MTH392

Modular Forms and Hecke Operators

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ABSTRACT

This report is the culmination of a semester-long reading project on Modular forms under the guidance of Dr. Saurabh Singh, IIT Kanpur. This is a report of what was learnt during course of the project.

Modular forms are functions on upper-half plane which have certain transformation properties. They come up unexpectedly in various number theoretic problems like Representation of integers as sum of k squares. They were the main objects in Andrew Wiles proof of Fermat's Last theorem in 1994.

The book *A First Course in Modular Forms* by Fred Diamond, Jerry Shurman was followed during the reading and most of the material in report is based on it. In Chapter 1, we define Modular forms on congruence subgroups, give examples and their connections with Complex Elliptic curves, Modular curves and their connection with modulli of Elliptic curves. In Chapter 2, we define Riemann surface structure on Modular curves and then add some missing points to compactify it. We give the summary of Chapter 3 of above book which defines differentials on compactified Modular curves which has nice relation with Automorphic forms (functions similar to Modular forms). Then uses Riemann-Roch theorem to exploit this relation and find dimension of space of Modular forms and Cusp forms. In Chapter 4, we define Hecke operators on space of Modular Forms and give their applications, defines L-function of a modular form and its functional equation. We also prove Ramanujan first two conjectures about his τ function which is straightforward corollary of a result at end of section 4.8.

This report is expository in nature and no new result is being claimed.

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§1. Modular Forms

§§1.1. Introduction

Modular forms are functions on upper-half plane which have certain transformation property. They come up naturally in various number theoretic problems like Representation of integers as sum of k squares. First traces of Modular forms can be found in work of Jacobi in early 19th century and further developed by Riemann, Hurwitz, Dedekind and Kronecker. But it was Ramanujan who in 1916, introduced his τ function and gave three conjectures about them. The first two were proved by Mordell in 1917. The third conjecture defied attempts of all mathematicians and was finally proven in 1974 by Deligne. In 1930s, Hecke worte a series of papers which we now call **Hecke theory of Modular forms** which gives us not only proof of first two conjectures but satisfactory expains why they are true and generalises it. We define Ramanjuan τ function in section 1.4 and state the conjectures related to it.

Modular forms appear unexpectedly in other number theoretic problem, most notably **Fermat's Last theorem**.

Theorem 1.1. All rational elliptic curves arise from Modular forms.

The above thm is called **Modularity theorem** was suggested by Taniyama and formulated by Shimura. Wiles proved it for semi-stable Elliptic curves in 1995 completing the proof Fermat's Last thm.

§§1.2. Modular group, congruence subgroups

Here we define important groups that we will be working with throughout our study.

Definition 1.2. The group $SL_2(\mathbb{Z})$ is called **modular group**.

Let $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any element of $SL_2(\mathbb{Z})$.Then modular group is generated by T and S. Also, it acts on **Riemann sphere** $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as Mobius transformation:

$$M \cdot \tau = \frac{a\tau + b}{c\tau + d} \quad \text{for } \tau \in \hat{\mathbb{C}}.$$

When this action is restricted to **upper half plane** $\mathcal{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$ gives action of $\mathcal{H}$.

Definition 1.3. Let $N \in \mathbb{N}$. The **principal congruence subgroup of level** N is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Definition 1.4. A subgroup Γ of $SL_2(\mathbb{Z})$ is **congruence subgroup** if there is some N s.t. $\Gamma(N) \subset \Gamma$ in which case it is said to be a congruence subgroup of level N.

Most important congruence subgroups besides principal ones are: (*-unspecified)

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Clearly, $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$ for any N. It can be easily seen that $\Gamma(N) \triangleleft \Gamma_1(N)$ and $\Gamma_1(N) \triangleleft \Gamma_0(N)$, $[\Gamma_1(N) : \Gamma(N)] = N$ and $[\Gamma_0(N) : \Gamma_1(N)] = \phi(N)$. It can be shown that $\Gamma_0(N)$ has finite index in $SL_2(\mathbb{Z})$. Thus every congruence subgroup is of finite index in $SL_2(\mathbb{Z})$. There exists finite index subgroups of $SL_2(\mathbb{Z})$ which are not congruence subgroups. We also define

$$GL_2(\mathbb{Q})^+ = \{\gamma \in GL_2(\mathbb{Q}) : \det(\gamma) > 0\}$$

§§1.3. Modular forms

Here we define our primary objects of study: Modular forms and cusp forms.

Definition 1.5. Let $M \in \mathrm{SL}_2(\mathbb{Z})$ then **factor of automorphy** is defined as

$$j(M, \tau) = c\tau + d \quad \text{for } \tau \in \mathcal{H}.$$

For any integer k, define **weight** k **operator** $[M]_k$ on functions $f : \mathcal{H} \to \mathbb{C}$ by

$$(f[\gamma]_k)(\tau) = j(\gamma, \tau)^{-k} f(\gamma \cdot \tau)$$

Definition 1.6. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and k an integer. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is called **weakly modular of weight** k **w.r.t** Γ if $f[\gamma]_k = f$ for all γ in Γ. i.e., it is weight k invariant w.r.t. Γ.

Let f be a weakly modular function f (of weight k) w.r.t Γ and suppose it is holomorphic on $\mathcal{H}$. Now we will define what it means for f to be **holomorphic at infinity**. If Γ is level N, then $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N) \subset \Gamma$. So there exists minimum h s.t. $\alpha = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$. Since f is weight k invariant under α, we have

$$f(\tau) = (f[\alpha]_k)(\tau) = j(\alpha, \tau)^{-k} f(\alpha \cdot \tau) = f(\tau + h), \tau \in \mathcal{H}.$$

Means f is periodic with period h. Let D be the open unit disk in $\mathbb{C}$ and $D' = D - \{0\}$. The function $\tau \mapsto e^{\frac{2\pi i \tau}{h}}$ takes $\mathcal{H}$ to D' and is also periodic with period h. Thus corresponding to f, the function $g : D' \to \mathbb{C}$ where $g(q) = f(h\log(q)/(2\pi \mathrm{i}))$ is well defined and $f(\tau) = g(e^{\frac{2\pi i\tau}{h}})$. Since f was assumed to be holomorphic on $\mathcal{H}$, the composition is g is holomorphic on punctured disk. By a thm from complex analysis, g has Laurent series expansion $g(q) = \sum_{n\in\mathbb{Z}} a_n q^n$, $q \in D'$. The relation $|q| = e^{\frac{-2\pi Im(\tau)}{h}}$ shows $q \to 0$ as $Im(\tau) \to \infty$. So define f to be **holomorphic at infinity** if g extends holomorphically to the puncture point, $q = 0$. Means f has fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \text{ where } q = e^{\frac{2\pi i\tau}{h}}$$

Definition 1.7. A Γ equivanlence class of $\mathbb{Q} \cup \{\infty\}$ is called a **cusp** of Γ.

Since each $s \in \mathbb{Q}$ takes the form $s = \alpha(\infty)$ for some $\alpha \in SL_2(\mathbb{Z})$, the number of cusps is atmost the number of cosets $\Gamma\alpha$ which is finite since $[SL_2(Z) : \Gamma]$ is finite. Now, we will define what it means for f to be **holomporphic at cusps**. Writing s $\in \mathbb{Q}$ as s=$\alpha(\infty)$, the holomorphy at s is naturally defined in terms of holomorphy at ∞ via $[\alpha]_k$ operator. Since $f[\alpha]_k$ is holomorphic on $\mathcal{H}$ and is weakly modular w.r.t. $\alpha^{-1}\Gamma\alpha$, again a congruent subgroup of $SL_2(\mathbb{Z})$, the notion of its holomorphy at ∞ makes sense. Now we are ready to define modular forms.

Definition 1.8. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and k an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is called **modular form of weight k w.r.t** Γ if

1. f is holomorphic
2. f is weakly modular of weight k w.r.t Γ
3. **Holomorphy condition:** $f[\alpha]_k$ holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$.

The vector space of modular forms of weight k w.r.t Γ is denoted by $\mathcal{M}_k(\Gamma)$.

To keep the vector spaces of modular forms *finite-dimensional*, modular forms need to be holomorphic not only on $\mathcal{H}$ but also at cusps. The modular forms which have $a_0 = 0$ play an important role in the subject.

