11/16/2023

Outline

1) Wald test
2) Score test
3) Generalized likelihood ratio test
4) Asymptotic Relative Efficiency

Likelihood-Based Inference

Setting $X_1, \ldots, X_n \overset{iid}{\sim} p_\theta(x)$, $p_\theta(x)$ "smooth" in θ

Assume $\mathbb{E}_\theta \nabla \ell_1(\theta; X_i) = 0$,

$$\mathrm{Var}_\theta\left[\nabla \ell_1(\theta; X_i)\right] = -\mathbb{E}_\theta \nabla^2 \ell_1(\theta; X_i) = J_1(\theta) \succ 0,$$

$$\hat{\theta}_{MLE} \xrightarrow{P_\theta} \theta \quad \text{(consistent)}$$

Then, if $\theta = \theta_0$:

$$\frac{1}{\sqrt{n}} \nabla \ell_n(\theta_0; X) \Rightarrow N_d(0, J_1(\theta_0))$$

$$\frac{1}{n} \nabla^2 J_n(\theta_0; X) \xrightarrow{P} J_1(\theta_0)$$

Used $0 = \nabla \ell_n(\hat{\theta}_n) \approx \nabla \ell_n(\theta_0) + \nabla^2 \ell_n(\theta_0)(\hat{\theta}_n - \theta_0)$

to get $\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, J_1(\theta_0)^{-1})$

Can use this for inference on θ_0!

Wald-Type Confidence Regions

Assume we have some estimator $\hat{J}_n \succ 0$ s.t. $\frac{1}{n}\hat{J}_n \xrightarrow{p} J_1(\theta_0) \succ 0$. Then we can plug in:

If $\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, J_1(\theta_0)^{-1})$

then $(J_1(\theta_0))^{1/2}\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, I_d)$

so $\hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, I_d)$ (Slutsky)

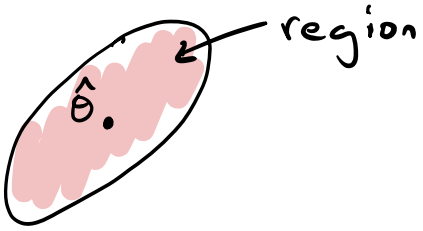
Leads to test of $H_0: \theta = \theta_0$:

$$\| \hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0) \|^2 \Rightarrow \chi^2_d \quad \text{(Reject if large)}$$

So, $\mathbb{P}_{\theta_0}\left(\hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0) \geq \underbrace{\chi^2_d(\alpha)}_{1-\alpha \text{ quantile}} \right) \to \alpha$

Note we reject θ_0 iff $\| \hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0) \|^2 > \chi^2_d(\alpha)$

$\Leftrightarrow$ reject θ_0 iff $\theta_0 \notin \underbrace{\hat{\theta}_n + \hat{J}_n^{-1/2} B(0)}_{\text{confidence ellipsoid}}$ $\chi_d(\alpha)$



More info $\Leftrightarrow$ smaller ellipse (shrinks like $1/\sqrt{n}$)

Options for $\hat{J}_n$:

1) Most obvious is to "plug in" the MLE:

(MLE for $J_n(\theta)$)

$$\hat{J}_n = J_n(\hat{\theta}_n)$$

$$= \text{Var}_\theta(\nabla \ell_n(\theta; X))\Big|_{\theta=\hat{\theta}_n}$$

(NB) $$\neq \text{Var}_{\hat{\theta}_n}(\nabla \ell_n(\hat{\theta}_n(X); X)) = 0$$

Or, $$\hat{J}_n = -\mathbb{E}_\theta \nabla^2 \ell_n(\theta)\Big|_{\theta=\hat{\theta}_n}$$

2) Observed Fisher info:

$$\hat{J}_n = -\nabla^2 \ell_n(\hat{\theta}_n; X)$$

Remarks:

