OLYMPIAD MATH

Made Simple

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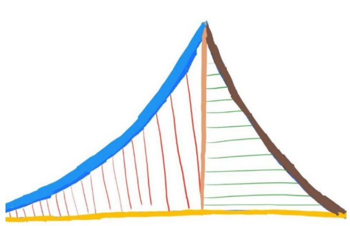

Contents

Introduction

This book links high school coordinate geometry to linear algebra and matrix analysis through solved problems.

Chapter 1

Linear Forms

Chapter 2

Circles

1. AB is tangent to the circles $CAMN$ and $NMBD$. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that $PE = QE$. (IMO 2000)

Chapter 3

Intersection of Conics

Chapter 4

Probability

1. Find the number of triples (a, b, c) of positive integers such that

   (a) ab is a prime;

   (b) bc is a product of two primes;

   (c) abc is not divisible by square of any prime and

   (d) $abc \leq 30$.

   (IOQM 2015)

2. We call a positive integer alternating if every two consecutive digits in its decimal representation are of different parity.Find all positive integers n such that n has a multiple which is alternating (IMO 2004)

3. Find the maximum value of x_0 for which there exists a sequence $x_0, x_1 \ldots x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that for $i = 1, \ldots, 1995$.

$$..x_{i-1.} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}.(IMO1995) \tag{4.1}$$

4. Let p be an odd prime number. How many p-element subsets A of $\{1, 2, ...2p\}$ are there, the sum of whose elements is divisible by p? (IMO1995)

5. Find all pairs (a, b) of integers $a, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.(IMO1997) \tag{4.2}$$

6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways;

$$4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1. \tag{4.3}$$

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.(IMO1997) \tag{4.4}$$

7. Let $x_1, x_2, \ldots, x_n$ be the real numbers satisfying the conditions

$$\left| x_1 + x_2 + \cdots + x_n \right| = 1 \tag{4.5}$$

and

$$\left|x_i\right| \leq \frac{n+1}{2} i = 1, 2, \ldots, n. \tag{4.6}$$

Show that there exists a permutation $y_1, y_2, \dot{s}, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$\left|y_1 + 2y_2 + \cdots + ny_n\right| \leq \frac{n+1}{2}. \tag{4.7}$$

(IMO 1997)

8. Let S denote the set of nonnegative integers. Find all functions f from s to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \forall m, n \epsilon S. (IMO 1996) \tag{4.8}$$

9. Let p, q, n be three positive integers with $p+q < n$. Let $(x_0, x_1, ..., x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditons:

   (a) $x_0 = x_n = 0$.

   (b) For each i with $1 \leq i \leq n$ either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$. Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$. (IMO 1996)

1. A postman has to deliver ive letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how rnany different way could he do

this so that exactly two of the five houses receive the correct letters?
(PRERMO 2012)

Chapter 5

permutation and combination

1. A positive integer $n > 1$ is called beautiful if n can be written in one and only one way as $n = a_1 + a_2 + \cdots + a_k = a_1 \cdot a_2 \cdots a_k$for some positive integers $a_1, a_2, \cdots, a_k$, where $k > 1$ and$a_1 \geq a_2 \geq \cdots \geq a_k$. (For example 6 is beautiful since$6 = 3 \cdot 2 \cdot 1 = 3 + 2 + 1$, and this is unique. But 8 is not beautiful since $8 = 4 + 2 + 1 + 1 = 4 \cdot 2 \cdot 1 \cdot 1$ as well as $8 = 2 + 2 + 2 + 1 + 1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$, souniqueness is lost.) Find the largest beautiful number less than 100. (IOQM 2015)

2. For $n \in N$, consider non-negative integer-valued functions f on $\{1, 2, \cdots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n} (i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case? (IOQM 2015)

3. In the land of Binary, the unit of currency is called Ben and currency notes areavailable in denominations $1, 2, 2^2, 2^3, \cdots$ Bens. The rules of the Government of Binary stipulate that one can not use more than

two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government. (IOQM 2015)

4. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a die that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed. (IOQM 2015)

5. Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling a 2×7 rectangle using some of these tiles.(IOQM 2015)

6. Consider the set

$$S = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\} \tag{5.1}$$

where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of S, find the largest integer less than or equal to D. (IOQM 2015)

7. Let P be a convex polygon with 50 vertices. A set F of diagonals of P is said to be minimally friendly if any diagonal $d \in F$ intersects at most one other diagonal in F at a point interior to P. Find the largest possible number of elements in a minimally friendly set F. (IOQM 2015)

8. Find all pairs (k, n) of positive integers such that

$$k! = (2n-1)(2n-2)(2n-4)\cdots(2n-2n+1). \tag{5.2}$$

(IMO 2019)

9. There are 4n pebbles of weights 1,2,3.....,4n.Each pebble is coloured in one of n colours and there are four pebbles of each colour.Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

   The total weights of both piles are the same. Each pile contains two pebbles of each colour. (IMO 2020)

10. Two squirrles,Bushy and jumpy,have collected 2021 walnuts for the winter .jumpy numbers the walnuts from 1 through 2021,and digs 2021 little holes in a circular pattern in the ground around their favourite tree.The next morning jumpy notices that bushy had placed one walnut into each hole ,but had paid no attention to the numbering .unhappy,Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves.In the k-th move,jumpy swaps the positions of the two walnuts adjacent to walnut k.Prove that there exists a value of k such

that ,on the k-th move,jumpy swaps some walnuts a and b such that $a < k < b$. (IMO 2021)

11. Twenty-one girls and twenty-one boys took part in a mathematical contest. Each contestant solved a t most six problems. For each girl and each boy, at least one problem was solved by both of them. Prove t hat there was a problem that was solved by at least three girls and at least three boys. (IMO 2001)

12. S is the set $\{1, 2, 3, \ldots, 1000000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $\{a + x_i a \in A\}$ are all pairwise disjoint. (IMO 2003)

13. S is the set of all (h, k) wi th h, k non-negative integers such that $h + k\ textlessn$. Each element of S is colored red or b lue, so that if (h, k) is red and $h' \leq h, k'\ leqk$, then (h', k') is also red. A type 1 subset of S has n blue elements with differen t first member and a type 2 subset of S has n blue elements with different second member. Show tha t there are the same number of type 1 and type 2 subsets. (IMO 2002)

14. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x,y,z respectively and $y < 0$ then the following operation is allowed: the numbers x,y,z are replaced by $x+y$,$-y$,$z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the live numbers is negative. Determine

whether this procedure necessarily comes to and end after a finite number of steps. (IMO 1986)

15. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1? (IMO 1986)

16. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all 0, such that $\left|a_i\right| \leq k - 1$ for all i and

$$\left|a_1 x_1 + a_2 x_2 + \cdots + a_n x_n\right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$$

(IMO 1987)

17. Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{n/3}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$ (IMO 1987)

18. Problem 4. Let $n \geq 3$ be an integer, and consider a circle with $n + 1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1,\ ldotsn$ such that each label is used exactly once, two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is

called beautiful if, for any four labels $a < b < c < d$ with $a+d = b+c$, the chord joining the points labelled a and d does not intersect the chord joi ning the points labelled b and c Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq nandgcd\,(x, y) = 1$. Prove that $m = n + 1$

19. An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots,$ 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country. (Imo 1978)

20. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there.. Let a be the number of distinct paths of exactly n jumps ending at E. Prove that

$$a_2n - 1 = 0, a_{2n} = \frac{1}{\sqrt{2}}\left(x^{n-1} - y^{n-1}\right) \tag{5.3}$$

, $n = 1, 2, 3, \ldots,$ where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. Note. A path of a jumps is a sequence of vertices $(P_0 \ldots P_n)$ such that

(a) $PA, P = E$

(b) for every $i, 0 \leq i \leq n-1, P$ is distinct from E;

(c) for every $i, 0 \leq i \leq n-1P$. and P_{i+1} are adjacent.

(Imo 1979)

Algebra

21. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5} kx_k = a, \sum k = 15k^3 x_k = a^2, \sum k = 15k^5 x_k = a^3 \qquad (5.4)$$

. (Imo 1979)

22. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $\frac{k}{a} \geq \frac{(b-1)}{(2b)}$.(IMO 1998)

23. Consider an $n \times n$ square board, where n is a fixed even positive integer.The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.N unit squares on the board are marked in such a way that every square ($markedorunmarked$) on the board is adjacent to at least one marked square.Determine the smallest possible value of N. (IMO 1999)

24. k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. *A move* is carried out as follows. Pick any

two points A and B which are not coincident. Suppose that A lies to the right of$B$.Replace B by another point B' to the rightof A such that $AB' = kBA$. For what values of k can we move the points arbitrarily far to the right by repeated moves? (IMO 2000)

25. 100 cards are numbered 1 to 100 ($each card different$) and placed in 3 boxes ($at least one card in each box$). How many wa ys can this be done so that if two boxes are selected and a card is taken from each, then the knowledgeof their sum alone is always sufficient to identify the third box? (IMO 2000)

26. The equation

$$(x-1)(x-2)....(x-2016) = (x-1)(x-2)...(x-2016) \tag{5.5}$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions? (IMO 2016)

27. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy- plane, respectively. Prove that (IMO 1992)

$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$,

where $|A|$ denotes the number of elements in the finite set $|A|$. (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

28. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Findthe smallest value of n such that whenever exactly n edges are coloured, the set of coloured edges necessarily contains a triangle all of whose edges have the same color. (IMO 1992)

29. On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an n by n block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board. (IMO 1993)

30. There are n lamps $L_0, \dots, L_{n-1}$ in a circle ($n > 1$), where we denote $L_{n+k} = L_k$. (A lamp at all times is either on or off.) Perform steps $s_0, s_1, \dots$ as follows: at step s_i, if L_{i-1} is lit, switch L_i from on to off or vice versa, otherwise do nothing. Initially all lamps are on. Show that: (IMO 1993)

   (a) There is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;

   (b) If $n = 2^k$, we can take $M(n) = n^2 - 1$;

   (c) If $n = 2^k + 1$, we can take $M(n) = n^2 - n + 1$.

31. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \dot{s}, 2k\}$ whose base 2 representation has precisely three $1s$.

    (a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.

    (b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$. (IMO 1994)

1. How many line segments have both their endpoints located at the vertices of a given cube? (PRERMO 2015)

2. Let $E(n)$ denote the sum of the even digits of n. For example, $E(1243) = 2 + 4 = 6$. What is the value of $E(1) + E(2) + E(3) + \cdots + E(100)$? (PRERMO 2015)

3. At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many women attended the party? (PRERMO 2015)

Chapter 6

Construction

Chapter 7

Optimization

Chapter 8

Algebra

1. Let x, y be positive integers such that

$$x^4 = (x-1)\left(y^3 - 23\right) - 1. \tag{8.1}$$

   Find the maximum possible value of $x + y$. (IOQM 2015)

2. The ex-radii of a triangle are $10\frac{1}{2}$, $12, 12$ and 14. If the sides of the triangle are the roots of the cubic

$$x^3 - px^2 + qx - r = 0, \tag{8.2}$$

   where p, q, r are integers, find the integer nearest to $\sqrt{\{p+q+r\}}$. (IOQM 2015)

3. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_{31} + p_{32} + p_{33} = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$. (IOQM 2015)

4. A positive integer m has the property that m^2 is expressible in the form

$4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$. (IOQM 2015)

5. Find the least positive integer n such that there are at least 1000 unordered pairs of diagonals in a regular polygon with n vertices that intersect at a right angle in the interior of the polygon. (IOQM 2015)

6. Let $d(m)$ denote the number of positive integer divisors of a positive integer m. If r is the number of integers $n \leq 2023$ for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r. (IOQM 2015)

7. Let Z be the set of integers. We want to determine all functions $f : Z \to Z$ such that for all integers a and b : $f(2a) + 2f(b) = f(f(a+b))$ (IMO 2019)

8. . A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time: Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged. Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user. (IMO 2019)

9. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right.

He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the $k^t h$ coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$, the process starting with the configuration THT would be: $THT \to HHT$ $\to HTT \to TTT$,which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

(IMO 2019)

10. A deck of n>1 cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal? (IMO 2020)

11. Let $n \geq 100$ be an integer.Ivan writes the numbers n,n+1,....,2n each on different cards.He then shuffles these n+1 cards,and divides them into two piles.prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square. (IMO 2021)

12. Let $m \geq 2$ be an integer,A be a finite set of (not necessarily positive)integers,and $B_1, B_2, B_3...B_m$ be subsets of A.Assume that for each

k=1,2,....,m the sum of the elements of B_k is m^k .Prove that A contains at least m/2 elements (IMO 2021)

13. The real numbers a,b,c,d are such that $a \geq b \geq c \geq d > 0$ and a+b+c+d=1. prove that

$$(a+2b+3c+4d)\, a^a b^b c^c d^d < 1 \tag{8.3}$$

(IMO 2020)

14. Show that the inequality $\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$ holds for all real numbers x1,....xn (IMO 2021)

15. Find all triples (a, b, p) of positive integers with (p) prime and Prove that:

$$\left(a^p = b! + p\right). \tag{8.4}$$

(IMO 2022)

16. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2. \tag{8.5}$$

(IMO 2022)

17. Let k be a positive integer and let S be a finite set of odd prime numbers. Prove there is at most one way (up to rotation and reflection) to place the elements of S around a circle such that the product of any two neighbours is of the form $x^2 + x + k$ for some positive integer x. (IMO 2022)

18. Determine all composite integers $n \geq 1$ that satisfy the following property: if $d_1, d_2, ..., d_k$ are all the positive divisors of n with $1 = d_1 \leq d_2 \leq \cdots \leq d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k-2$. (IMO 2023)

19. For each integer $k \geq 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$ where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k} \tag{8.6}$$

(IMO 2023)

20. Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)} \tag{8.7}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$. (IMO 2023)

21. Determine all real numbers such that, for every positive integer n, the integer

$$[\alpha] + [2\alpha] + \cdots + [\alpha] \tag{8.8}$$

is a multiple of n. Note that $[z]$ denotes the greatest integer less than or equal to z. For example $[-\pi] = -4$ and $[2] = [2.9] = 2$. (IMO 2024)

22. Let $\mathbb{Q}$ be the set of rational numbers. A function $f : \mathbb{Q} \to \mathbb{Q}$ is called aquaesulian if the following property holds: for every $x, y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \quad \text{or} \quad f(f(x) + y) = x + f(y). \tag{8.9}$$

Show that there exists an integer c such that for any aquaesulian function f there are at most c different rational numbers of the form $f(r) + f(-r)$ for some rational number r, and find the smallest possible value of c. (IMO 2024)

23. Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$. What is the value of $\sum_{n=1}^{90} \frac{1}{S_n+S_{n-1}}$? (Prermo 2013)

24. There are $n-1$ red balls, n green balls, and $n+1$ blue balls in a bag. The number of ways of choosing two balls from the bag that have different colours is 299. What is the value of n? (Prermo 2013)

25. To each element of the set $S = \{1, 2, \ldots, 1000\}$ a color is assigned.

Suppose that for any two elements a, b of S, if 15 divides $a+b$, then they are both assigned the same color. What is the maximum possible number of distinct colors used? (Prermo 2013)

26. Let Akbar and Birbal together have n marbles, where $n > 0$. Akbar says to Birbal, "If I give you some marbles, then you will have twice as many marbles as I will have." Birbal says to Akbar, "If I give you some marbles, then you will have thrice as many marbles as I will have." What is the minimum possible value of n for which the above statements are true? (Prermo 2013)

27. Carol was given three numbers and was asked to add the largest of the three to the product of the other two. Instead, she multiplied the largest with the sum of the other two, but still got the right answer. What is the sum of the three numbers? (Prermo 2013)

28. Three real numbers x, y, z are such that $x^2 + 6y = -17$, $y^2 + 4z = 1$, and $x^2 + 2x = 2$. What is the value of $x^2 + y^2 + z^2$? (Prermo 2013)

29. Let $f(x) = x^3 - 3x + b$ and $g(x) = x^2 + bx - 3$, where b is a real number. What is the sum of all b for which $f(x) = 0$ and $g(x) = 0$ have a common root? (Prermo 2013)

30. Find all pairs (m, n) of positiv e integers such that $\frac{m^2}{2mn^2 - n^3 + 1}$ is a positive integer. (IMO 2003)

31. Given n>2 and reals $x_1 \leq x_2 \leq \ldots \leq x_n$, show that $\left(\sum_{ij} \left|x_i x_j\right|^2\right) \leq \frac{2}{3}\left(n^2-1\right) \sum_{ij} \left(x_i x_j\right)^2$Show that we have equality iff the sequence is an arithmetic progressi on. (IMO 2003)

32. Show that for each prime p, there exists a p rime q such that $n^p - p$ is not divisible by q for any positive integer n. (IMO 2003)(IMO 2003)

33. Let a, b, c, d be integers with $a<b<c<d<0$. Suppose that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime. (IMO 2001)

34. Let n be an odd integer greater then 1, an d let $k_1, k_2 \ldots, k_n$ be given integers. For each of the n! permutations $a = (a_1, a_2 \ldots, a_n)$ of 1,2,. . . , n, let $S(a) = \sum_{i=1}^{n} k_i a_i$. 83 Prove that there are two permutations b and c, $b \neq$, such that n! is a divisor of $S(b) - S(c)$. (IMO 2001)

35. Prove that 79 $\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$ for all posi tive real numbers a, b and c. (IMO 2001)

36. Find all pairs of integer $m>2, n>2$ such that there are infinetely many positive integers k for which $k^n + k^2 - 1$ divides $k^m + k - 1$. (IMO 2002)

37. The positive divisors of the integer $n \geq 1$ are $d_1 \leq d_2 \leq \ldots \leq d_k$ so that $d_1 = 1, d_k = n$. Let $d = d_1 d_2 + d_2 d_3 + \ldots d_k - d_k$. Show that $d \leq n^2$ and find all n for which d divides n^2. 1(IMO 2002)

38. Find all real-valued functions on the reals such that $(f(x) + f(y))(f\underline{r}aku + f(v)) = f(xu - yv) = f(xv - yu)$ for a 11 x, y, u, v. (IMO 2002)

39. Let a, b and c be the lengths of the sides of a triangle. Prove that.

$$a^2 b(a-b) + b^2 c(b-c) + c^2 a(c-a) \geq 0$$

Determine when quality occurs. (IMO 1983)

40. Let ABC be an equilateral triangle and ϵ the set of aLl points contained in the three segments AB, BC, and CA (including A, B, and C). Determine whether for every partition of ϵn into two disjoint subsets, at least one of the two subsets that contains the vertices of a right-angled triangle. Justify your answer. (IMO 1983)

41. For any polynomial $P(x) = a_0 + a_1x + ... + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, ...,$let $Q_i(x) = (1 + x)^i$. Prove that if $i_1 i_2, ..., i_n$ are integers such that $0 \leq i_1 < i_2 < ...... < i_n$, then

$$w(Q_{i1} + Q_{i2}, +.... + Q_{in}) \geq w(Q_{i1})$$

(IMO 1985)

42. Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients. (IMO 1993)

1. A man walks a certain distance and rides back in $3\frac{3}{4}$ hours; he could ride both ways in $2\frac{1}{2}$ hours. How many hours would it take him to walk both ways? (PRERMO 2015)

2. Positive integers a and b are such that $a + b = \frac{a}{b} + \frac{b}{a}$. What is the value of $a^2 + b^2$? (PRERMO 2015)

3. The equations $x^2 - 4x + k = 0$ and $x^2 + kx - 4 = 0$, where k is a

real number, have exactly one common root. What is the value of k? (PRERMO 2015)

