Linear Regression

Machine Learning

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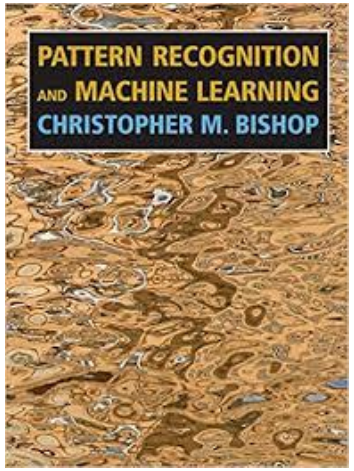
POLITECNICO
MILANO 1863

References

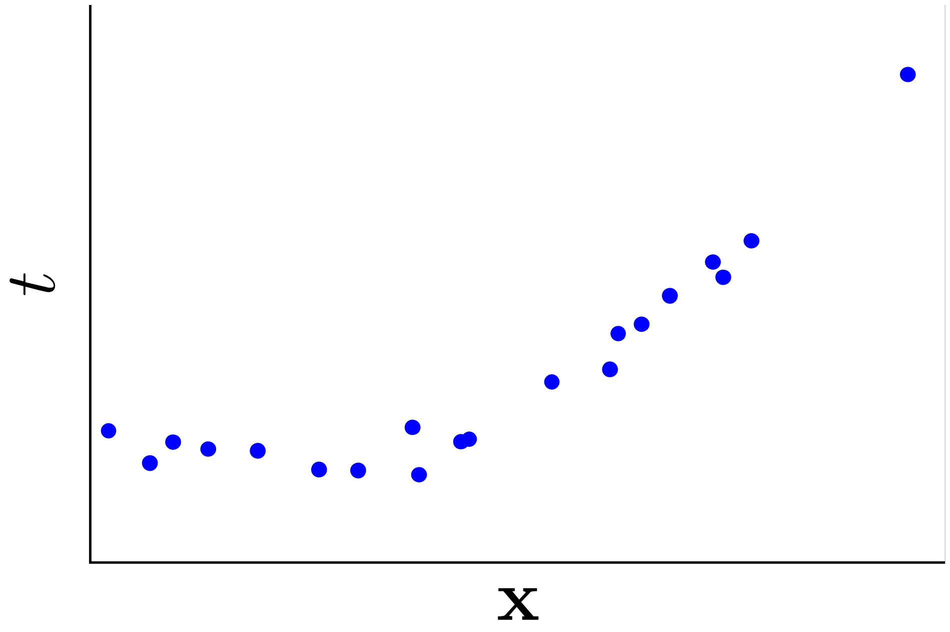
- *Pattern Recognition and Machine Learning*, Bishop
  - Chapter 1 (1.1, 1.2, 1.3)
  - Chapter 3 (3.1, 3.3)



What is regression?

- Learn an **approximation** of function $f(x)$ that maps input x to a continuous output t from a dataset $\mathcal{D}$

$$\mathcal{D} = \{\langle x, t \rangle\} \Rightarrow t = f(x)$$



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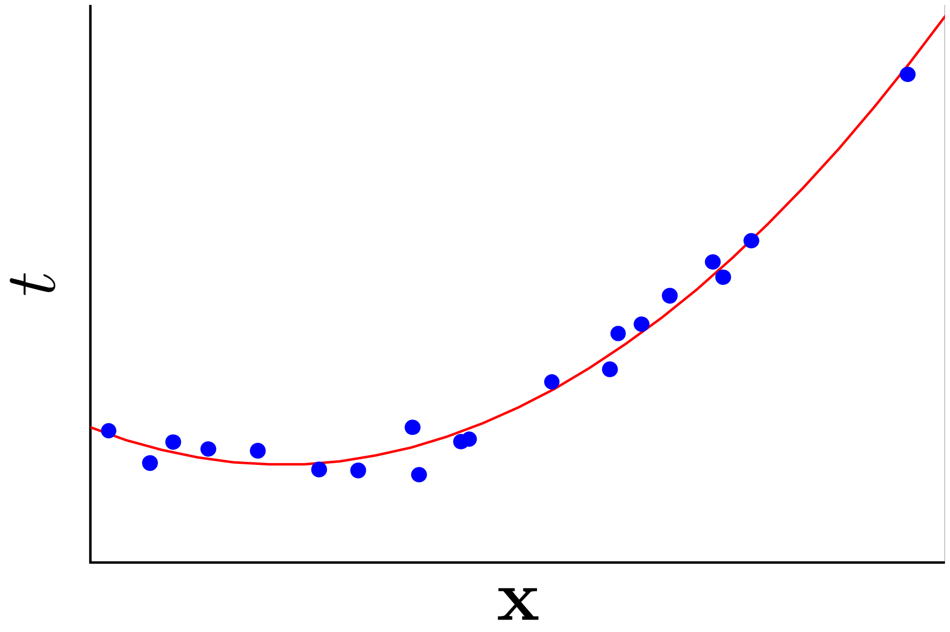
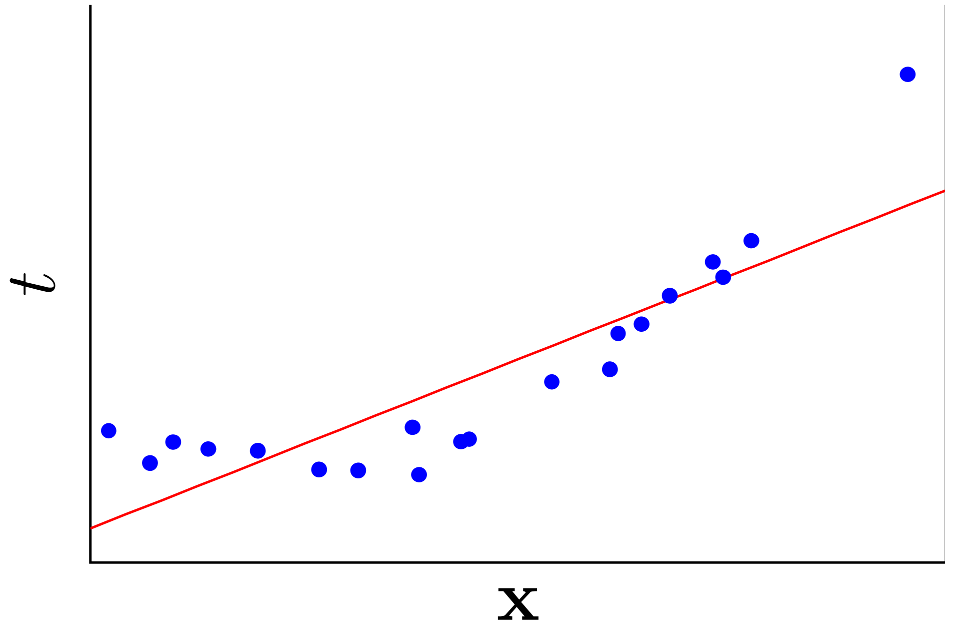
$$\mathcal{D} = \{\langle x, t \rangle\} \Rightarrow t = f(x)$$

  - How do we model f?
  - How do we evaluate our approximation?
  - How do we optimize our approximation?