Definition 1.9. Let $f \in \mathcal{M}_k(\Gamma)$ then f is called a **cusp form of weight** k **w.r.t** Γ if

- $a_0 = 0$ in Fourier expansion of $f[\gamma]_k$ for all $\gamma \in SL_2(\mathbb{Z})$

The space of cusp forms of weight k w.r.t. Γ is denoted by $\mathcal{S}_k(\Gamma)$.

$$\mathcal{M}(\Gamma) = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_k(\Gamma) \qquad \mathcal{S}(\Gamma) = \bigoplus_{k\in\mathbb{Z}} \mathcal{S}_k(\Gamma) \tag{1.1}$$

$\mathcal{M}(\Gamma)$ is a graded ring (in fact an algebra) with obivious addition and multiplication and $\mathcal{S}(\Gamma)$ is its graded ideal.

§§1.4. Examples

1. The **zero** function on $\mathcal{H}$ is modular form of every weight and w.r.t. every congruence subgroup. The **constant** functions on $\mathcal{H}$ are modular forms of weight 0 w.r.t. every congruence subgroup.
   Here we give non-trivial examples of modular forms w.r.t full modular group $SL_2(\mathbb{Z})$. Later we will see modular forms w.r.t. other congruence subgroups.

Definition 1.10. Let $k > 2$ be an even integer. Define **Eisenstein series of weight** k denoted by G_k as

$$G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2-(0,0)} \frac{1}{(c\tau+d)^k},\ \tau \in \mathcal{H}$$

G_k is easily seen to weakly modular w.r.t. $SL_2(Z)$. It can be shown that sum is absolutely convergent and uniformly convergent on compact subsets of $\mathcal{H}$ (so G_k is holomorphic on $\mathcal{H}$ and its terms can be rearranged) and it satisfies holomorphy condition. To compute the Fourier series of G_k, we use Poission Summation Formula or the identity

$$\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau-d} + \frac{1}{\tau+d}\right) = \pi\cot(\pi\tau) = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m, q = e^{2\pi i\tau}$$

to establish the following:

$$G_k(\tau) = 2\zeta(\tau) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{c=1}^{\infty}\sum_{m=1}^{\infty} m^{k-1}q^{cm}$$

After rearranging the double summation, we get that

$$G_k(\tau) = 2\zeta(\tau) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \text{ where } \sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$$

Dividing G_k by the leading coefficient gives a series having rational coefficients with a common denominator. This **normalized Eisenstein series** $G_k(\tau)/2\zeta(k)$ is denoted by $E_k(\tau)$.

2. Let $g_2(\tau) = 60G_4(\tau)$ and $g_3(\tau) = 60G_6(\tau)$ and define the **discriminant function**

$$\Delta : \mathcal{H} \to \mathbb{C} \text{ given by } \Delta(\tau) = (g_2(\tau))^3 - 27(g_3(\tau))^2$$

Then δ is weakly modular of weight 12 and holomorphic on $\mathcal{H}$ and $a_0 = 0, a_1 = (2\pi)^{12}$. So $\Delta \in \mathcal{S}_{12}(SL_2(\mathbb{Z}))$. It will be shown (section 1.5) that $\Delta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$. The Fourier coefficients of Δ have special significance in number theory.

Theorem 1.11.

$$\Delta(\tau) = (2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)q^n, q = e^{2\pi i\tau}$$

be fourier expansion of Δ, where $\tau(n)$ are integers.

The arithmetical function $\tau(n)$ is known as **Ramanujan's tau function**. Ramanujan made three conjectures about them:

(a) $\tau(mn) = \tau(m)\tau(n)$ if (m, n)=1

(b) $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ for prime p and $r > 0$

(c) $|\tau(p)| \leq p^{11/2}$ for all primes p

The first two are proved by using theory of Hecke operators at the end of Section 4.8. The third one turned out to be very deep result. It power $11/2 = (12-1)/2$ has to do with the fact that Δ is weight 12 cusp form. This result generalises to other cusp forms as well.

3. It follows that the following function is holomorphic on $\mathcal{H}$.

$$j : \mathcal{H} \to \mathbb{C} \text{ given by } j(\tau) = 1728\frac{(g_2(\tau))^3}{\Delta(\tau)}$$

Since numerator and denominator of j have same weight, it is $SL_2(\mathbb{Z})$ invariant and is called the **modular invariant**. Since δ have order 1 zero at ∞, j has simple pole at ∞ and is normalized (factor 1728) to have residue 1 at it.

4. The **Dedekind eta function** is the infinite product

$$\eta(\tau) = q_{24}\prod_{n=1}^{\infty}(1-q^n),\ q_{24} = e^{\frac{2\pi i\tau}{24}},\ q = e^{2\pi i\tau}$$

Since the series $S(\tau) = \sum_{n=1}^{\infty} log(1-q^n)$ is absolutely and uniformly convergent on compact subsets of $\mathcal{H}$, a thm from complex analysis says that η is holomorphic on $\mathcal{H}$ and satisfies the lograthmic differentiation formula for products, $(\log\prod_n f_n)' = \sum_n f_n'/f_n$. The eta function satisfies a relation similar to that satisfied by the theta function.

Proposition 1.12. The Dedekind eta function satisfies the transformation law

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$$

Hence the function $\eta^{24} \in \mathcal{S}_{12}(SL_2(\mathbb{Z}))$.

It will be shown that $\mathcal{S}_{12}(SL_2(\mathbb{Z})$ which contains Δ is one dimensional. Comparing the leading terms of the Fourier expansion, we get the identity, $\Delta = (2\pi)^{12}\eta^{24}$. Ramanujan studied his tau function starting from this defn of Δ. i.e.,

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n, q = e^{2\pi i\tau}$$

§§1.5. Complex tori and elliptic curves

Here we state important properties of complex torus and elliptic curves.

Definition 1.13. A **lattice** in $\mathbb{C}$ is the set $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\{\omega_1, \omega_2\}$ a $\mathbb{C}$ basis of $\mathbb{R}$.

We make the normalizing convention $\omega_1/\omega_2 \in \mathcal{H}$, but this still does not specify a basis given a lattice. Instead,

Lemma 1.14. Consider two lattices $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and $\Lambda' = \omega_1'\mathbb{Z} + \omega_2'\mathbb{Z}$ with $\omega_1/\omega_2 \in \mathcal{H}$ and $\omega_1'/\omega_2' \in \mathcal{H}$. Then $\Lambda = \Lambda'$ if and only if

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Definition 1.15. A **complex torus** is quotient of the complex plane by a lattice,

$$\mathbb{C}/\Lambda = \{z + \Lambda : z \in \mathbb{C}\}$$

It is easy to see that $\mathbb{C}/\Lambda$ is a Riemann surface (a connected set that looks like $\mathbb{C}$ in small neighbourhood of each point). The notion of holomorphic maps makes sense in Riemann surfaces because it is local notion and Open mapping thm is also valid because it is local result. Any holomorphic maps between two compact Riemann surfaces is surjective or constant. Using this for non-constant holomorphic map between complex tori, we get

Theorem 1.16. Let $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be a holomorphic map between complex tori. Then there exists complex numbers m and b with $m\Lambda \subset \Lambda'$ s.t. $\phi(z + \Lambda) = mz + b + \Lambda'$. The map is invertible if and only if $m\Lambda = \Lambda'$.

Corollary 1.17. Let $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be a holomorphic map between complex tori, with $\phi(z + \Lambda) = mz + b + \Lambda'$. Then the following are equivalent:

- ϕ is a group homomorphism
- $b \in \Lambda'$, means $\phi(z + \Lambda) = mz + \Lambda'$
- $\phi(0) = 0$

In particular, there exists a nonzero holomorphic group homomorphism between the complex tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ if and only if there exists some nonzero $m \in \mathbb{C}$ such that $m\Lambda \subset \Lambda'$, and a holomorphic group isomorphism $\iff$ there exists some $m \in \mathbb{C}$ such that $m\Lambda = \Lambda'$.

Definition 1.18. A non-constant holomorphic homomorphism between complex tori is called an **isogeny**.

Almost all complex tori have multiply-by-N maps the only endomorphism. But some complex tori have endomorphisms other than the multiply-by-N maps $[N]$. If endomorphism ring is isomorphic to ring of integers of imaginary number field, then complex tori are said have **complex multiplication**.

Now we see that how complex tori can be viewed as cubic curves over $\mathbb{C}$. These cubic curves are called **Elliptic Curves**. Given a lattice Λ, the meromorphic functions $f : \mathbb{C}/\Lambda \to \hat{\mathbb{C}}$ can be naturally identified with Λ periodic meromorphic function on complex plane.