4. Let $P(x)$ be a non-zero polynomial with integer coefficients. If $P(n)$ is divisible by n for each positive integer n, what is the value of $P(0)$? (PRERMO 2015)

5. Let a, b, and c be real numbers such that $a-7b+8c=4$ and $8a+4b-c=7$. What is the value of $a^2-b^2+c^2$? (PRERMO 2015)

6. Let a, b, and c be such that $a+b+c=0$ and $P=\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}$ is defined. What is the value of P? (PRERMO 2015)

1. If real numbers a, b, c, d, e satisfy a + 1 = b + 2 = c + 3 = d + 4 = e + 5 = a + b + c + d + e + 3, what is the value of $a^2+b^2+c^2+d^2+e^2$? (PRERMO 2014)

1. For how many pairs of positive integers (x, y) is x + 3y = 1007(PRE-RMO 2012)

2. Rama was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3. She got 43 as the answer. What would have been her answer if she had solved the problem correctly? (PRERMO 2012)

3. The letters R, M, and O represent whole numbers. If $R\times M\times O=240$, $R\times O+M=46$, and $R+M\times O=64$, what is the value of $R+M+O$? (PRERMO 2012)

4. Let $P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$ What is the largest integer that is a divisor of P(n) for all positive even integers n?(PRE-RMO 2012)

5. How many integer pairs (x, y) satisfy $x^2+4y^2-2xy-2x-4y-8=0$? (PRERMO 2012)

6. Let $S_n = n^2+20n+12$, n a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?(PRERMO 2012)

7. Suppose that $4^{x_1} = 5$, $5^{x_2} = 6$, $6^{x_3} = 7, \ldots, 126^{x_{123}} = 127$, $127^{x_{124}} = 128$. What is the value of the product $x_1 x_2 \ldots x_{124}$? (PRERMO 2012)

8. If $\frac{1}{\sqrt{2011}+\sqrt{2012}} = \frac{\sqrt{m}-\sqrt{n}}{\sqrt{m+n}}$, where m and n are positive integers, what is the value of $m+n$? (PRERMO 2012)

9. If $a = b-c$, $b = c-d$, $c = d-a$, and $abcd \neq 0$, then what is the value of $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$? (PRERMO 2012)

10. How many non-negative integral values of x satisfy the equation

$$\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]? \tag{8.10}$$

(Here $[x]$ denotes the greatest integer less than or equal to x. For example, $[3.4] = 3$ and $[-2.3] = -3$.) (PRERMO 2012)

11. Let x_1, x_2, x_3 be the roots of the equation $x^3+3x+5=0$. What is the value of the expression $\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)\left(x_3+\frac{1}{x_3}\right)$? (PRERMO 2012)

12. What is the sum of the squares of the roots of the equation

$$x^2 - 7[x] + 5 = 0? \tag{8.11}$$

Here $[x]$ denotes the greatest integer less than or equal to x. For example, $[3.4] = 3$ and $[-2.3] = -3$. (PRERMO 2012)

Chapter 9

Geometry

1. On each side of an equilateral triangle with side length n units, where n is an integer,$1 \leq n \leq 100$, consider $n-1$ points that divide the side into n equal segments.Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up. Two coins are said to be adjacent if the distance between them is 1 unit. A move consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.(IOQM 2015)

2. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment.If N denotes the number of ways we can choose the pegs such that the drawn linesegments divide the interior of the triangle into exactly nine

regions, find the sum ofthe squares of the digits of N. (IOQM 2015)

3. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF, respectively. If the area of triangle ABC is 480, find the area of triangle GYZ. (IOQM 2015)

4. The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \leq i < j < k \leq 6$, has at least one redside, find the sum of the squares of the digits of N . (IOQM 2015)

5. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X . (IOQM 2015)

6. Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD, respectively, such that the perimeter of triangle MCN is 2. Let O be the circumcenter of triangle MAN, and P be the circumcenter of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of $m+n$. (IOQM 2015)

7. Let ABC be a triangle in the xy-plane, where B is at the origin $(0,0)$. Let BC be produced to D such that $BC : CD = 1 : 1$, CA be produced to E such that $CA : AE = 1 : 2$, and AB be produced to F such that $AB : BF = 1 : 3$. Let $G\,(32, 24)$ be the centroid of triangle ABC and K be the centroid of triangle DEF. Find the length GK. (IOQM

2015)

8. In the coordinate plane, a point is called a lattice point if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right-angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$. (IOQM 2015)

9. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set $\{5, 6, 7, 8, 9, 10\}$. (IOQM 2015)

10. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $PQ \parallel AB$.

    Let P_1 be a point on line PB_1 such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1 such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$. Prove that points P, Q, P_1, and Q_1 are concyclic. (IMO 2019)

11. Let I be the in center of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at points D, E, and F, respectively.

    The line through D perpendicular to EF meets ω again at R. Line AR meets *omega* again at P. The circumcircles of triangles PCE and

PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A that is perpendicular to AI. (IMO 2019)

12. consider the convex quadrilateral ABCD.The point P is the interior of ABCD.The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC. \tag{9.1}$$

prove that the following three lines meet in a point:the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB (IMO 2020)

13. Prove that there exists a positive constant c such that the following statement is true: Consider an integer n>1, and a set S of n points in the plane such that the distance between any two different points in S is at least 1. It follows that there is a line l separating S such that the distance from any point of S to l is at least $cn^{\frac{-1}{3}}$ (A line l separates a set of points S if some segment joining two points in S crosses l.) Note. Weaker results with replaced by cn^{α} may be awarded points depending on the value of the constant α >1/3. (IMO 2020)

14. Let D be an interior point of the acute triangle ABC with AB >AC so that $\angle DAB = \angle CAD$.The point E on the segment AC satisfies $\angle ADE = \angle BCD$,the point F on the segment AB satisfies $\angle FDA =$

$\angle DBC$,and the point X on the line AC satisfies CX=BX. let O_1 and O_2 be the circumcentres of the triangles ADC and EXD,respectively.Prove that the lines BC,EF,and O_1O_2 are concurrent (IMO 2021)

15. Let r be a circle with centre I,and ABCD a convex quadrilateral such that each of the segments AB,BC,CD and DA is a tangent to r.Let Ωbe the circumcircle of the triangle AIC.The extension of BA beyond A meets Ωat X,and the extension of BC beyond C meets Ωat Z.The extensions of AD and CD beyond D meet Ωat Y and T,respectively.Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC \tag{9.2}$$

(IMO 2021)

16. Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle. (IM0 2022)

17. Let ABC be an acute-angled triangle with $AB \leq AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets

Ω again at$E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$.Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$. (IMO 2023)

18. Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C, CB_1 = B_1A, AC_1 = C_1B$, and $\angle BAC + \angle CB_1A + \angle AC_1B = 480°$. Let BC_1 and CB_1 meet at A_2, let CA_1 and AC_1 meet at B_2, and let AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points. (Note: no 2 sides have equal length.) (IMO 2023)

19. Let ABC be a triangle with $AB \leq AC \leq BC$. Let the incentre and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively. Prove that $\angle KIL + \angle YPX = 180°$. (IMO 2024)

20. Three points X, Y, Z are on a straight line such that $XY = 10$ and $XZ = 3$. What is the product of all possible values of YZ? (Prermo 2013)

21. Let AD and BC be the parallel sides of a trapezium $ABCD$. Let P

and Q be the midpoints of the diagonals AC and BD. If $AD = 16$ and $BC = 20$, what is the length of PQ? (Prermo 2013)

22. In a triangle ABC, let H, I, and O be the orthocenter, incenter, and circumcenter, respectively. If the points B, H, I, and C lie on a circle, what is the magnitude of $\angle BOC$ in degrees? (Prermo 2013)

23. Let ABC be an equilateral triangle. Let P and S be points on AB and AC, respectively, and let Q and R be points on BC such that $PQRS$ is a rectangle. If $PQ = \sqrt{3} \times PS$ and the area of $PQRS$ is $\frac{28}{3}$, what is the length of PC? (Prermo 2013)

24. Let A_1, B_1, C_1, D_1 be the midpoints of the sides of a convex quadrilateral $ABCD$ and let A_2, B_2, C_2, D_2 be the midpoints of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of $ABCD$? (Prermo 2013)

25. Let S be a circle with center O. A chord AB, not a diameter, divides S into two regions R_1 and R_2. Let S_1 be a circle with center in R_1 touching AB, the circle S internally. Let S_2 be a circle with center in R_2 touching AB at Y, the circle S internally, and passing through the center of S. The point X lies on the diameter passing through the center of S_2, and $\angle YXO = 30°$. If the radius of S_2 is 100, then what is the radius of S? (Prermo 2013)

26. In a triangle ABC with $\angle BCA = 90°$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. If the

ratio of the area of quadrilateral $BXYC$ to the area of triangle ABC is 13:18 and $BC = 12$, then what is the length of AC? (Prermo 2013)

27. A convex hexagon has the property that for a ny pair of opposite sides the distance between their midpoints is $\frac{\sqrt{3}}{2}$ times the sum of their lengths Show that all the hexagon's angles are equal. (IMO 2003)

28. $ABCD$ is cyclic. The feet of the perpendicula r from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff $RP = RQ$. hfill(IMO 2003)

29. Let ABC be an acute-angled triangle with circumcentre 0. Let P on BC be the foot o f the altitude from A.
Suppose that $<BCS \leq \angle ABC + 30^0$.
Prove that $<CAB+ \leq cop\angle 90^o$. (IMO 2001)

30. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^0$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC? (IMO 2001)

31. BC is a diameter of a circle center 0. A is any point on the circle with $\angle AOC > 60^0$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through 0 parallel to AD meets AC at J. Show that J is the inc enter of triangle CEF. (IMO 2002)

32. $n > 2$ circlesof radius 1 are drawn in the plane so that no line meets

more th an two of the circles. Their centers are $0_1, 0_2 \ldots 0_n$. Show that $\sum_{i<} 1/0_i 0_j \leq (n-1)\frac{\pi}{4}$. (IMO 2002)

33. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $a(X)$ the measure of the angle between OA and OX in radians countrclockwise from OA $(O \leq a(X) < 2\pi)$. Let $C(X)$ be the circle with center O and radius of length $\frac{OX + a(X)}{OX}$. each point of the plane is colored by one of a finite number of colors. Proveoint Y for which $a(y) > 0$ such that color appears on the circumference of the circle $C(Y)$. (IMO 1984)

34. Let $ABCD$ be a convex quadrilateral such tha the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel. (IMO 1984)

35. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices $(n > 3)$,and let p be its perimeter.Prove that.

$$In - 3 < \frac{2d}{p} < \left(\frac{n}{2}\right)\left(\frac{n+1}{2}\right) - 2,$$

Where $\left(x\right)$ denotes the gratest integer not excee ding x (IMO 1984)

36. let A be one of the two distinct points of intersection of two unequal coplanar tangents to the circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at

P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1,M_2 be the midpoint of P_2Q_2 prove that $\angle O_1AO_2 = \angle M_1AM_2$. (IMO1983)

37. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$. (IMO 1985)

38. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. The circumscribed circle of the triangle ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle. (IMO 1985)

39. P is a point inside a given triangle $ABC.D, E, F$ are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

    $\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$ is least. (IMO 1981)

40. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear (IMO 1981)

41. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the sid e opposite A_i). For all $i = 1, 2, 3, M_i$ is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection. of T_i in the interior bisector of anngle A_i. Prove that the

lines M_1, S_1,M_2S_2 and M_3S_3 are concurrent. (IMO 1982)

42. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear. (IMO 1982)

43. Let S be a square with sides of length 100, and let L be a path with in S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, ...A_{n-1}A_1$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $\frac{1}{2}$. Prove that there are two points X and Y in & such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198. (IMO 1982)

44. A triangle $A_1A_2A_3$ and a point $P_0$are given in the plane.We define $A_s = A_s - 3$ for all $s \geq 4$ .We construct a set of points P_1, P_2,$P_3 \ldots$,such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle 120° clockwise $(fork = 0, 1, 2, 3 \ldots)$ .Prove that if P_{1986}=P_0, then thetriangle $A_1A_2A_3$ is equilateral . (IMO 1986)

45. Let A, B be adjacent vertices of a regular n-gon $(n \leq 5)$ in the plane having center at O. A triangle XYZ, which is congruent to and initially conincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside

the polygon. Find the locus of X. (IMO 1986)

46. In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circumcircle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and Mrespectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas. (IMO 1987)

47. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n. (IMO 1987)

48. Consider two coplanar circles of radii R and r $(R > r)$ with the same center. Let P be a fixed point on the smaller circle and B a variable point on the lar ger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then $A = P$) (i) Find the set of values of $BC^2 + CA^2 + AB^2$ (ii) Find the locus of the midpoint of BC. (IMO 1988)

49. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$. (IMO 1988)

50. Problem 5. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three

of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied: * no line passes through any point of the configuration; * no region contains points of both colours

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines (Imo 2013)

51. Problem 6. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1, on CA and C_1, on AB analogously, using the excircles opposite B and C. respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$, lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled. (Imo 2013)

    The excircle of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined. (Imo 2013)

52. problem7 Let ABC be an acute-angled triangle with orthocentre$H$, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the fect of the altitudes from B and C, respectively. Denote by w_1 the circumcircle of BWN, and let X be the point on wy such that WX is a diameter of w_1 Analogously, denote by w_2 the circumcircle of CWM. and let Y be the point on such that WY is a diameter of Prove that X, Y and Hare collinear. (Imo 2013)

53. Problem 8. Let $Q_{>0}$ be the set of positive rational mumbers. Let $f : Q_{>0} \to R$ be a function satisfying the following three conditions:

   (a) for all $x, y \epsilon Q > 0$, we have $f(x) f(y) \geq f(xy)$

   (b) for all $x, y \epsilon Q > 0$,we have$f(x+y) \geq f(x) + f(y)$

   (c) there exists a rational number $a > 1$ such that $f(a) = a$.

   prove that $F(x) = x$ for all $x \epsilon Q > 0$.

   (Imo 2013)

54. Problem 9. let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares. (Imo 2014)

55. Problem 10. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$ Point His the foot of the perpendicular from A to BD. Points S and T lie on sides $ABandAD$, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90°, \angle THC - \angle DTC = 90°$. Prove that line BD is tangent to the circumcircle of triangle TSH. (Imo 2014)

56. Problem 4. Points $PandQ$lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines $BMandCN$ intersect on circumcircle of triangle ABC (Imo 2014)

57. Problem 11. A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cats the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n. in any set of a lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

    Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c. (Imo 2014)

58. Problem 12. We say that a finite set S of points in the plane is balanced if, for any two different points $AandB$ in S, there is a point Cin Ssuch that $AC = BC$. We say that S is centre-free if for any three different points A, B and C in S, there is no point P in S such that $PA = PB = PC$

    (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

    (b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

    (Imo 2015)

59. Problem 13. Determine all triples (a, b, c) of positive integers such that each of the numbers $ab - c, bc - a, ca - b$

    is a power of 2

(A power of 2 is an integer of the form 2^n,Where n is a non-negative integer). (Imo 2015)

60. Problem 14. Let ABC be an acute triangle with $AB>AC$ Let I be its circumcircle, H its orthocentre, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q he the point on T such that $\angle HQA = 90$, and let K be the point on T such that $\angle HKQ = 90°$. Assume that the points$A, B, C, KandQ$ are all different, and lie on T in this order.

    Prove that the circumcircles of triangles KQH and FKM are tangent to each other. (Imo2015)

61. Problem 15. Triangle ABC has circumcircle Ω and circumcentre O. A circle T with centre. A intersects the segment BC at points $DandE$, such that $B, D, E$and Care all different and lie on line BC in this onter. Let $FandG$be the points of intersection of $Tand\Omega$. such that $A.FB.CandG$lie on Ωin this order. Let K he the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA Suppose that the lines $FKandGL$ are different and intersect at the point X. Prove that X lies on the line AO. (Imo 2015)

62. Problem 16. Let R be the set of real numbers. Determine all functions

$f : R \to R$ satisfying the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x) \qquad (9.3)$$

for all real numbers x and y (Imo2015)

63. problem17 the sequence $a_1, a_2, \ldots$ of an integers satisfies the following conditions;

    (a) $1 \leq a_j \leq 2015$ for all $j \geq 1$;

    (b) $k + a_k \neq l + a_l$ for all $1 \leq k<l$.

    prove that there exist two positive integers $bandN$ such that

    $\left|\sum_{j=m+1}^{n}(aj - b)\right| \leq 1007^2$

    for all integers $mandn$ satisfying $n > m \geq N$ (Imo 2015)

64. Prove that the set $\{1, 2, ..., 1989\}$ can be expressed as the disjoint union of subsets $A_i (i = 1, 2, ..., 117)$ such that : (i) Each A_i contains 17 elements ; (ii) The sum of all the elements in each A_i is the same . (IMO 1989)

65. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

    (i) The area of the triangle $A_0\ B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$

(ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC. (IMO 1989)

66. Let n and k be positive integers and let S be a set of n points in the plane such that

(i) No three points of S are collinear, and

(ii) For any point P of S there are at least k points of S equidistant from P. (IMO 1989)

Prove that:

$$k < \frac{1}{2} + \sqrt{2n}.$$

67. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD+BC$. There exists a point. P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

. (IMO 1989)

68. Chords AB and CD of a circle imersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangen t line at E to the circle through D, E. and M intersects the lines BC and AC

at F and G. respectively, If

$$\frac{AM}{AB} = t$$

find

$$\frac{EG}{EF}$$

in terms of t . (IMO 1990)

69. Let n_3 and consider a set E of 2_{n-1} distinct points on a circle. Suppose that exactly k of these points are to he colored black. Such a coloring is "*good*" if there is at least one pair of black points such that the interior of one of the ares between them contains exactly in points from E. Find the smallest value of k so that every such coloring of k points of E is good (IMO 1990)

70. Given an initial integer $n_0 > 1$, two players. A a nd B, choose integers $n_1, n_2, n_3, ...$ alternately accordi ng to the following rules: Knowing n_{2k}, A chooses any integer n_{2k+2} such that

$$n_{2k} \leq n_{2k+1} \leq n_2^2 k$$

Knowing n_{2k+1} , B chooses any integer n_{2k+2} such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power. Plaver A wins the game by choosing the number 1990: player B wins by choosing the number 1. For which n_0 does: $(a)A$ have a winning strategy? (b) B have a winning strategy? (c) Neither player have a winning strategy? (IMO 1990)

71. Prove that there exists a convex 1990-gon with the following two properties (a) All angles are equal. (b) The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2$,.....,1990^2 in some order. (IMO 1990)

72. Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles $\angle PAB, \angle PBC, \angle PCA$ is less than or equal to $30°$. (IMO 1991)

73. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AKBK$, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon. (Imo 1977).

