OLYMPIAD MATH

Made Simple

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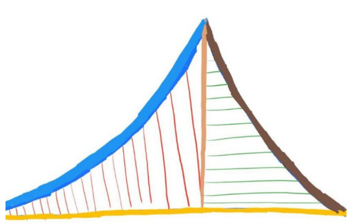

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This book links high school coordinate geometry to linear algebra and matrix analysis through solved problems.

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Probability

1. Find the number of triples (a, b, c) of positive integers such that

   (a) ab is a prime;

   (b) bc is a product of two primes;

   (c) abc is not divisible by square of any prime and

   (d) $abc \leq 30$.

(IOQM 2015)

1. A postman has to deliver ive letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how rnany different way could he do this so that exactly two of the five houses receive the correct letters? (PRERMO 2012)

Chapter 5

permutation and combination

1. A positive integer $n > 1$ is called beautiful if n can be written in one and only one way as $n = a_1 + a_2 + \cdots + a_k = a_1 \cdot a_2 \cdots a_k$for some positive integers $a_1, a_2, \cdots, a_k$, where $k > 1$ and$a_1 \geq a_2 \geq \cdots \geq a_k$. (For example 6 is beautiful since$6 = 3 \cdot 2 \cdot 1 = 3 + 2 + 1$, and this is unique. But 8 is not beautiful since $8 = 4 + 2 + 1 + 1 = 4 \cdot 2 \cdot 1 \cdot 1$ as well as $8 = 2 + 2 + 2 + 1 + 1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$, souniqueness is lost.) Find the largest beautiful number less than 100. (IOQM 2015)

2. For $n \in N$, consider non-negative integer-valued functions f on $\{1, 2, \cdots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n} (i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case? (IOQM 2015)

3. In the land of Binary, the unit of currency is called Ben and currency notes areavailable in denominations $1, 2, 2^2, 2^3, \cdots$ Bens. The rules of the Government of Binary stipulate that one can not use more than

two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government. (IOQM 2015)

4. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a die that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed. (IOQM 2015)

5. Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling a 2×7 rectangle using some of these tiles.(IOQM 2015)

6. Consider the set

$$S = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\} \tag{5.1}$$

where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of S, find the largest integer less than or equal to D. (IOQM 2015)

7. Let P be a convex polygon with 50 vertices. A set F of diagonals of P is said to be minimally friendly if any diagonal $d \in F$ intersects at most one other diagonal in F at a point interior to P. Find the largest possible number of elements in a minimally friendly set F. (IOQM 2015)

8. Find all pairs (k, n) of positive integers such that

$$k! = (2n-1)(2n-2)(2n-4)\cdots(2n-2n+1)\,. \tag{5.2}$$

(IMO 2019)

9. There are 4n pebbles of weights 1,2,3.....,4n.Each pebble is coloured in one of n colours and there are four pebbles of each colour.Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

   The total weights of both piles are the same. Each pile contains two pebbles of each colour. (IMO 2020)

10. Two squirrles,Bushy and jumpy,have collected 2021 walnuts for the winter .jumpy numbers the walnuts from 1 through 2021,and digs 2021 little holes in a circular pattern in the ground around their favourite tree.The next morning jumpy notices that bushy had placed one walnut into each hole ,but had paid no attention to the numbering .unhappy,Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves.In the k-th move,jumpy swaps the positions of the two walnuts adjacent to walnut k.Prove that there exists a value of k such

that ,on the k-th move,jumpy swaps some walnuts a and b such that a<k<b. (IMO 2021)

11. Twenty-one girls and twenty-one boys took part in a mathematical contest. Each contestant solved a t most six problems. For each girl and each boy, at least one problem was solved by both of them. Prove t hat there was a problem that was solved by at least three girls and at least three boys. (IMO 2001)

12. S is the set $\{1, 2, 3, \ldots, 1000000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $\{a + x_i a \in A\}$ are all pairwise disjoint. (IMO 2003)

13. S is the set of all (h, k) wi th h, k non-negative integers such that $h + k\ textlessn$. Each element of S is colored red or b lue, so that if (h, k) is red and $h' \leq h, k'\ leqk$, then (h', k') is also red. A type 1 subset of S has n blue elements with differen t first member and a type 2 subset of S has n blue elements with different second member. Show tha t there are the same number of type 1 and type 2 subsets. (IMO 2002)

14. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x,y,z respectively and $y < 0$ then the following operation is allowed: the numbers x,y,z are replaced by $x + y$,$-y$,$z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the live numbers is negative. Determine

whether this procedure necessarily comes to and end after a finite number of steps. (IMO 1986)

15. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1? (IMO 1986)

16. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all 0, such that $\left| a_i \right| \leq k - 1$ for all i and

$$\left| a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$$

(IMO 1987)

17. Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{n/3}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$ (IMO 1987)

18. Problem 4. Let $n \geq 3$ be an integer, and consider a circle with $n + 1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1,\ ldotsn$ such that each label is used exactly once, two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is

called beautiful if, for any four labels $a < b < c < d$ with $a+d = b+c$, the chord joining the points labelled a and d does not intersect the chord joi ning the points labelled b and c Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq nandgcd\,(x, y) = 1$. Prove that $m = n + 1$

1. How many line segments have both their endpoints located at the vertices of a given cube? (PRERMO 2015)

2. Let $E(n)$ denote the sum of the even digits of n. For example, $E(1243) = 2 + 4 = 6$. What is the value of $E(1) + E(2) + E(3) + \cdots + E(100)$? (PRERMO 2015)

3. At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many women attended the party? (PRERMO 2015)

Chapter 6

Construction

Chapter 7

Optimization

Chapter 8

Algebra

1. Let x, y be positive integers such that

$$x^4 = (x-1)\left(y^3 - 23\right) - 1. \tag{8.1}$$

Find the maximum possible value of $x + y$. (IOQM 2015)

2. The ex-radii of a triangle are $10\frac{1}{2}$, $12, 12$ and 14. If the sides of the triangle are the roots of the cubic

$$x^3 - px^2 + qx - r = 0, \tag{8.2}$$

where p, q, r are integers, find the integer nearest to $\sqrt{\{p+q+r\}}$. (IOQM 2015)

3. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_{31} + p_{32} + p_{33} = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$. (IOQM 2015)

4. A positive integer m has the property that m^2 is expressible in the form

$4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$. (IOQM 2015)

5. Find the least positive integer n such that there are at least 1000 unordered pairs of diagonals in a regular polygon with n vertices that intersect at a right angle in the interior of the polygon. (IOQM 2015)

6. Let $d(m)$ denote the number of positive integer divisors of a positive integer m. If r is the number of integers $n \leq 2023$ for which $\sum_{i=1}^{n} d\left(i\right)$ is odd, find the sum of the digits of r. (IOQM 2015)

7. Let Z be the set of integers. We want to determine all functions f : $Z \rightarrow Z$ such that for all integers a and b : $f\left(2a\right)+2f\left(b\right) = f\left(f\left(a+b\right)\right)$ (IMO 2019)

8. . A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time: Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged. Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user. (IMO 2019)

9. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right.

