A}+\mathbf{B}+\mathbf{C}\| = \sqrt{9} = 3 \tag{18}$$

Finding angles:

$$\cos\theta_1 = \frac{\begin{pmatrix}3\\-3\\0\end{pmatrix}\begin{pmatrix}-2 & -2 & 1\end{pmatrix}}{\sqrt{9}\sqrt{9}} \tag{19}$$

$$\cos\theta_1 = \frac{1}{9} \tag{20}$$

$$\theta_1 = \cos^{-1}\left(\frac{1}{9}\right) \implies 84^\circ \tag{21}$$

$$\cos\theta_2 = \frac{\begin{pmatrix}0\\3\\2\end{pmatrix}\begin{pmatrix}-2 & -2 & 1\end{pmatrix}}{\sqrt{13}\sqrt{9}} \tag{22}$$

$$\cos\theta_2 = \frac{-4}{3\sqrt{13}} \tag{23}$$

$$\theta_2 = \cos^{-1}\left(\frac{-4}{3\sqrt{13}}\right) \implies 111^\circ \tag{24}$$

$$\cos\theta_3 = \frac{\begin{pmatrix}-5\\-2\\-1\end{pmatrix}\begin{pmatrix}-2 & -2 & 1\end{pmatrix}}{\sqrt{30}\sqrt{9}} \tag{25}$$

$$\cos\theta_3 = \frac{13}{\sqrt{30}\sqrt{9}} \tag{26}$$

$$\theta_3 = \cos^{-1}\left(\frac{13}{\sqrt{30}\sqrt{9}}\right) \implies 38^\circ \tag{27}$$

$$\therefore \theta_1 \neq \theta_2 \neq \theta_3$$

Hence we can say that $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is not equally inclined to $\mathbf{A}, \mathbf{B}$and$\mathbf{C}$