Introduction to Quantum Computing

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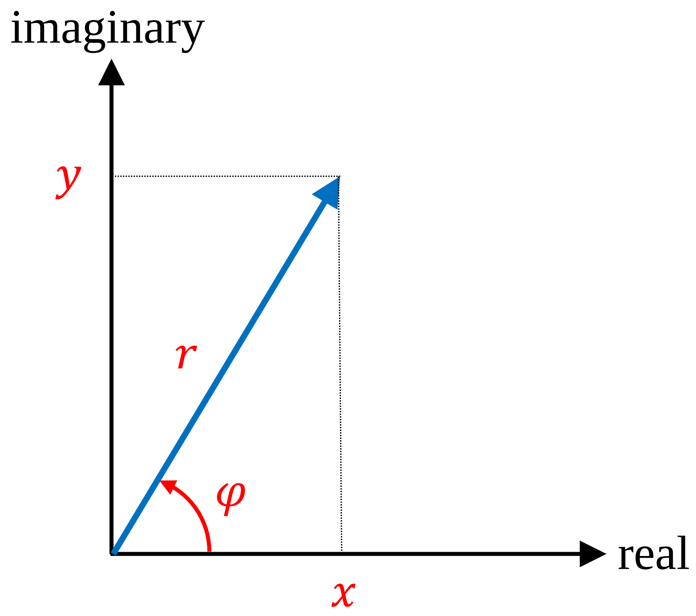
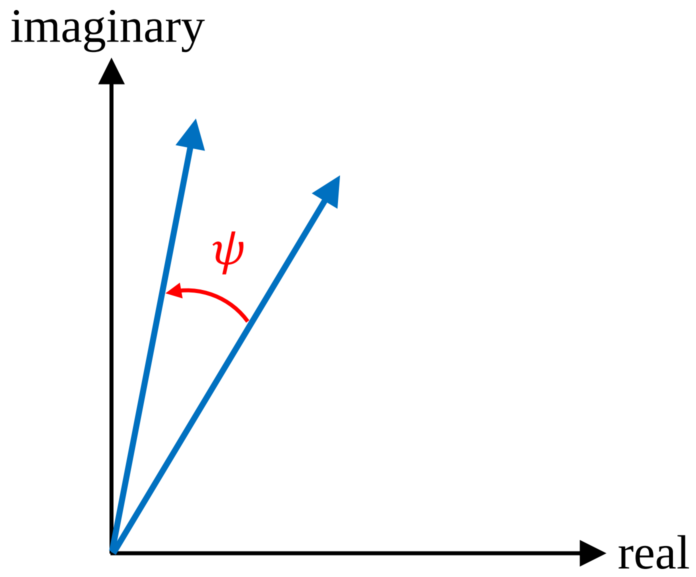
Complex numbers mini recap

- Complex number
$$z = x + iy = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$$
with $r = \sqrt{x^2 + y^2}$
- A complex number can be **rotated by an angle $\boldsymbol{\psi}$** by multiplying it with $\boldsymbol{e^{i\psi}}$
- The **conjugate** of complex number z is
$$\bar{z} = x - iy = re^{-i\varphi}$$
- The **Hermitian** of a vector of complex numbers
$$z = \begin{bmatrix} a \\ b \end{bmatrix}$$
is its **conjugate transpose**
$$z^H = [\bar{a} \quad \bar{b}]$$





Dirac’s ket and bras

- In Dirac's notation (also know as **bra-ket** notation), a **ket** is a unitary column vector denoted by
$$|v\rangle \leftrightarrow \vec{v}$$
  - it is used to represent a quantum state, that is, a complex unit vector (a vector of length 1)
  - Example: if we have a quantum system with two possible states, which we label as $|0\rangle$ and $|1\rangle$, then any state of the system can be written as a linear combination of these two states:
  $$|v\rangle = a|0\rangle + b|1\rangle$$
  where *a* and *b* are complex numbers (amplitudes), and $|a|^2 + |b|^2 = 1$ to ensure normalization
  - The ket $|v\rangle$ represents the state of the quantum system
- Kets are used in conjunction with **bras**, which are row vectors denoted by
$$\langle v| \leftrightarrow \vec{v}^H$$
The bra $\langle v|$ represents the conjugate transpose of the ket $|v\rangle$