He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^th coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$, the process starting with the configuration THT would be: $THT \to HHT$ $\to HTT \to TTT$,which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

(IMO 2019)

10. A deck of n>1 cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal? (IMO 2020)

11. Let $n \geq 100$ be an integer.Ivan writes the numbers n,n+1,....,2n each on different cards.He then shuffles these n+1 cards,and divides them into two piles.prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square. (IMO 2021)

12. Let $m \geq 2$ be an integer,A be a finite set of (not necessarily positive)integers,and $B_1, B_2, B_3...B_m$ be subsets of A.Assume that for each

k=1,2,....,m the sum of the elements of B_k is m^k .Prove that A contains at least m/2 elements (IMO 2021)

13. The real numbers a,b,c,d are such that $a \geq b \geq c \geq d > 0$ and a+b+c+d=1. prove that

$$(a + 2b + 3c + 4d)\, a^a b^b c^c d^d < 1 \tag{8.3}$$

(IMO 2020)

14. Show that the inequality $\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}$ holds for all real numbers x1,....xn (IMO 2021)

15. Find all triples (a, b, p) of positive integers with (p) prime and Prove that:

$$(a^p = b! + p).$$

(IMO 2022)

16. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2.$$

(IMO 2022)

17. Let k be a positive integer and let S be a finite set of odd prime numbers. Prove there is at most one way (up to rotation and reflection) to place the elements of S around a circle such that the product of any two neighbours is of the form $x^2 + x + k$ for some positive integer x. (IMO 2022)

18. Determine all composite integers $n \geq 1$ that satisfy the following property: if $d_1, d_2, ..., d_k$ are all the positive divisors of n with $1 = d_1 \leq d_2 \leq \cdots \leq d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k-2$. (IMO 2023)

19. For each integer $k \geq 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$ where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2} \cdots a_{n+k}$$

(IMO 2023)

20. Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$. (IMO 2023)

21. Determine all real numbers such that, for every positive integer n, the integer

$$[\alpha] + [2\alpha] + \cdots + [\alpha]$$

is a multiple of n. Note that $[z]$ denotes the greatest integer less than or equal to z. For example $[-\pi] = -4$ and $[2] = [2.9] = 2$. (IMO 2024)

22. Let $\mathbb{Q}$ be the set of rational numbers. A function $f : \mathbb{Q} \to \mathbb{Q}$ is called aquaesulian if the following property holds: for every $x, y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \quad \text{or} \quad f(f(x) + y) = x + f(y).$$

Show that there exists an integer c such that for any aquaesulian function f there are at most c different rational numbers of the form $f(r) + f(-r)$ for some rational number r, and find the smallest possible value of c. (IMO 2024)

23. Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$. What is the value of $\sum_{n=1}^{90} \frac{1}{S_n+S_{n-1}}$? (Prermo 2013)

24. There are $n-1$ red balls, n green balls, and $n+1$ blue balls in a bag. The number of ways of choosing two balls from the bag that have different colours is 299. What is the value of n? (Prermo 2013)

25. To each element of the set $S = \{1, 2, \ldots, 1000\}$ a color is assigned. Suppose that for any two elements a, b of S, if 15 divides $a + b$, then they are both assigned the same color. What is the maximum possible number of distinct colors used? (Prermo 2013)

26. Let Akbar and Birbal together have n marbles, where $n > 0$. Akbar says to Birbal, "If I give you some marbles, then you will have twice as many marbles as I will have." Birbal says to Akbar, "If I give you some marbles, then you will have thrice as many marbles as I will have." What is the minimum possible value of n for which the above statements are true? (Prermo 2013)

27. Carol was given three numbers and was asked to add the largest of the three to the product of the other two. Instead, she multiplied the largest with the sum of the other two, but still got the right answer. What is the sum of the three numbers? (Prermo 2013)

28. Three real numbers x, y, z are such that $x^2 + 6y = -17$, $y^2 + 4z = 1$, and $x^2 + 2x = 2$. What is the value of $x^2 + y^2 + z^2$? (Prermo 2013)

29. Let $f(x) = x^3 - 3x + b$ and $g(x) = x^2 + bx - 3$, where b is a real number. What is the sum of all b for which $f(x) = 0$ and $g(x) = 0$ have a common root? (Prermo 2013)

30. Find all pairs (m, n) of positiv e integers such that $\frac{m^2}{2mn^2 - n^3 + 1}$ is a positive integer. (IMO 2003)

31. Given n>2 and reals $x_1 \leq x_2 \leq \ldots \leq x_n$, show that $\left(\sum_{ij} \left|x_i x_j\right|^2\right) \leq \frac{2}{3}\left(n^2 - 1\right)\sum_{ij}\left(x_i x_j\right)^2$Show that we have equality iff the sequence is an arithmetic progressi on. (IMO 2003)

32. Show that for each prime p, there exists a p rime q such that $n^p - p$ is not divisible by q for any positive integer n. (IMO 2003)(IMO 2003)

33. Let a, b, c, d be integers with $a<b<c<d<0$. Suppose that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime. (IMO 2001)

34. Let n be an odd integer greater then 1, an d let $k_1, k_2 \ldots, k_n$ be given integers. For each of the n! permutations $a = (a_1, a_2 \ldots, a_n)$ of 1,2,. . . , n, let $S(a) = \sum_{i=1}^{n} k_i a_i$. 83 Prove that there are two permutations b and c, $b \neq$, such that n! is a divisor of $S(b) - S(c)$. (IMO 2001)

35. Prove that 79 $\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$ for all posi tive real numbers a, b and c. (IMO 2001)

36. Find all pairs of integer $m>2, n>2$ such that there are infinetely many positive integers k for which $k^n + k^2 - 1$ divides $k^m + k - 1$. (IMO 2002)

37. The positive divisors of the integer $n \geq 1$ are $d_1 \leq d_2 \leq \ldots \leq d_k$ so that $d_1 = 1, d_k = n$. Let $d = d_1 d_2 + d_2 d_3 + \ldots d_k - d_k$. Show that $d \leq n^2$ and find all n for which d divides n^2. 1(IMO 2002)

38. Find all real-valued functions on the reals such that $(f(x) + f(y))(f\underline{r}aku + f(v)) = f(xu - yv) = f(xv - yu)$ for a 11 x, y, u, v. (IMO 2002)

39. Let a, b and c be the lengths of the sides of a triangle. Prove that.

$$a^2 b(a - b) + b^2 c(b - c) + c^2 a(c - a) \geq 0$$

Determine when quality occurs. (IMO 1983)