Definition 1.19. The Weierstrass-$\wp$ function is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

It converges absolutely and uniformly on compact subsets of $\mathbb{C} - \Lambda$. It is an example of Λ-periodic function on $\mathbb{C}$: define $f_\lambda(\tau) = \wp(\tau + \lambda) - \wp(\tau)$ for some $\lambda \in \Lambda$. Since $f'_\lambda(\tau) = 0$ so $f_\lambda = const$. This constant for each λ is same (using continuity). Since $\wp$ is even function, constant is 0.

Eisenstein series generalize to functions of a variable lattice

$$G_k(\Lambda) = \sum_{\omega\in\Lambda}{}' \frac{1}{\omega^k}$$

So $G_k(\tau)$ from before is $G_k(\Lambda_\tau)$. It satisfies **homogenity condition** $G_k(m\Lambda) = m^{-k}G_k(\Lambda)$. The next result shows that these lattice Eisenstein series appear in Laurent expansion of Weierstrass $\wp$ function.

Theorem 1.20. Let $\wp$ be Weierstrass function w.r.t lattice Λ. Then

1. The Laurent expansion of $\wp$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (n+1)G_{n+2}(\Lambda)z^n$$

for all z s.t. $0 < |z| < inf\{|\omega| : \omega \in \Lambda - 0\}$

2. The functions $\wp$ and $\wp'$ satisfies the equation

$$(\wp(z)')^2 = 4(\wp(z))^3 - 60G_4(\Lambda)z - 140G_6(\Lambda)$$

3. Let $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and $\omega_3 = \omega_1 + \omega_2$. Then the cubic satisfied by $\wp$ and $\wp'$ is

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3) \text{ where } e_i = \wp(\omega_i/2)$$

This equation is non-singular meaning right side has distinct roots.

The appearance of Eisenstein series as coefficients of a non-singular curve lets us prove the following as a cor to above prop.

Corollary 1.21. The function Δ is non-vanishing on $\mathcal{H}$. i.e. $\Delta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$.

Proposition 1.22. Given an elliptic curve $y^2 = 4x^3 - a_2x - a_3$, there exists a lattice Λ such that $a_2 = 60G_4(\Lambda)$ and $a_3 = 140G_6(\Lambda)$.

Thus complex tori (Riemann surfaces, complex analytic objects) and elliptic curves (solution sets of cubic polynomials, algebraic objects) are interchangeable. let the term complex elliptic curve be a synonym for "complex torus".

Definition 1.23. A meromorphic function $f : \mathbb{C} \to \mathbb{C}$ with period Λ is called **elliptic function** with respect to Λ. These are just meromorphic functions of complex torus $\mathbb{C}/\Lambda$.

§§1.6. Modular curves and Moduli space

We have seen that two complex elliptic curves $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are equivalent if and only if $m\Lambda = \Lambda'$. Now define quotient set of equivalence classes of elliptic curves.
View $\tau, \tau' \in \mathcal{H}$ equivalent if there is $\gamma \in SL_2(\mathbb{Z})$ and consider the resulting quotient as well. This section shows there is bijection from first quotient to second quotient. In fact it shows generalization of that. i.e. For $\Gamma(N), \Gamma_0(N)$ and $\Gamma_1(N)$ for every N. For that we define elliptic curves with some extra data called **torsion data**.

- An **enhanced elliptic curve** for $\Gamma_0(N)$ is the pair (E, C) where E is complex elliptic curve and C is cyclic group of E of order N. Define $(E, C) \sim (E', C')$ if some isomorphism E $\to E'$ takes C to C'.

$$S_0(N) = \{\text{Enhanced elliptic curves for } \Gamma_0(N)\}/\sim$$

An element of $S_0(N)$ is denoted by $[E, C]$.

- An **enhanced elliptic curve** for $\Gamma_1(N)$ is the pair (E, P) where E is complex elliptic curve and P is point on E of order N. Define $(E, P) \sim (E', P')$ if some isomorphism $E \to E'$ takes P to P'.

$$S_1(N) = \{\text{Enhanced elliptic curves for } \Gamma_0(N)\}/\sim$$

An element of $S_1(N)$ is denoted by $[E, P]$.

- An **enhanced elliptic curve** for $\Gamma(N)$ is the pair $(E,(P,Q))$ where E is complex elliptic curve and P,Q are points on E of order N which generate $E[N]$. Define $(E,(P,Q)) \sim (E',(P',Q'))$ if some isomorphism $E \to E'$ takes P to P' and Q to Q'.

$$S(N) = \{\text{Enhanced elliptic curves for } \Gamma(N)\}/\sim$$

  An element of $S(N)$ is denoted by $[E,(P,Q)]$.

Each of $S_0(N)$, $S_1(N)$, and $S(N)$ is a **space of moduli** or **moduli space** of isomorphism classes of complex elliptic curves and N-torsion data.

Definition 1.24. Let Γ be congruence subgroup of $SL_2(\mathbb{Z})$. The orbit space of action of Γ on $\mathcal{H}$ is known as **modular curve** $Y(\Gamma)$.

Modular curves for $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$ are denoted by

$$Y_0(N) = \mathcal{H}/\Gamma_0(N), \qquad Y_1(N) = \mathcal{H}/\Gamma_1(N), \qquad Y(N) = \mathcal{H}/\Gamma(N)$$

Proving the following thm amounts to checking that the torsion data defining the moduli spaces match the conditions defining the congruence subgroups.

Theorem 1.25. Let N be a positive integer. The moduli space for $\Gamma_0(N)$ is

$$S_0(N) = \{[E_\tau, \langle 1/N + \Lambda_\tau\rangle] : \tau \in \mathcal{H}\}$$

Two points $[E_\tau, \langle 1/N + \Lambda_\tau\rangle]$ and $[E_{\tau'}, \langle 1/N + \Lambda_{\tau'}\rangle]$ are equal if and only if $\Gamma_0(N)\tau = \Gamma_0(N)\tau'$. Thus there is a bijection

$$\psi_0 : S_0(N) \to Y_0(N) \qquad [E_\tau, < 1/N + \Lambda_\tau\rangle] \mapsto \Gamma_0(N)\tau$$

Similar statement is true for $S_1(N)$ and $S(N)$.

§2. Modular curves as Riemman surfaces

Let Γ be a congruence subgroup. Here we will define Riemann surface structure on modular curve $Y(\Gamma)$ and then compactify it denoted by $X(\Gamma)$. Compact Riemann surfaces are described by polynomial equations. Thus modular curves have complex analytic and algebraic characterizations like complex elliptic curves.

A Riemann surface is connected, Hausdorff and second countable space endowed with atlas of complex charts (to be defined later).

§§2.1. Topology on $Y(\Gamma)$

We have canonical map $\mathcal{H} \to Y(\Gamma)$ taking τ to $\Gamma\tau$. Since $\mathcal{H}$ is also a topological space, it induces quotient topology on $Y(\Gamma)$ which makes quotient map continuous. Since $\mathcal{H}$ is connected and second countable, it follows $Y(\Gamma)$ is connected and second countable. Now we will prove certain property of action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ which will imply that $Y(\Gamma)$ is also Hausdorff.

Theorem 2.1. The action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ is *properly discontinuous*. i.e. Given any τ_1, τ_2 of $\mathcal{H}$, we can find neighbourhood U_1, U_2 of τ_1, τ_2 respectively with property

$$\forall \gamma \in SL_2(\mathbb{Z}), \text{if } \gamma(U_1) \cap U_2 \neq \emptyset \text{ then } \gamma(\tau_1) = \tau_2.$$

Corollary 2.2. For any congruence subgroup Γ of $SL_2(\mathbb{Z})$, the modular curve $Y(\Gamma)$ is Hausdorff.

§§2.2. Charts

Now we give Riemann surface structure on modular curve $Y(\Gamma)$ which roughly means that we can view locally view $Y(\Gamma)$ as a open set in $\mathbb{C}$ and then can define notion of holomorphic and meromorphic functions on $Y(\Gamma)$ (because they are defined locally). But first we define few terms.

Definition 2.3. Let X be a topological space. A **complex chart** is a homeomorphism $\phi : U \to V$ where U is open in X and V is open in $\mathbb{C}$.

Definition 2.4. Two complex charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ are said to be **compatible charts** if either $U_1 \cap U_2 = \emptyset$ or the function $\phi_2 \circ \phi_1^{-1} : V_1 \to V_2$ is holomorphic.

Now we define precisely what we mean by Riemann surface structure.

Definition 2.5. A **complex atlas** $\mathcal{A}$ on X is collection of compatible charts whose domains covers X.

Now we come to defining complex atlas on $Y(\Gamma)$. Let $\tau \in \mathcal{H}$. If stabilizer of τ in Γ is trivial group of transformations then we can use the neighbourhood U of τ from thm 2.1 which will be homeomorphic to its image $\pi(U)$ in natural projection. Hence we have defined complex chart around $\pi(\tau) \in Y(\Gamma)$.