Dirac's notation: multiplications

- The inner (scalar) product between two kets $|x\rangle$ and $|y\rangle$ (that is, between two vectors $\vec{x}^H$ and $\vec{y}$) is represented with the following notation
$$\langle x|y\rangle \leftrightarrow \vec{x}^H \cdot \vec{y}$$

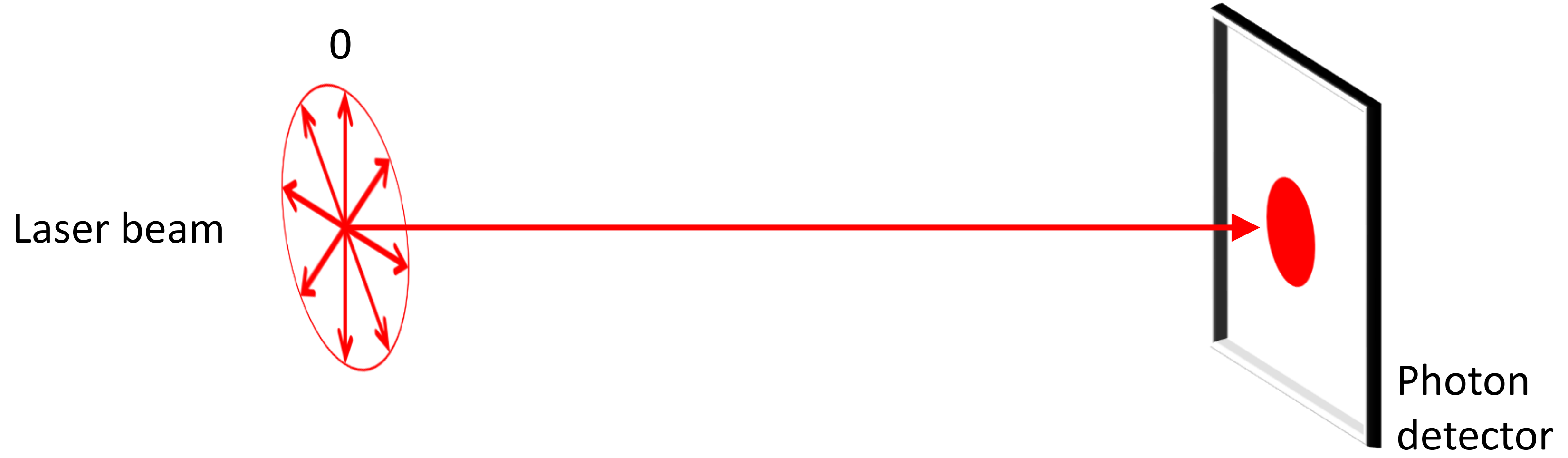
- We often need to multiply a ket $\vec{v}$ (a column vector) with a matrix M. This is the notation
$$M|v\rangle \leftrightarrow M \cdot \vec{v}$$

- We can concatenate multiplications, for example
$$\langle x|M|y\rangle \leftrightarrow \vec{x}^H \cdot M \cdot \vec{y}$$