- Both have $\frac{1}{n}\hat{J}_n \xrightarrow{p} J_1(\theta_0)$ in "nice" iid sampling setting
- Both make sense outside of iid setting
- Heuristically, plug-in measures info about θ in "typical" data set but obs. info. measures info about θ in "this" data set

Wald interval for θ_j:

If $\hat{\theta}_n \approx N_d(\theta_0, J_n(\theta_0)^{-1})$

then $\hat{\theta}_{n,j} \approx N_d(\theta_{0,j}, \underbrace{(J_n(\theta_0)^{-1})_{jj}}_{s.e.(\hat{\theta}_{n,j})^2})$

Leads to univariate interval:

$$C_j = \hat{\theta}_{nj} \pm \hat{s.e.}(\hat{\theta}_{n,j}) \cdot z_{\alpha/2}$$

$$= \hat{\theta}_{nj} \pm \sqrt{(\hat{J}_n^{-1})_{jj}} \cdot z_{\alpha/2}$$

glm function in R uses these intervals / p-values, with $\hat{J}_n = -\nabla^2 \ell(\hat{\theta}_n)$

Conf. ellipsoid for $\theta_{0,s} = (\theta_{0,j})_{j \in s}$: $(|s| = k)$

$$\hat{\theta}_{n,s} \approx N_k(\theta_{0,s}, (J_n(\theta_0)^{-1})_{ss})$$

$$\rightsquigarrow C_s = \hat{\theta}_{n,s} + ((\hat{J}_n^{-1})_{ss})^{1/2} B_{\chi_k(\alpha)}(0)$$

More generally, if $\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, \Sigma(\theta_0))$

and $\hat{\Sigma}_n \xrightarrow{P_{\theta_0}} \Sigma(\theta_0)$ ($\hat{\theta}_n$ not nec. MLE)

then we can do the same things

Ex Generalized linear model with fixed x

$x_1, \dots, x_n \in \mathbb{R}^d$ fixed

$$Y_i \overset{\text{ind.}}{\sim} p_{\eta_i}(y) = e^{\eta_i y_i - A(\eta_i)} h(y_i)$$

$$\eta_i = \beta' x_i \quad (\underline{\text{canonical form}})$$

Let $\mu_i(\beta) = \mathbb{E}_\beta Y_i$ $\quad (= \mu(\eta_i(\beta)))$

(more general: $f(\mu_i) = \beta' x_i$ for $\underline{\text{link fcn}}$ f)

Most common examples:

Logistic regression: $Y_i \overset{\text{ind.}}{\sim} \text{Bern}\left(\frac{e^{x_i'\beta}}{1+e^{x_i'\beta}}\right)$

Poisson log-linear model: $Y_i \overset{\text{ind.}}{\sim} \text{Pois}(e^{x_i'\beta})$

$$\ell_n(\beta; Y) = \sum_i (x_i'\beta) y_i - A(x_i'\beta) - \log h(y_i)$$

$$\nabla \ell_n(\beta; Y) = \sum_i y_i x_i - \dot{A}(x_i'\beta) \cdot x_i$$

$$= \sum_i (y_i - \mu_i(\beta)) x_i$$

$$-\nabla^2 \ell_n(\beta; Y) = \sum_i \ddot{A}(x_i'\beta) \cdot x_i x_i'$$

$$= \sum_i \text{Var}_\beta(y_i) \cdot x_i x_i'$$

$$= \text{Var}_\beta(\nabla \ell_n(\beta; Y)) \quad \text{(Not random)}$$

$$\left(-\nabla^2 \ell_n(\beta)\right)^{-1/2} \nabla \ell_n(\beta_0) \sim (0, I_d) \quad \text{in finite samples}$$

$$\overset{*}{\Rightarrow} N_d(0, I_d)$$

* Under regularity cond. on $X = \begin{pmatrix} - x_1^T - \\ \vdots \\ - x_n^T - \end{pmatrix}$

Taylor expansion of ℓ_n leads to

$$\hat{J}_n^{1/2} (\hat{\beta}_n - \beta) \Rightarrow N_d(0, I_d)$$

Advantages of Wald test:

1) Easy to invert, simple conf. regions
2) Asymptotically correct

Disadvantages:

1) Have to compute MLE
2) Depends on parameterization
3) Relies on two approximations:
   $\nabla \ell_n \approx$ Normal <u>and</u> $\ell_n \approx$ quadratic
4) Need MLE to be consistent
5) Confidence interval/ellipsoid might go outside (H)!