40. Let ABC be an equilateral triangle and ϵ the set of aLl points contained in the three segments AB, BC, and CA (including A, B, and C). Determine whether for every partition of ϵn into two disjoint subsets, at least one of the two subsets that contains the vertices of a right-angled triangle. Justify your answer. (IMO 1983)

41. For any polynomial $P(x) = a_0 + a_1 x + ... + a_k x^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, ...,$let $Q_i(x) = (1+x)^i$. Prove that if $i_1 i_2, ..., i_n$ are integers such that $0 \leq i_1 < i_2 < ...... < i_n$, then

$$w(Q_{i1} + Q_{i2}, +.... + Q_{in}) \geq w(Q_{i1})$$

(IMO 1985)

FUNCTION EQUATIONS

42. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:
(i) $f(xf(y)) = yf(x)$ for all positive x, y;
(ii) $f(x) \to 0$ as $x \to \infty$. (IMO 1983)

43. Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, x, y and z are non-negaive real numbers for which $x + y + z = 1$. (IMO 1984)

1. A man walks a certain distance and rides back in $3\frac{3}{4}$ hours; he could ride both ways in $2\frac{1}{2}$ hours. How many hours would it take him to

walk both ways? (PRERMO 2015)

2. Positive integers a and b are such that $a + b = \frac{a}{b} + \frac{b}{a}$. What is the value of $a^2 + b^2$? (PRERMO 2015)

3. The equations $x^2 - 4x + k = 0$ and $x^2 + kx - 4 = 0$, where k is a real number, have exactly one common root. What is the value of k? (PRERMO 2015)

4. Let $P(x)$ be a non-zero polynomial with integer coefficients. If $P(n)$ is divisible by n for each positive integer n, what is the value of $P(0)$? (PRERMO 2015)

5. Let a, b, and c be real numbers such that $a-7b+8c = 4$ and $8a+4b-c = 7$. What is the value of $a^2 - b^2 + c^2$? (PRERMO 2015)

6. Let a, b, and c be such that $a+b+c = 0$ and $P = \frac{a^2}{2a^2+bc} + \frac{b^2}{2b^2+ca} + \frac{c^2}{2c^2+ab}$ is defined. What is the value of P? (PRERMO 2015)

1. If real numbers a, b, c, d, e satisfy

$$a + 1 = b + 2 = c + 3 = d + 4 = e + 5 = a + b + c + d + e + 3,$$

what is the value of $a^2 + b^2 + c^2 + d^2 + e^2$? (PRERMO 2014)

2. Let $x_1, x_2, \cdots, x_{2014}$ be real numbers different from 1, such that$x_1 + x_2 + \cdots + x_{2014} = 1$ and

$$\frac{x_1}{1-x_1} + \frac{x_2}{1-x_2} + \cdots + \frac{x_{2014}}{1-x_{2014}} = 1.$$

What is the value of

$$\frac{x_1^2}{1-x_1}+\frac{x_2^2}{1-x_2}+\frac{x_3^2}{1-x_3}+\cdots+\frac{x_{2014}^2}{1-x_{2014}}?$$

(PRERMO 2014)

1. For how many pairs of positive integers (x, y) is x + 3y = 1007(PRERMO 2012)

2. Rama was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3. She got 43 as the answer. What would have been her answer if she had solved the problem correctly? (PRERMO 2012)

3. The letters R, M, and O represent whole numbers. If $R \times M \times O = 240$, $R \times O + M = 46$, and $R + M \times O = 64$, what is the value of $R + M + O$? (PRERMO 2012)

4. Let $P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$ What is the largest integer that is a divisor of P(n) for all positive even integers n?(PRERMO 2012)

5. How many integer pairs (x, y) satisfy $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$? (PRERMO 2012)

6. Let $S_n = n^2 + 20n + 12$, n a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?(PRERMO 2012)

7. Suppose that $4^{x_1} = 5$, $5^{x_2} = 6$, $6^{x_3} = 7, \ldots, 126^{x_{123}} = 127$, $127^{x_{124}} = 128$. What is the value of the product $x_1 x_2 \ldots x_{124}$? (PRERMO 2012)

8. If $\frac{1}{\sqrt{2011}+\sqrt{2012}} = \frac{\sqrt{m}-\sqrt{n}}{\sqrt{m+n}}$, where m and n are positive integers, what is the value of $m+n$? (PRERMO 2012)

9. If $a = b - c$, $b = c - d$, $c = d - a$, and $abcd \neq 0$, then what is the value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$? (PRERMO 2012)

10. How many non-negative integral values of x satisfy the equation

$$\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]? \tag{8.4}$$

(Here $[x]$ denotes the greatest integer less than or equal to x. For example, $[3.4] = 3$ and $[-2.3] = -3$.) (PRERMO 2012)

11. Let x_1, x_2, x_3 be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression $\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)$? (PRERMO 2012)

12. What is the sum of the squares of the roots of the equation

$$x^2 - 7[x] + 5 = 0? \tag{8.5}$$

(Here $[x]$ denotes the greatest integer less than or equal to x. For example, $[3.4] = 3$ and $[-2.3] = -3$. (PRERMO 2012)

Chapter 9

Geometry

1. On each side of an equilateral triangle with side length n units, where n is an integer, $1 \leq n \leq 100$, consider $n-1$ points that divide the side into n equal segments.Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up. Two coins are said to be adjacent if the distance between them is 1 unit. A move consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.(IOQM 2015)

2. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment.If N denotes the number of ways we can choose the pegs such that the drawn linesegments divide the interior of the triangle into exactly nine

regions, find the sum ofthe squares of the digits of N. (IOQM 2015)

3. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF, respectively. If the area of triangle ABC is 480, find the area of triangle GYZ. (IOQM 2015)

4. The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \leq i < j < k \leq 6$, has at least one redside, find the sum of the squares of the digits of N . (IOQM 2015)

5. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X . (IOQM 2015)

6. Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD, respectively, such that the perimeter of triangle MCN is 2. Let O be the circumcenter of triangle MAN, and P be the circumcenter of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of $m+n$. (IOQM 2015)

7. Let ABC be a triangle in the xy-plane, where B is at the origin $(0,0)$. Let BC be produced to D such that $BC : CD = 1 : 1$, CA be produced to E such that $CA : AE = 1 : 2$, and AB be produced to F such that $AB : BF = 1 : 3$. Let $G\,(32, 24)$ be the centroid of triangle ABC and K be the centroid of triangle DEF. Find the length GK. (IOQM

2015)

8. In the coordinate plane, a point is called a lattice point if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right-angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$. (IOQM 2015)

9. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set $\{5, 6, 7, 8, 9, 10\}$. (IOQM 2015)

10. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $PQ \parallel AB$.