The points with non-trivial group of stabilizing transformations are special enough to have their own name. Defining complex charts around them is a bit difficult.

Definition 2.6. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. For any $\tau \in \mathcal{H}$, let Γ_τ denote the stabilizer subgroup of τ. i.e. $\Gamma_\tau = \{\gamma \in \tau : \gamma \cdot \tau = \tau\}$. A point $\tau \in \mathcal{H}$ is called **elliptic point for** Γ if Γ_τ is non-trivial group of transformations. i.e. $\{\pm I\} \subsetneq \{\pm I\}\Gamma_\tau$.

We now state a prop which will be proved in next section and holds the key for defining complex charts around an elliptic point.

Proposition 2.7. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. For each elliptic point τ of Γ the stabilizer subgroup Γ_τ is finite cyclic.

Thus each point $\tau \in \mathcal{H}$ has a associated positive integer,

$$h_\tau = [\{\pm I\}\Gamma_\tau : \{\pm I\}] = |\Gamma_\tau|/2 \text{ if -I } \in \Gamma$$
$$= |\Gamma_\tau| \text{ otherwise}$$

Let $\delta_\tau = \begin{pmatrix} 1 & \tau \\ 1 & \bar{\tau} \end{pmatrix}$. Then stabilizer subgroup of 0 in conjugated transformation group $(\delta_\tau\{\pm I\}\Gamma\delta_\tau^{-1})_0/\{\pm$ is the conjugate of the stabilizer of τ, $\delta_\tau(\{\pm I\}\Gamma_\tau/\{\pm I\})\delta_\tau^{-1}$. This group is cyclic of order h_τ by prop 2.7. Since it fixes 0 and ∞, it is group of rotations of circle through angular multiples of $2\pi/h_\tau$ about the origin.

This suggests that neighbourhood of $\pi(\tau)$ in $Y(\Gamma)$ is roughly π-image of circular sector of angle $2\pi/h_\tau$ about τ in $\mathcal{H}$ and identifying action of π is the wrapping action of h_τth power map.

To make the above discussion precise, we put $\tau_1 = \tau_2 = \tau$ in thm 2.1 and obtain following cor.

Corollary 2.8. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Each point $\tau \in \mathcal{H}$ has a neighborhood U in $\mathcal{H}$ such that

$$\forall \gamma \in SL_2(\mathbb{Z}) \text{ , if } \gamma(U) \cap U \neq \emptyset \text{ then } \gamma \in \Gamma_\tau$$

Such a neighborhood has no elliptic points except possibly τ.

Let $\pi(\tau) \in Y(\Gamma)$ and U be neighbourhood as in cor. Define $\psi : U \to \mathbb{C}$ by $\psi = \rho \circ \delta$ where $\delta = \delta_\tau$ and $\rho(z) = z^h$, $h = h_\tau$. $V = \psi(U)$, an open set by Open Mapping thm. One easily checks that for $\tau_1, \tau_2 \in \mathcal{H}$

$$\pi(tau_1) = \pi(\tau_2) \iff \psi(\tau_1) = \psi(\tau_2)$$

Thus there exists an injection $\varphi : \pi(U) \to V$ s.t. $\psi = \varphi \circ \pi$. By defn of V=$\psi(U)$, φ is surjective also. By Open Mapping thm of Riemann surfaces (note φ is holomorphic map on $\pi(U)$), φ is homeomorphism.

Thus we defined charts on $Y(\Gamma)$. We also check that the above charts are compatible.

§§2.3. Elliptic points

It remains to give proof idea of prop 2.7, that each elliptic point τ of Γ has finite cyclic stabilizer subgroup . In process of doing so we will show that $Y(\Gamma)$ has only finitely many elliptic points.

The simplest case is $Y(1) = SL_2(\mathbb{Z})$. The next two lems will show that Y(1) can essentially be identified with the set

$$\mathcal{D} = \{\tau \in \mathcal{H} : |Re(\tau)| \leq 1/2, |\tau| \geq 1\}$$

Lemma 2.9. The map $\pi : \mathcal{D} \to Y(1)$ surjects where π is natural projection $\pi(\tau) = SL_2(\mathbb{Z})\tau$.

Lemma 2.10. Suppose τ_1 and τ_2 are distinct points in $\mathcal{D}$ and that $\tau_2 = \gamma\tau_1$ for some $\gamma \in SL_2(\mathbb{Z})$. Then either

1. $Re(\tau_1) = \pm 1/2$ and $\tau_2 = \tau_1 \mp 1$, or
2. $|\tau_1|$ and $\tau_2 = -1/\tau_1$.

So with suitable boundary identification the set $\mathcal{D}$ is a model for Y(1), also called **fundamental domain** of $SL_2(\mathbb{Z})$. The next prop gives all finite order matrices of $SL_2(\mathbb{Z})$. Its proof is non-trivial and uses algebraic number theory.

Proposition 2.11. Let $\gamma \in SL_2(\mathbb{Z})$

1. If γ has order 3 then γ is conjugate to $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^{\pm}$ in $SL_2(\mathbb{Z})$
2. If γ has order 4 then γ is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\pm}$ in $SL_2(\mathbb{Z})$
3. If γ has order 6 then γ is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{\pm}$ in $SL_2(\mathbb{Z})$

Corollary 2.12. The elliptic points for $SL_2(\mathbb{Z})$ are $SL_2(\mathbb{Z})$i and $SL_2(\mathbb{Z})\mu_3$ where $\mu_3 = e^{2\pi i/3}$. The modular curve Y(1) has two elliptic points. The stabilizer subgroups of i and μ_3 are

$$SL_2(\mathbb{Z})_i = < \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} >, SL_2(\mathbb{Z})_{\mu_3} = < \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} >$$

For each elliptic point τ of $SL_2(\mathbb{Z})$ the stabilizer subgroup $SL_2(\mathbb{Z})\tau$ is finite cyclic.

Since Γ is finite index in $SL_2(\mathbb{Z})$ and $\Gamma \leq SL_2(\mathbb{Z})$, we prove prop 2.7.

Corollary 2.13. The modular curve Y(Γ) has finitely many elliptic points. For each elliptic point τ of Γ the stabilizer subgroup Γ_τ is finite cyclic.

Now we have defined complex atlas on $Y(\Gamma)$.

§§2.4. Cusps

The modular curve we defined above have finitely points missing corresponing to cusps. When we adjoin and define appropriate local chart in neighbourhood of cusps, the modular curve becomes **compact Riemann surface** denoted by $X(\Gamma) = \Gamma\backslash\mathcal{H}^*$ and also called modular curve. ($\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ is extended upper half plane).

§3. Dimension Formulas

§§3.1. Genus

Every compact Riemann surface is isomorphic to a Riemann surface of genus g for some non-negative integer g. Since $X(\Gamma)$ is compact RS, how to find its genus? The Riemann Hurwitz formula comes to rescue. It says if $F : X \to Y$ is a non-zero map between compact RS, then

$$2g(X) - 2 = deg(F)(2g(Y) - 2) + \sum_{x \in X} [mult_p(F) - 1]$$

There we take $X = X(\Gamma)$ and $Y = X(1)$ and F to be natural projection. Then $deg(F) = [SL_2(\mathbb{Z}) : \{\pm I\}\Gamma]$. Let ε_2, ε_3 denotes number of period 2, 3 elliptic points resp. and ε_∞ denotes number of cusps. Then

$$g = 1 + \frac{deg(F)}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}$$

§§3.2. Overview of rest of the chapter

1. Then we define differentials of degree k on the compact Riemann surface $X(\Gamma)$ by taking a collection of differentials of degree k on local charts with certain compatibility conditions. It is easy to see that they form complex vector space and is denoted by $\Omega^{\otimes k}(X(\Gamma))$.
2. It turns out that when we pullback this differential in $\mathcal{H}$, it gives a well-defined global differential $f(\tau)(d\tau)^k$ of degree k.
3. It is easy to see that this function f is weakly modular of weight 2k w.r.t. Γ. It turns out that this function is meromorphic at all cusps. Such type of functions are called **Automorphic forms** of weight 2k w.r.t Γ and are denoted by $\mathcal{A}_{2k}(\Gamma)$.
4. It is easy to see that $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ are subspaces of $\mathcal{A}_k(\Gamma)$.
5. It turns out that for k even, $\mathcal{A}_k(\Gamma) \cong \Omega^{\otimes k/2}(X(\Gamma))$ as complex vector spaces. Now the images of $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ under this isomorphism are subspaces of $\Omega^{\otimes k/2}(X(\Gamma))$. It turns out that these subspaces can be defined entirely in terms of **linear space** of **canonical divisors** on $X(\Gamma)$.
6. Now we use Riemann-Roch thm to find the dimension of linear space of these divisors. So we get the dimension formula of $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ in terms of genus of surface $X(\Gamma)$, number of elliptic and cusp points.