Score Test

Test $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

We can bypass quadratic approximation entirely by using score as test stat

$$\frac{1}{\sqrt{n}} \nabla \ell_n(\theta_0; X) \overset{P_{\theta_0}}{\Rightarrow} N_d(0, J_1(\theta_0))$$

(or $J_n(\theta_0)^{-1/2} \nabla \ell_n(\theta_0; X) \overset{P_{\theta_0}}{\Rightarrow} N_d(0, I_d)$)

So, we can reject $H_0: \theta = \theta_0$ if

$$\| J_n(\theta_0)^{-1/2} \nabla \ell_n(\theta_0; X) \|_2^2 \geq \chi_d^2(\alpha)$$

$d=1$: $\dot{\ell}_n(\theta_0) / \sqrt{J_n(\theta_0)} \Rightarrow N(0,1)$,

can do 1-sided tests

Remarks

- No quadratic approx., no MLE
- No need to estimate Fisher info at θ_0

Can be generalized to case with nuisance params

Typically estimate via MLE on H_0

Score test is invariant to reparameterization*:

Assume $d=1$, $\theta = g(s)$, $\dot{g}(s) > 0 \ \forall s$

$$q_s(x) = p_{g(s)}(x)$$

$$\dot{\ell}^{(s)}(s;x) = \frac{d}{ds} \log p_{g(s)}(x)$$

$$= \dot{\ell}^{(\theta)}(g(s);x) \cdot \dot{g}(s)$$

$$J^{(s)}(s) = J^{(\theta)}(g(s)) \cdot \dot{g}(s)^2$$

$$\text{So} \quad \dot{\ell}^{(s)}(s_0;x) \Big/ \sqrt{J^{(s)}(s_0)} \overset{a.s.}{=} \dot{\ell}^{(\theta)}(\theta_0;x) \Big/ \sqrt{J^{(\theta)}(\theta_0)}$$

if $\theta_0 = g(s_0)$

Ex s-parameter exp. fam:

$$X_1, \dots, X_n \overset{iid}{\sim} e^{\eta^T T(x) - A(\eta)} h(x)$$

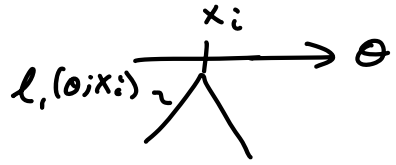
$$\nabla \ell_n(\eta;X) = \Sigma T(X_i) - n\mu(\eta)$$

$$\| J_n(\eta_0)^{-1/2} (\Sigma T(X_i) - n\mu(\eta_0)) \|^2 \Rightarrow \chi^2_d$$

$$\frac{\Sigma T(X_i) - n\mu(\eta_0)}{\sqrt{n \operatorname{Var}_{\eta_0}(T(X_i))}} \overset{P_{\eta_0}}{\Rightarrow} N(0,1)$$

Ex $X_1, \ldots, X_n \overset{iid}{\sim} \text{Laplace}(\theta) = \frac{1}{2} e^{-|x-\theta|}$

