Online Learning Applications

Part 3: Adversarial expert problems and MABs

Adversarial expert problems

Adversarial expert problem

At each time $t = 1, \ldots, T$:

1. The environment chooses a loss function $\ell_t : A \to [0, 1]$
2. The learner chooses an arm $a_t \in A$
3. The learner receives a loss $\ell_t(a_t)$
4. The learner observes the loss $\ell_t(a)$ of **all** arms $a \in A$

Goal

Design an algorithm that achieves sublinear regret ($\lim_{T \to \infty} \frac{R_T}{T} = 0$).

What can we hope to achieve?

Theorem

In the adversarial expert problem, any algorithm suffers regret at least

$$\mathcal{R}_T \geq \Omega\left(\sqrt{\log(K)T}\right).$$

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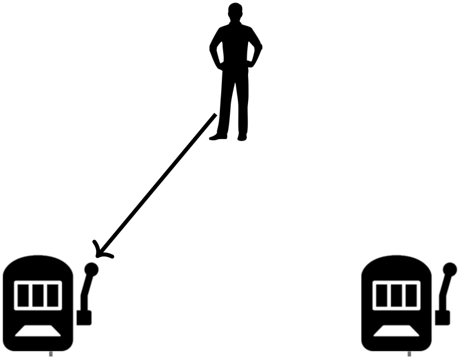

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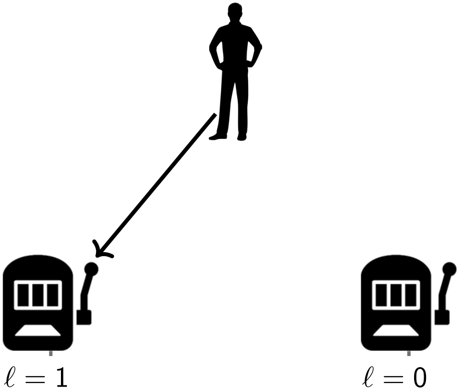

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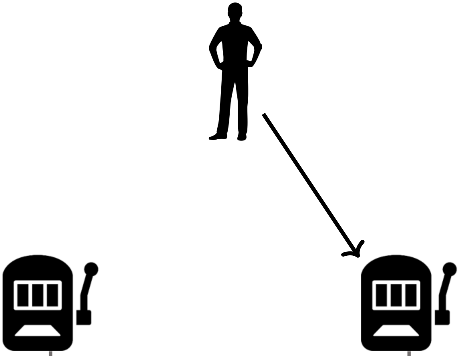

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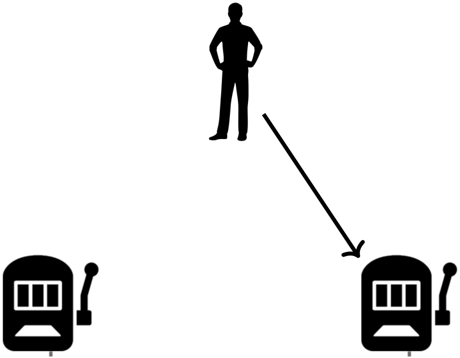

$\ell = 0$ $\ell = 1$

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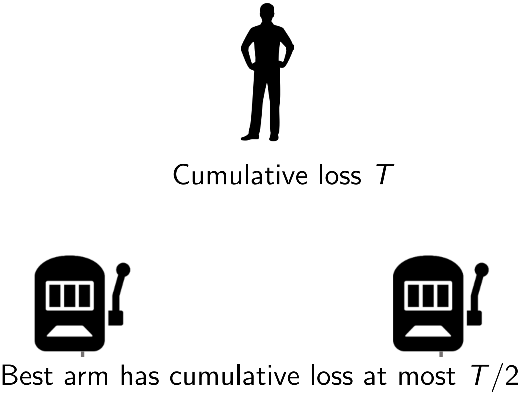

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Deterministic Algorithm Regret

Any deterministic algorithm suffers regret $\Omega(T)$.

Randomization

A randomized algorithm chooses a **distribution** over arms.

Idea:

- Assign a weight $w(a)$ to each arm
- Play each arm with probability proportional to the weight

- We should assign a large weight to "good" arms
- We observe only the past losses
- We decrease the weight exponentially in the loss: essential to obtain optimal bounds

Hedge

Algorithm: Hedge

Set of arm A, number of rounds T, learning rate η;
Initialization: $w_1 \leftarrow (1, 1, \ldots, 1)$;
for $t = 1, \ldots, T$ **do**
  $x_t(a) \leftarrow \frac{w_t(a)}{\sum_{a' \in A} w_t(a')}$ for every $a \in A$;
  sample arm $a_t \sim x_t \in \Delta_A$;
  play a_t;
  observe loss vector $\ell_t \in [0, 1]^K$;
  suffer **expected** loss $\langle \ell_t, x_t \rangle$;
  update weights: $w_{t+1}(a) \leftarrow w_t(a) e^{-\eta \ell_t(a)}$ for every $a \in A$;

Hedge

Theorem

Hedge with learning rate $\eta = \sqrt{\frac{\log K}{T}}$ *achieves regret:*

$$R_T \leq O\left(\sqrt{T \log K}\right).$$

Proof.