7. E.g. For $SL_2(\mathbb{Z})$, the full modular group and even k$\geq$ 4,

$$\dim(\mathcal{M}_k(SL_2(\mathbb{Z}))) = \begin{cases} \lfloor \frac{k}{12} \rfloor \text{ if k } \equiv 2 mod(12) \\ \lfloor \frac{k}{12} \rfloor + 1 otherwise \end{cases}$$

   $\dim(\mathcal{S}_k(SL_2(\mathbb{Z}))) = \dim(\mathcal{M}_k(SL_2(\mathbb{Z}))) - 1$. In particular, $\dim(\mathcal{S}_{12}(SL_2(\mathbb{Z}))) = 1$. We use this in section 4.8 to prove Ramanujan first two conjectures from sect 1.4.

§4. Hecke Operators

It turns out that basis of $\mathcal{M}_k(\Gamma)/\mathcal{S}_k(\Gamma)$ can be constructed explicitly. They are generalisation of the Eisenstein series that we have seen above. Therefore this quotient is known as **Eisenstein space** and is denoted by $\mathcal{E}_k(\Gamma)$. Then one uses the theory of Hecke operators to obtain basis of $\mathcal{S}_k(\Gamma)$ hence obtaining a complete information about the space $\mathcal{M}_k(\Gamma)$.

First we decompose the space $\mathcal{M}_k(\Gamma)$ that is useful for later purposes.

§§4.1. Dirichlet characters and Eigenspaces

Let $G_N = (\mathbb{Z}/N\mathbb{Z})^*$.

Definition 4.1. A **Dirichlet character modulo** N is homomorphism $\chi : G_N \to \mathbb{C}^*$.

The set of Dirichlet characters modulo N forms a group under pointwise multiplication with pointwise complex conjugation as inverse operation and called **dual group** of G denoted by $\widehat{G}$. The identity element is called **trivial character modulo** N denoted by $\mathbf{1}_N$.

Proposition 4.2. The group $\widehat{G_N}$ is isomorphic to G_N. In particular, the number of Dirichlet characters modulo N is $\phi(N)$.

The two groups are noncanonically isomorphic, meaning that constructing an actual isomorphism from G_N to $\widehat{G_N}$ involves arbitrary choices of which elements map to which characters.

Let $d, N \in \mathbb{Z}^+$ s.t. $d|N$ and χ be Dirichlet character modulo d. Let $\pi_{N,d} : G_N \to G_d$ be canonical projection. Then χ lifts to Dirichlet character modulo N to $\chi_N = \chi \circ \pi_{N,d}$.

Definition 4.3. For a Dirichlet character χ mod N, the smallest $d \in \mathbb{Z}^+$ s.t. $\chi = \chi_d \circ \pi_{N,d}$ is called **conductor** of χ. If conductor is N, then χ is called **primitive character** modulo N.

Every Dirichlet character χ mod N can be extended to completely multiplicative arithmtic function (abusing the notation) $\chi : \mathbb{Z} \to \mathbb{C}^*$. It satisfies

$$\chi(0) = \begin{cases} 1 \text{ if } N = 1 \\ 0 \text{ if } N > 1 \end{cases}$$

Definition 4.4. Let χ be a Dirichlet character. Then χ is said to be **even character** if $\chi(-1) = 1$ and if $\chi(-1) = -1$ then **odd**.

Lemma 4.5. If $N = 1$ or $N = 2$ then $\chi(-1) = 1$. If $N > 3$ then there are exactly $\phi(N)/2$ odd character and same number of even characters.

Definition 4.6. Let χ be Dirichlet character mod N. Then χ**-eigenspace** of $\mathcal{M}_k(N, \chi)$

$$\mathcal{M}_k(N, \chi) = \{f \in \mathcal{M}_k(\Gamma_1) : f[\gamma]_k = \chi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0\}$$

Using Maschke's thm from Representation theory, we have following decomposition and similar decomposition for cusp form and Eisenstein spaces:

$$\mathcal{M}_k(\Gamma_1(N)) = \oplus_\chi \mathcal{M}_k(N, \chi)$$

§§4.2. The double coset operator

Now we start the theory of Hecke operators which will be used to give canonical basis of space $\mathcal{S}_k(\Gamma_1(N))$. For given congruence subgroups Γ_1, Γ_2 of $SL_2(\mathbb{Z})$, we define a family of linear double coset operators from vector spaces $\mathcal{M}_k(\Gamma_1)$ to $\mathcal{M}_k(\Gamma_2)$, taking subspace of cusp forms of Γ_1 to that of Γ_2.

Remark 4.7. Let G be a group and X be a set on which G acts. Then $X\backslash G$ denotes the orbits and the action is left. $G\backslash X$ also denotes orbit space but here action is right. In particular, when H is subgroup of G then G/H denotes set of left cosets of H in G. $H\backslash G$ denotes the set of right cosets of H in G.

Clearly, $\Gamma_1, \Gamma_2 \leq GL_2(\mathbb{Q})^+$. For each $\alpha \in GL_2(\mathbb{Q})^+$, the set

$$\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$$

is **double coset** in $GL_2(\mathbb{Q})^+$. The group Γ_1 acts on $\Gamma_1\alpha\Gamma_2$ by left multiplication partitioning it into orbits $\Gamma_1 \backslash \Gamma_1\alpha\Gamma_2$. We prove that orbit space is finite.

Lemma 4.8. $\alpha^{-1}\Gamma_1\alpha \cap SL_2(\mathbb{Z})$ is again a congruence subgroup of $SL_2(\mathbb{Z})$.

Lemma 4.9. Let $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. Then left multiplication by α map

$$\Gamma_1 \to \Gamma_1\alpha\Gamma_2 \qquad \text{given by} \qquad \gamma_2 \mapsto \alpha\gamma_2$$

induces a natural bijection between coset space $\Gamma_3 \backslash \Gamma_2$ and orbit space $\Gamma_1 \backslash \Gamma_1\alpha\Gamma_2$.

Any two congruence subgroups G_1 and G_2 of $SL_2(\mathbb{Z})$ are **commensurable** meaning that the indices $[G_1 : G_1 \cap G_2]$ and $[G_2 : G_1 \cap G_2]$ are finite. So by using lem 1 and lem 2 we get that orbit space $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ is finite.

Definition 4.10. Let $\beta \in GL_2(\mathbb{Q})^+$ and $k \in \mathbb{Z}$, then **weight-k β operator** on functions $f : \mathcal{H} \to \mathbb{C}$ is given by

$$(f[\beta]_k)(\tau) = (\det(\beta))^{k-1} j(\beta, \tau)^{-k} f(\beta \cdot \tau), \tau \in \mathcal{H}$$

Definition 4.11. The **weight-k $\Gamma_1 \alpha \Gamma_2$ operator** on $f \in \mathcal{M}_k(\Gamma_1)$ defined by

$$f[\Gamma_1 \alpha \Gamma_2]_k = \sum_j f[\beta_j]_k$$

where $\{\beta_j\}$ are orbit representatives, i.e., $\Gamma_1 \alpha \Gamma_2 = \cup_j \Gamma_1 \beta_j$

Now we want to show the following:

1. **The double coset operator is well defined:**
   If $\{\beta_j\}$ and $\{\beta'_j\}$ are two representatives then there are $\{\gamma_{1,j}\}$ in Γ_1 s.t. $\beta'_j = \gamma_{1,j}\beta_j$. Now, using the fact that f is Γ_1 invariant, we have

$$f[\beta'_j]_k = f[\gamma_{1,j}\beta_j]_k = (f[\gamma_{1,j}]_k)[\beta_j]_k = f[\beta_j]_k$$

2. **It takes modular forms w.r.t Γ_1 to modular forms w.r.t Γ_2:**
   To show that $f[\Gamma_1 \alpha \Gamma_2]_k$ is Γ_2 invariant, we first note that any $\gamma_2 \in \Gamma_2$ permutes the orbit space $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ by right multiplication. So if $\{\beta_j\}$ are set of orbit representatives of $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ then $\{\beta_j \gamma_2\}$ are orbit representatives as well. Thus,

$$(f[\Gamma_1 \alpha \Gamma_2]_k)[\gamma_2]_k = \sum_j f[\beta_j \gamma_2]_k = f[\Gamma_1 \alpha \Gamma_2]_k$$

   To show holomorphy at cusps, we note that for any $f \in \mathcal{M}_k(\Gamma_1)$ and any $\gamma \in GL_2(\mathbb{Q})^+$, the function $f[\gamma]_k$ is holomorphic at infinity. Second, note that if $g_1, g_2, ...g_d : \mathcal{H} \to \mathbb{C}$ then so is their sum $g_1 + g_2 + ... + g_d$. Now if $\delta \in SL_2(\mathbb{Z})$, the function $(f[\Gamma_1 \alpha \Gamma_2]_k)[\delta]_k$ is sum of functions $g_j = f[\gamma_j]_k$ where $\gamma_j = \beta_j \delta \in GL_2(\mathbb{Q})^+$.