Linear Regression

- In linear regression, $f(x)$ is modeled with linear functions
  - Linear models can be easily **explained**
  - A linear regression problem can be solved **analytically**
  - Linear functions can be extended to model also **non-linear relationships**
  - More sophisticated methods are based on linear regression

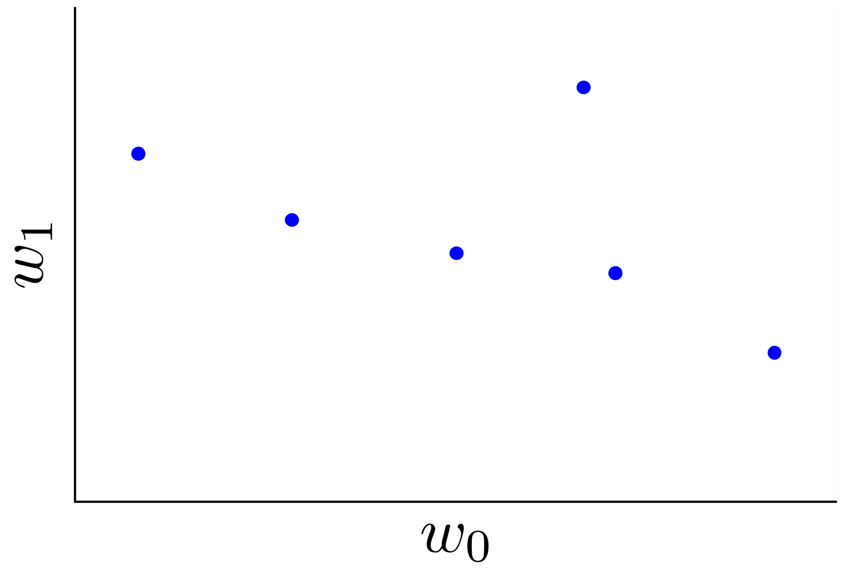


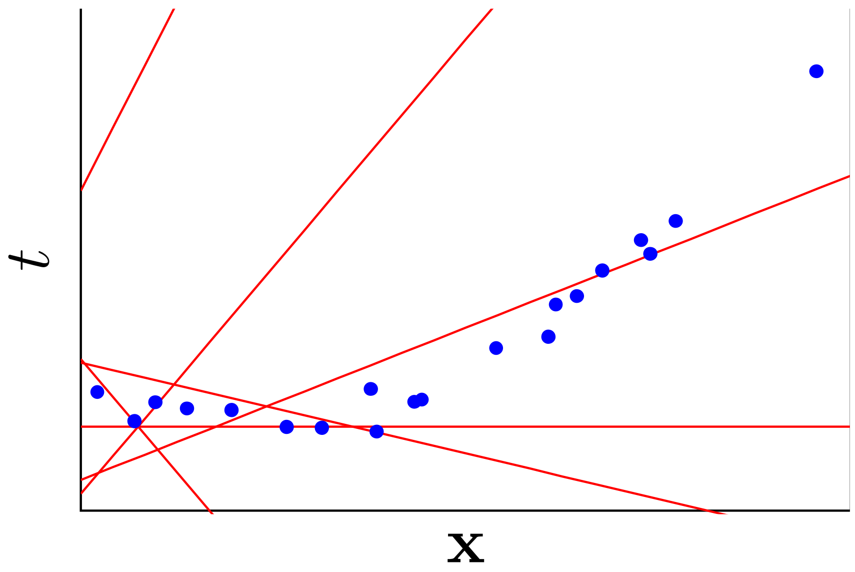
Linear Regression: model

- The simplest linear model can be defined as:

$$y(\mathbf{x}, \mathbf{w}) = w_0 + \sum_{j=1}^{D-1} w_j x_j = \mathbf{w}^T \mathbf{x}$$

  - $\mathbf{x} = (1, x_1, \ldots, x_{D-1})$
  - w_0 is called **bias parameter**





Linear Regression: loss function and optimization

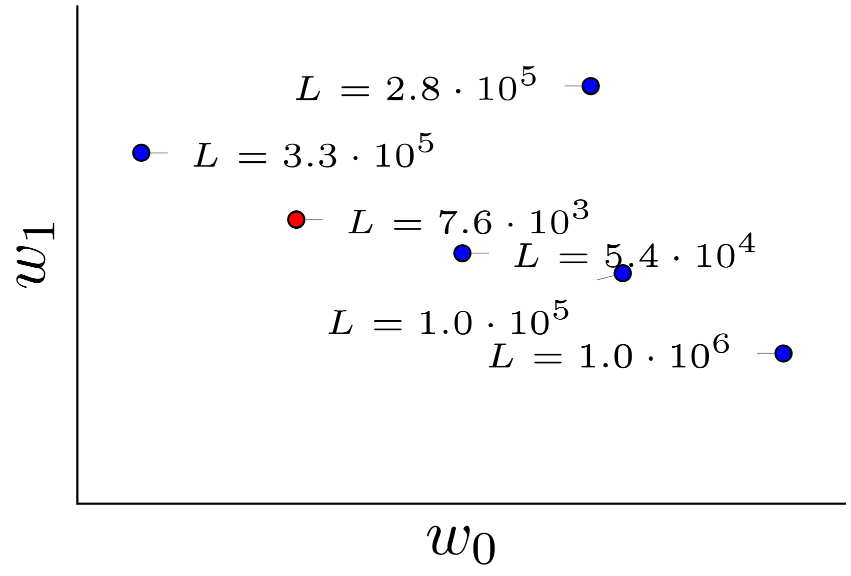
- A convenient **error loss function** is the **sum of squared errors** (SSE):

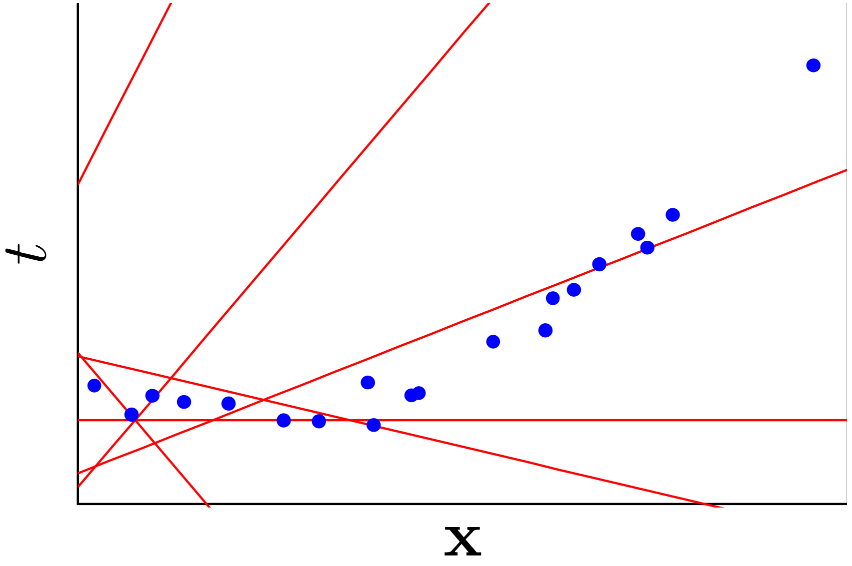
$$L(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}(y(x_n, \mathbf{w}) - t_n)^2$$