- Let $\ell_t^2 \in \mathbb{R}^K$ be the vector of squared losses
- Let $z_t = \sum_{a \in A} w_t(a)$ be the sum of the weights at round t

For each $t \in \{1, \dots, T\}$:

$$
\begin{aligned}
z_{t+1} = \sum_{a \in A} w_{t+1}(a) = \sum_{a \in A} w_t(a) e^{-\eta \ell_t(a)} &= z_t \sum_{a \in A} x_t(a) e^{-\eta \ell_t(a)} \\
&\leq z_t \sum_{a \in A} x_t(a) \left(1 - \eta \ell_t(a) + \eta^2 \ell_t(a)^2\right) = z_t \left(1 - \eta \langle \ell_t, x_t \rangle + \eta^2 \langle \ell_t^2, x_t \rangle\right) \\
&\leq z_t e^{-\eta \langle \ell_t, x_t \rangle + \eta^2 \langle \ell_t^2, x_t \rangle},
\end{aligned}
$$

where

- The first inequality comes from $e^{-x} \leq 1 - x + x^2$ for each $x \geq 0$
- The second inequality comes from $1 + x \leq e^x$ for each $x \in \mathbb{R}$

Proof.

By induction, we get

$$z_{T+1} \leq z_1 \prod_{t=1}^{T} e^{-\eta\langle \ell_t, x_t\rangle + \eta^2 \langle \ell_t^2, x_t\rangle} = K e^{-\eta \sum_{t=1}^{T}\langle \ell_t, x_t\rangle + \eta^2 \sum_{t=1}^{T}\langle \ell_t^2, x_t\rangle}.$$

Moreover, by induction we get

$$w_{T+1}(a) = \prod_{t=1}^{T} e^{-\eta \ell_t(a)} = e^{-\eta \sum_{t=1}^{T} \ell_t(a)}.$$

Let $a^* \in A$ be the best arm in hindsight, that is, $a^* \in \arg\min_{a \in A} \sum_{t=1}^{T} \ell_t(a)$. Then,

$$e^{-\eta \sum_{t=1}^{T} \ell_t(a^*)} = w_{T+1}(a^*) \leq z_{T+1} \leq K e^{-\eta \sum_{t=1}^{T}\langle \ell_t, x_t\rangle + \eta^2 \sum_{t=1}^{T}\langle \ell_t^2, x_t\rangle}.$$

Proof.

Taking logs on both sides, we get

$$-\eta \sum_{t=1}^{T} \ell_t(a^*) \leq \log K - \eta \sum_{t=1}^{T} \langle \ell_t, x_t \rangle + \eta^2 \sum_{t=1}^{T} \langle \ell_t^2, x_t \rangle.$$

Hence, we have

$$R_T = \sum_{t=1}^{T} \langle \ell_t, x_t \rangle - \sum_{t=1}^{T} \ell_t(a^*) \leq \frac{\log K}{\eta} + \eta \sum_{t=1}^{T} \langle \ell_t^2, x_t \rangle \leq \frac{\log K}{\eta} + \eta T \tag{1}$$

$$= 2\sqrt{T \log K}, \tag{2}$$

where the last inequality comes from $\ell_t^2 \leq \mathbf{1}$ and the last equality from the definition of η. □

Adversarial MABs

Adversarial MABs

At each time $t = 1, \dots, T$:

1. The environment chooses a loss function $\ell_t : A \to [0, 1]$
2. The learner chooses an arm a_t
3. The learner receives a loss $\ell_t(a_t)$
4. ⚠ The learner observes **only** the loss $\ell_t(a_t)$ of arm a_t

Goal

Design an algorithm that achieves sublinear regret ($\lim_{T\to\infty} \frac{\mathcal{R}_T}{T} = 0$).

What can we hope to achieve?

The lower bound is slightly worse that under full information. The dependence on the number of arms is $\sqrt{K}$ instead of $\sqrt{\log(K)}$.

Theorem

In the adversarial MABs, any algorithm suffers regret at least

$$\mathcal{R}_T \geq \Omega\left(\sqrt{KT}\right).$$

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In the adversarial MABs, any algorithm suffers regret at least

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⚠ It doesn't make much sense to talk about instance-dependent bounds (e.g., dependence on Δ_a) since we are not making any assumption about the losses.

Working with limited information

- We cannot directly apply Hedge since we have access only to the loss of the arm we played
- The chosen arm a_t has consequences on the received loss but also on the **information gathered**

How can we adapt Hedge to this feedback model?