    Let P_1 be a point on line PB_1 such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1 such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$. Prove that points P, Q, P_1, and Q_1 are concyclic. (IMO 2019)

11. Let I be the in center of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at points D, E, and F, respectively.

    The line through D perpendicular to EF meets ω again at R. Line AR meets *omega* again at P. The circumcircles of triangles PCE and

PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A that is perpendicular to AI. (IMO 2019)

12. consider the convex quadrilateral ABCD.The point P is the interior of ABCD.The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC. \tag{9.1}$$

prove that the following three lines meet in a point:the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB (IMO 2020)

13. Prove that there exists a positive constant c such that the following statement is true: Consider an integer n>1, and a set S of n points in the plane such that the distance between any two different points in S is at least 1. It follows that there is a line l separating S such that the distance from any point of S to l is at least $cn^{\frac{-1}{3}}$ (A line l separates a set of points S if some segment joining two points in S crosses l.) Note. Weaker results with replaced by cn^{α} may be awarded points depending on the value of the constant α >1/3. (IMO 2020)

14. Let D be an interior point of the acute triangle ABC with AB >AC so that $\angle DAB = \angle CAD$.The point E on the segment AC satisfies $\angle ADE = \angle BCD$,the point F on the segment AB satisfies $\angle FDA =$

$\angle DBC$,and the point X on the line AC satisfies CX=BX. let O_1 and O_2 be the circumcentres of the triangles ADC and EXD,respectively.Prove that the lines BC,EF,and O_1O_2 are concurrent (IMO 2021)

15. Let r be a circle with centre I,and ABCD a convex quadrilateral such that each of the segments AB,BC,CD and DA is a tangent to r.Let Ωbe the circumcircle of the triangle AIC.The extension of BA beyond A meets Ωat X,and the extension of BC beyond C meets Ωat Z.The extensions of AD and CD beyond D meet Ωat Y and T,respectively.Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC \tag{9.2}$$

(IMO 2021)

16. Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle. (IM0 2022)

17. Let ABC be an acute-angled triangle with $AB \leq AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets

Ω again at$E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$.Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$. (IMO 2023)

18. Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C, CB_1 = B_1A, AC_1 = C_1B$, and $\angle BAC + \angle CB_1A + \angle AC_1B = 480°$. Let BC_1 and CB_1 meet at A_2, let CA_1 and AC_1 meet at B_2, and let AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points. (Note: no 2 sides have equal length.) (IMO 2023)

19. Let ABC be a triangle with $AB \leq AC \leq BC$. Let the incentre and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively. Prove that $\angle KIL + \angle YPX = 180°$. (IMO 2024)

20. Three points X, Y, Z are on a straight line such that $XY = 10$ and $XZ = 3$. What is the product of all possible values of YZ? (Prermo 2013)

21. Let AD and BC be the parallel sides of a trapezium $ABCD$. Let P

and Q be the midpoints of the diagonals AC and BD. If $AD = 16$ and $BC = 20$, what is the length of PQ? (Prermo 2013)

22. In a triangle ABC, let H, I, and O be the orthocenter, incenter, and circumcenter, respectively. If the points B, H, I, and C lie on a circle, what is the magnitude of $\angle BOC$ in degrees? (Prermo 2013)

23. Let ABC be an equilateral triangle. Let P and S be points on AB and AC, respectively, and let Q and R be points on BC such that $PQRS$ is a rectangle. If $PQ = \sqrt{3} \times PS$ and the area of $PQRS$ is $\frac{28}{3}$, what is the length of PC? (Prermo 2013)

24. Let A_1, B_1, C_1, D_1 be the midpoints of the sides of a convex quadrilateral $ABCD$ and let A_2, B_2, C_2, D_2 be the midpoints of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of $ABCD$? (Prermo 2013)

25. Let S be a circle with center O. A chord AB, not a diameter, divides S into two regions R_1 and R_2. Let S_1 be a circle with center in R_1 touching AB, the circle S internally. Let S_2 be a circle with center in R_2 touching AB at Y, the circle S internally, and passing through the center of S. The point X lies on the diameter passing through the center of S_2, and $\angle YXO = 30°$. If the radius of S_2 is 100, then what is the radius of S? (Prermo 2013)

26. In a triangle ABC with $\angle BCA = 90°$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. If the

ratio of the area of quadrilateral $BXYC$ to the area of triangle ABC is 13:18 and $BC = 12$, then what is the length of AC? (Prermo 2013)

27. A convex hexagon has the property that for a ny pair of opposite sides the distance between their midpoints is $\frac{\sqrt{3}}{2}$ times the sum of their lengths Show that all the hexagon's angles are equal. (IMO 2003)

28. $ABCD$ is cyclic. The feet of the perpendicula r from D to the lines AB, BC, CA are P, Q, R respectivel$y$. Show that the angle bisectors of ABC and CDA meet on the line AC iff $RP = RQ$. hfill(IMO 2003)

29. Let ABC be an acute-angled triangle with circumcentre 0. Let P on BC be the foot o f the altitude from A.

    Suppose that $<BCS \leq \angle ABC + 30^0$.

    Prove that $<CAB+ \leq cop\angle 90^o$. (IMO 2001)

30. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^0$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC? (IMO 2001)

31. BC is a diameter of a circle center 0. A is any point on the circle with $\angle AOC > 60^0$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through 0 parallel to AD meets AC at J. Show that J is the inc enter of triangle CEF. (IMO 2002)

32. $n > 2$ circlesof radius 1 are drawn in the plane so that no line meets

more th an two of the circles. Their centers are $0_1, 0_2 \ldots 0_n$. Show that $\sum_{i<} 1/0_i 0_j \leq (n-1)\frac{\pi}{4}$. (IMO 2002)

33. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $a(X)$ the measure of the angle between OA and OX in radians countrclockwise from OA $(O \leq a(X) < 2\pi)$. Let $C(X)$ be the circle with center O and radius of length $\frac{OX+a(X)}{OX}$. each point of the plane is colored by one of a finite number of colors. Proveoint Y for which $a(y) > 0$ such that color appears on the circumference of the circle $C(Y)$. (IMO 1984)

34. Let $ABCD$ be a convex quadrilateral such tha the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel. (IMO 1984)

35. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices $(n > 3)$,and let p be its perimeter.Prove that.

$$In - 3 < \frac{2d}{p} < \left(\frac{n}{2}\right)\left(\frac{n+1}{2}\right) - 2,$$

Where $\left(x\right)$ denotes the gratest integer not excee ding x (IMO 1984)

36. let A be one of the two distinct points of intersection of two unequal coplanar tangents to the circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at

P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1,M_2 be the midpoint of P_2Q_2 prove that $\angle O_1AO_2 = \angle M_1AM_2$. (IMO1983)

37. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD+BC = AB$. (IMO 1985)

38. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. The circumscribed circle of the triangle ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle. (IMO 1985)

39. P is a point inside a given triangle $ABC.D, E, F$ are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which
$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$ is least. (IMO 1981)

40. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear (IMO 1981)

41. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the sid e opposite A_i). For all $i = 1, 2, 3, M_i$ is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection. of T_i in the interior bisector of anngle A_i. Prove that the

lines M_1, S_1,M_2S_2 and M_3S_3 are concurrent. (IMO 1982)

42. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear. (IMO 1982)

43. Let S be a square with sides of length 100, and let L be a path with in S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, ...A_{n-1}A_1$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $\frac{1}{2}$. Prove that there are two points X and Y in & such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198. (IMO 1982)

44. A triangle $A_1A_2A_3$ and a point $P_0$are given in the plane.We define $A_s = A_s{-}3$ for all $s \geq 4$ .We construct a set of points P_1, P_2,$P_3 \ldots$,such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle 120° clockwise $(fork = 0, 1, 2, 3 \ldots)$ .Prove that if P_{1986}=P_0, then thetriangle $A_1A_2A_3$ is equilateral . (IMO 1986)

45. Let A, B be adjacent vertices of a regular n-gon $(n \leq 5)$ in the plane having center at O. A triangle XYZ, which is congruent to and initially conincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside

the polygon. Find the locus of X. (IMO 1986)

46. In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circumcircle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and Mrespectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas. (IMO 1987)

47. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n. (IMO 1987)

48. Consider two coplanar circles of radii R and r $(R > r)$ with the same center. Let P be a fixed point on the smaller circle and B a variable point on the lar ger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then $A = P$) (i) Find the set of values of $BC^2 + CA^2 + AB^2$ (ii) Find the locus of the midpoint of BC. (IMO 1988)

49. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$. (IMO 1988)

50. Problem 5. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three

of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied: * no line passes through any point of the configuration; * no region contains points of both colours

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines (Imo 2013)

51. Problem 6. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1, on CA and C_1, on AB analogously, using the excircles opposite B and C. respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$, lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled. (Imo 2013)

    The excircle of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined. (Imo 2013)

52. problem7 Let ABC be an acute-angled triangle with orthocentre$H$, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the fect of the altitudes from B and C, respectively. Denote by w_1 the circumcircle of BWN, and let X be the point on wy such that WX is a diameter of w_1 Analogously, denote by w_2 the circumcircle of CWM. and let Y be the point on such that WY is a diameter of Prove that X, Y and Hare collinear. (Imo 2013)

53. Problem 8. Let $Q_{>0}$ be the set of positive rational mumbers. Let $f : Q_{>0} \to R$ be a function satisfying the following three conditions:

   (a) for all $x, y \epsilon Q > 0$, we have $f(x) f(y) \geq f(xy)$

   (b) for all $x, y \epsilon Q > 0$,we have$f(x+y) \geq f(x) + f(y)$

   (c) there exists a rational number $a > 1$ such that $f(a) = a$.

   prove that $F(x) = x$ for all $x \epsilon Q > 0$.

   (Imo 2013)

54. Problem 9. let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares. (Imo 2014)

55. Problem 10. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$ Point His the foot of the perpendicular from A to BD. Points S and T lie on sides $ABandAD$, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90°, \angle THC - \angle DTC = 90°$. Prove that line BD is tangent to the circumcircle of triangle TSH. (Imo 2014)

56. Problem 4. Points $PandQ$lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines $BMandCN$ intersect on circumcircle of triangle ABC (Imo 2014)

57. Problem 11. A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cats the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n. in any set of a lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

    Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c. (Imo 2014)

58. Problem 12. We say that a finite set S of points in the plane is balanced if, for any two different points $AandB$ in S, there is a point Cin Ssuch that $AC = BC$. We say that S is centre-free if for any three different points A, B and C in S, there is no point P in S such that $PA = PB = PC$

    (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

    (b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

    (Imo 2015)

59. Problem 13. Determine all triples (a, b, c) of positive integers such that each of the numbers $ab - c, bc - a, ca - b$

    is a power of 2

(A power of 2 is an integer of the form 2^n,Where n is a non-negative integer). (Imo 2015)

60. Problem 14. Let ABC be an acute triangle with $AB>AC$ Let I be its circumcircle, H its orthocentre, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q he the point on T such that $\angle HQA = 90$, and let K be the point on T such that $\angle HKQ = 90°$. Assume that the points$A, B, C, KandQ$ are all different, and lie on T in this order.

    Prove that the circumcircles of triangles KQH and FKM are tangent to each other. (Imo2015)

61. Problem 15. Triangle ABC has circumcircle Ω and circumcentre O. A circle T with centre. A intersects the segment BC at points $DandE$, such that $B, D, E$and Care all different and lie on line BC in this onter. Let $FandG$be the points of intersection of $Tand\Omega$. such that $A.FB.CandG$lie on Ωin this order. Let K he the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA Suppose that the lines $FKandGL$ are different and intersect at the point X. Prove that X lies on the line AO. (Imo 2015)

62. Problem 16. Let R be the set of real numbers. Determine all functions

$f : R \to R$ satisfying the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x) \qquad (9.3)$$

for all real numbers x and y (Imo2015)

63. problem17 the sequence $a_1, a_2, \ldots$ of an integers satisfies the following conditions;

   (a) $1 \leq a_j \leq 2015$ for all $j \geq 1$;

   (b) $k + a_k \neq l + a_l$ for all $1 \leq k<l$.

   prove that there exist two positive integers $bandN$ such that

   $\left| \sum_{j=m+1}^{n} (aj - b) \right| \leq 1007^2$

   for all integers $mandn$ satisfying $n > m \geq N$ (Imo 2015)

64. Prove that the set $\{1, 2, ...,1989\}$ can be expressed as the disjoint union of subsets $A_i (i = 1, 2, ...,117)$ such that : (i) Each A_i contains 17 elements ; (ii) The sum of all the elements in each A_i is the same . (IMO 1989)

65. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

   (i) The area of the triangle $A_0\ B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$

(ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC. (IMO 1989)

66. Let n and k be positive integers and let S be a set of n points in the plane such that

(i) No three points of S are collinear, and

(ii) For any point P of S there are at least k points of S equidistant from P. (IMO 1989)

Prove that:

$$k < \frac{1}{2} + \sqrt{2n}.$$

67. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD+BC$. There exists a point. P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

. (IMO 1989)

68. Chords AB and CD of a circle imersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangen t line at E to the circle through $D, E.$ and M intersects the lines BC and AC

at F and G. respectively, If

$$\frac{AM}{AB} = t$$

find

$$\frac{EG}{EF}$$

in terms of t . (IMO 1990)

69. Let n_3 and consider a set E of 2_{n-1} distinct points on a circle. Suppose that exactly k of these points are to he colored black. Such a coloring is "*good*" if there is at least one pair of black points such that the interior of one of the ares between them contains exactly in points from E. Find the smallest value of k so that every such coloring of k points of E is good (IMO 1990)