  - the sum in $\mathcal{L}$ is also called **residual sum of squares** (RSS) and can be written as the sum of **residual errors**:

$$RSS(\mathbf{w}) = \|\boldsymbol{\epsilon}\|_2^2 = \sum_{i=1}^{N}\epsilon_i^2$$

  - closed-form optimization of $\mathcal{L}$ can be easily obtained





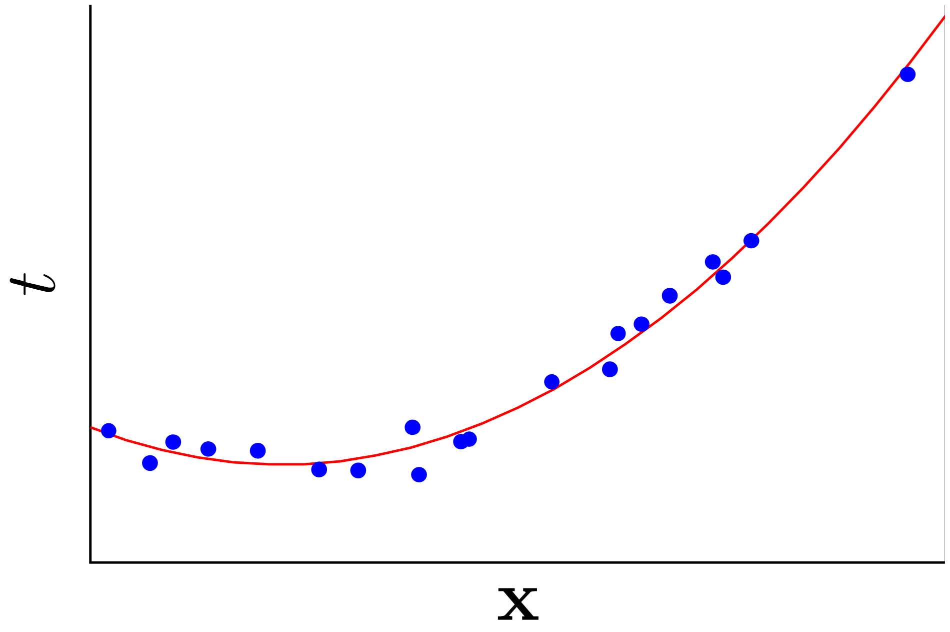
Linear Models and Basis Functions

Linear Models

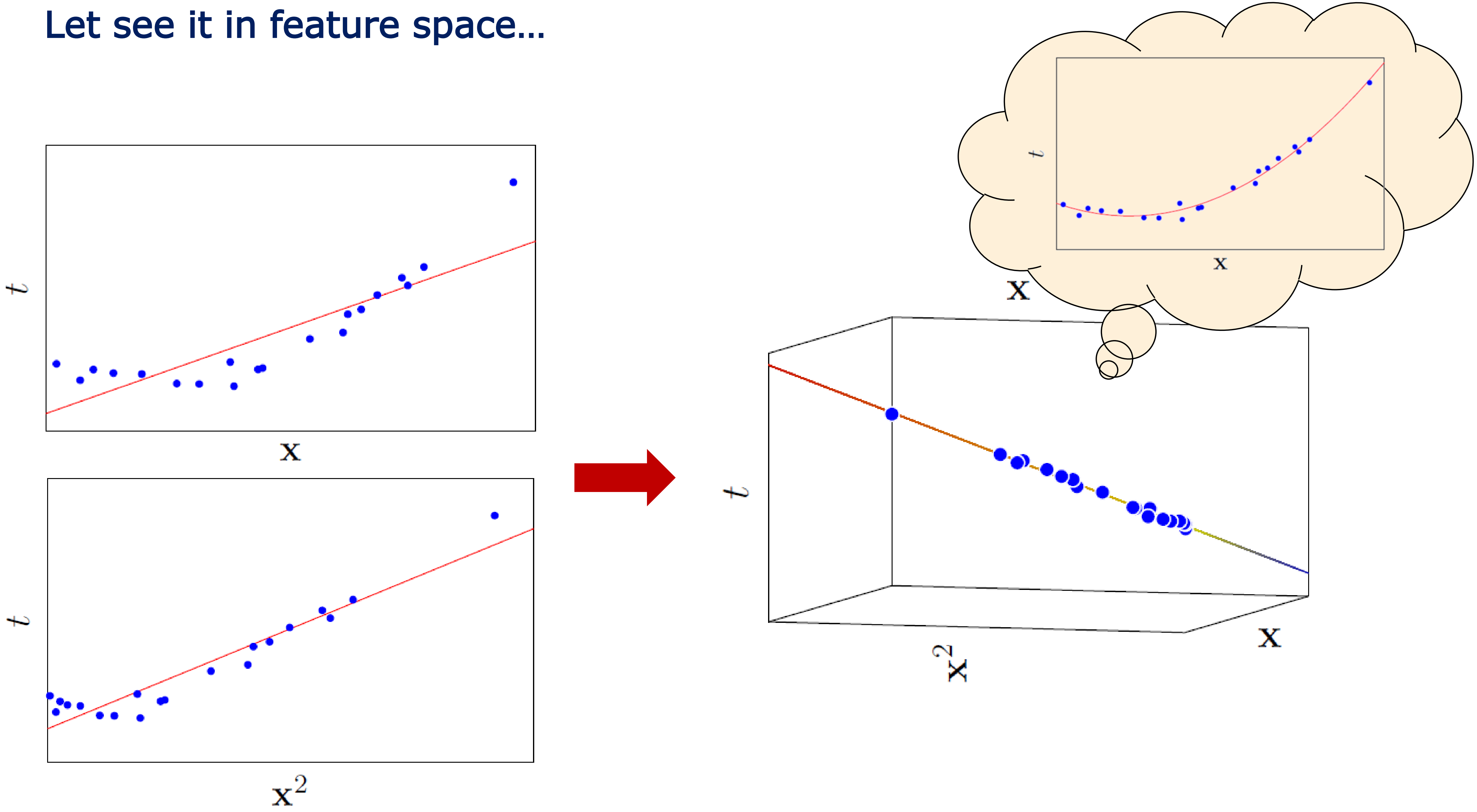
- A linear combination of the input variables is not enough to model data…
- … but we just need a regression model that is **linear in the parameters**
- We can define a model using non-linear **basis functions**:

$$y(\mathbf{x}, \mathbf{w}) = w_0 + \sum_{j=1}^{M-1} w_j \phi_j(\mathbf{x}) = \mathbf{w}^T \boldsymbol{\phi}(\mathbf{x})$$

  - $\boldsymbol{\phi}(\mathbf{x}) = (1, \phi_1(\mathbf{x}), \ldots, \phi_{M-1}(\mathbf{x}))^T$



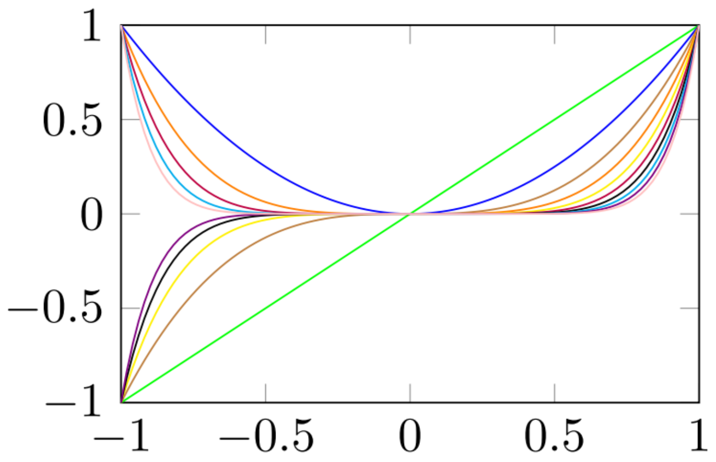
Let see it in feature space…



Basis Functions

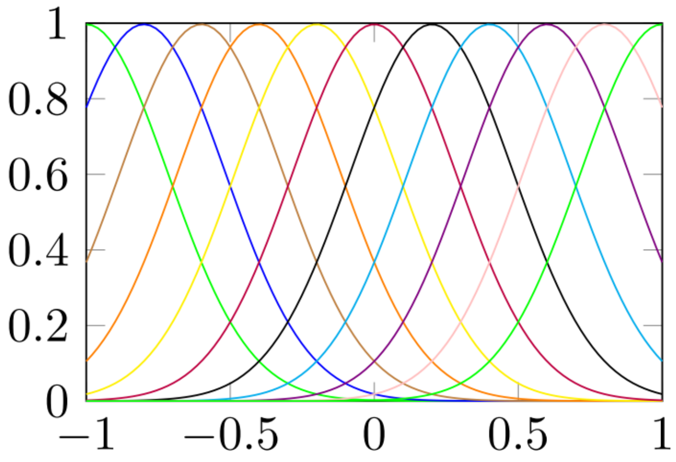
- Some examples of basis function (assuming single-variable input)

