Modular Forms

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Overview

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and conjectured three results. Hecke, Atkin and Lehner developed this theory further.
4. Elliptic curves enter into the picture in 1960's. (Taniyama-Shimura-Weil conjecture)

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The element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $\mathcal{H}$. Hence we can consider the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$.

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Notation: The image of an element $g \in \mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{PSL}_2(\mathbb{Z})$ will be denoted by the same symbol g. Let

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Fundamental Domain of the Modular Group

Let us define the set

$$\mathcal{D} = \{\tau \in \mathcal{H} : |Re(\tau)| \leq 1/2, |\tau| \geq 1\}$$

and also define $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$.

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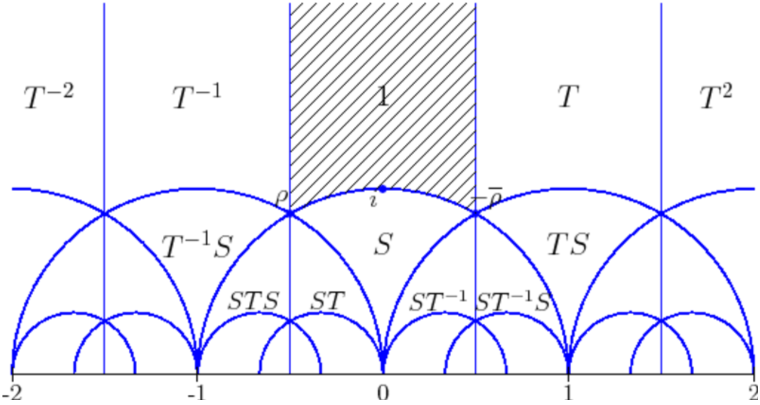

Figure: Fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$

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be the stabilizer of τ in G. One has $I(\tau) = \{1\}$ except in the following three cases:
    1. $\tau = i$, in which case $I(\tau)$ is the group of order 2 generated by S;
    2. $\tau = \rho = e^{2\pi i/3}$, in which case $I(\tau)$ is the group of order 3 generated by ST
    3. $\tau = -\bar{\rho} = e^{\pi i/3}$, in which case $I(\tau)$ is the group of order 3 generated by TS.

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5. Otherwise $|z'| < 1$, the element $-1/z'$ would have an imaginary part strictly larger than $\mathrm{Im}\,(z')$,

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4. If $c = 0$, we have $d = \pm 1$ and g is the translation by $\pm b$. Since $R(z)$ and $R(gz)$ are both between $-\frac{1}{2}$ and $\frac{1}{2}$, this implies either $b = 0$ and $g = 1$ or $b = \pm 1$ in which case one of the numbers $R(z)$ and $R(gz)$ is $\frac{1}{2}$ and the other is $-\frac{1}{2}$.

Motivation

The Modular Group

Modular Functions

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6. The case $d = 0$ gives $|z| \leq 1$ hence $|z| = 1$; on the other hand, $ad - bc = 1$ implies $b = -1$, hence $gz = a - 1/z \implies a = 0$ except if $R(z) = \pm\frac{1}{2}$, i.e. if $z = \rho$ or $-\tilde{\rho}$ in which case we have $a = 0, -1$ or $a = 0, 1$.

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7. When $z = \rho$: $d = 1$ gives $a - b = 1$ and $g\rho = a - 1/(1 + \rho) = a + \rho$, hence $a = 0, 1$;

Similarly when $z = -\bar{\rho}, d = -1$.
Finally the case $c = -1$ leads to the case $c = 1$ by changing the signs of a, b, c, d (which does not change g, viewed as an element of G).

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3. By (2) and (3), it follows that these points coincide and that $g'g = 1$. Hence we have $g \in G'$.

Overview

Definition

Let k be an integer. We say a function $f : \mathcal{H} \longrightarrow \widehat{\mathbb{C}}$ is weakly modular of weight $2k$ if f is meromorphic on the half plane $\mathcal{H}$ and verifies the relation

$$f(\tau) = (c\tau + d)^{-2k} f\left(\frac{a\tau + b}{c\tau + d}\right) \qquad \text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

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Proposition

Let f be meromorphic on $\mathcal{H}$. The function f is a weakly modular function of weight $2k \iff$ it satisfies the two relations:

$$\begin{aligned} f(\tau + 1) &= f(\tau) \\ f(-1/\tau) &= \tau^{2k} f(\tau) \end{aligned} \tag{1}$$

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3. It means that the "differential form of weight k", $f(z)dz^k$ is invariant under G.
4. Since G is generated by the elements S and T, it suffices to check the invariance by S and by T.

Suppose that $f(\tau+1)=f(\tau)$. Let D be the open unit disk in $\mathbb{C}$ and let $D'=D-\{0\}$. The function $\tau\mapsto e^{2\pi i\tau}$ takes $\mathcal{H}$ to D' and is also periodic. Thus corresponding to f, the function $g:D'\to\mathbb{C}$ where

$$g(q)=f\left(\frac{\log(q)}{2\pi i}\right) \tag{2}$$

is well defined and $f(\tau)=g(e^{2\pi i\tau})$.