74. P is a given point inside a given sphere.Three mutually perpendic ular rays from Pintersect the sphere at points U, V, and W; Q denotes the vertex diagonally opposite to P in the parallelepiped determined by PU, PV, and PW. Find the locus of Q for all such triads of rays from P (Imo 1978)

75. In triangle ABC, $AB = AC$. A circle is t angent internally to the circumcircle of triangle ABC and also to sides AB, AC at $P.Q$, re-

spectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC. (Imo 1978)

76. A prism with pentagons $A1A2A3A4A5$ and $B1B2B3B4B5$, as top and bottom faces is given. Each side of the two pentagons and each of the line- segments A, B for all $i, j = 1, \ldots, 5$, is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color. (Imo 1979)

77. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal. (Imo 1979)

78. Given a plane π, a point P in this plane and a point Q not in π, find all points$R$ in π such that the ratio $(QP + PA)/QR$ is a maximum. (Imo 1979)

79. Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC,CA, and AB at K, L, and M, respectively.The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute. (IMO 1998)

80. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S. (IMO 1999)

81. Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the center of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D, respectively. Prove that CD is tangent to G_2. (IMO 1999)

82. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle. (IMO 2000)

83. In the convex quadrilateral $ABCD$,the diagonals AC and BD are perpendicular and the oppositesides AB and DC are not parallel.Suppose that thepoint P, where the perpendicular bisectors of $AB$and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral ifand only if the trianglesABP and CDP have equalareas. (IMO 1998)

84. Let ABC be an acute-angled triangle with $AB \neq AC$.The circle with diameter BC intersects the sides AB and AC at M and N respectively.Denote by O the midpoint of the side BC.The bisector of the angles $<BAC$ and $<MON$ intersects at R.Prove that the circumcir-

cles of the triangles BMR and CNR have a common point on the side BC (IMO 2004)

85. In a convex quadrilateral ABCD the diagonal BD does not bisect the angles ABC and CDA.The point P lies inside ABCD and satisfies

$$\angle PBC = \angle DBA and \angle PDC = \angle BDA.$$

Prove that ABCD is a cyclic quadrilateral if and only if AP=CP (IMO 2004)

86. Six points are chosen on the sides of an equilateral triangle ABC: A_1,A_2 on BC,B_1,B_2 on CA and C_1,C_2 on AB, such that they are the vertices of a convex hexagon A_1A_2 B_1B_2 C_1C_2 with equal side lengths.Prove that the line A_1B_2,B_1C_2 and C_1A_2 are concurrent.(IMO 2005)

87. prove that x, y, z be three positve real such that $xyz \geq 1$.Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$

(IMO 2005)

88. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BEDF$. The lines $AC$and BD meet at P,the lines BD and EF meet at Q, the lines EF and AC meet at R.Prove that the circumcircles of the triangles PQR, as E and F

vary,have a common point other than P. (IMO 2005)

89. In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each. (IMO 2005)

90. Let P be a regular 2006-gon. A diagonal of P is called good if its endpoints divide the boundary of P into two parts, each composed of an odd mumber of sides of P.The sides of Pare also called good.Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration (IMO 2006)

91. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P. (IMO 2006)

92. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cycle quadrilateral.Let l be a line passing through A. suppose that l intersts the interior of the segment DC at F and intersects line BC at G.suppose also that $EF = EG = EC$. Prove that l is the bisector of angle DAB. (IMO 2007)

93. In triangle ABC the bisector of angle BCA intersects the circumcircle

again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoin of AC is L.Prove that the triangles RPK and RQL have the same area. (IMO 2007)

94. An acute-angled triangle ABC has orthocentre H.The circle passing through H withcentre the midpoint of BC intersects the line BC at $A1$ and $A2$. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at $B1$ and $B2$,and the circle passing throughH with centre the midpoint of AB intersects the line AB at $C1$ and $C2$. Show that $A1$, $A2$, $B1$, $B2$,$C1$, $C2$ lie on a circle. (IMO 2008)

95. Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$.Denote the incircles of triangles $ABCandADC$ by $\omega_1 and \omega_2$ respectively.Suppose that there exists a circle ω tangent to the ray BA beyond A and tothe ray BC beyond C, which is also tangent to the lines AD and CD.Prove that the common external tangents of ω_1 and ω_2 intersect on ω.(IMO 2008)

96. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides $CAandAB$, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let $\lceil$ bethe circle passing through K, L and M. Suppose that the line PQ is tangent to the circle $\lceil$. Prove that $OP = OQ$. (IMO 2009)

97. Let ABC be a triangle with $AB = AC$. Theangle bisectors of $\angle CAB$

and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incentre of triangle ADC.Suppose that $\angle BEK$=45°. Find all possible values of $angleCAB$. (IMO 2009)

98. Let A,B,C,D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent. (IMO 1995)

99. We are given a positive interger r and a rectangular board $ABCD$ with dimensions $\left|AB\right| = 20$, $\left|BC\right| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

    (a) Show that the task cannot be done if r is divisible by 2 or 3.

    (b) Prove that the task is possible when $r = 73$.

    (c) Can the task be done when $r = 97$? (IMO 1996)

100. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and

whose legs, of lengths m and n, lie along edges of the square s. Let S_1 be the total area of the black part of triangle ans S_2 be the total area of white part. Let

$$f(m,n) = \left| S_1 - S_2 \right|. \tag{9.4}$$

(a) calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \leq \frac{1}{2}max\left\{m,n\right\}$ for all m and n

(c) Show that there is no constant C such that $f(m,n) < c$ for all m and n. (IMO 1997)

101. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle BC. \tag{9.5}$$

Let D, E be the incenters of triangles APB ,APC, respectively. Show that AP ,BD, CE meet at a point. (IMO 1996)

102. Let $ABCDEF$ be a convex hexagon such that A is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{p}{2}.(IMO1996) \tag{9.6}$$

103. The angle at A is the smallest angle of triangle ABC. The point B

and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respctively. The lines BV and CW meet at T. Show that

$$AU = TB + TC.(IMO1997) \tag{9.7}$$

104. Determine all integers $n > 3$ for which there exist n points $A_1 \ldots, A_n$ in the plane, no three collinear, and real numbers $r_1, \ldots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_i A_j A_k$ is $r_i + r_j + r_k$. (IMO 1995)

105. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$. (IMO 1995)

106. Let a, b, c be positive real numbers such that $abc = 1$. Prove that.

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.(IMO1995) \tag{9.8}$$

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107. Triangle BCF has a right angle at B. Let A be the point on line CF such that

$$FA = FBandF \tag{9.9}$$

lies between A and C. Point D is chosen such that

$$DA = DCandAC \tag{9.10}$$

is the bisector of $\angle DAB$. Point E is chosen such that

$$EA = EDandAD \tag{9.11}$$

is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram

$$(whereAM||EXandAE||MX) \tag{9.12}$$

. Prove that lines

$$BD, FX, andME \tag{9.13}$$

are concurrent. (IMO 2016)

108.

$$LetP = A_1A_2...A_k \tag{9.14}$$

be a convex polygon in the plane. The vertices

$$A_1, A_2, ...A_k \tag{9.15}$$

have integral coordinates and lie on a circle. Let S be the area of P.

An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n. (IMO 2016)

109. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, Ag, and the hunter's starting point, Bo, are the same. After $n-1$ rounds of the game, the rabbit is at point $An-$ and the hunter is at point $B-1$. In the nth round of the game, three things occur in order. (IMO 2017)

(i) The rabbit moves invisibly to a point A, such that the distance between $An-1$ and A,, is exactly 1. (ii) A tracking device reports a point P, to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P and A, is at most 1. (iii) The hunter moves visibly to a point B, such that the distance between $Bu-1$ and Bn is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10 rounds she can ensure that the distance between her and the rabbit is at most 1002.

(i) The rabbit moves invisibly to a point An such that the distance between $An-1$ and An is exactly 1.

(ii) A tracking device reports a point Pa to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P, and An, is at most 1.

(iii) The hunter moves visibly to a point B, such that the distance between $B-1$ and B, is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10 rounds she can ensure that the distance between her and the rabbit is at most 100? (IMO 2017)

110. Let Rand S be different points on a circle and such that RS is not a diameter. Let E be the tange nt line to 2 at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter are RS of Q so that the circumcircle I of triangle JST intersects (at two distinct points. Let A be the common point of I and that is closer to R. Line AJ meets again at K. Prove that the line KT is tangent to γ. (IMO 2017)

111. An integer $N \leq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same h eight, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following V conditions hold. (IMO 2017)

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players.

(3) no one stands between the two shortest players.

Show that this is always possible.

112. Let I be the circumcircle of acute-angled triangle ABC. Points D and

E lie on segments

$$ABandAc, \tag{9.16}$$

respectively, such that $AD = AE$. The per pendicular bisectors of BD and CE intersect the minor arcs AB and AC of I at point s F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line). (IMO 2018)

113. An anti-Pascal triangle is an equilateral triangular array of numbers such that, excep t for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10. Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from

$$1to1 + 2 + ... + 2018? \tag{9.17}$$

(IMO 2018)

114. A convex quadrilateral $ABCD$ satisfies

$$AB.CD = BC.DA. \tag{9.18}$$

Point X lies inside. $ABCD$ so that

$$\angle XAB = \angle XCD and \angle XBC = \angle XDA. \tag{9.19}$$

Prove that

$$\angle BXA + \angle DXC = 180^\circ \tag{9.20}$$

. (IMO 2018)

115. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR. (IMO 1992)

116. Let D be a point inside acute triangle ABC such that $\angle ADB = \angle ACB$ $+ \pi/2$ and $AC \cdot BD = AD \cdot BC$.

   (a) Calculate the ratio $(AB \cdot CD)/(AC \cdot B)$.