Polynomial



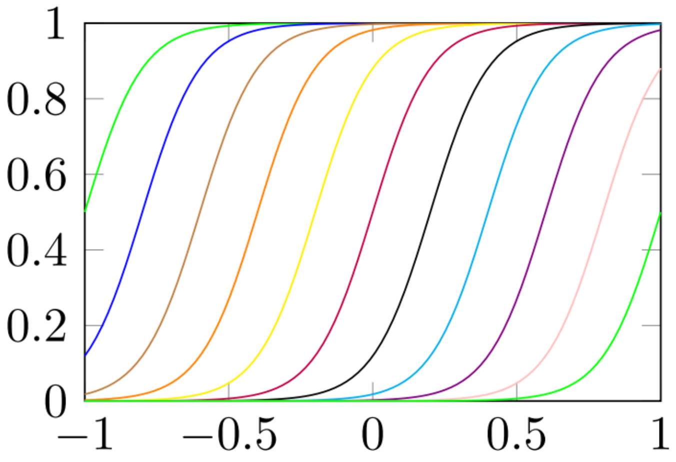
$$\phi_j(x) = x^j$$

Gaussian



$$\phi_j(x) = \exp\left(-\frac{(x-\mu_j)^2}{2\sigma^2}\right)$$

Sigmoidal

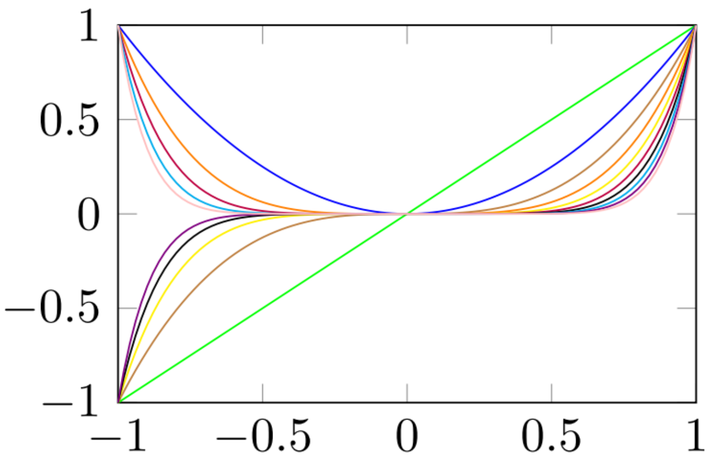


$$\phi_j(x) = \frac{1}{1+\exp\left(\frac{\mu_j - x}{\sigma}\right)}$$

Basis Functions

- Some examples of basis function (assuming single-variable input)

Polynomial



$$\phi_j(x) = x^j$$

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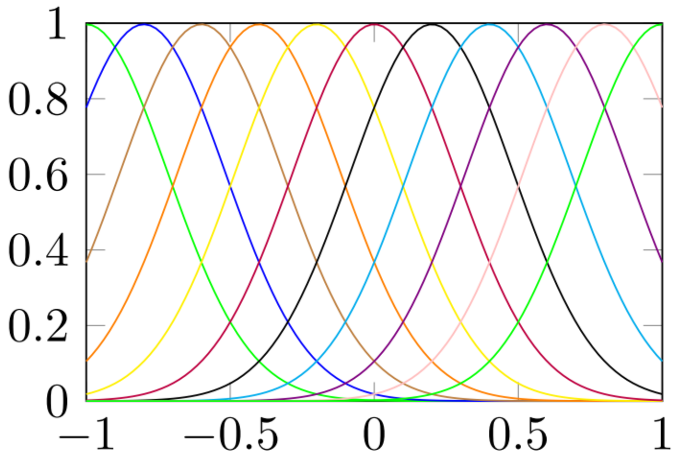

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LOCAL

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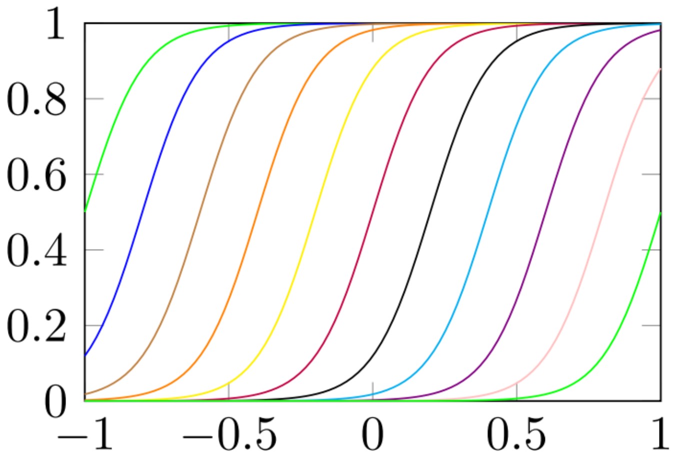

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Least Squares

Ordinary Least Squares

- For linear models, a closed-form optimization of the RSS, known as **least squares**, starting from the matrix form of the loss function:

$$L(\mathbf{w}) = \frac{1}{2}RSS(\mathbf{w}) = \frac{1}{2}(\mathbf{t} - \mathbf{\Phi w})^T (\mathbf{t} - \mathbf{\Phi w})$$

  - where $\mathbf{\Phi} = (\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N))^T$ and $\mathbf{t} = (t_1, \ldots, t_N)^T$

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- We can compute first and second derivative of $\mathcal{L}(w)$ to find the optimal w

$$\frac{\partial L(\mathbf{w})}{\partial \mathbf{w}} = -\mathbf{\Phi}^T (\mathbf{t} - \mathbf{\Phi w}) \qquad \frac{\partial^2 L(\mathbf{w})}{\partial \mathbf{w} \partial \mathbf{w}^T} = \mathbf{\Phi}^T \mathbf{\Phi}$$

$$\Rightarrow \quad \hat{\mathbf{w}}_{OLS} = \left(\mathbf{\Phi}^T \mathbf{\Phi} \right)^{-1} \mathbf{\Phi}^T \mathbf{t}$$

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Assuming $(\boldsymbol{\Phi}^T \boldsymbol{\Phi})$ non singular
Complexity $O(NM^2 + M^3)$

Sequential Learning

- Closed-form optimization (OLS) is not feasible with large dataset
- Instead, a **stochasitcs** (or **sequential**) gradient descent is possible
- **Least Mean Square** (LMS) algorithm:

$$L(\mathbf{x}) = \sum_n L(x_n)$$

$$\Rightarrow \quad \mathbf{w}^{(n+1)} = \mathbf{w}^{(n)} - \alpha^{(n)} \nabla L(x_n)$$