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Also g is meromorphic on D'. Hence g has a Laurent series expansion

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When f is holomorphic at infinity, we set $f(\infty) := \tilde{f}(0)$.

Definition

A modular function which is holomorphic everywhere (including infinity) is called a modular form; if such a function is zero at infinity, it is called a cusp form.

Lattices and Modular functions

Definition

A lattice in a finite dimensional $\mathbb{R}$ vector space V is a subgroup Γ of V verifying one of the following equivalent conditions:

1. Γ is discrete and V/Γ is compact;
2. Γ is discrete and generates the $\mathbb{R}$-vector space V;
3. There exists an $\mathbb{R}$-basis $(e_1, \ldots, e_n)$ of V which is a $\mathbb{Z}$-basis of Γ (i.e. $\Gamma = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$)

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$$\mathcal{R} := \{\Lambda : \Lambda \text{ is a lattice of } \mathbb{C} \text{ considered as an } \mathbb{R} - \text{vector space}\}$$

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Lattices and Modular functions

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A lattice in a finite dimensional $\mathbb{R}$ vector space V is a subgroup Γ of V verifying one of the following equivalent conditions:

1. Γ is discrete and V/Γ is compact;
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Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and let $(\omega_1, \omega_2) \in \mathcal{M}$. We put

$$\omega_1' = a\omega_1 + b\omega_2 \text{ and } \omega_2' = c\omega_1 + d\omega_2$$

It is clear that $\{\omega_1', \omega_2'\}$ is a basis of $\Gamma(\omega_1, \omega_2)$.

Moreover, if we set $\tau = \omega_1/\omega_2$ and $\tau' = \omega_1'/\omega_2'$, we have

$$\tau' = \frac{a\tau + b}{c\tau + d} = g\tau$$

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For two elements of $\mathcal{M}$ to define the same lattice it is necessary and sufficient that they are congruent modulo $\mathrm{SL}_2(\mathbb{Z})$.

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Hence $\mathcal{R} \cong \mathcal{M}/\,\mathrm{SL}_2(\mathbb{Z})$. Make now $\mathbb{C}^*$ act on $\mathcal{R}$ (resp. on $\mathcal{M}$) by:

$$\Gamma \longmapsto \lambda\Gamma \qquad \text{(resp. } (\omega_1, \omega_2) \mapsto (\lambda\omega_1, \lambda\omega_2)), \qquad \lambda \in \mathbb{C}^*$$

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The quotient $\mathcal{M}/\mathbb{C}^*$ is identified with $\mathcal{H}$ by $(\omega_1, \omega_2) \mapsto z = \omega_1/\omega_2$, and this identification transforms the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{M}$ into that of $G = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ on $\mathcal{H}$.

Relation with Elliptic Curves

Proposition

The map $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$ gives by passing to the quotient, a bijection of $\mathcal{R}/\mathbb{C}^*$ onto $\mathcal{H}/G$. (Thus, an element of $\mathcal{H}/G$ can be identified with a lattice of $\mathbb{C}$ defined up to a homothety.)

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Theorem [Diamond and Shurman 1.3.2]

Suppose $\varphi : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda'$ is a holomorphic map between complex tori. Then there exist complex numbers m, b with $m\Lambda \subset \Lambda'$ such that $\varphi(z + \Lambda) = mz + b + \Lambda'$.

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This gives a third description of $\mathcal{H}/G = \mathcal{R}/\mathbb{C}^*$: it is the set of isomorphism classes of elliptic curves.

Proof

1. We can lift φ to a holomorphic map $\tilde{\varphi} : \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{\varphi}"] \arrow[d] & \mathbb{C} \arrow[d] \\ \frac{\mathbb{C}}{\Lambda} \arrow[r, "\varphi"] & \frac{\mathbb{C}}{\Lambda'} \end{tikzcd}$$

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$$f_\lambda(z) = \tilde{\varphi}(z + \lambda) - \tilde{\varphi}(z).$$

Since $\tilde{\varphi}$ lifts a map between the quotients, the continuous function f_λ maps to the discrete set Λ' and is therefore constant.

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5. Now $\tilde{\varphi}$ is a first degree polynomial $\tilde{\varphi}(z) = mz + b$, and again since this lifts a map between quotients, necessarily $m\Lambda \subset \Lambda'$.