3. **It takes cusp forms w.r.t Γ_1 to cusp forms w.r.t Γ_2:**
   For any f $\in \mathcal{S}_k(\Gamma_1)$ and any $\gamma \in GL_2(\mathbb{Q})^+$, the function g=$f[\gamma]_k$ vanishes at infinity and proof of holomorphy condition shows that f $\in \mathcal{S}_k(\Gamma_2)$.

Special cases of double coset operator arises when:

1. $\Gamma_1 \supset \Gamma_2$. Taking $\alpha = I$ makes the double coset operator $f[\Gamma_1 \alpha \Gamma_2]_k = f$, the natural inclusion of the subspace $\mathcal{M}_k(\Gamma_1)$ into $\mathcal{M}_k(\Gamma_2)$, an injection.

2. $\alpha^{-1}\Gamma_1\alpha = \Gamma_2$. Here the double coset operator $f[\Gamma_1 \alpha \Gamma_2]_k = f[\alpha]_k$, the natural translation of the subspace $\mathcal{M}_k(\Gamma_1)$ into $\mathcal{M}_k(\Gamma_1)$, an isomorphism.

3. $\Gamma_1 \subset \Gamma_2$. Taking $\alpha = I$ and letting $\{\gamma_{2,j}\}$ be the set of coset representatives of $\Gamma_1 \backslash \Gamma_2$ makes the double coset operator $f[\Gamma_1 \alpha \Gamma_2]_k = \sum_j f[\gamma_{2,j}]_k$, the natural **trace map** that projects $\mathcal{M}_k(\Gamma_1)$ onto its subspace $\mathcal{M}_k(\Gamma_2)$ by symmetrizing over quotient.

Infact, it can be proved that any double coset operator is composition of these.

§§4.3. The operators $\langle d \rangle$ and T_p

Here we define two kinds of double coset operators on $\mathcal{M}_k(\Gamma_1(N))$ which are called **Hecke operators**. Later we find transformation formula for one of these operators.

The map

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^*, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d(\operatorname{mod} N)$$

is surjective homomorphism with kernel $\Gamma_0(N)$. This shows $\Gamma_1(N) \triangleleft \Gamma_0(N)$ and induces an isomorphism $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$.

Now take any $\alpha \in \Gamma_0(N)$ and set $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$ and consider the double coset operation $[\Gamma_1 \alpha \Gamma_2]_k$. We are in case (2) of special cases discussed above. Hence, this operator translates each function $f \in \mathcal{M}_k(\Gamma_1(N))$ to

$$f[\Gamma_1 \alpha \Gamma_2]_k = f[\alpha]_k \in \mathcal{M}_k(\Gamma_1(N))$$

Thus the group $\Gamma_0(N)$ acts on $\mathcal{M}_k(\Gamma_1(N))$ and its subgroup $\Gamma_1(N)$ acts trivially, so this is really an action of the quotient $(\mathbb{Z}/N\mathbb{Z})^*$. The action of $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is determined by $d(\operatorname{mod} N)$, denoted $\langle d \rangle$ and given by

$$\langle d \rangle f = f[\alpha]_k \text{ for any } \alpha = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(N) \text{ with } \delta \equiv d(\operatorname{mod} N)$$

This is first type of Hecke operator and called **diamond operator**. For any character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$, the space $\mathcal{M}_k(N, \chi)$ is precisely the χ-eigenspace of the diamond operators.

$$\mathcal{M}_k(N, \chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f \ \forall d \in (\mathbb{Z}/N\mathbb{Z})^*\}$$

For second type of Hecke operators, take again $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$ but $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, p prime. This operator is denoted by T_p. Thus,

$$T_p : \mathcal{M}_k(\Gamma_1(N) \to \mathcal{M}_k(\Gamma_1(N) \text{ given by } T_p f = f[\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N)]_k$$

The next prop gives an explict representation of T_p. Recall that it is specified by orbit representatives of $\Gamma_1 \alpha \Gamma_2 / \Gamma_1$ which are coset representatives for Γ_2/Γ_3 left multiplied by α (lem 2 of section 1). The main idea of proof is that using some (non-trivial) computations, we compute the coset representatives of Γ_2/Γ_3.

Proposition 4.12. Let N $\in \mathbb{Z}^+$, let $\Gamma_1, \Gamma_2 = \Gamma_1(N)$ and let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ where p is prime. The operator $T_p = [\Gamma_1 \alpha \Gamma_2]_k$ on $\mathcal{M}_k(\Gamma_1(N))$ is given by

$$T_p f = \begin{cases} \sum_{j=0}^{p-1} = f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right]_k & p \mid N \\ \sum_{j=0}^{p-1} = f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right]_k + f\left[\begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k & p \nmid N, mp - nN = 1 \end{cases}$$

The next prop given fourier series expansion of $T_p f$ in terms of fourier series expansion of f. This is easily proved by applying previous proposition.

Proposition 4.13. Let f $\in \mathcal{M}_k(\Gamma_1(N))$ and let its Fourier expansion be

$$f(\tau) = \sum_{m=0}^{\infty} a_m(f) q^m$$

a) $T_p f$ has fourier expansion

$$\begin{aligned} T_p f &= \sum_{n=0}^{\infty} a_{np}(f) q^n + \mathbf{1}_N(p) p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p \rangle f) q^{np} \\ &= \sum_{n=0}^{\infty} (a_{np}(f) + \mathbf{1}_N(p) p^{k-1} a_{n/p}(\langle p \rangle f)) q^n \end{aligned}$$

That is, $a_n(T_p f) = a_{np}(f) + \mathbf{1}_N(p) p^{k-1} a_{n/p}(\langle p \rangle f)$ for $f \in \mathcal{M}_k(\Gamma_1(N))$
b) Let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a character. Then

$$a_n(T_p f) = a_{np}(f) + \chi(p) p^{k-1} a_{n/p}(\langle p \rangle f) \text{ for } f \in \mathcal{M}_k(N, \chi)$$

The next result is that Hecke operators commute with each other.

Proposition 4.14. Let $d, e \in (\mathbb{Z}/N\mathbb{Z})^*$ and let p, q be primes. Then

1. $\langle d \rangle T_p = T_p \langle d \rangle$
2. $\langle d \rangle \langle e \rangle = \langle e \rangle \langle d \rangle$
3. $T_p T_q = T_q T_p$

§§4.4. The operators $\langle n \rangle$ and T_n

So far Hecke operators $\langle d \rangle$ and T_p are defined for $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and p prime. Now we want to extend the definitionss to $\langle n \rangle$ and T_n for all $n \in \mathbb{Z}^+$.

Extension of diamond operators: If $(n, N) = 1$ then $\langle n \rangle$ is determined by $n(\bmod N)$. If $(n, N) > 1$, define $\langle n \rangle = 0$, the zero operator on $\mathcal{M}_k(\Gamma_1(N))$.

Extension of T_p operators: Set $T_1 = 1$. T_p is already defined for primes p. For primes powers, define inductively

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} < p > T_{p^{r-2}}, r \geq 2$$

Using induction on r and s and using the fact that T_p and T_q commutes for different primes p and q, it can easily be seen that $T_{p^r} T_{q^s} = T_{q^s} T_{p^r}$.
Extend the defn multiplicatively to T_n to all n,

$$T_n = \prod T_{p^{e_i}} \text{ where } n = \prod p^{e_i}$$

so that T_n commute for all n and $T_{nm} = T_n T_m$ if $(n, m) = 1$.

The next prop detemines Fourier coefficients of $T_n f$ in terms of Fourier coefficients of f.