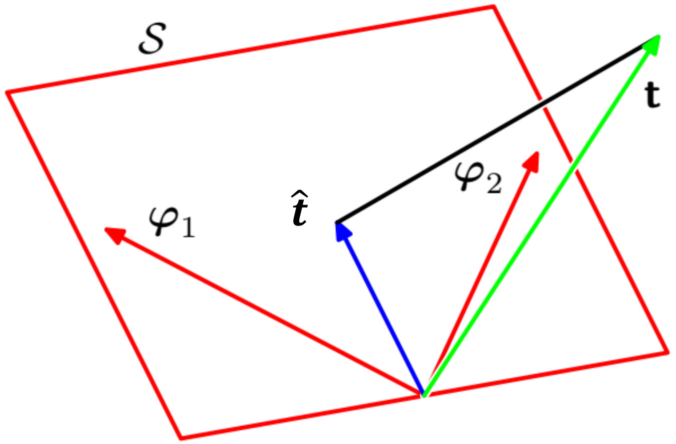
$$\Rightarrow \quad \mathbf{w}^{(n+1)} = \mathbf{w}^{(n)} - \alpha^{(n)} \left({\mathbf{w}^{(n)}}^T \phi(\mathbf{x}_n) - t_n \right) \phi(\mathbf{x}_n)$$

- α is called learning rate and to guarantee convergence:

$$\sum_{n=0}^{\infty} \alpha^{(n)} = +\infty \qquad \sum_{n=0}^{\infty} {\alpha^{(n)}}^2 < +\infty$$

Geometric Interpreation of OLS

- Let $\boldsymbol{t}$ be the N-dimensional target vector
- Let φ_j be the j-th column of matrix $\boldsymbol{\Phi}$
  - $\varphi_1, \dots, \varphi_M$ identify a linear subspace $\mathcal{S}$
- Let $\hat{\boldsymbol{t}}$ be the N-dimensional vector computed as $\boldsymbol{\Phi w}$
  - $\hat{\boldsymbol{t}}$ is a linear combination of φ_j and lies in $\mathcal{S}$
- OLS finds $\hat{\boldsymbol{t}}$ minimizing the SSE with respect to $\boldsymbol{t}$
  - $\hat{\boldsymbol{t}}$ represents the projection of $\boldsymbol{t}$ onto the subspace $\mathcal{S}$



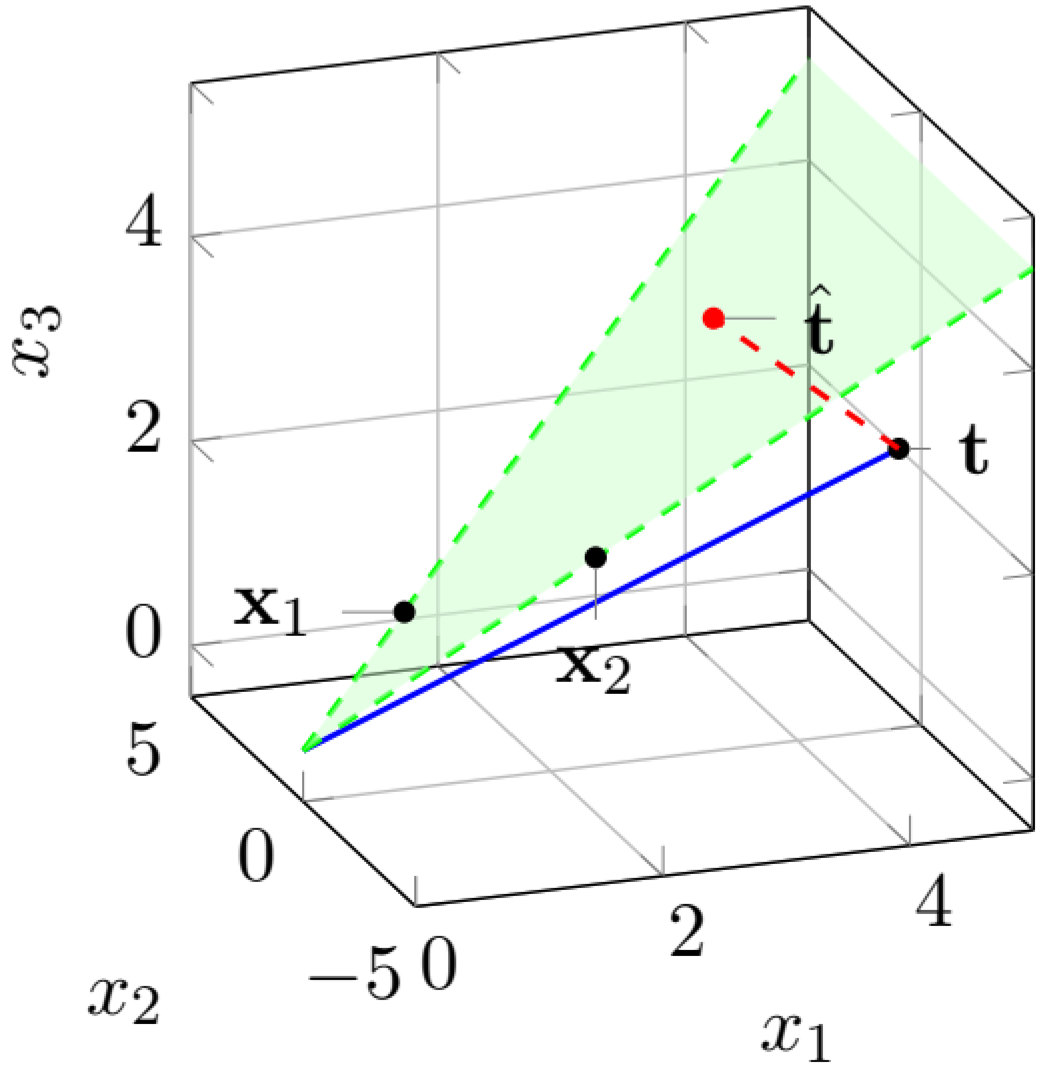
$$\hat{\mathbf{t}} = \mathbf{\Phi}\hat{\mathbf{w}} = \underbrace{\mathbf{\Phi}\left(\mathbf{\Phi}^T\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^T}_{\text{Hat Matrix (H)}}\mathbf{t}$$

Geometric Interpreation of OLS: an example

- Let N=3 and M=2

$$\boldsymbol{\Phi} = \mathbf{X} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \\ 1 & 2 \end{pmatrix}$$

$$\mathbf{t} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} \qquad \hat{\mathbf{t}} = \begin{pmatrix} 3.5 \\ 1 \\ 3.5 \end{pmatrix}$$



Multiple Outputs

- What happens if our regression problem has multiple outputs, i.e., **t** is not scalar
- It is possible to solve independently a regression problem for each problem
- Yet, it is possible to use the same set of basis functions:

$$\hat{\mathbf{W}} = \left(\mathbf{\Phi}^T \mathbf{\Phi}\right)^{-1} \mathbf{\Phi}^T \mathbf{T}$$