70. Given an initial integer $n_0 > 1$, two players. A a nd B, choose integers $n_1, n_2, n_3, ...$ alternately accordi ng to the following rules: Knowing n_{2k}, A chooses any integer n_{2k+2} such that

$$n_{2k} \leq n_{2k+1} \leq n_2^2 k$$

Knowing n_{2k+1} , B chooses any integer n_{2k+2} such that

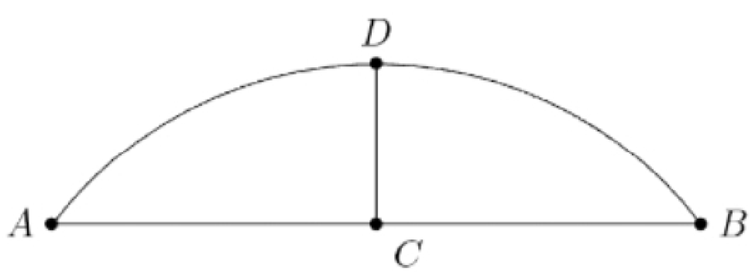
$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power. Plaver A wins the game by choosing the number 1990: player B wins by choosing the number 1. For which n_0 does: $(a)A$ have a winning strategy? (b) B have a winning strategy? (c) Neither player have a winning strategy? (IMO 1990)

71. Prove that there exists a convex 1990-gon with the following two properties (a) All angles are equal. (b) The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2$,.....,1990^2 in some order. (IMO 1990)

72. Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles $\angle PAB, \angle PBC, \angle PCA$ is less than or equal to $30°$. (IMO 1991)

1. The figure below shows a broken piece of a circular plate made of glass.



C is the midpoint of AB, and D is the midpoint of arc AB. Given that $AB = 24$ cm and $CD = 6$ cm, what is the radius of the plate in centimeters? (The figure is not drawn to scale.) (PRERMO 2015)

2. A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any interior point, and the sides of the square

are parallel to the sides of the two given rectangles. What is the smallest possible area of the square? (PRERMO 2015)

3. What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12? (PRERMO 2015)

4. In rectangle $ABCD$, $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\angle BPC = 90°$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC, and CPD, what is the value of $r_1 + r_2 + r_3$? (PRERMO 2015)

5. In the acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\angle BAE = 40°$. If $\angle DAE = \angle DFE$, what is the magnitude of $\angle ADF$ in degrees? (PRERMO 2015)

6. The circle ω touches the circle Ω internally at P. The center O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume $PY > PX$. Let PY intersect ω at Z. If $YZ = 2PZ$, what is the magnitude of $\angle LPYX$ in degrees? (PRERMO 2015)

1. Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. If $AB = 20$, $BC = 70$, and $CD = 90$, then what is the value of DA? (PRERMO 2014)

2. In a triangle with integer side lengths, one side is three times as long

as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle? (PRERMO 2014)

3. In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX : XB = 1 : 2$ and $AY : YC = 2 : 1$.If the area of triangle AXY is 10, then what is the area of triangle ABC? (PRERMO 2014)

4. Let XOY be a triangle with $\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Suppose that $XN = 19$ and $YM = 22$. What is XY? (PRERMO 2014)

1. PS is a line segment of length 4 and O is the midpoint of PS. A semicircular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semicircular arc drawn with QR as diameter is tangent to PS. What is the area of the region $QXROQ$ bounded by the two semicircular arcs? (PRERMO 2012)

2. O and I are the circumcentre and incentre of $\triangle ABC$ respectively. Suppose O lies in the interior of $\triangle ABC$ and I lies on the circle passing through B, O, and C. What is the magnitude of $\angle BAC$ in degrees? (PRERMO 2012)

3. In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of the segment BD? (PRERMO 2012)

4. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. What is the area of $\triangle ABE$? (PRERMO 2012)

5. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle? (PRERMO 2012)

6. $ABCD$ is a square and $AB = 1$ Equilateral triangles AYB and CXD are drawn such that X and Y are inside the square.What is the length of XY? (PRERMO 2012)

Chapter 10

Discrete

1. What is the number of ordered pairs (A, B) where A and B are subsets of $\{1, 2, \ldots, 5\}$ such that neither $A \subseteq B$ nor $B \subseteq A$?(PRERMO 2014)

2. The Bank of Oslo issues two types of coin: aluminium (*denoted* A) and bronze (*denoted* B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Marianne repeatedly performs the following operation: she identifies the longest chain containing the k^{th} coin from the left, and movees all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be

$$AA\underline{B}BBABA \to BBB\underline{A}AABA \to AA\underline{A}BBBBA \to$$
$$BBB\underline{B}AAAA \to ....$$

Find all pairs (n, k) with $(1 \leq k \leq 2n)$ such that for every initial ordering, at some moment during the process, the leftmost (n) coins will all be of the same type. (IMO 2022)

3. Let n be a positive integer. A Nordic square is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. Two different cells are considered adjacent if they share a common side. Every cell that is adjacent only to cells containing la Rger numbers is called a valley. An uphill path is a sequence of one or more cells such that:

   (a) The first cell in the sequence is a valley,

   (b) Each subsequent cell in the sequence is adjacent to the previous cell, and

   (c) The numbers written in the cells in the sequence are in increasing order.

   Find as a function of n, the smallest possible total number of uphill paths in a Nordic square. (IMO 2022)

4. Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$ the i^{th} row contains exactly i circles, exactly one of which is coloured red. A ninja path in a *Japanese triangle* is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a *Japanese triangle* with $n = 6$ along with a *ninja path* in that tr iangle containing two red circles. In terms of n, find the greatest k such that in each *Japanese triangle* there is a *ninja path* containing

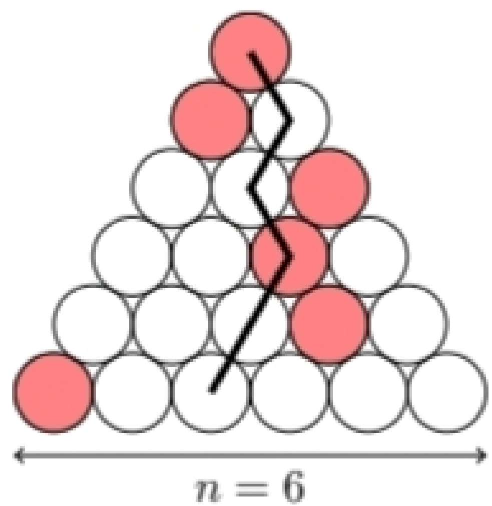


Figure 10.1: Image 1

at least k red circles. (IMO 2023)

5. Determine all pairs (a, b) of positive integers for which there exist positive integers g and N Such that

$$\gcd\,(a^n + b, b + a) = g$$

Holds for all integers $n \geq N$. Note that $\gcd\,(x, y)$ denotes the greatest common divisor of integers x and y. (IMO 2024)

6. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let N be a positive integer. Suppose that, for each $n \geq N$, an is equal to the number of times an appears in the list $a_1, a_2, \ldots, a_{n-1}$.