Single Qubit States

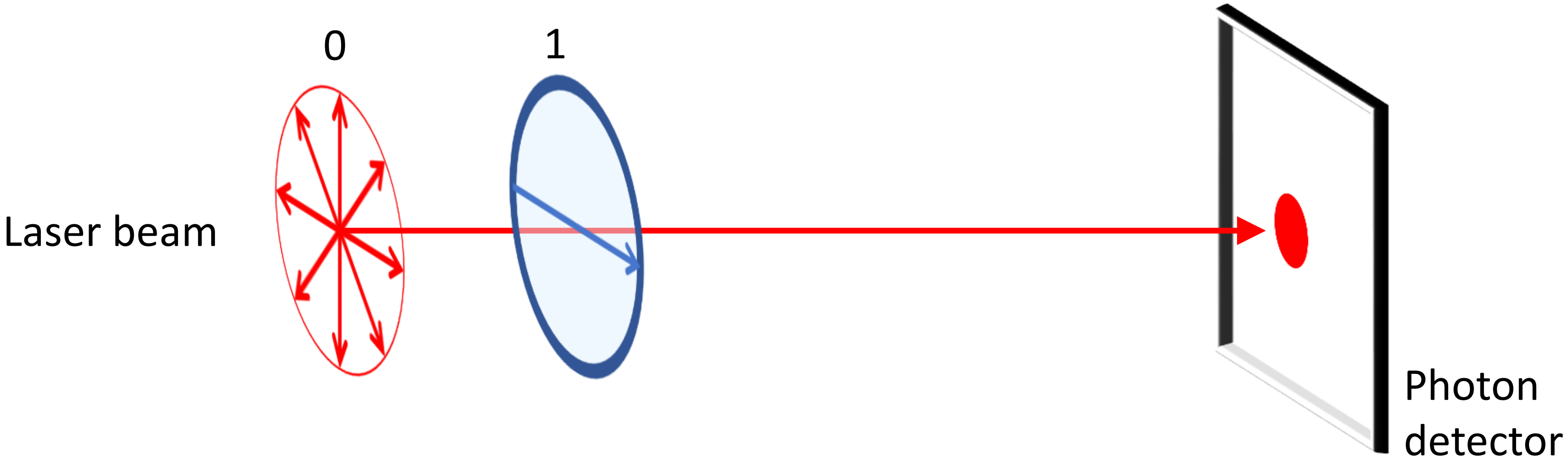
Build and measure qubits

- A toy example



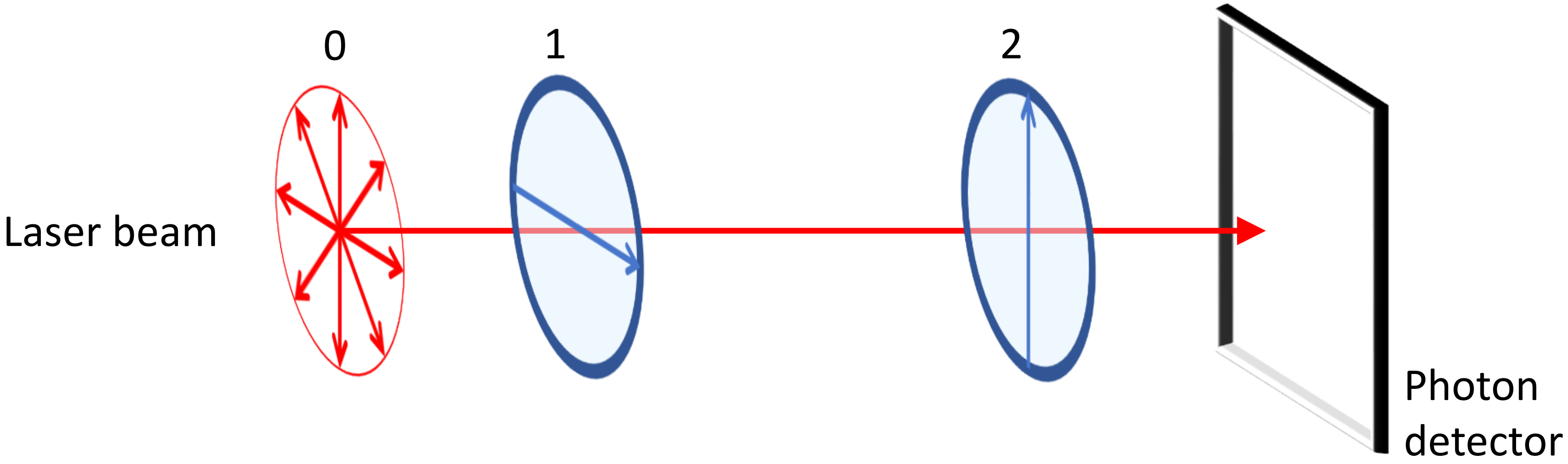
Build and measure qubits

- A toy example



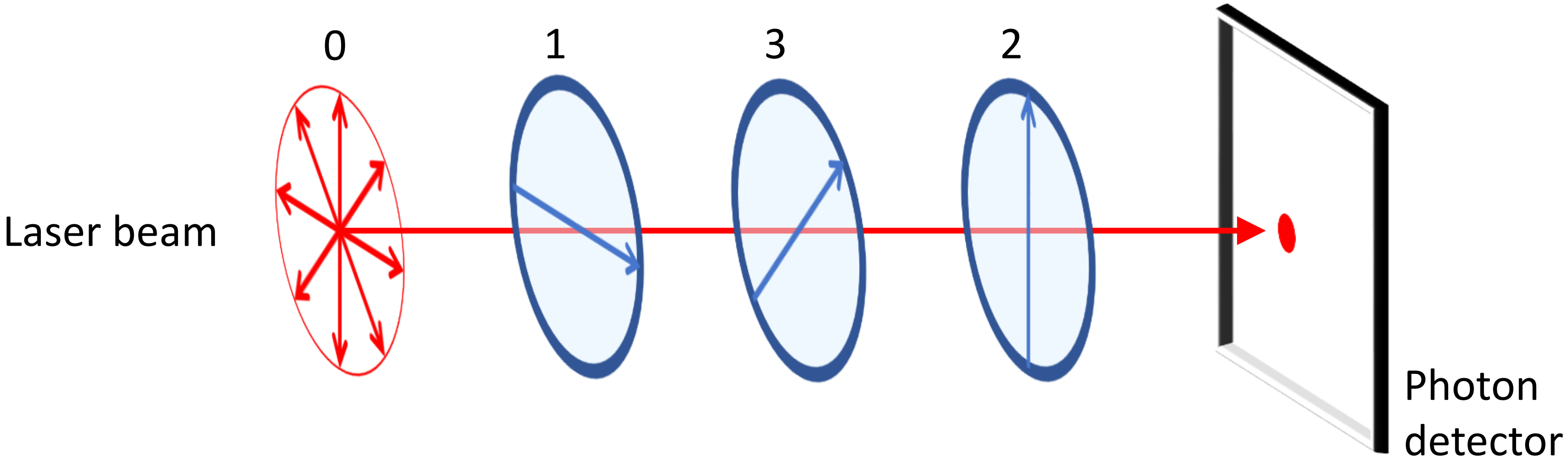
Build and measure qubits

- A toy example



Build and measure qubits

- A toy example

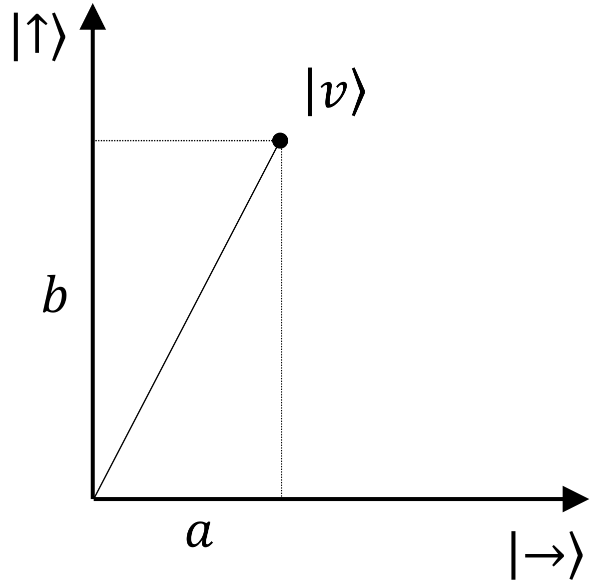


Photon's polarization

- Photon's polarization is described with a **unit** vector $|v\rangle$
  - *unit vector*: norm of the vector equal to one
- We write $|\rightarrow\rangle$ and $|\uparrow\rangle$ for unit vectors that represent horizontal and vertical polarization
- An arbitrary polarization can be expressed as a linear combination

$$|v\rangle = a|\rightarrow\rangle + b|\uparrow\rangle$$

- Example: $|\nearrow\rangle = \frac{1}{\sqrt{2}}|\rightarrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\rangle$
  is a unit vector representing polarization of 45° degrees



Photon’s polarization and qubits

- The state of a photon can be represented by a unit vector of two complex numbers
$$|v\rangle = \ a|{\rightarrow}\rangle + b|{\uparrow}\rangle$$
- Coefficients ***a*** and ***b*** are **complex numbers** and are called the **amplitudes** of $|v\rangle$ in the directions $|{\rightarrow}\rangle$ and $|{\uparrow}\rangle$
  - a^2 is the probability to **measure** the photon in polarization $|{\rightarrow}\rangle$
  - b^2 is the probability to **measure** the photon in polarization $|{\uparrow}\rangle$
- When *a* and *b* are both non-zero, $|v\rangle$ is said to be a **superposition** of $|{\rightarrow}\rangle$ and $|{\uparrow}\rangle$
- In the photon polarization example, $|{\rightarrow}\rangle$ and $|{\uparrow}\rangle$ are call the **basis**