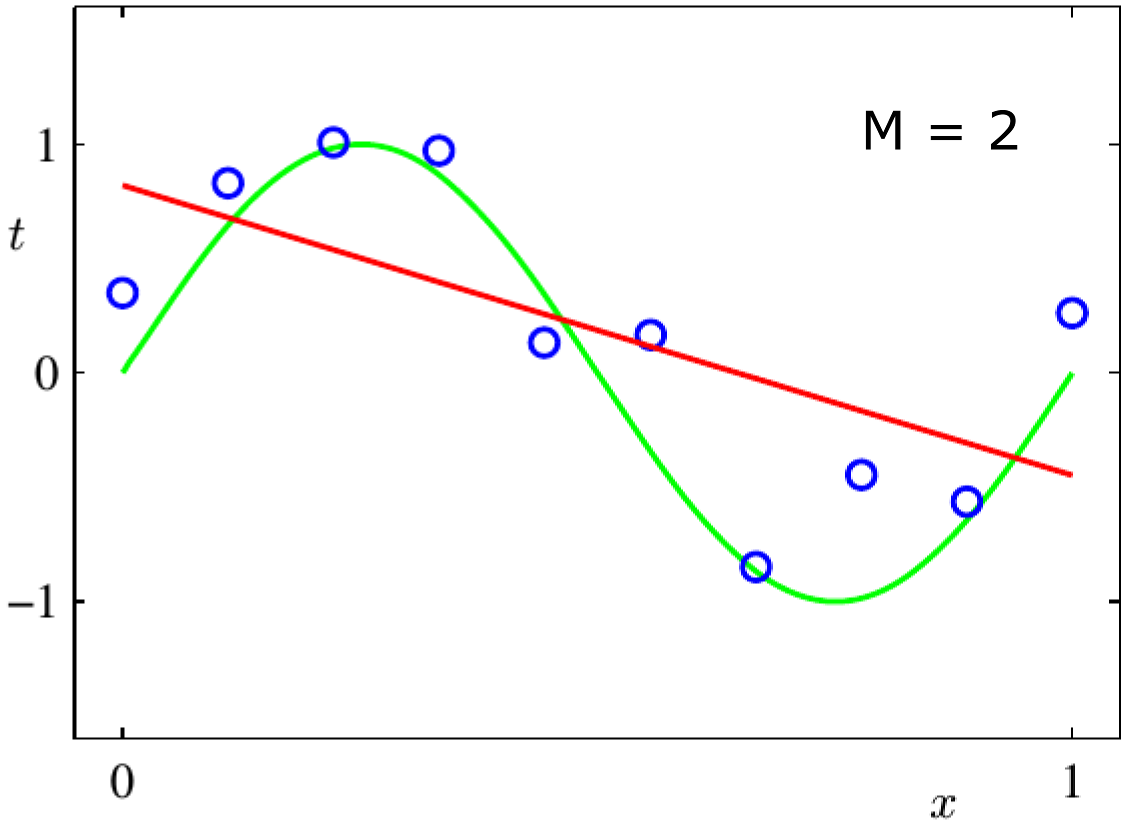
  - Where each column of matrix **T** and $\widehat{W}$ are respectively the target vector of each and the weight vector for each output
- The solution above can be easily **decoupled** for each output **k**:

$$\hat{\mathbf{w}}_k = \left(\mathbf{\Phi}^T \mathbf{\Phi}\right)^{-1} \mathbf{\Phi}^T \mathbf{t}_k$$

  - as a benefit $(\boldsymbol{\Phi}^T \boldsymbol{\Phi})^{-1}$ can be computed only **once**

