NCERT 12.10.5.14

1. If $\mathrm{A}, \mathrm{B}, \mathrm{C}$ are mutually perpendicular texttors of equal magnitudes,show that the $\mathrm{A}+\mathrm{B}+\mathrm{C}$ is equally inclined to $\mathrm{A}, \mathrm{B} and \mathrm{C}$.

   Suppose we have the following vectors:

$$\mathbf{v}_1 = [3, -3, 0] \tag{1}$$

$$\mathbf{v}_2 = [0, 3, 2] \tag{2}$$

$$\mathbf{v}_3 = [-5, -2, -1] \tag{3}$$

   Step 1: Initialize

   Set $\mathbf{u}_1 = \mathbf{v}_1$:

   $\mathbf{u}_1 = [3, -3, 0]$

   Step 2: Orthogonalization

   For $\mathbf{v}_2$:

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 \tag{4}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(\mathbf{u}_1^\top \mathbf{v}_2 \right) \mathbf{u}_1 \tag{5}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(-\frac{3}{2} \right) \mathbf{u}_1 \implies [1.5, 1.5, 2] \tag{6}$$

For $\mathbf{v}_3$:

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 \tag{7}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \left(\mathbf{u}_2^\top \mathbf{v}_3\right)\mathbf{u}_2 - \left(\mathbf{u}_1^\top \mathbf{v}_3\right)\mathbf{u}_1 \tag{8}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - (-2.121)\,\mathbf{u}_1 - (-4.28) \cdot \mathbf{u}_2 \tag{9}$$

$$\implies [-1.302, -1.302, 1.93] \tag{10}$$

Step 3: Normalization

Normalize each vector:

$$\mathbf{u}_1 = \frac{\mathbf{u}_1}{\mathbf{u}_1} \tag{11}$$

$$\mathbf{u}_2 = \frac{\mathbf{u}_2}{\mathbf{u}_2} \tag{12}$$

$$\mathbf{u}_3 = \frac{\mathbf{u}_3}{\mathbf{u}_3} \tag{13}$$

The final orthonormal basis is:

$$\mathbf{u}_1 = [0.707, -0.707, 0] \tag{14}$$

$$\mathbf{u}_2 = [0.514, 0.514, 0.685] \tag{15}$$

$$\mathbf{u}_3 = [-0.487, -0.487, -0.724] \tag{16}$$

$$\tag{17}$$

Step 4: QR Decoposition

we calculate Q by means of Gram–Schmidt process

Q is an orthogonal matrix

$$Q = \begin{pmatrix} 0.707 & 0.514 & -0.487 \\ -0.707 & 0.514 & -0.487 \\ 0 & 0.685 & -0.724 \end{pmatrix}$$

To verify it as a orthonormal matrix we have to check this property i.e, $Q^\top . Q = I$

$$\implies Q^\top Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence we can say that $\mathrm{A} + \mathrm{B} + \mathrm{C}$ is equally inclined to $\mathrm{A}, \mathrm{B} and \mathrm{C}$