Test $H_0: \theta \le 0$ vs $H_1: \theta > 0$ (right-tailed)



$$\ell_n(\theta; x) = -\sum_{i=1}^n |x_i - \theta| - n \log(2)$$

$$\dot{\ell}_n(\theta; x) = \sum_{i=1}^n \text{sgn}(x_i - \theta) \qquad \text{sgn}(z) = \begin{cases} +1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases}$$

$$\dot{\ell}_n(0; x) = \sum_{i=1}^n \text{sgn}(x_i)$$

$$= \#\{i : x_i > 0\} - \#\{i : x_i < 0\}$$

$$= 2\#\{i : x_i > 0\} - n$$

$Y \sim \text{Binom}(n, 1/2)$ sign test

Note: this test is $\approx$ the exact NP LRT for $H_0: \theta = 0$ vs $H_0: \theta = \varepsilon$, for $\varepsilon \downarrow 0$

Intuition: Maximize power for nearby alternatives, since we'll have power ≈ 1 for $\theta \gg 1/\sqrt{n}$

More generally, one-sided score test is "almost" UMP for nearby alternatives.

$$\log \frac{p_{\theta_0+\varepsilon}(x)}{p_{\theta_0}(x)} \approx \varepsilon \dot{\ell}_n(\theta_0; x) \quad \text{for small } \varepsilon > 0$$

Ex Pearson's χ^2 test (goodness of fit)

$$N = (N_1, \dots, N_d) \sim \text{Multinom}(n, (\pi_1, \dots, \pi_d))$$

$$= \frac{n!\, \pi_1^{N_1} \cdots \pi_d^{N_d}}{N_1! \cdots N_d!} \, 1\{\textstyle\sum N_i = n\}$$

Note $\sum \pi_j = 1$ so this is a full-rank $(d-1)$-parameter exp. family, e.g.

$$\pi_j = \begin{cases} \dfrac{1}{1 + \sum_{k>1} e^{\eta_k}} & j = 1 \\ \dfrac{e^{\eta_j}}{1 + \sum_{k>1} e^{\eta_k}} & j > 1 \end{cases}$$

$$\nabla \ell_n(\eta; N) = (N_2, \dots, N_d) - (n\pi_2, \dots, n\pi_d)$$

$$\text{Var}_\eta(\nabla \ell(\eta)) = \begin{pmatrix} n\pi_2(1-\pi_2) & \cdots & -n\pi_i\pi_j \\ \vdots & \ddots & \vdots \\ -n\pi_i\pi_j & \cdots & n\pi_d(1-\pi_d) \end{pmatrix}$$

$$= n\left(\text{diag}(\pi_{2:d}) - \pi_{2:d}\pi_{2:d}'\right)$$

$$\Rightarrow J_n(\eta)^{-1} = \frac{1}{n} \cdot \left(\text{diag}(\pi_{2:d})^{-1} - \pi_1^{-1} 1 1'\right)$$

(uses $(A + uv')^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}$)

Score test of $H_0: \pi = \pi_0$: (don't really need asy. approx)

$$\nabla \ell_n(\eta_0) J_n^{-1}(\eta_0) \nabla \ell_n(\eta_0) \overset{\text{(algebra)}}{=} \sum_{j=1}^{d} \frac{(N_j - n\pi_{0j})^2}{n\pi_{0j}} \overset{P_{\pi_0}}{\Rightarrow} \chi^2_{d-1}$$

Generalized LRT

Test $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

Taylor expand around $\hat{\theta}_n$:

$$\ell_n(\theta_0) - \ell_n(\hat{\theta}_n) = \nabla \ell(\hat{\theta}_n)^{0} + \frac{1}{2}(\theta_0 - \hat{\theta}_n)^T \nabla^2 \ell_n(\tilde{\theta}_n)(\theta_0 - \hat{\theta}_n)$$

$$= \frac{-1}{2} \cdot \left\| \left(\underbrace{\frac{-1}{n} \nabla^2 \ell_n(\tilde{\theta}_n)}_{\xrightarrow{p} J_1(\theta_0)} \right)^{1/2} \underbrace{(\sqrt{n}(\theta_0 - \hat{\theta}_n))}_{\Rightarrow N(0, J_1(\theta_0)^{-1})} \right\|_2^2$$

$$\Rightarrow \frac{-1}{2} \chi^2_d$$

Test stat: $2\left(\ell_n(\hat{\theta}_n; X) - \ell_n(\theta_0; X)\right) \overset{P_{\theta_0}}{\Rightarrow} \chi^2_d$