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How can we adapt Hedge to this feedback model?

Idea:

- Update the weight only for the played arm a_t
- Modify the loss $\ell_t(a_t)$ to $\tilde{\ell}_t(a_t) := \ell_t(a_t)/x_t(a_t)$
- Set all the other $\tilde{\ell}_t(a) = 0$ (i.e., the weights are not updated)
- This is an **unbiased estimator** of the loss ($\mathbb{E}[\tilde{\ell}_t(a)] = \ell_t(a)$)

EXP3

Algorithm: EXP3

set of arms A, number of rounds T, learning rate η;
$w_1 \leftarrow (1, 1, \ldots, 1)$;
for $t = 1, \ldots, T$ **do**
    $x_t(a) \leftarrow \frac{w_t(a)}{\sum_{a' \in A} w_t(a')}$ for every $a \in A$;
    sample arm $a_t \sim x_t \in \Delta_A$;
    play a_t;
    observe loss $\ell_t(a_t)$;
    suffer **expected** loss $\langle \ell_t, x_t \rangle$;
    compute $\tilde{\ell}_t(a) \leftarrow \frac{\ell_t(a_t)}{x_t(a_t)} \mathbb{I}[a_t = a]$ for every $a \in A$;
    update weights: $w_{t+1,i} \leftarrow w_{t,i} e^{-\eta \tilde{\ell}_t(a)}$ for every $a \in A$;

EXP3

Theorem

EXP3 with learning rate $\eta = \sqrt{\frac{\log K}{KT}}$ *achieves regret:*

$$R_T \leq O\left(\sqrt{K \log(K) T}\right).$$

- Regret proportional to $\sqrt{T}$ as in the full-info setting with Hedge
- Worse dependence from the number of arms w.r.t Hedge

EXP3

Theorem

EXP3 with learning rate $\eta = \sqrt{\frac{\log K}{KT}}$ *achieves regret:*

$$R_T \leq O\left(\sqrt{K \log(K) T}\right).$$

- Regret proportional to $\sqrt{T}$ as in the full-info setting with Hedge
- Worse dependence from the number of arms w.r.t Hedge

Relation to Stochastic bandit:

- Same worst-case regret bound under weaker assumptions!
- No instance-dependent regret bounds
- Worse performance in "easy" stochastic instances (no instance-dependent bounds)

Proof.

EXP3 is equivalent to Hedge with losses $\tilde{\ell}_t$. Two challenges to extend the analysis to EXP3:

- $\tilde{\ell}_t \neq \ell_t$
- $\tilde{\ell}_t$ is no more bounded in $[0,1]^K$

We can follow the proof of Hedge to show:

$$\sum_{t=1}^{T}\langle\tilde{\ell}_t, x_t\rangle - \sum_{t=1}^{T}\tilde{\ell}_t(a^*) \leq \frac{\log K}{\eta} + \eta\sum_{t=1}^{T}\langle\tilde{\ell}_t^2, x_t\rangle.$$

Proof.

Taking expectation on both sides:

$$\mathbb{E}\left[\sum_{t=1}^{T}\langle\tilde{\ell}_t, x_t\rangle - \sum_{t=1}^{T}\tilde{\ell}_t(a^*)\right] \leq \frac{\log K}{\eta} + \eta\sum_{t=1}^{T}\sum_{a\in A} x_t(a)\frac{\ell_t(a)^2}{x_t(a)}$$
$$\leq \frac{\log K}{\eta} + \eta\sum_{t=1}^{T}\sum_{a\in A}\ell_t(a)^2 \leq \frac{\log K}{\eta} + \eta TK.$$

Since $\tilde{\ell}_t(a)$ is an unbiased estimator of $\ell_t(a)$:

$$\sum_{t=1}^{T}\langle\ell_t, x_t\rangle - \sum_{t=1}^{T}\ell_t(a^*) = \mathbb{E}\left[\sum_{t=1}^{T}\langle\tilde{\ell}_t, x_t\rangle - \sum_{t=1}^{T}\tilde{\ell}_t(a^*)\right] \leq \frac{\log K}{\eta} + \eta TK = 2\sqrt{K\log(K)T},$$

where the last equality follows from the definition of η.

□

References

Nick Littlestone and M. K. Warmuth. The weighted majority algorithm. *Information and Computation*, 108(2): 212–261, 1994.

Yoav Freund and Robert Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of Computer and System Sciences*, 55(1):119–139, 1997.

N. Cesa-Bianchi, Y. Freund, D. Haussler, D. P. Helmbold, R. E. Schapire, and M. K. Warmuth. How to use expert advice. *Journal of the ACM*, 44:427–485, 1997.

Peter Auer, Nicolo Cesa-Bianchi, Yoav Freund, and Robert E. Schapire. The nonstochastic multiarmed bandit problem. *SIAM Journal of Computing*, 32:48–77, 2002.