   (b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular. (IMO 1993)

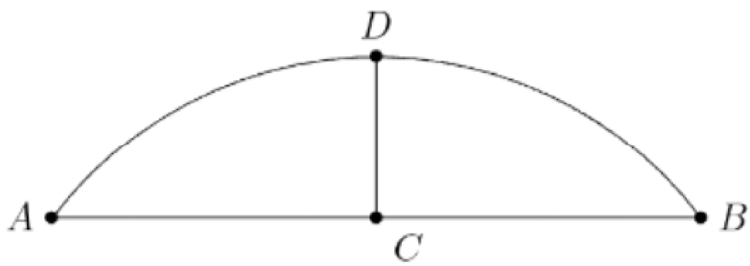
117. For three points P, Q, R in the plane, we define $m(PQR)$ as the minimum length of the three a ltitudes of $\triangle PQR$. (If the points are collinear, we set $m(PQR) = 0$.) Prove that for points A, B, C, X in the plane, $m(ABC) \leq m(ABX) + m(AXC) + m(XBC)$. (IMO 1993)

118. ABC is an isosceles triangle with $AB = AC$. Suppose that 1. M is the midpoint of BC and O is the point on the line AM such that OB is

perpendicular t o AB; 2. Q is an arbitrary point on the segment BC different from B and C; 3. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$. (IMO 1994)

1. The figure below shows a broken piece of a circular plate made of glass.



C is the midpoint of AB, and D is the midpoint of arc AB. Given that $AB = 24$ cm and $CD = 6$ cm, what is the radius of the plate in centimeters? (The figure is not drawn to scale.) (PRERMO 2015)

2. A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square? (PRERMO 2015)

3. What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12? (PRERMO 2015)

4. In rectangle $ABCD$, $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\angle BPC = 90°$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC, and CPD, what is the value of $r_1 + r_2 + r_3$? (PRERMO 2015)

5. In the acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\angle BAE = 40°$. If $\angle DAE = \angle DFE$, what is the magnitude of $\angle ADF$ in degrees? (PRERMO 2015)

6. The circle ω touches the circle Ω internally at P. The center O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume $PY > PX$. Let PY intersect ω at Z. If $YZ = 2PZ$, what is the magnitude of $\angle LPYX$ in degrees? (PRERMO 2015)

1. Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. If $AB = 20$, $BC = 70$, and $CD = 90$, then what is the value of DA? (PRERMO 2014)

2. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle? (PRERMO 2014)

3. In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX : XB = 1 : 2$ and $AY : YC = 2 : 1$.If the area of triangle AXY is 10, then what is the area of triangle ABC? (PRERMO 2014)

4. Let XOY be a triangle with $\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Suppose that $XN = 19$ and $YM = 22$. What is XY? (PRERMO 2014)

1. PS is a line segment of length 4 and O is the midpoint of PS. A semicircular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semicircular arc drawn with QR as diameter is tangent to PS. What is the area of the region $QXROQ$ bounded by the two semicircular arcs? (PRERMO 2012)

2. O and I are the circumcentre and incentre of $\triangle ABC$ respectively. Suppose O lies in the interior of $\triangle ABC$ and I lies on the circle passing through B, O, and C. What is the magnitude of $\angle BAC$ in degrees? (PRERMO 2012)

3. In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of the segment BD? (PRERMO 2012)

4. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. What is the area of $\triangle ABE$? (PRERMO 2012)

5. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle? (PRERMO 2012)

6. $ABCD$ is a square and $AB = 1$ Equilateral triangles AYB and CXD are drawn such that X and Y are inside the square.What is the length of XY? (PRERMO 2012)

Chapter 10

Discrete

1. What is the number of ordered pairs (A, B) where A and B are subsets of $\{1, 2, \ldots, 5\}$ such that neither $A \subseteq B$ nor $B \subseteq A$?(PRERMO 2014)

2. The Bank of Oslo issues two types of coin: aluminium (*denoted* A) and bronze (*denoted* B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Marianne repeatedly performs the following operation: she identifies the longest chain containing the k^{th} coin from the left, and movees all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be

$$AA\underline{B}BBABA \to BBB\underline{A}AABA \to AA\underline{A}BBBBA \to BBB\underline{B}AAAA \to .... \tag{10.1}$$

Find all pairs (n, k) with $(1 \leq k \leq 2n)$ such that for every initial ordering, at some moment during the process, the leftmost (n) coins will all be of the same type. (IMO 2022)

3. Let n be a positive integer. A Nordic square is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. Two different cells are considered adjacent if they share a common side. Every cell that is adjacent only to cells containing la Rger numbers is called a valley. An uphill path is a sequence of one or more cells such that:

   (a) The first cell in the sequence is a valley,

   (b) Each subsequent cell in the sequence is adjacent to the previous cell, and

   (c) The numbers written in the cells in the sequence are in increasing order.

   Find as a function of n, the smallest possible total number of uphill paths in a Nordic square. (IMO 2022)

4. Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$ the i^{th} row contains exactly i circles, exactly one of which is coloured red. A ninja path in a *Japanese triangle* is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a *Japanese triangle* with $n = 6$ along with a *ninja path* in that tr iangle containing two red circles. In terms of n, find the greatest k such that in each *Japanese triangle* there is a *ninja path* containing

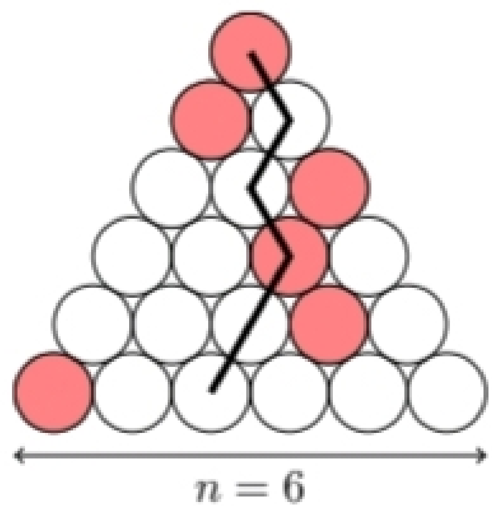


Figure 10.1: Image 1

at least k red circles. (IMO 2023)

5. Determine all pairs (a, b) of positive integers for which there exist positive integers g and N Such that

$$gcd\,(a^n + b, b + a) = g \tag{10.2}$$

Holds for all integers $n \geq N$. Note that $\gcd(x, y)$ denotes the greatest common divisor of integers x and y. (IMO 2024)

6. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let N be a positive integer. Suppose that, for each $n \geq N$, an is equal to the number of times an appears in the list $a_1, a_2, \ldots, a_{n-1}$.

Prove that at least one of the sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ is eventually periodic. An infinite sequence $b_1, b_2 b_3, \ldots$ is eventually periodic if there exist positive integers p and M such that $b_{m+p} = b_m$ for all $m \geq M$. (IMO 2024)

7. Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there Is exactly one monster in each row except the first row and the last row, and That each column contains at most one monster. Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common Turbo the Tortoise is on a quest to escape from a rectangular grid of cells. Starting on any ce Ll in the first row, Turbo repeatedly moves to an adjacent cell sharing a common side.(He is allowed to return to a previously) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo r emembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over. Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n^{th} attempt or earlier, regardless of the locations of the monsters. (IMO 2024)

8. Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$. What is the value of $\sum_{n=1}^{90} \frac{1}{S_n+S_{n-1}}$? (Pre-rmo 2013)

9. An infinite sequence $x_0, x_1, x_2, ...$of real numbers is said to be bounded

if there is a constant C such that $\left|x_i\right| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, ...$ Such that

$$\left|x_i - x_j\right| \left|i - j\right|^a \geq 1$$

for every pair of distinct nonnegative integers i, j. (IMO 1991)

10. Let n be a fixed integer, with $n \geq 2.(a)$ Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_{1 \leq i \leq n} x_i\right)^4$$

holds for all real numbers $x_1, ..., x_n \geq 0$. (b) For this constant C, determine when equality holds. (IMO 1999)

11. A,B, C are positive reals with product1. Prove that $\left(A - 1 + \frac{1}{B}\right)\left(B - 1 + \frac{1}{C}\right)\left(C - 1 + \frac{1}{A}\right) \leq 1.$ (IMO 2000)

12. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n, n \geq 1.$$

(IMO 2005)

13. In a mathematical compentition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two

of tem are friends.(In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room. (IMO 2007)

14. Let n and k be positive integers with $k \geq n$ and $k-n$ an even number. Let $2n$ lamps labelled $1, 2, \ldots, 2n$ be given, each of which can be either on or off. Initially all the lamps are off.We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 thr ough n are all on, and lamps $n+1$ through $2n$ are all off. Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

15. Find all functions $f : (0, \infty) \to (0), \infty$ so, (f is a function from the positive real numbers to the positive real numbers) such that $\frac{(f(w))^2+(f(x))^2}{f(y)^2+f(z)^2}$ for all positive real numbers w, x, y, z, satisfying $wx = yz$.(IMO 2008)

16. Let n be a positive integer $a_1, \ldots, a_k$ $(k \geq 2)$ be distinct integer in the set $\{1, \ldots, n\}$ such that n divides $a_i\,(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k\,(a_i - 1)$. (IMO 2009)

17. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive

integers such th at the subsequences

$$s_{s1}, s_{s2}, s_{s3}, \ldots and s_{s1+1}, s_{s2+1}, s_{s3+1}, \ldots \tag{10.3}$$

are both arithmetic progressions. Prove that the sequence s_1, s_2, s_3,...is itself an arithmetic progession. (IMO 2009) Determine the ratio $\frac{N}{M}$. (IMO 2008)

18. Find all integers $n \leq 3$ for which there exist real numbers

$$01.02.02, \tag{10.4}$$

such that

$$an + 1 = a1andan + 2 = a2, and \tag{10.5}$$

$$i = 1, 2, ..., n.aiai + 1 + 1 = ai + 2fori = 1, 2, ...n \tag{10.6}$$

. (IMO 2018)

19. A site is any point (x, y) in the plane such that z and y are both positive integers less than or equal to 20.Initially, each of the 400 sites is unoccupied. Amy and Been take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $sqrt(5)$ On his turn, Ben places a new blue stone on

any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone. Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones. (IMO 2018)

20. Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \epsilon A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \geq 2$. (IMO 1994)

Chapter 11

Number Systems

1. Let n be a positive integer such that $1 \leq n \leq 1000$. Let M_n be the number of integers in the set $X_n = \left\{\sqrt{4n+1}, \sqrt{4n+2}, \ldots, \sqrt{4n+1000}\right\}$. Let

$$a = \max M_n : 1 \leq n \leq 1000, \tag{11.1}$$

and

$$b = \min M_n : 1 \leq n \leq 1000. \tag{11.2}$$

Find $a - b$. (IOQM 2015)

2. Find the number of elements in the set

$$(a, b) \in \{N\} : 2 \leq a, b \leq 2023, \log_a(b) + 6\log_b(a) = 5. \tag{11.3}$$

(IOQM 2015)

3. Let α and β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}. \tag{11.4}$$

Find the smallest possible value of β. (IOQM 2015)