   Prove that at least one of the sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ is eventually periodic. An infinite sequence $b_1, b_2 b_3, \ldots$ is eventually periodic if there exist positive integers p and M such that $b_{m+p} = b_m$ for all $m \geq M$. (IMO 2024)

7. Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there Is exactly one monster in each row except the first row and the last row, and That each column contains at most one monster. Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common Turbo the Tortoise is on a quest to escape from a rectangular grid of cells. Starting on any ce Ll in the first row, Turbo repeatedly moves to an adjacent cell sharing a common side.(He is allowed to return to a previously) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo r emembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over. Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n^{th} attempt or earlier, regardless of the locations of the monsters. (IMO 2024)

8. Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$. What is the value of $\sum_{n=1}^{90} \frac{1}{S_n+S_{n-1}}$? (Prermo 2013)

9. An infinite sequence $x_0, x_1, x_2, ...$of real numbers is said to be bounded

if there is a constant C such that $\left|x_i\right| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, ...$ Such that

$$\left|x_i - x_j\right|\left|i - j\right|^a \geq 1$$

for every pair of distinct nonnegative integers i, j. (IMO 1991)

Chapter 11

Number Systems

1. Let n be a positive integer such that $1 \leq n \leq 1000$. Let M_n be the number of integers in the set $X_n = \left\{\sqrt{4n+1}, \sqrt{4n+2}, \ldots, \sqrt{4n+1000}\right\}$. Let

$$a = \max M_n : 1 \leq n \leq 1000, \tag{11.1}$$

and

$$b = \min M_n : 1 \leq n \leq 1000. \tag{11.2}$$

Find $a - b$. (IOQM 2015)

2. Find the number of elements in the set

$$(a, b) \in \{N\} : 2 \leq a, b \leq 2023, \log_a(b) + 6\log_b(a) = 5. \tag{11.3}$$

(IOQM 2015)

3. Let α and β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}. \tag{11.4}$$

Find the smallest possible value of β. (IOQM 2015)

4. For $n \in N$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n . Consider the set

$$A = \{n \in N : P(n)\, isnon - zero, squarefreeandS(n)\, isaproperdivisorofP(n)\}. \tag{11.5}$$

Find the maximum possible number of digits of the numbers in A . (IOQM 2015)

5. For any finite non-empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

$$\max(A) \times |B| = 12 \quad \text{and} \tag{11.6}$$

$$|A| \times \max(B) = 11, \tag{11.7}$$

and N can be written as $100a + b$ where a, b are positive integers less than 100, find $a + b$. (IOQM 2015)

6. The sequence $\langle a_n \rangle_{n\geq 0}$ is defined by $a_0 = 1$, $a_1 = -4$, and $a_{n+2} = -4a_{n+1} - 7a_n$ for $n \geq 0$. Find the number of positive integer divisors of $a_{250} - a_{49}a_{51}$. (IOQM 2015)

7. A quadruple (a, b, c, d) of distinct integers is said to be balanced if $a + b = c + d$ and $a < b < c < d$. Find the number of balanced quadruples of distinct integers in the set $\{1, 2, \cdots, 12\}$. (IOQM 2015)

8. There is an integer n>1. There are n2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can star using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positit from the lower station and reach the higher one byve integer k for which one can guarantee that there are two stations that are linked by both companies. (IMO 2020)

9. Find the smallest positive integer k such that $k(3^3 + 4^3 + 5^3) = a^n$ for some positive integers a and n, with $n > 17$. (Prermo 2013)

10. Let $S(M)$ denote the sum of the digits of a positive integer M written in base 10. Let N be the smallest positive integer such that $S(N) = 2013$. What is the value of $S(5N + 2013)$? (Prermo 2013)

11. Let m be the smallest odd positive integer for which $1+2+\cdots+m$ is a square of an integer and let n be the smallest even positive integer for which $1+2+\cdots+n$ is a square of an integer. What is the value of $m+n$? (Prermo 2013)

12. What is the maximum possible value of k for which 2013 can be written as a sum of k consecutive positive integers? (Prermo 2013)

13. Let a, b and c be positive integers, no two of which have a common divisor grater than 1. Show that $2abc-ab-bc-ca$ is the largest integer which cannot be expressed in the form $xbc+yca+zab$, where x, y and z are non-negative integers. (IMO 1983)

14. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression ? justify your answer. (IMO 1983)

15. Find one pair of positive integers a and b such that : (i) $ab\,(a+b)$ is not divisible by 7; $(ii)(a+b)^7 - a^7 - b^7$ is divisible by 7^7 (IMO 1984)

16. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a+d = 2^k$ and $b+c = 2^m$ for some integers k and m, then $a = 1$ (IMO 1984)

17. Let n and k be given relatively prime natural numbers $k < n$.Each number in the set $M = 1, 2, ...n-1$ is colored either blue or white. It is given that (i) for each $i \epsilon M$, both i and $n-i$ have the same color; (ii) for each $i \epsilon M$, $i \neq k$, both i and $\left| i-k \right|$ have the same color. Prove that all numbers in M must have the same color. (IMO 1985)

18. Given a set M of 1985 distinct positive integers, none of which has a prime divisor grater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer. (IMO 1985)

19. For every real number x_1, construct the sequence $x_1, x_2, ..116.$by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{4} \right)$$

for each $n \geq 1$ Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n. (IMO 1985)

20. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, ..., n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that $F(n, r) = \frac{n+1}{r+1}$ (IMO 1981)

21. (a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers (b)For which values of $n > 2$ is there exactly one set having the stated property? (IMO 1981)

22. . The function $f(n)$ is defined for all positive integers n and takes on

non-negative integer values. Also, for all m, n

$$f(m+n) - f(m) - f(n) = 0\,(or)\,1$$

$$f(2) = 0, f(3) > 0, and f(9999) = 3333.$$

Determine $f(1982)$. (IMO 1982)

23. Prove that if n is a positive integer such that the equation.

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y), then it has at least three such solutions. Sh w that the equation has no solutions in integers when $n = 2891$. (IMO 1982)

ALGEBRA

24. Determine the maximum value of $m^3 + n^3$, where m and n are integers satisfying $m, n \epsilon \{1, 2, ..., 1981\}$ and $\left(n^2 - mn - m^2\right)^2 = 1$ (IMO 1981)

25. The function $f(x, y)$ satisfies (1) $f(0, y) = y + 1$, (2) $f(x+1, 0) = f(x, 1)$, (3) $f(x+1, y+1) = f(x, f(x+1, y))$, for all non-negative integers x, y. Determine $f(4, 1981)$. (IMO 1981)

MATHEMATICAL ANALYSIS

26. Consider the infinite sequences $\{x_n\}$ of positive real numbers with following properties: $x_0 = 1$, and for all $i \geq 0, x_{i+1} \leq x_i$. (a) Prove that for every such sequence, there is $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ... + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ... + \frac{x_{n1}^2}{x_n} < 4.$$

(IMO 1982)

27. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13.d\}$ such that $ab - 1$ is not a perfect square. (IMO 1986)

28. Let $p_n\left(k\right)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$,which have exactly k fixed points.Prove that

$$\sum_{k=0}^{n} k \cdot p_n\left(k\right) = n$$

(Remark:A permtation f of a set S is one-to-one mapping of S onto itself.An element i in S is called a fixed point of the the permutation f if f(i)=i.) (IMO 1987)

29. Let n be a positive integer and let $A_1, A_2, \ldots, A_{2n+1}$ be subsets o f a

set B. Suppose that (a) Each A_i has exactly $2n$ elements, (b) Each $A_i \cap A_j$ $(1 \leq i \leq j \leq 2n+1)$contains exactl y one element, and (c) Every element of B belongs to at least two of the A_i.