- The possible polarization states of a photon is an example of **qubit**

Photon's polarization and qubits

- What happens when a photon with no polarization $|v_0\rangle = a|{\rightarrow}\rangle + b|{\uparrow}\rangle$ emitted by light source (0) meets the polaroid (1) with axis $|{\rightarrow}\rangle$
  - the photon gets through with probability $|a|^2$
  - is absorbed with probability $|b|^2$
  - photons that passe through the polaroid are polarized in the direction of the polaroid's axis $|{\rightarrow}\rangle$
    $$|v_1\rangle = 1|{\rightarrow}\rangle + 0|{\uparrow}\rangle = |{\rightarrow}\rangle$$
  - (the photon is not in superposition any more with respect to $|{\rightarrow}\rangle$ and $|{\uparrow}\rangle$)
- What happens once polaroid (2) with polarization axis $|{\uparrow}\rangle$ is inserted?
  - any photons that pass through polaroid (1) will leave polarized in the direction of polaroid (1)'s axis, in this case horizontal, $|{\rightarrow}\rangle$
  - a horizontally polarized photon has no amplitude in the vertical direction, so it has no chance of passing through polaroid (2), which has a vertical orientation
  - no light reaches the screen

Photon's polarization and qubits

- What happens if polaroid (3) with polarization $|\nearrow\rangle$ is inserted between (1) and (2)?
  - the horizontally polarized photon's state $|v_1\rangle = |\rightarrow\rangle$ can be written as (think to this as a change of axis)

$$|v_1\rangle = |\rightarrow\rangle = \frac{1}{\sqrt{2}}|\nearrow\rangle + \frac{1}{\sqrt{2}}|\searrow\rangle$$

  - the horizontally polarized photon passes through polaroid (3) with probability $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$
  - photons that pass through polaroid (3) have polarization $|\nearrow\rangle$
  - the diagonally polarized photon's state $|\nearrow\rangle$ can be written as

$$|v_1\rangle = |\nearrow\rangle = \frac{1}{\sqrt{2}}|\rightarrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\rangle$$

  - when these photons hit polaroid (2), they have amplitude in the vertical direction
    - some of them will pass thorough polaroid (2) and hit the screen with probability 1/2

Qubits

- A **qubit** is the **unit of quantum information**, analogous to a classical bit.
  - a qubit has an infinite possible values, differently from classical bits that can be either 0 or 1
- Its state is represented by a unit vector $|v\rangle$ of two complex numbers a and b

$$|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle$$

- where we define the two vectors $|0\rangle$ and $|1\rangle$, also called the **standard basis**, as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- The complex numbers a and b are called the **amplitudes**
  - a^2 is the probability to **measure** the qubit in state $|0\rangle$
  - b^2 is the probability to **measure** the photon in state $|1\rangle$
- We will later explain what does it means to **measure** a qubit …

Single qubits basis

- When describing the state qubit, it is possible to use any pair of **orthonormal basis** $|u\rangle$ and $|u^{\perp}\rangle$

$$|v\rangle = a|u\rangle + b|u^{\perp}\rangle$$

  - two vectors $|u\rangle$ and $|u^{\perp}\rangle$ are orthonormal if they are orthogonal and both with unitary norm

- The most used basis are the standard basis denoted with labels $|0\rangle$ and $|1\rangle$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Another set of useful basis are the Hadamard basis with labels $|+\rangle$ and $|-\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Single qubits measurement

- Any device that measures a qubit must have two preferred states $|u\rangle$ and $|u^\perp\rangle$
  - $|u\rangle$ and $|u^\perp\rangle$ form an orthonormal basis
    - throughout the course, when not specified differently, measurement is done with respect to the standard basis $|u\rangle = |0\rangle$ and $|u^\perp\rangle = |1\rangle$
  - the measurement outcome is always one of the two basis vectors $|u\rangle$ or $|u^\perp\rangle$
  - The state of qubit $|v\rangle = a|u\rangle + b|u^\perp\rangle$ is measured as $|u\rangle$ with probability $|a|^2$ and as $|u^\perp\rangle$ with probability $|b|^2$
- **Measurement** of a qubit **changes its state**
  - if qubit with state $|v\rangle = a|u\rangle + b|u^\perp\rangle$ is measured as$|u\rangle$, then the state changes to
$$|v\rangle = 1|u\rangle + 0|u^\perp\rangle = |u\rangle$$
  - a second measurement with respect to the same basis will return $|u\rangle$ with probability 1
  - the same if the qubit is measured as $|u^\perp\rangle$
- This behavior of measurement is an **axiom** of quantum mechanics