Proposition 4.15. Let $f \in \mathcal{M}_k(\Gamma_1(N))$ have fourier expansion

$$f(\tau) = \sum_{m=0}^{\infty} a_m(f) q^m \text{ where } q = e^{2\pi i \tau}$$

then for all $n \in \mathbb{Z}^+$, fourier coefficients of $T_n f$ is given by

$$a_m(T_n f) = \sum_{d|(m,n)} d^{k-1} a_{mn/d^2}(< d > f)$$

§§4.5. The Petersson inner product

To study the space of cusp forms $\mathcal{S}_k(\Gamma_1(N))$ further, we make it into an inner product space. The inner product will be defined as an integral. The first few results will establish that the integral in question converges and is well defined. It does not converge on the larger space $\mathcal{M}_k(\Gamma_1(N))$, so the inner product structure is restricted to the cusp forms.

Definition 4.16. The **hyperbolic measure** on upper half plane is defined as

$$d\mu(\tau) = \frac{dxdy}{y^2}, \tau = x + iy \in \mathcal{H}$$

This measure is invariant under the automorphism group $GL_2(\mathbb{R})^+$ of $\mathcal{H}$ and thus $SL_2(\mathbb{Z})$ invariant. Since the set $\mathbb{Q} \cup \{\infty\}$ is countable and thus have measure zero, so $d\mu$ suffices integrating over extended half plane $\mathcal{H}^*$. Recall that fundamental domain of $\mathcal{H}^*$ under action of $SL_2(\mathbb{Z})$ is

$$\mathcal{D}^* = \{\tau \in \mathcal{H} : |Re(\tau)| \leq 1/2, |\tau| \geq 1\} \cup \{\infty\}$$

The next thm says that integrating certain nice functions w.r.t hyperbolic measure over fundamental domain is well defined.

Theorem 4.17. For any continuous, bounded function $f : \mathcal{H} \to \mathbb{C}$ and any $\alpha \in SL_2(\mathbb{Z})$, the integral $\int_{\mathcal{D}^*} f(\alpha(\tau))d\mu(\tau)$ converges.

Let Γ be congruence subgroup and let $\{\alpha_j\}$ represents the coset space $\{\pm I\}\Gamma \setminus SL_2(\mathbb{Z})$. If φ is Γ-invariant then the sum $\sum_j \int_{\mathcal{D}^*} \varphi(\alpha_j(\tau))d\mu(\tau)$ is independent of set of coset representatives $\{\alpha_j\}$. Also $\cup_j \alpha_j(\mathcal{D}^*)$ represents the modular curve $X(\Gamma)$ upto some boundary identification. So naturally, the defn of $\int_{X(\Gamma)}$ should be

$$\int_{X(\Gamma)} \varphi(\tau)d\mu(\tau) = \int_{\cup_j \alpha_j(\mathcal{D})^*} \varphi(\tau)d\mu(\tau) = \sum_j \int_{\mathcal{D}^*} \varphi(\alpha_j(\tau))d\mu(\tau)$$

In particular, setting $f = 1$, we get **volume** of $X(\Gamma)$ to be

$$V_\Gamma = \int_{X(\Gamma)} d\mu(\tau) = [SL_2(\mathbb{Z}) : \{\pm I\}\Gamma]V_{SL_2(\mathbb{Z})}$$

Definition 4.18. Let Γ be a congruence subgroup. The **Petterson inner product**

$$\langle \cdot, \cdot \rangle_\Gamma : \mathcal{S}_k(\Gamma) \to \mathcal{S}_k(\Gamma)$$

is given by

$$\langle f, g \rangle_\Gamma = \frac{1}{V_\Gamma} \int_{X(\Gamma)} f(\tau)\overline{g(\tau)}(Im(\tau))^k d\mu(\tau)$$

Clearly, the inner product is linear in f, conjugate-linear in g, Hermitian-symmetric and positive definite. It can be shown that the integral in above defn converges absolutely hence $\langle f, g \rangle$ is well defined. The normalising factor ensures that $1/V_\Gamma$ ensures that when $\Gamma' \subset \Gamma$ then $\langle , \rangle_{\Gamma'} = \langle , \rangle_\Gamma$ on $\mathcal{S}_k(\Gamma)$.

§§4.6. Adjoints of the Hecke operators

Here show that the Hecke operators $\langle n \rangle$ and T_n are normal w.r.t Petersson inner product. To do so, we explicitely compute adjoints of these operators. The next lem establishes few technalities involved.

Lemma 4.19. Let Γ be a congruence subgroup and $\alpha \in GL_2(\mathbb{Q})^+$.

1. If $\varphi : \mathcal{H} \to \mathbb{C}$ be a continuous, bounded and Γ-invariant then

$$\int_{\alpha^{-1}\Gamma\alpha\backslash\mathcal{H}^*} \varphi(\alpha(\tau))d\mu(\tau) = \int_{X(\Gamma)} \varphi(\tau)d\mu(\tau)$$

2. If $\alpha^{-1}\Gamma\alpha \subset SL_2(\mathbb{Z})$ then $V_{\alpha^{-1}\Gamma\alpha} = V_\Gamma$ and $[SL_2(\mathbb{Z}) : \alpha^{-1}\Gamma\alpha] = [SL_2(\mathbb{Z}) : \Gamma]$.
3. There exists $\beta_1, \beta_2, ...\beta_n \in GL_2(\mathbb{R})^+$ where n=$[\Gamma : \alpha^{-1}\Gamma\alpha \cap \Gamma]$ s.t.

$$\Gamma\alpha\Gamma = \cup\Gamma\beta_j = \cup\beta_j\Gamma$$

The next prop shows how to compute adjoints.

Proposition 4.20. Let Γ and α as above and let $\alpha' = \det(\alpha)\alpha^{-1}$. Then

1. If $\alpha^{-1}\Gamma\alpha \subset SL_2(\mathbb{Z})$ then for all $f \in \mathcal{S}_k(\Gamma)$ and $g \in \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)$

$$\langle f[\alpha]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\alpha']_k\rangle_\Gamma$$

2. If $f, g \in \mathcal{S}_k(\Gamma)$, then

$$\langle f[\Gamma\alpha\Gamma]_k, g\rangle_{\alpha^{-1}\Gamma\alpha} = \langle f, g[\Gamma\alpha'\Gamma]_k\rangle_\Gamma$$

In particular, if $\alpha^{-1}\Gamma\alpha = \Gamma$ then $[\alpha]_k^* = [\alpha]_k$. In any case $[\Gamma\alpha\Gamma]_k^* = [\Gamma\alpha'\Gamma]_k$

Theorem 4.21. In inner product space $\mathcal{S}_k(\Gamma_1(N))$, the Hecke operators $\langle p\rangle$ and T_p for p $\nmid$ N have adjoints

$$\langle p\rangle^* = \langle p\rangle^{-1} \text{ and } T_p^* = \langle p\rangle^{-1}T_p$$

Thus the Hecke operators $\langle n\rangle$ and T_n for n relatively prime to N are normal.

From the Spectral thm of linear algebra, given a commuting family of normal operators on a finite-dimensional inner product space, the space has an orthogonal basis of simultaneous eigenvectors for the operators. Since each such vector is a modular form we say **eigenform** instead, and the result is

Theorem 4.22. The space $\mathcal{S}_k(\Gamma_1(N))$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators $\{\langle n\rangle, T_n : (n, N) = 1\}$

§§4.7. Newforms and Oldforms

Here we're going to see results that take lower level $M|N$ modular forms to level N modular forms. The most trivial way is that if $M|N$ the $\mathcal{S}_k(\Gamma_1(M)) \subset \mathcal{S}_k(\Gamma_1(N))$. Another way is to let $d|(N/M)$ and $\alpha_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ so that

$$f[\alpha_d]_k(\tau) = d^{k-1} f(d\tau)$$

By exercise 1.2.11, the injective linear map takes $\mathcal{S}_k(\Gamma_1(M))$ to $\mathcal{S}_k(\Gamma_1(N))$.

So we have two types of modular forms: ones that come from lower levels called **oldforms** and ones that are truely level N called **newforms**. These have strong with characters mod N: primitive characters which are true mod N characters and non-primitive characters which arises from proper divisors of N.

It is important to distinguish between oldforms and newforms as our study will suggest newforms have certain nice properties (Th 4.27) and to study space of cusp forms, it is sufficient to study space of newforms. We make the defn of oldforms and newforms precise.

