

## NCERT 12.10.5.14

1. If  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are mutually perpendicular vectors of equal magnitudes, show that the  $\mathbf{A} + \mathbf{B} + \mathbf{C}$  is equally inclined to  $\mathbf{A}, \mathbf{B}$  and  $\mathbf{C}$ .

**Solution:**

Suppose we have the following vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

**Step 1: Initialize**

Set  $\mathbf{u}_1 = \mathbf{v}_1$ :

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

**Step 2: Orthogonalization**

For  $\mathbf{v}_2$  :

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 \quad (1)$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left( \mathbf{u}_1^\top \mathbf{v}_2 \right) \mathbf{u}_1 \quad (2)$$

$$\mathbf{u}_2 \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad (3)$$

For  $\mathbf{v}_3$ :

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 \quad (4)$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \left( \mathbf{u}_2^\top \mathbf{v}_3 \right) \mathbf{u}_2 - \left( \mathbf{u}_1^\top \mathbf{v}_3 \right) \mathbf{u}_1 \quad (5)$$

$$\mathbf{u}_3 \Rightarrow \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \quad (6)$$

### Step 3: Normalization

Normalize each vector:

$$\mathbf{u}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \quad (7)$$

$$\mathbf{u}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \quad (8)$$

$$\mathbf{u}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \quad (9)$$

The final orthonormal basis is:

$$\begin{aligned}\mathbf{u}_1 &= \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \Rightarrow \begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix} \\ \mathbf{u}_2 &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix} \\ \mathbf{u}_3 &= \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} \Rightarrow \begin{pmatrix} -0.408 \\ -0.408 \\ 0.816 \end{pmatrix}\end{aligned}$$

#### Step 4: QR Decoposition

we calculate Q by means of Gram-Schmidt process

Q is an orthogonal matrix

$$Q = \begin{pmatrix} 0.577 & 0.707 & -0.408 \\ 0.577 & -0.707 & -0.408 \\ 0.577 & 0 & 0.816 \end{pmatrix}$$

To verify it as a orthonormal matrix we have to check this property i.e,  $Q^\top \cdot Q = I$

$$\Rightarrow Q^\top Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Step 5: Findings angles between  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  and  $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$**

$$\mathbf{u}_1 = \begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix} \quad (10)$$

$$\mathbf{u}_2 = \begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix} \quad (11)$$

$$\mathbf{u}_3 = \begin{pmatrix} -0.408 \\ -0.408 \\ 0.816 \end{pmatrix} \quad (12)$$

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \implies \mathbf{y} = \begin{pmatrix} 0.876 \\ -0.538 \\ 1.393 \end{pmatrix} \quad (13)$$

Normalize each vector:

$$\|\mathbf{u}_1\| = 0.9987 \quad (14)$$

$$\|\mathbf{u}_2\| = 0.9998 \quad (15)$$

$$\|\mathbf{u}_3\| = 0.9987 \quad (16)$$

$$\|\mathbf{y}\| = 2.9972 \quad (17)$$

Finding angles:

$$\cos \theta_1 = \frac{\begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix} \begin{pmatrix} 0.876 & -0.538 & 1.393 \end{pmatrix}}{(0.9987) (2.9972)} \quad (18)$$

$$\cos \theta_1 = \frac{1}{3} \quad (19)$$

$$\theta_1 = \cos^{-1} \left( \frac{1}{3} \right) \Rightarrow 70^\circ \quad (20)$$

$$\cos \theta_1 = \frac{\begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix} \begin{pmatrix} 0.876 & -0.538 & 1.393 \end{pmatrix}}{(0.99967) (2.9972)} \quad (21)$$

$$\cos \theta_1 = \frac{1}{3} \quad (22)$$

$$\theta_1 = \cos^{-1} \left( \frac{1}{3} \right) \Rightarrow 70^\circ \quad (23)$$

$$\cos \theta_1 = \frac{\begin{pmatrix} -0.408 \\ -0.408 \\ 0.816 \end{pmatrix} \begin{pmatrix} 0.876 & -0.538 & 1.393 \end{pmatrix}}{(0.9987) (2.9972)} \quad (24)$$

$$\cos \theta_1 = \frac{1}{3} \quad (25)$$

$$\theta_1 = \cos^{-1} \left( \frac{1}{3} \right) \Rightarrow 70^\circ \quad (26)$$

$$\therefore \theta_1 = \theta_2 = \theta_3$$

Hence we can say that  $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$  is equally inclined to  $\mathbf{u}_1, \mathbf{u}_2$  and  $\mathbf{u}_3$