Composite vs. Composite:

$H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta \setminus \Theta_0$,

Assume
- $\Theta = \mathbb{R}^d$, Θ_0 d_0-dim manifold
- $\theta_0 \in \text{relint}(\Theta_0)$
- $\hat{\theta}_n \xrightarrow{P_{\theta_0}} \theta_0$
- Likelihood "smooth"

Then $2(\ell_n(\hat{\theta}_n) - \ell_n(\hat{\theta}_0)) \Rightarrow \chi^2_{d-d_0}$

where $\hat{\theta}_0 = \underset{\theta \in \Theta_0}{\text{argmin}}\ \ell_n(\theta; X)$

Why? Assume wlog $\theta_0 = 0$, $J_1(0) = I_d$ (reparam.)

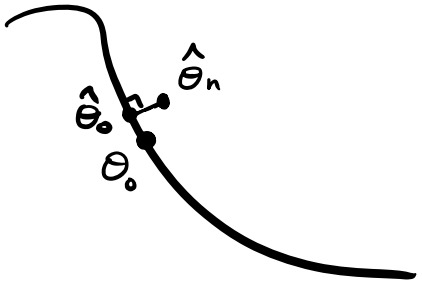
Then $\hat{\theta}_n \approx N_d(\theta_0, \frac{1}{n} I_d)$

And locally, $\nabla^2 \ell_n(\theta) \approx n I_d$ near θ_0

$$\ell_n(\theta) - \ell_n(\hat{\theta}_n) \approx \frac{n}{2} \|\theta - \hat{\theta}_n\|^2$$

$$\hat{\theta}_0 \approx \underset{\theta \in \Theta_0}{\text{argmin}} \|\theta - \hat{\theta}_n\| = \text{Proj}_{\Theta_0}(\hat{\theta}_n)$$

$$2(\ell_n(\hat{\theta}_0) - \ell_n(\hat{\theta}_n)) \approx n \|\hat{\theta}_n - \text{Proj}_{\Theta_0}(\hat{\theta}_n)\|^2$$