Definition 4.23. For each divisor d of N, let ß$_d$ be the map

$$i_d : (\mathcal{S}_k(\Gamma_1(Nd^{-1})))^2 \to \mathcal{S}_k(\Gamma_1(N)) \text{ given by } (f, g) \mapsto f + g[\alpha_d]_k$$

The subspace **oldforms at level** N is

$$\mathcal{S}_k(\Gamma_1(N))^{old} = \sum_{p|N} i_p(\mathcal{S}_k(\Gamma_1(N))^2)$$

and the subspace of **newforms at level** N is the orthogonal complement w.r.t Petterson inner product,

$$\mathcal{S}_k(\Gamma_1(N))^{new} = (\mathcal{S}_k(\Gamma_1(N))^{old})^{\perp}$$

The Hecke operators respect the decomposition of $\mathcal{S}_k(\Gamma_1(N))$ into old and new.

Proposition 4.24. The spaces $\mathcal{S}_k(\Gamma_1(N))^{old}$ and $\mathcal{S}_k(\Gamma_1(N))^{new}$ is stable under the Hecke operators T_n and $\langle n \rangle$ for all $n \in \mathbb{Z}^+$.

The following thm says that if f satisfies certain conditions on coefficients then it is an oldform. It is a very important theorem for results in next section. The proof is very hard. Doing some non-trivial calculations, first it reduces the thm to various versions and then uses representation theory to prove the final version.

Theorem 4.25. Main lemma Let $f \in \mathcal{S}_k(\Gamma_1(N))$ has Fourier expansion $f(\tau) = \sum a_n(f) q^n$ with $a_n(f)$=0 whenever $(n, N) = 1$ then f takes the form $f = \sum_{p|N}$ ß$_p f_p$ where each $f_p \in \mathcal{S}_k(\Gamma_1(N/p))$.

§§4.8. Eigenforms and proof of Ramanujan Conjectures

Definition 4.26. A non-zero modular form $\mathcal{M}_k(\Gamma_1(N))$ that is an eigenform for the Hecke operators T_n and $\langle n \rangle$ for all $n \in \mathbb{Z}^+$ is called **Hecke eigenform** or simply **eigenform**. The eigenform

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n$$

is **normalized** if $a_1(f) = 1$. A **newform** is a normalized eigenform in $\mathcal{S}_k(\Gamma_1(N))^{new}$.

Lemma 4.27. Suppose $f \in \mathcal{S}_k(\Gamma_1(N))$ be an eigenform for the Hecke operators T_n and $\langle n \rangle$ with $(n, N) = 1$. Then if $a_1(f) = 0$ then $a_n(f) = 0$ for all n with $(n, N) = 1$.

The above lem combined with Main lem gives that if $f \in \mathcal{S}_k(\Gamma_1(N))^{new}$ then $a_1(f) \neq 0$.

Theorem 4.28. Let $f \in \mathcal{S}_k(\Gamma_1(N))^{new}$ be a non-zero eigenform for the Hecke operators T_n and $\langle n \rangle$ away from N. i.e. For all n with $(n, N) = 1$. Then

1. f is a Hecke eigenform i.e. an eigenform for T_n and $\langle n \rangle$ for all $n \in \mathbb{Z}^+$. A suitable scalar multiple of f is a newform i.e. $a_1(f) \neq 0$
2. If g satisfies the same conditions as f and has the same T_n eigenvalues, then $g = cf$ for some constant c.

The set of newforms in the space $\mathcal{S}_k(\Gamma_1(N))^{new}$ is an orthogonal basis of the space. Each such newform lies in an eigenspace $\mathcal{S}_k(N, \chi)$ and satisfies $T_n f = a_n(f) f$ for all $n \in \mathbb{Z}^+$. That is, its Fourier coefficients are its T_n-eigenvalues.

The next thm can be used to find basis of $\mathcal{S}_k(\Gamma_1(N))^{old}$. It says that in order to study $\mathcal{S}_k(\Gamma_1(N))$, it is enough to study $\mathcal{S}_k(\Gamma_1(N))^{new}$ because other basis functions essentially arise from newforms of proper divisors of N.

Theorem 4.29. The set $\mathcal{B}$ is a basis of $\mathcal{S}_k(\Gamma_1(N))$

$$\mathcal{B} = \{f(n\tau) : f \text{ is a eigenform of level } M \text{ and } nM|N\}$$

The next thm gives analogy between newforms and primitive characters. The proof uses a result called **Strong Multiplicity One** which is also used in the proof of previous theorem.

Theorem 4.30. Let $g \in \mathcal{S}_k(\Gamma_1(N))$ be a normalised eigenform. Then there is a newform $f \in \mathcal{S}_k(\Gamma_1(M))^{new}$ for some $M|N$ such that $a_p(f) = a_p(g)$ for all $p \nmid N$.

Proposition 4.31. Let $f \in \mathcal{M}_k(N, \chi)$. Then f is a normalized eigenform $\iff$ its Fourier coefficients satisfy following

1. $a_1(f) = 1$
2. $a_{p^{r+1}}(f) = a_p(f)a_{p^r}(f) - \chi(p)p^{k-1}a_{p^{r-1}}(f)$ for prime p and $r > 0$
3. $a_m(f)a_n(f) = a_{mn}(f)$ when $(m, n) = 1$

This prop does not say that if any function $f(\tau) = \sum_{n=0}^{\infty} a_n(f)q^n$ with coefficients satisfying (a), (b) and (c) is a normalized eigenform. The function need not be a modular form at all.

Because $\mathcal{S}_{12}(SL_2(\mathbb{Z}))$ is 1-dimensional (formula from end of section 3.2) and Δ from section 1.4 is in it, Δ is eigenform for Hecke operators on $\mathcal{M}_{12}(SL_2(\mathbb{Z}))$. Normalizing Δ we get

Corollary 4.32. The first two conjectures of Ramanujan about τ function are true.

§§4.9. Connection with L-functions and functional equations

Each modular form $f \in \mathcal{M}_k(\Gamma_1(N))$ has an associated Dirichlet series, its L-function.

Definition 4.33. Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$, let s $\in \mathbb{C}$ be a complex variable, then $L(s, f)$ the L-function of f defined by

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$$

Convergence of $L(s, f)$ in a half plane of s-values follows from estimating the Fourier coefficients of f.

Proposition 4.34. Let $f \in \mathcal{M}_k(\Gamma_1(N)$ be a cusp form then $L(s, f)$ converges for all s with $Re(s) > k/2 + 1$. If f is not a cusp form then $L(s, f)$ converges absolutely for all s with $Re(s) > k$.

The condition of f being normalised eigenform is equivalent to its L-function having an **Euler product**.

Theorem 4.35. Let $f \in \mathcal{M}_k(N, \chi)$, $f = \sum_{n=0}^{\infty} a_n q^n$. The following are equivalent

- f is normalised eigenform
- $L(s, f)$has an Euler product expansion

$$L(s, f) = \prod_p (1 - a_p p^{-s} + \chi(p)p^{k-1-2s})^{-1}$$

Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_1(N))$. The **Mellin transform** of f is

$$g(s) = \int_0^{\infty} f(it) t^s \frac{dt}{t}$$

for values of s such that integral converges absolutely.

Proposition 4.36. The Mellin transform of f is

$$g(s) = (2\pi)^{-s}\Gamma(s)L(s,f)\,,\, Re(s) > k/2+1$$

Define an operator called **Atkin-Lehner involution** on space of cusp forms:

$$W_N : \mathcal{S}_k(\Gamma_1(N)) \to \mathcal{S}_k(\Gamma_1(N))\,,\, f \mapsto i^k N^{1-k/2} f[\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}]_k$$

The operator W_N is (a) idempotent, meaning W_N^2 is identity and (b) self-adjoint w.r.t Petterson inner product. Letting $\mathcal{S}_k(\Gamma_1(N))^+$ and $\mathbb{S}_k(\Gamma_1(N))^-$ denote the eigenspaces

$$\mathcal{S}_k(\Gamma_1(N))^{\pm} = \{f \in \mathcal{S}_k(\Gamma_1(N)) : W_N(f) = \pm f\}$$

gives an orthogonal decomposition of $\mathbb{S}_k(\Gamma_1(N))$

$$\mathcal{S}_k(\Gamma_1(N)) = \mathcal{S}_k(\Gamma_1(N))^+ \oplus \mathcal{S}_k(\Gamma_1(N))^-$$

The next result states that $\Lambda_N(s) = N^{s/2}g(s)$ satisfies a functional equation:

Theorem 4.37. Suppose $f \in \mathcal{S}_k(\Gamma_1(N))^{\pm}$. Then the Mellin transform $\Lambda_N(s)$ extends to an entire function satisfying the functional equation

$$\Lambda_N(s) = \pm\Lambda_N(k-s)$$

Consequently, $L(s,f)$ has analytic continuation to the full s-plane.