Exercise

- Show that the Hadamard basis are orthogonal

  - we need to show that $\langle + | - \rangle = 0$

  - remember that $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

  - $\langle + | - \rangle = \frac{1}{\sqrt{2}}(1 \quad 1) \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

Exercise

- How to pass from one basis (standard) to another one (Hadamard)?
  - use a system of linear equations where the amplitudes of the basis states are the unknowns
- Standard:

$$a|0\rangle + b|1\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$$

- Hadamard:

$$x|+\rangle + y|-\rangle = x\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + y\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}x+y\\x-y\end{pmatrix}$$

$$\begin{cases} a = \dfrac{x+y}{\sqrt{2}} \\ b = \dfrac{x-y}{\sqrt{2}} \end{cases} \rightarrow \begin{cases} x = \dfrac{a+b}{\sqrt{2}} \\ y = \dfrac{a-b}{\sqrt{2}} \end{cases}$$

Single qubits measurement and superposition

- Superposition is basis-dependent: all states are in superpositions with respect to some bases and not with respect to others
  - example: $|v\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ is in superposition with respect to the basis $|0\rangle$ and $|1\rangle$
  - but not with respect to the basis $|+\rangle$ and $|-\rangle\}$ for which we have $|v\rangle = 1|+\rangle + 0|-\rangle$
  - more in general, $|v\rangle = a|0\rangle + b|1\rangle$ (with $a \neq 0$ and $b \neq 0$)
    - is in superposition with respect to the basis $|0\rangle$ and $|1\rangle$
    - but not with respect to the basis $|x\rangle = a|0\rangle + b|1\rangle$ and $|y\rangle = \bar{b}|0\rangle - \bar{a}|1\rangle$
- If you measure qubit $|v\rangle$
  - with respect to the basis $|0\rangle$, $|1\rangle$ you have a probabilistic results
    - 50% of being $|0\rangle$
    - 50% of being $|1\rangle$
  - with respect to the basis $|+\rangle$, $|-\rangle$ you have a deterministic results
    - 100% of being $|+\rangle$

Exercise

- Show that $|x\rangle = a|0\rangle + b|1\rangle$ is orthogonal to $|y\rangle = \bar{b}|0\rangle - \bar{a}|1\rangle$
- An easy way to see why $|x\rangle$ is orthogonal to $|y\rangle$ is to compute their inner product $\langle x|y\rangle$ and check that it is zero
- The bra $\langle y|$ corresponding to $|y\rangle$ is
$$\langle y| = b\langle 0| - a\langle 1|$$
  - since $\bar{\bar{a}} = a$ and $\bar{\bar{b}} = b$
    - why? $a = |a|e^{i\alpha}$, $\bar{a} = |a|e^{-i\alpha}$, $\bar{\bar{a}} = |a|e^{i\alpha}$
- Now compute the inner product
$$\langle w|v\rangle = (b\langle 0| - a\langle 1|)(a|0\rangle + b|1\rangle)$$
- Expand term by term
$$\langle w|v\rangle = ba\langle 0|0\rangle + bb\langle 0|1\rangle - aa\langle 0|1\rangle - ab\langle 1|1\rangle = ba - ab = 0$$
  - remember that $\langle 0|0\rangle = \langle 1|1\rangle = 1$, and $\langle 0|1\rangle = \langle 1|0\rangle = 0$

Interpretation of superposition

- Superposition is not just a probabilistic mixture of $|0\rangle$ and $|1\rangle$
  - an **erroneous interpretation** is that the state is either $|0\rangle$ or $|1\rangle$ and that we just do not happen to know which
  - $|v\rangle = a|0\rangle + b|1\rangle$ is a definite state, which, when measured in certain bases, gives deterministic results, while in others it gives random results
- States that are combinations of $|0\rangle$ and $|1\rangle$ in similar proportions but with different amplitudes represent different states that behave differently in many situations
  - the following fours states

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$
$$\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \quad \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

    if measured with standard basis, give the same 0.5 probabilities for $|0\rangle$ and $|1\rangle$, although they behave differently with other measurement basis

Information carried by single qubits

- Qubits can take one of infinitely many states
- One might hope that a single qubit could store lots of classical information
- However …
  - a single measurement yields at most a single classical bit of information
  - because measurement changes the state, one cannot make two measurements on the original state of a qubit
- If you have a qubit in state $|v\rangle = a|0\rangle + b|1\rangle$ with a and b unknown, **it is impossible to measure $\boldsymbol{a}$ and $\boldsymbol{b}$**
- Quantum state cannot be cloned
  - it is not possible to measure a qubit's state twice, even indirectly by copying the qubit's state and measuring the copy