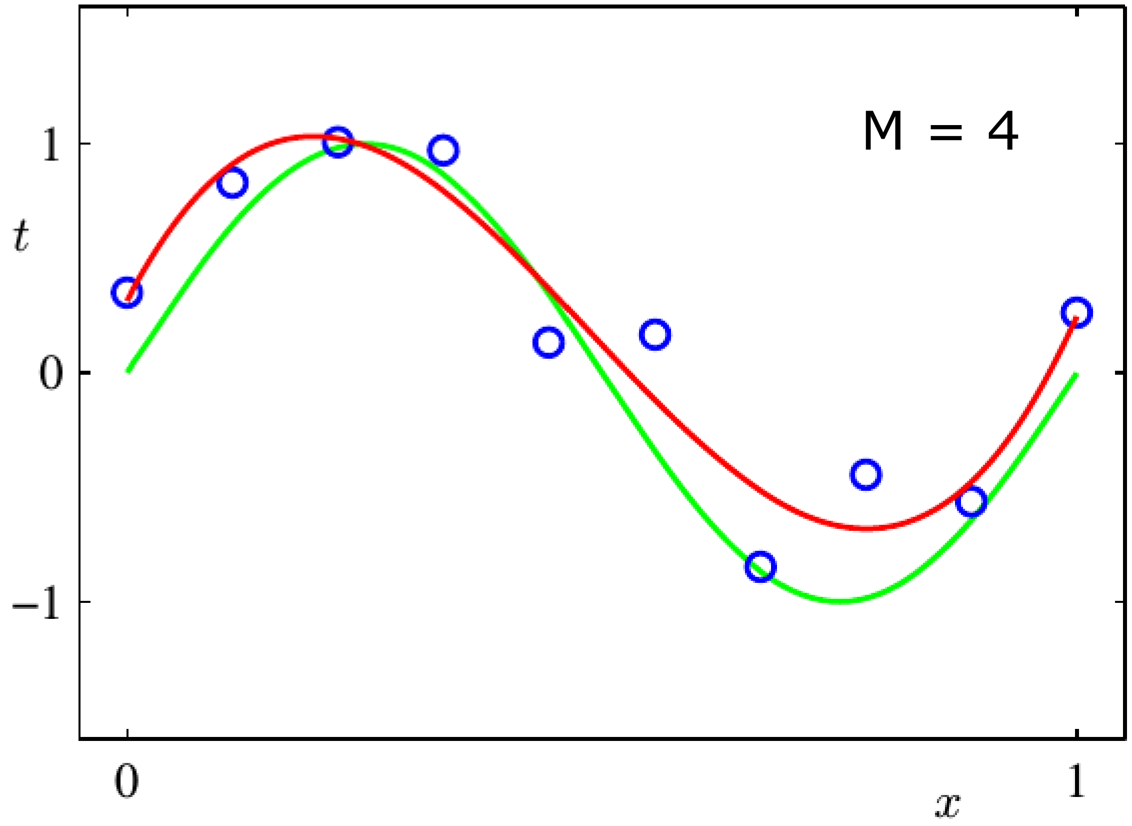
Regularization

How do we design the linear models? An example



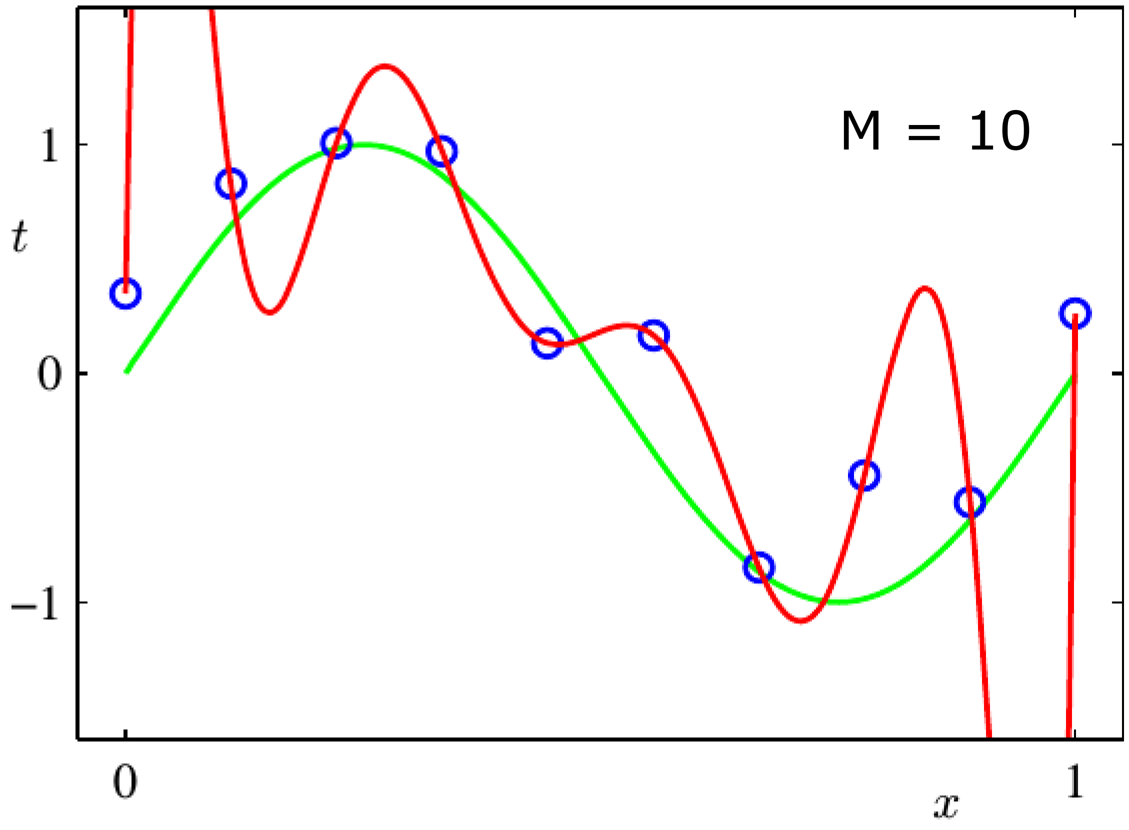
1-order polynomial model

How do we design the linear models? An example



3-order polynomial model

How do we design the linear models? An example



9-order polynomial model

What is regularization?

- What happens to model parameters when complexity increases?

	$M=1$	$M=2$	$M=3$	$M=10$
$\hat{w}_0$	0.19	0.82	0.31	0.35
$\hat{w}_1$		-1.27	7.99	232.37
$\hat{w}_2$			-25.43	-5321.83
$\hat{w}_3$				48568.31
$\hat{w}_4$				-231639.30
$\hat{w}_5$				640042.26
$\hat{w}_6$				-1061800.52
$\hat{w}_7$				1042400.18
$\hat{w}_8$				-557682.99
$\hat{w}_9$				125201.43

Regularization

- How do we extend the loss function?

$$L(\mathbf{w}) = L_D(\mathbf{w}) + \lambda L_W(\mathbf{w})$$

  - $\mathcal{L}_D(w)$ is the usual loss function (e.g., RSS)
  - $\mathcal{L}_w(w)$ accounts for model complexity
  - λ is the **regularization** coefficient

- How do we design $\mathcal{L}_w(w)$?
  - Ridge Regression
  - Lasso

Ridge Regression

- In **ridge regression**:

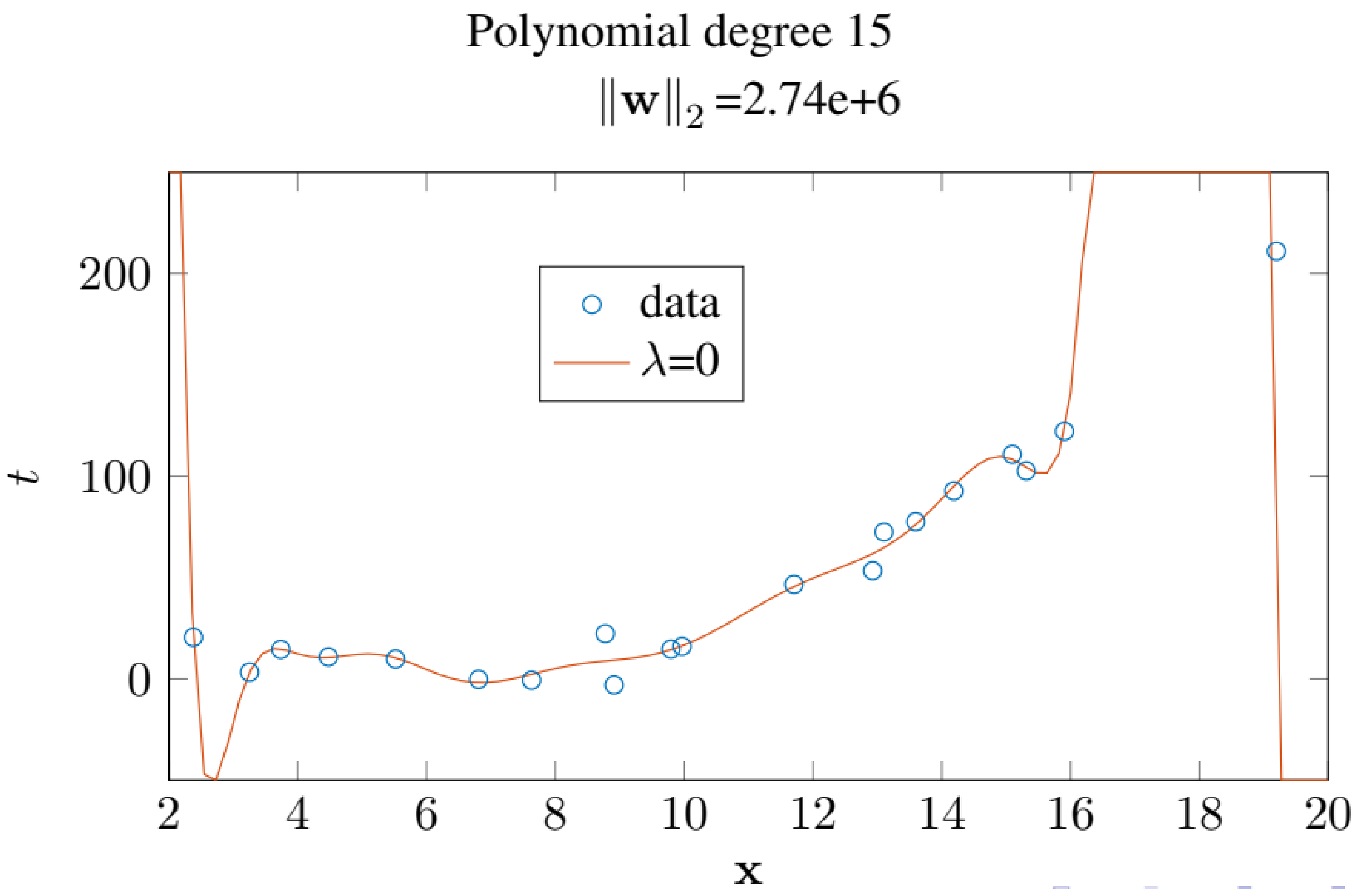
$$L_W(\mathbf{w}) = \frac{1}{2}\mathbf{w}^T\mathbf{w} = \frac{1}{2}\|\mathbf{w}\|_2^2$$

$$\Rightarrow \quad L(\mathbf{w}) = \frac{1}{2}\sum_{i=1}^{N}\left(t_i - \mathbf{w}^T\phi(\mathbf{x}_i)\right)^2 + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$

- The loss function is still quadratic with respect to w and closed-form optimization is still possible:

$$\hat{\mathbf{w}}_{ridge} = \left(\lambda\mathbf{I} + \mathbf{\Phi}^T\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^T\mathbf{t}$$