4. For $n \in N$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n . Consider the set

$$A = \{n \in N : P(n)\, isnon - zero, squarefreeandS(n)\, isaproperdivisorofP(n)\}. \tag{11.5}$$

Find the maximum possible number of digits of the numbers in A . (IOQM 2015)

5. For any finite non-empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

$$\max(A) \times |B| = 12 \quad \text{and} \tag{11.6}$$

$$|A| \times \max(B) = 11, \tag{11.7}$$

and N can be written as $100a + b$ where a, b are positive integers less than 100, find $a + b$. (IOQM 2015)

6. The sequence $\langle a_n \rangle_{n \geq 0}$ is defined by $a_0 = 1$, $a_1 = -4$, and $a_{n+2} = -4a_{n+1} - 7a_n$ for $n \geq 0$. Find the number of positive integer divisors of $a_{250} - a_{49}a_{51}$. (IOQM 2015)

7. A quadruple (a, b, c, d) of distinct integers is said to be balanced if $a + b = c + d$ and $a < b < c < d$. Find the number of balanced quadruples of distinct integers in the set $\{1, 2, \cdots, 12\}$. (IOQM 2015)

8. There is an integer $n > 1$. There are n2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can star using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positit from the lower station and reach the higher one byve integer k for which one can guarantee that there are two stations that are linked by both companies. (IMO 2020)

9. Find the smallest positive integer k such that $k(3^3 + 4^3 + 5^3) = a^n$ for some positive integers a and n, with $n > 17$. (Prermo 2013)

10. Let $S(M)$ denote the sum of the digits of a positive integer M written in base 10. Let N be the smallest positive integer such that $S(N) = 2013$. What is the value of $S(5N + 2013)$? (Prermo 2013)

11. Let m be the smallest odd positive integer for which $1+2+\cdots+m$ is a square of an integer and let n be the smallest even positive integer for which $1+2+\cdots+n$ is a square of an integer. What is the value of $m+n$? (Prermo 2013)

12. What is the maximum possible value of k for which 2013 can be written as a sum of k consecutive positive integers? (Prermo 2013)

13. Let a, b and c be positive integers, no two of which have a common divisor grater than 1. Show that $2abc-ab-bc-ca$ is the largest integer which cannot be expressed in the form $xbc+yca+zab$, where x, y and z are non-negative integers. (IMO 1983)

14. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression ? justify your answer. (IMO 1983)

15. Find one pair of positive integers a and b such that : (i) $ab\,(a+b)$ is not divisible by 7; $(ii)(a+b)^7 - a^7 - b^7$ is divisible by 7^7 (IMO 1984)

16. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a+d = 2^k$ and $b+c = 2^m$ for some integers k and m, then $a = 1$ (IMO 1984)

17. Let n and k be given relatively prime natural numbers $k < n$.Each number in the set $M = 1, 2, ...n-1$ is colored either blue or white. It is given that (i) for each $i \epsilon M$, both i and $n-i$ have the same color; (ii) for each $i \epsilon M$, $i \neq k$, both i and $\left| i-k \right|$ have the same color. Prove that all numbers in M must have the same color. (IMO 1985)

18. Given a set M of 1985 distinct positive integers, none of which has a prime divisor grater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer. (IMO 1985)

19. For every real number x_1, construct the sequence $x_1, x_2, ..116.$by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{4} \right)$$

for each $n \geq 1$ Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n. (IMO 1985)

20. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, ..., n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that $F(n, r) = \frac{n+1}{r+1}$ (IMO 1981)

21. (a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers (b)For which values of $n > 2$ is there exactly one set having the stated property? (IMO 1981)

22. . The function $f(n)$ is defined for all positive integers n and takes on

non-negative integer values. Also, for all m, n

$$f(m+n) - f(m) - f(n) = 0\,(or)\,1$$

$$f(2) = 0, f(3) > 0, and f(9999) = 3333.$$

Determine $f(1982)$. (IMO 1982)

23. Prove that if n is a positive integer such that the equation.

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y), then it has at least three such solutions. Sh w that the equation has no solutions in integers when $n = 2891$. (IMO 1982)

24. Consider the infinite sequences $\{x_n\}$ of positive real numbers with following properties: $x_0 = 1$, and for all $i \geq 0, x_{i+1} \leq x_i$. (a) Prove that for every such sequence, there is $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ... + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ... + \frac{x_{n1}^2}{x_n} < 4.$$

(IMO 1982)

25. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13.d\}$ such that $ab - 1$ is not a perfect square. (IMO 1986)

26. Let $p_n(k)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$,which have exactly k fixed points.Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n$$

(Remark:A permtation f of a set S is one-to-one mapping of S onto itself.An element i in S is called a fixed point of the the permutation f if f(i)=i.) (IMO 1987)

27. Let n be a positive integer and let $A_1, A_2, \ldots, A_{2n+1}$ be subsets o f a set B. Suppose that (a) Each A_i has exactly $2n$ elements, (b) Each $A_i \cap A_j$ $(1 \leq i \leq j \leq 2n+1)$contains exactl y one element, and
(c) Every element of B belongs to at least two of the A_i.

For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements? (IMO 1988)

28. Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer. (IMO 1988)

29. problem 1 Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, m_3, \ldots$ (not necessarily different) such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\ldots\left(1+\frac{1}{m_k}\right) \tag{11.8}$$

(Imo 2013)

30. problem2 let $a_{0<}a_{1<}a_{2<}\ldots$ be an infinite sequence of positive integers.prove that there exists a unique integer $n \geq$ 1such that

$$a_{n<}\frac{a_0+a1+\ldots+a_n}{n}<a_{n+1}. \tag{11.9}$$

(Imo2014) (Imo 2014)

31. Problem 3. For each positive integer n, the Bank of Cape Town ienes coins of denomination$\frac{1}{n}$ Given a finite collection of such coins (of not necessarily differ ent denominations) with total value at most $99+\frac{1}{2}$ prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1. (Imo2014)

32. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number. (IMO 1989)

33. A permutation $(x_1, x_2, ..., x_m)$ of the set $\{$1,2.....,2n$\}$. where a is a positive integer, is said to have property P if $\left|x_i - x_{i+1}\right| = n$ for at least one in$\{$1,2,....,2n-1$\}$. Show that, for each n, there are more permitations

with property P than without. (IMO 1989)

34. Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is integer. (IMO 1990)

35. Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI.BI.CI.}{AA'.BB'.CC'.} \leq \frac{8}{27}$$

. (IMO 1991)

36. Let $n > 6$ be an integer and $a_1, a_2, ..., a_k$ be all the natura numbers less than n and relatively prime to n If

$$a_2 - a_1 = a_3 - a_2 = ... = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2. (IMO 1991)

37. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence. (Imo 1977)

38. Let n be a given integer ¿ 2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, \ldots A$ number $m\epsilon V_n$ is called indecomposable in V_n, if there do not exist numbers p ,$q\epsilon V_n$ such that $pq = m$. Prove that there exists a number $r\epsilon V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (products which differ only in the order of their factors will be considered the same). (Imo 1977)

39. Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$. (Imo 1977)

40. Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n+1) > f(f(n)) \tag{11.10}$$

for each positive integer n, then

$$f(n) = n \tag{11.11}$$

for each n (Imo 1977)

41. m and n are natural numbers with $1 \leq m<n$ In their decimal representations, the last three digits of 1978 are equal, respectively, to the last three digits of 1978". Find m and n such that $m + n$ has its least value. (Imo 1978)

42. The set of all positive integers is the union of two disjoint subsets

$$f(1), f(2), \ldots, f(n), \ldots, g(1), g(2), \ldots, g(n), \ldots \tag{11.12}$$

,where

$$f(1) < f(2) < \ldots < f(n) < \ldots, \tag{11.13}$$

$$g(1) < g(2) < \ldots < g(n) < \ldots \tag{11.14}$$

$$, and, g(n) = f(f(n)) + 1 \tag{11.15}$$

for all $n \geq 1$ and Determine (240). (Imo 1978)

43. Let a_k $(k = 1, 2, 3. \ldots, n, \ldots)$ be a sequece of distinct positive integers. Prove that for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k} \tag{11.16}$$

(Imo 1978)

44. Let p and q be natural numbers such that

$$\frac{p}{q} = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319} \tag{11.17}$$

.Prove that p is divisible by 1979. (Imo 1979)

45. For any positive integer n, let d(n) denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $\frac{d(n^2)}{d(n)} = k$ for some n. (IMO 1998)

46. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$. (IMO 1998)

47. Consider all functions f from the set N of all positive integers into itself satisfying $f\left(t^2 f\left(s\right)\right) = s\left(f\left(t\right)\right)^2$ for all s and t in N. Determine the least possible value of $f\left(1998\right)$. (IMO 1998)

48. Determine all pairs (n, p) of positive integers such that$p$ is a prime,n not exceeded $2p$,and$(p-1)^n + 1$ is divisible by n^{p-1}. (IMO 1999)

49. Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.] (IMO 2000)

50. Let ABC be a triangle with incentre I.A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB$$

Show that $AP \geq AI$,and that equality holds if only if $P = I$.(IMO2006)

51. Determine all pairs (x,y) of integers such that

$$1 + 2^x + 2^{x+1} = y^2$$

(IMO 2006)

52. Let N be the set of positive integers. Determine all functions $g : N \rightarrow$

N such that

$$(g(m)+n)(m+g(n)) \tag{11.18}$$

is a perfect square for all $m, n \in N$. (IMO2010)

53. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed: Type 1: Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1}. Type 2: Choose a nonempty box B_k with $1 \leq k \leq 4$. Remove one coin from B_k and exchange the contents of (possible empty) boxes B_{k+1} and B_{k+2}. Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{(b^c)}$.) (IMO2010)

54. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have

$$a_n = max\{a_k + a_{n-k} | 1 \leq k \leq n-1\} \tag{11.19}$$

for all $n > s$. Prove that there exist positive integers l and N, with $l \leq s$ and such that $a_n = a_l + a_{n-l}$ for all $n \leq N$. (IMO2010)