Definition

Let $F : \mathcal{R} \longrightarrow \mathbb{C}$ be a function, and let $k \in \mathbb{Z}$. We say that F is of weight $2k$ if

$$F(\lambda\Gamma) = \lambda^{-2k} F(\Gamma) \text{ for all lattices } \Gamma \text{ and all } \lambda \in \mathbb{C}^*. \tag{4}$$

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3. Above formula shows that the product $\omega_2^{2k} F(\omega_1, \omega_2)$ depends only on $\tau = \omega_1/\omega_2$. There exists then a function f on $\mathcal{H}$ s.t.
$$F(\omega_1, \omega_2) = \omega_2^{-2k} f(\omega_1/\omega_2)$$

1. Since F is invariant by $\mathrm{SL}_2(\mathbb{Z})$, f satisfies the identity:
$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

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2. Conversely, if f satisfies above formula, formula (9) associates to it a function F on $\mathcal{R}$ which is of weight $2k$.
3. We can thus identify modular functions of weight $2k$ with some lattice functions of weight $2k$.

Examples of modular functions- Eisenstein series

Lemma

Let Γ be a lattice in $\mathbb{C}$. The series $\sum'_{\gamma\in\Gamma} 1/|\gamma|^{\sigma}$ is convergent for $\sigma > 2$.

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Proof

1. We can proceed as with the series $\Sigma 1/n^\alpha$, i.e. majorize the series under consideration by a multiple of the double integral $\iint \frac{dxdy}{(x^2+y^2)^{\sigma/2}}$ extended
2.

Definition

Let k be an integer > 1. If Γ is a lattice of $\mathbb{C}$, put

$$G_k(\Gamma) = \sum_{\gamma\in\Gamma}{}' \frac{1}{\gamma^{2k}}$$

It is called the Eisenstein series of index k.

Hence we can view G_k as a function on $\mathcal{M}$, given by:

$$G_k\left(\omega_1, \omega_2\right) = \sum_{m,n}{}' \frac{1}{\left(m\omega_1 + n\omega_2\right)^{2k}}$$

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Proof

1. We have shown that $G_k(z)$ is weakly modular of weight $2k$.
2. We have to show that G_k is everywhere holomorphic (including infinity).

1. First suppose that $z \in \mathcal{D}$. Then
$$\begin{aligned} |mz+n|^2 &= m^2 z\bar{z} + 2mnR(z) + n^2 \\ &\geq m^2 - mn + n^2 = |m\rho - n|^2 \end{aligned}$$
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2. This shows that the series $G_k(z)$ converges normally in $\mathcal{D}$, thus also in each of the transforms $g\mathcal{D}$ of $\mathcal{D}$ by G. Since these cover $\mathcal{H}$, we see that G_k is holomorphic in $\mathcal{H}$. (compact normal convergence)

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4. In view of the uniform convergence in $\mathcal{D}$, we can make the passage to the limit term by term. The terms $1/(mz+n)^{2k}$ relative to $m \neq 0$ give 0; the others give $1/n^{2k}$. Thus
$$\lim G_k(z) = \sum{}' \frac{1}{n^{2k}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2\zeta(2k)$$

Ramanujan Δ function

The Eisenstein series of lowest weights are G_2 and G_3, which are of weight 4 and 6. It is convenient to replace these by multiples:

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If we put

$$\Delta = g_2^3 - 27g_3^2$$

we have $\Delta(\infty) = 0$; that is to say, Δ is a cusp form of weight 12.

Relation with elliptic curves

Let Γ be a lattice of $\mathbb{C}$ and let

$$\wp_\Gamma(u) = \frac{1}{u^2} + \sum_{\gamma \in \Gamma}{}' \left(\frac{1}{(u-\gamma)^2} - \frac{1}{\gamma^2} \right)$$

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Up to a numerical factor, $\Delta = g_2^3 - 27 g_3^2$ is equal to the discriminant of the polynomial $4x^3 - g_2 x - g_3$.

Relation with elliptic curves

Let Γ be a lattice of $\mathbb{C}$ and let

$$\wp_\Gamma(u) = \frac{1}{u^2} + \sum_{\gamma\in\Gamma}{}' \left(\frac{1}{(u-\gamma)^2} - \frac{1}{\gamma^2} \right)$$

be the corresponding Weierstrass function. The $G_k(\Gamma)$ occur into the Laurent expansion of $\wp_\Gamma$:

$$\wp_\Gamma(u) = \frac{1}{u^2} + \sum_{k=2}^{\infty} (2k-1) G_k(\Gamma) u^{2k-2}$$

If we put $x = \wp_\Gamma(u), y = \wp'_\Gamma(u)$, we have

$$y^2 = 4x^3 - g_2 x - g_3$$

with $g_2 = 60G_2(\Gamma), g_3 = 140G_3(\Gamma)$ as above.
Up to a numerical factor, $\Delta = g_2^3 - 27g_3^2$ is equal to the discriminant of the polynomial $4x^3 - g_2x - g_3$.
We can prove that the cubic in the projective plane is isomorphic to the elliptic curve $\mathbb{C}/\Gamma$. In particular, it is a nonsingular curve, and this shows that Δ is $\neq 0$.