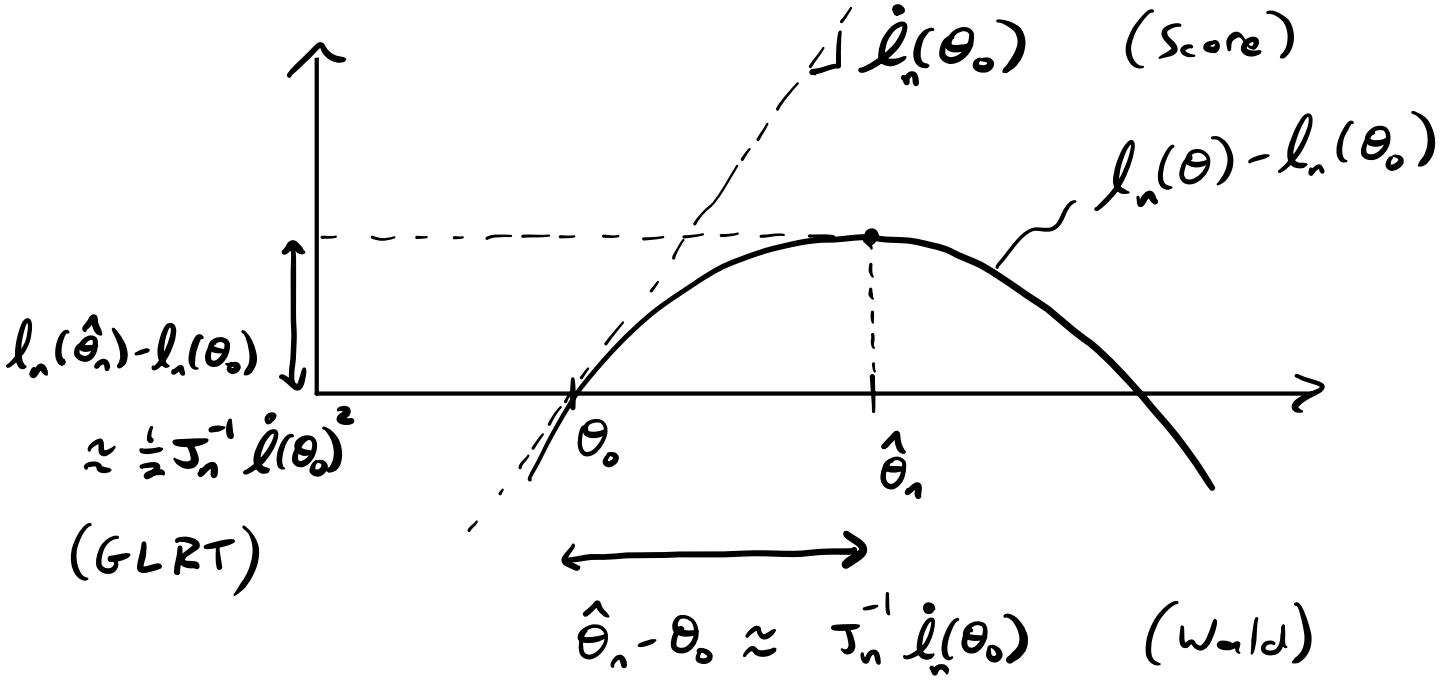
$$= n \|\text{Proj}_{\Theta_0^{\perp_\theta}}(\hat{\theta}_n)\|^2$$

$$\Rightarrow \chi^2_{d-d_0}$$

Asymptotic Equivalence

Recall quadratic approx. picture ($d=1$):

$$\ell_n(\theta) - \ell_n(\theta_0) \approx \dot{\ell}_n(\theta_0)(\theta - \theta_0) + \frac{1}{2} J_n(\theta_0)(\theta - \theta_0)^2$$



For large n,

$$\underset{\text{(GLRT)}}{\ell_n(\hat{\theta}_n) - \ell_n(\theta_0)} \approx \|J_n(\theta_0)^{1/2}(\hat{\theta}_n - \theta_0)\|^2 \approx \|J_n(\theta_0)^{-1/2} \nabla \ell_n(\theta_0)\|^2$$

$$\approx \|\hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0)\|^2$$

(Wald)

(score)

Asymptotic Relative Efficiency (ARE)

Suppose $\hat{\theta}_n^{(i)}$ $i = 1, 2$ are two asy. Normal estimators of $\theta_0 \in \mathbb{R}$, with

$$\sqrt{n}\left(\hat{\theta}_n^{(i)} - \theta_0\right) \Rightarrow N(0, \sigma_i^2)$$

The ARE of $\hat{\theta}^{(2)}$ wrt $\hat{\theta}^{(1)}$ is σ_1^2/σ_2^2

e.g. if $\sigma_2^2 = 2\sigma_1^2$ then $\hat{\theta}^{(2)}$ is 50% as efficient

Interpretation: Suppose $\sigma_1^2/\sigma_2^2 = \gamma \in (0, 1)$

Then for large n,

$$\hat{\theta}^{(1)}_{\lfloor \gamma n \rfloor}(X_1, \ldots, X_{\lfloor \gamma n \rfloor}) \overset{D}{\approx} \hat{\theta}^{(2)}_n(X_1, \ldots, X_n) \approx N\left(\theta, \frac{\sigma_2^2}{n}\right)$$

Using $\hat{\theta}^{(2)}$ is like throwing away $100(1-\gamma)\%$ of the data and then using $\hat{\theta}^{(1)}$