55. Given any $setA = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i \leq j \leq 4$ for which $a_i + a_j$ divides s_A. Find

all sets A of four distinct positive integers which achieve the largest possible value of n_A. (IMO2011)

56. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$. (IMO2011)

57. Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \ldots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \ldots (1 + a_n)^n > n^n.(IMO2012) \tag{11.20}$$

58. Find all functions $f : Z \to Z$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a). \tag{11.21}$$

(Here Z denotes the set of integers.) (IMO2012) (a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ $\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}$. $(*)$ (b) Show that there are real numbers x_1 $leqx_2 \leq \cdots \leq x_n$ such that equality holds in $(*)$. (IMO 2007)

59. Let $aandb$ be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$. (IMO 2007)

60. Let n be a positive integer. Consider $S = (x, y, z) : x, y, z \epsilon i0, 1, \dot{;} n,$

$x + y + z > 0$ as a set of $(n+1)^3 - 1$ points in three-dimensional space.Determine the smallest possible number of planes, the union of which contains S but doesnot include $(0, 0, 0)$. (IMO 2007)

61. Prove that 7 $\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$ for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1.(b)$ Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz = 1$. (IMO 2008)

62. Prove that there exist infinitely many po sitive integers n such that $n^2 + 1$ has a prime divis or which is greater than $2n + \sqrt{2}n$. (IMO 2008)

63. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths
$a, f(b)$ and $f(b + f(a) - 1)$.
$(Atriangleisnon-degenerateifitsverticesarenotcollinear)$. (IMO 2009)

64. Let a_1, a_2,..., a_n be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0and making n jumps to the right with lengths $a_1, a_2 \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M. (IMO 2009)

65. Find all positive integers n for which each cell of an $n \times n$ table can

be filled with one of the letters

$$I, M and O \tag{11.22}$$

in such a way that: in each row and each column,one third of the entries are I,one third are M and one third are O;and in any diagonal,if the number of entries on the diagonal is a multiple of three,the$n$ one third of the entries are I, one thirdv are M and one third are O. (IMO 2016)

66. A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$ What is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set

$$P(a + 1), P(a + 2), ..., P(a + b) \tag{11.23}$$

is fragrant?(IMO 2016) (a) Prove that Geoff can always fulfil his wish if n is odd. (b) Prove that Geoff can never fulfil his wish if n is even.

67. An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of r and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers ao, 41, 4 such that, for each (x, y)4 in S, we have (IMO 2017)

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1. \tag{11.24}$$

68. Let $a1, a2$,... be an infinite sequence of positive integers. Suppose that there is an integer N 1 such that, for each $n \neq N$, the number 01 is an integer. Prove that there is a positive integer M such that for all $m1 >= M$. (IMO 2018)

69. Find all integers a, b, c with $1 < a < b < c$ such that (IMO 1992) $(a-1)(b-1)(c-1)$ is a divisor of abc - 1.

70. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares. (IMO 1992)

    (a) Prove that $S(n) \leq n^2$ - 14 f or each $n \geq 4$.

    (b) Find an integer n such that $S(n) = n^2 - 14$.

    (c) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.

71. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some $i, j, 1 \leq i \leq j \leq m$, there exists $k, 1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that $\frac{a_1+a_2+\cdots+a_m}{m} \geq \frac{n+1}{2}$. (IMO 1994)

72. Determine all ordered pairs (m, n) of positive integers such that $\frac{n^3+1}{mn-1}$ is an integer. (IMO 1994)

1. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square? (PRERMO 2015)

2. Let n be the largest integer that is the product of exactly 3 distinct prime numbers, x, y, and $10x + y$, where x and y are digits. What is the sum of the digits of n? (PRERMO 2015)

3. A subset B of the set of first 100 positive integers has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B? (PRERMO 2015)

1. A natural number k is such that $k^2 < 2014, (k+1)^2$. What is the largest prime factor of k? (PRERMO 2014)

2. The first term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014^{th} term of the sequence? (PRERMO 2014)

3. What is the smallest possible natural number n for which the equation $x^2 - nx + 2014 = 0$ has integer roots? (PRERMO 2014)

4. If $x^{\left(x^4\right)} = 4$, what is the value of $x^{\left(x^2\right)} + x^{\left(x^8\right)}$? (PRERMO 2014)

5. Let S be a set of real numbers with mean M. If the means of the sets $S \cup \{15\}$ and $S \cup \{15, 1\}$ are $M + 2$ and $M + 1$, respectively, then how many elements does S have?

6. Natural numbers k, l, p, and q are such that a and b are roots of the equation $x^2 - kx + l = 0$ such that $a + \frac{1}{b}$ and $b + \frac{1}{a}$.What is the sum of all possible values of q? (PRERMO 2014)

7. For natural numbers x and y, let (x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with

$x \leq y$ satisfy the equation $xy = x + y + (x, y)$? (PRERMO 2014)

8. For how many natural numbers n between 1 and 2014 $(both inclusive)$ is $\frac{8n}{9999-n}$ an integer? (PRERMO 2014)

9. For a natural number b, let $N(b)$ denote the number of natural numbers a for which the equation $x^2 + ax + b = 0$ has integer roots. What is the smallest value of b for which $N(b) = 20$? (PRERMO 2014)

10. One morning, each member of Manjul's family drank an 8-ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Manjul drank $\frac{1}{7}$-th of the total amount of milk and $\frac{2}{17}$-th of the total amount of coffee. How many people are there in Manjul's family? (PRERMO 2014)

Chapter 12

Differentiation

Chapter 13

Integration

Chapter 14

Functions

1. Let f be a one-to-one function from the set of natural numbers to itself such that $f(mn) = f(m) f(n)$ for all natural numbers m and n. What is the least possible value of $f(999)$? (PRERMO 2014)

1. Let N be the set of natural numbers. Suppose $f : N \to N$ is a function satisfying the following conditions:

   (a) $f(mn) = f(m) f(n)$,

   (b) $f(m) < f(n)$ if $m < n$,

   (c) $f(2) = 2$.

   What is the value of $\sum_{k=1}^{20} f(k)$? (PRERMO 2012)

2. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1? (IMO 1986)

3. Let n be an integer greater than or equal to 2. Prove that if k^2+k+n is prime for all integers k such that $0 \le k \le \sqrt{n/3}$, then k^2+k+n is prime for all integers k such that $0 \le k \le n-2$ (IMO 1987)

4. A function f is defined on the positive integers by

$$f(1) = 1, f(3) = 3,$$
$$f(2n) = f(n),$$
$$f(4n+1) = 2f(2n+1) - f(n),$$
$$f(4n+3) = 3f(2n+1) - 2f(n),$$

for all positive integers n. Determine the number of positive integers n, less than or equal to 1988,for which $f(n) = n$. (IMO 1988)

5. Show that set of real numbers x which satisfy the in equality

$$\sum k = 1^{70} \frac{k}{x-k} \ge \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988(IMO 1988)

6. Let Q^+ be the set of positive rational numbers. Construct a function $f : Q^+ \to Q^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in Q^+. (IMO 1990)

COMBINATOMICS

7. Let $S = \{1, 2, 3, ..., 280\}$. Find the smallest integer n such that each $n-$ element subset of S contains five numbers which are pairwise relatively prime. (IMO 1991)

GRAPH THEORY

8. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2.....k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1. [A graph consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices. u, v belongs to at most one edge. The graph G is connected if for cach pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, ..., v_m = y$ such that each pair v_i, v_{i+1} $(0 \le i < m)$ is joined by an edge of G.] (IMO

1991)

9. Let Q^+ be the set of positive rational numbers. Construct a function $f : Q^+ \to Q^+$ such that

$$f\left(xf\left(y\right)\right) = \frac{f\left(x\right)}{y}$$

for all x, y in Q^+. (IMO 1990)

10. Determine all functions $f : \mathbf{R} \to \mathbf{R}$ such that
$f\left(x - f\left(y\right)\right) = f\left(f\left(y\right)\right) + xf\left(y\right) + f\left(x\right) - 1$
for all real numbers x,y. (IMO 1999)

11. Determine all functions $f : R \to R$ such that the euality

$$f(\lfloor x \rfloor y) = f(x)\lfloor f(y) \rfloor \tag{14.1}$$

holds for all $x, y \in R$.(Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.) (IMO2010)

12. Let $f : R \to R$ be a real-valued function defined on the set of real numbers that satisfies

$$f\left(x\right) + y \leq yf\left(x\right) + f\left(f\left(x\right)\right) \tag{14.2}$$

for all real numbers x and y. Prove that $f\left(x\right) = 0$ for all $x \leq 0$.

(IMO2011)

13. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, unwtil all of the weights have been placed. Determine the number of ways in which this can be done. (IMO2011)

14. The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players. At the start of the game$A$ chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful. After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that: 1. If $n \geq 2^k$, then B can guarantee a win. 2. For all sufficiently large k, there exists an integer

$n \geq 1.99^k$ such that B cannot guarantee a win. (IMO2012)

15. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{1}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.(IMO2012) \tag{14.3}$$

Chapter 15

Matrices

1. A $n \times n$ matrix whose entires come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, \ldots, n$, the ith row and ith column together contain all elements of S. Show that

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of n. (IMO 1997)

2. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares? (IMO 1996)

Chapter 16

Trignometry

1. In a triangle ABC, let I denote the incenter. Let the lines AI, BI, and CI intersect the incircle at P, Q, and R, respectively. If $\angle BAC = 40°$, what is the value of $\angle QPR$ in degrees? (PRERMO 2014)

2. Four real constants a, b, A, B are given, and

$$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta \tag{16.1}$$

. Prove that if

$$f(\theta) >= 0 \tag{16.2}$$

,for all real θ, then

$$a^2 + b^2 \leq 2 and A^2 + B^2 \geq= 1 \tag{16.3}$$

(Imo 1977)