For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements? (IMO 1988)

30. Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer. (IMO 1988)

31. problem 1 Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, m_3, \ldots$ (not necessarily different) such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\ldots\left(1+\frac{1}{m_k}\right) \tag{11.8}$$

(Imo 2013)

32. problem2 let $a_{0<}a_{1<}a_{2<}\ldots$ be an infinite sequence of positive integers.prove that there exists a unique integer $n \geq 1$such that

$$a_{n<}\frac{a_0+a1+\ldots+a_n}{n}<a_{n+1}. \tag{11.9}$$

(Imo2014) (Imo 2014)

33. Problem 3. For each positive integer n, the Bank of Cape Town ienes coins of denomination$\frac{1}{n}$ Given a finite collection of such coins (of not necessarily differ ent denominations) with total value at most $99 + \frac{1}{2}$ prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1. (Imo2014)

34. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number. (IMO 1989)

35. A permutation $(x_1, x_2, ..., x_m)$ of the set {1,2.....,2n}. where a is a positive integer, is said to have property P if $\left| x_i - x_{i+1} \right| = n$ for at least one in{1,2,....,2n-1}. Show that, for each n, there are more permitations with property P than without. (IMO 1989)

36. Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is integer. (IMO 1990)

37. Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI.BI.CI.}{AA'.BB'.CC'.} \leq \frac{8}{27}$$

. (IMO 1991)

38. Let $n > 6$ be an integer and $a_1, a_2, ..., a_k$ be all the natura numbers less than n and relatively prime to n If

$$a_2 - a_1 = a_3 - a_2 = ... = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2. (IMO 1991)

1. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square? (PRERMO 2015)

2. Let n be the largest integer that is the product of exactly 3 distinct prime numbers, x, y, and $10x + y$, where x and y are digits. What is the sum of the digits of n? (PRERMO 2015)

3. A subset B of the set of first 100 positive integers has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B? (PRERMO 2015)

1. A natural number k is such that $k^2{<}2014{<}(k{+}1)^2$. What is the largest prime factor of k? (PRERMO 2014)

2. The first term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014^{th} term of the sequence? (PRERMO 2014)

3. What is the smallest possible natural number n for which the equation $x^2 - nx + 2014 = 0$ has integer roots? (PRERMO 2014)

4. If $x^{\left(x^4\right)} = 4$, what is the value of $x^{\left(x^2\right)} + x^{\left(x^8\right)}$? (PRERMO 2014)

5. Let S be a set of real numbers with mean M. If the means of the sets $S \cup \{15\}$ and $S \cup \{15, 1\}$ are $M + 2$ and $M + 1$, respectively, then how many elements does S have?

6. Natural numbers k, l, p, and q are such that a and b are roots of the equation $x^2 - kx + l = 0$ such that $a + \frac{1}{b}$ and $b + \frac{1}{a}$.What is the sum of all possible values of q? (PRERMO 2014)

7. For natural numbers x and y, let (x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with $x \leq y$ satisfy the equation $xy = x + y + (x, y)$? (PRERMO 2014)

8. For how many natural numbers n between 1 and 2014 ($bothinclusive$) is $\frac{8n}{9999-n}$ an integer? (PRERMO 2014)

9. For a natural number b, let $N\left(b\right)$ denote the number of natural numbers a for which the equation $x^2 + ax + b = 0$ has integer roots. What is the smallest value of b for which $N\left(b\right) = 20$? (PRERMO 2014)

10. One morning, each member of Manjul's family drank an 8-ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Manjul drank $\frac{1}{7}$-th of the total amount of milk and $\frac{2}{17}$-th of the total amount of coffee. How many people are there in Manjul's family? (PRERMO 2014)

Chapter 12

Differentiation

Chapter 13

Integration

Chapter 14

Functions

1. Let f be a one-to-one function from the set of natural numbers to itself such that $f(mn) = f(m) f(n)$ for all natural numbers m and n. What is the least possible value of $f(999)$? (PRERMO 2014)

1. Let N be the set of natural numbers. Suppose $f : N \to N$ is a function satisfying the following conditions:

   (a) $f(mn) = f(m) f(n)$,

   (b) $f(m) < f(n)$ if $m < n$,

   (c) $f(2) = 2$.

   What is the value of $\sum_{k=1}^{20} f(k)$? (PRERMO 2012)

2. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1? (IMO 1986)

3. Let n be an integer greater than or equal to 2. Prove that if k^2+k+n is prime for all integers k such that $0 \leq k \leq \sqrt{n/3}$, then k^2+k+n is prime for all integers k such that $0 \leq k \leq n-2$ (IMO 1987)

4. A function f is defined on the positive integers by

$$f(1) = 1, f(3) = 3,$$
$$f(2n) = f(n),$$
$$f(4n+1) = 2f(2n+1) - f(n),$$
$$f(4n+3) = 3f(2n+1) - 2f(n),$$

for all positive integers n. Determine the number of positive integers n, less than or equal to 1988,for which $f(n) = n$. (IMO 1988)

5. Show that set of real numbers x which satisfy the in equality

$$\sum k = 1^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988(IMO 1988)

6. Let Q^+ be the set of positive rational numbers. Construct a function $f : Q^+ \to Q^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in Q^+. (IMO 1990)

COMBINATOMICS

7. Let $S = \{1, 2, 3, ..., 280\}$. Find the smallest integer n such that each $n-$ element subset of S contains five numbers which are pairwise relatively prime. (IMO 1991)

GRAPH THEORY

8. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2.....k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1. [A graph consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices. u, v belongs to at most one edge. The graph G is connected if for cach pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, ..., v_m = y$ such that each pair v_i, v_{i+1} $(0 \leq i < m)$ is joined by an edge of G.] (IMO

1991)

Chapter 15

Matrices

Chapter 16

Trignometry

1. In a triangle ABC, let I denote the incenter. Let the lines AI, BI, and CI intersect the incircle at P, Q, and R, respectively. If $\angle BAC = 40°$, what is the value of $\angle QPR$ in degrees? (PRERMO 2014)