Global phase of a qubit

- Let's consider two qubits

$$|v\rangle = a|0\rangle + b|1\rangle$$

- and

$$|v'\rangle = e^{i\varphi}|v\rangle$$

  - remember that $e^{i\varphi}$ is a unitary complex number (a complex number of length 1) and therefore the amplitudes of v' and v have the same "lengths"

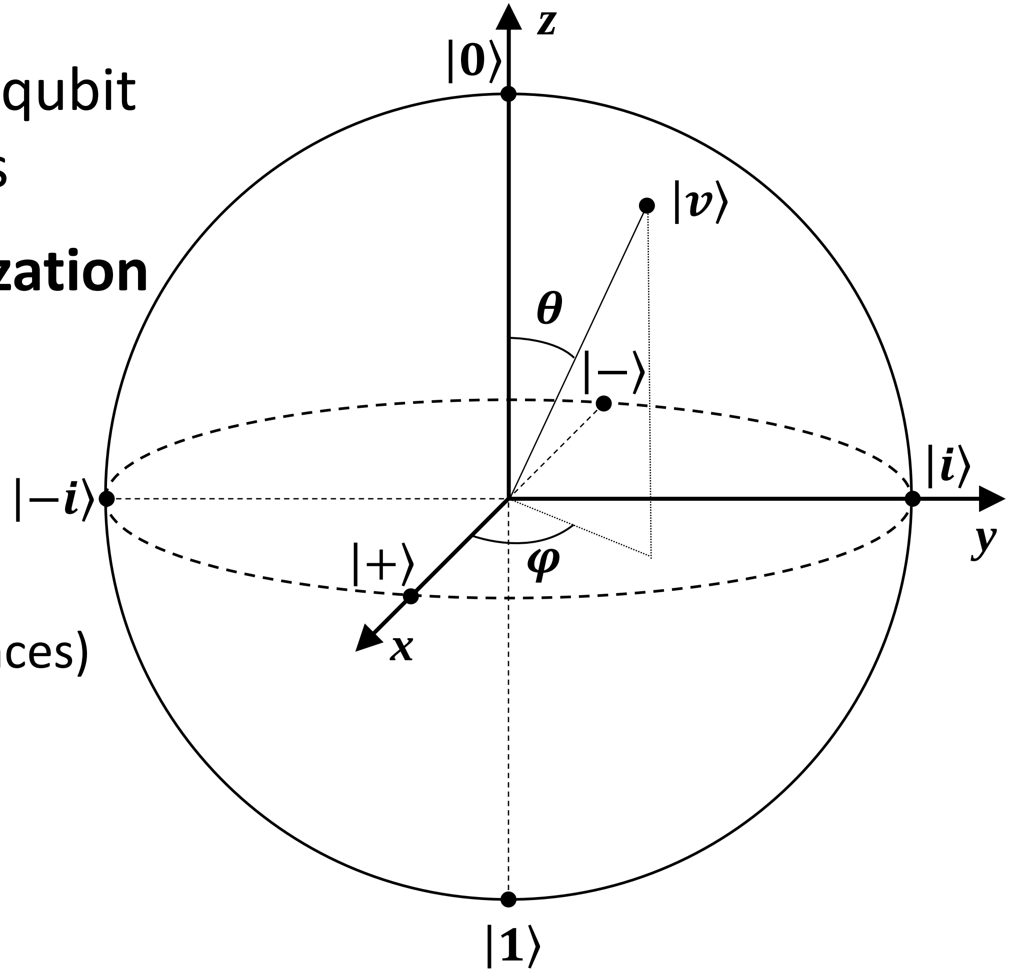
- Two vectors $|v\rangle$ and $|v'\rangle$ describe **the same qubit**. We write

$$|v\rangle \sim |v'\rangle$$

- $|v\rangle$ and $|v'\rangle$ differ only from a rotation φ which is called the **global phase**
- Demonstration (we will see this later more in detail)
  - measuring the rotated qubit v' leads to the same result as measuring the original qubit v
    $\langle v'|M_k|v'\rangle = \left\langle e^{i\varphi}v|M_k|ve^{i\varphi}\right\rangle = e^{-i\varphi}e^{i\varphi}\langle v|M_k|v\rangle = \langle v|M_k|v\rangle$

Bloch sphere representation of qubit

- Apparently, there are four degrees of freedom in a qubit $|v\rangle = a|0\rangle + b|1\rangle$ as a and b are complex numbers
- One degree of freedom is removed by the **normalization constraint** $|a|^2 + |b|^2 = 1$
- We can rewrite the qubit in this way $|v\rangle = e^{i\alpha}\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\beta}\sin\left(\frac{\theta}{2}\right)|1\rangle$
  - we can multiply by $e^{-i\alpha}$ (**global phase** has no consequences)
- We can rewrite qubit $|v\rangle$ in **spherical coordinates** $|v\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$
- The **relative phase $\varphi = \beta - \alpha$ is meaningful**



Exercise

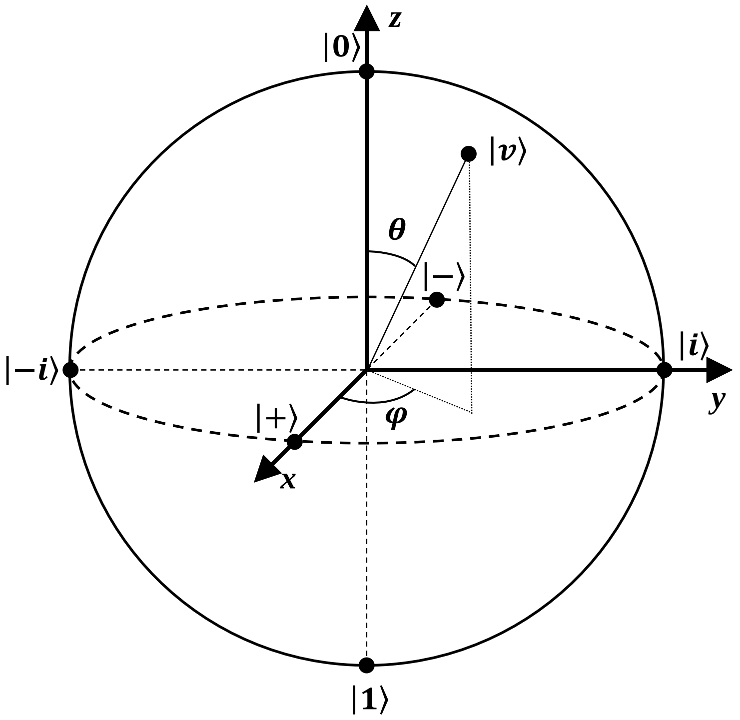
- Describe qubits $|i\rangle$ and $|-i\rangle$ using the standard basis

    - $|i\rangle$ requires $\varphi = \frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$
    - replacing in $\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$ and remembering that $e^{i\frac{\pi}{2}} = i$ we obtain

$$|i\rangle = \frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}i|1\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\ i\end{pmatrix}$$

    - $|-i\rangle$ requires $\varphi = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$
    - replacing in $\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$ and remembering that $e^{-i\frac{\pi}{2}} = -i$ we obtain

$$|-i\rangle = \frac{\sqrt{2}}{2}|0\rangle - \frac{\sqrt{2}}{2}i|1\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\ -i\end{pmatrix}$$



Exercise

- Show that two qubits $|v\rangle$ and $|w\rangle$ on the opposite side of the Bloch sphere are orthogonal
  - $|v\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$
    - the opposite point on the sphere has coordinates $(\pi - \theta, \pi + \varphi)$
  - $|w\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\pi+\varphi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$
  - we can rewrite as
  - $|w\rangle = \sin\left(\frac{\theta}{2}\right)|0\rangle - e^{i\varphi}\cos\left(\frac{\theta}{2}\right)|1\rangle$
    - $\cos\left(\frac{\pi-\theta}{2}\right) = \sin\left(\frac{\theta}{2}\right)$, $\sin\left(\frac{\pi-\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)$ and $e^{i(\pi+\varphi)} = -e^{i\varphi}$
  - the bra of $|w\rangle$ is
  - $\langle w| = \sin\left(\frac{\theta}{2}\right)\langle 0| - e^{-i\varphi}\cos\left(\frac{\theta}{2}\right)\langle 1|$
  - now compute the inner product
  - $\langle w|v\rangle = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 0$
    - remember that $\langle 0|0\rangle = \langle 1|1\rangle = 1$, and $\langle 0|1\rangle = \langle 1|0\rangle = 0$