Ridge Regression: quadratic example



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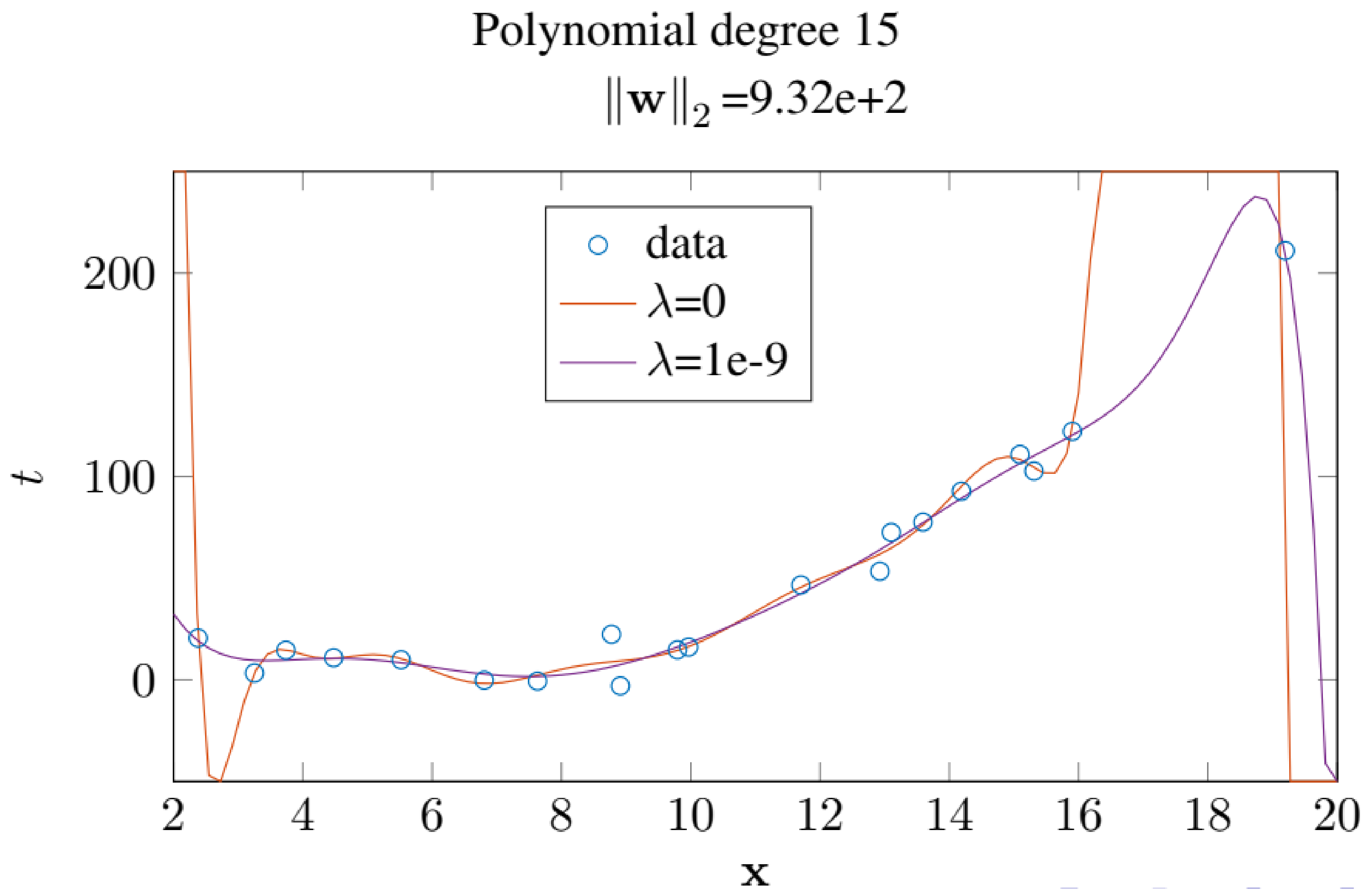

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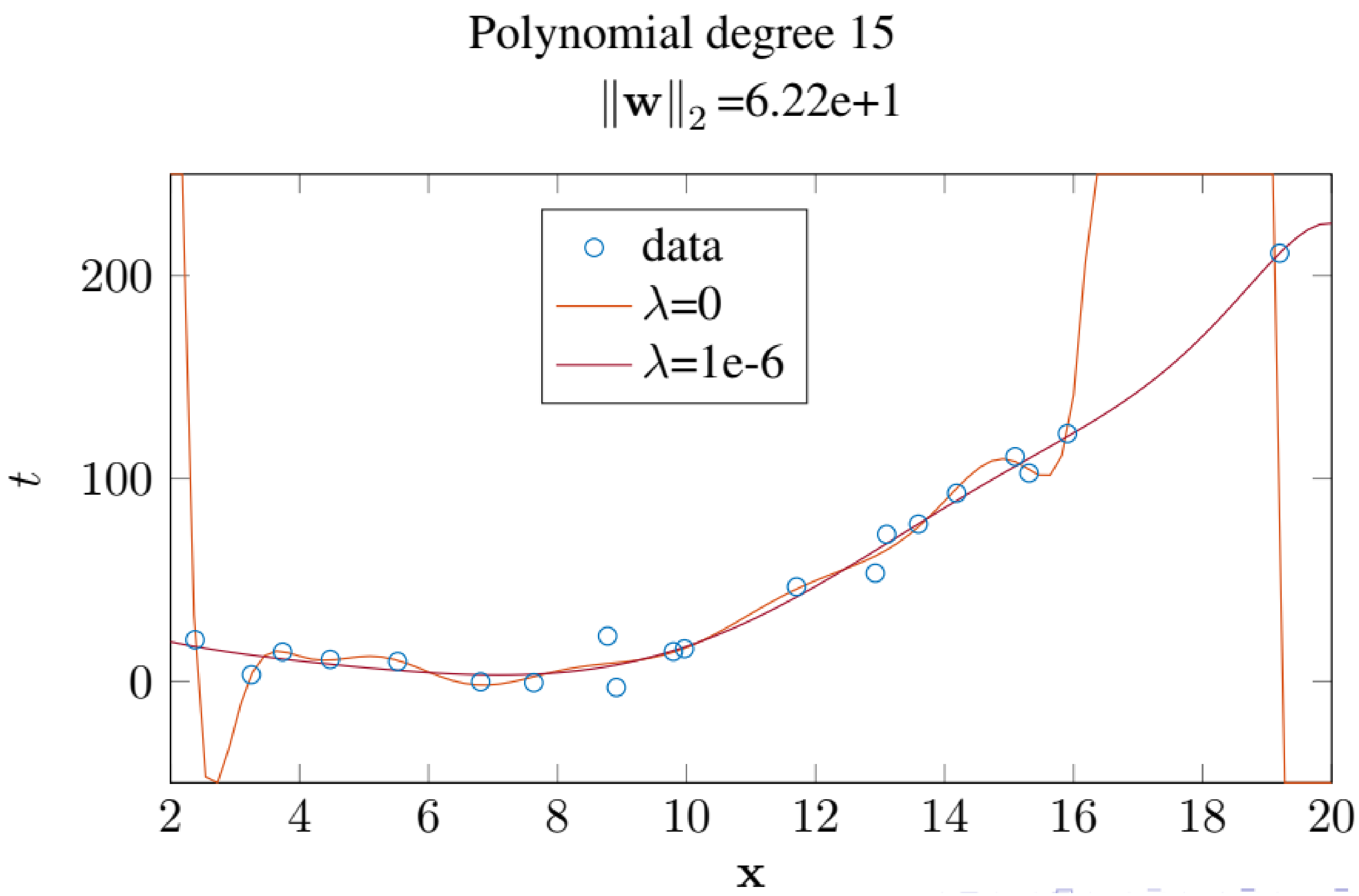

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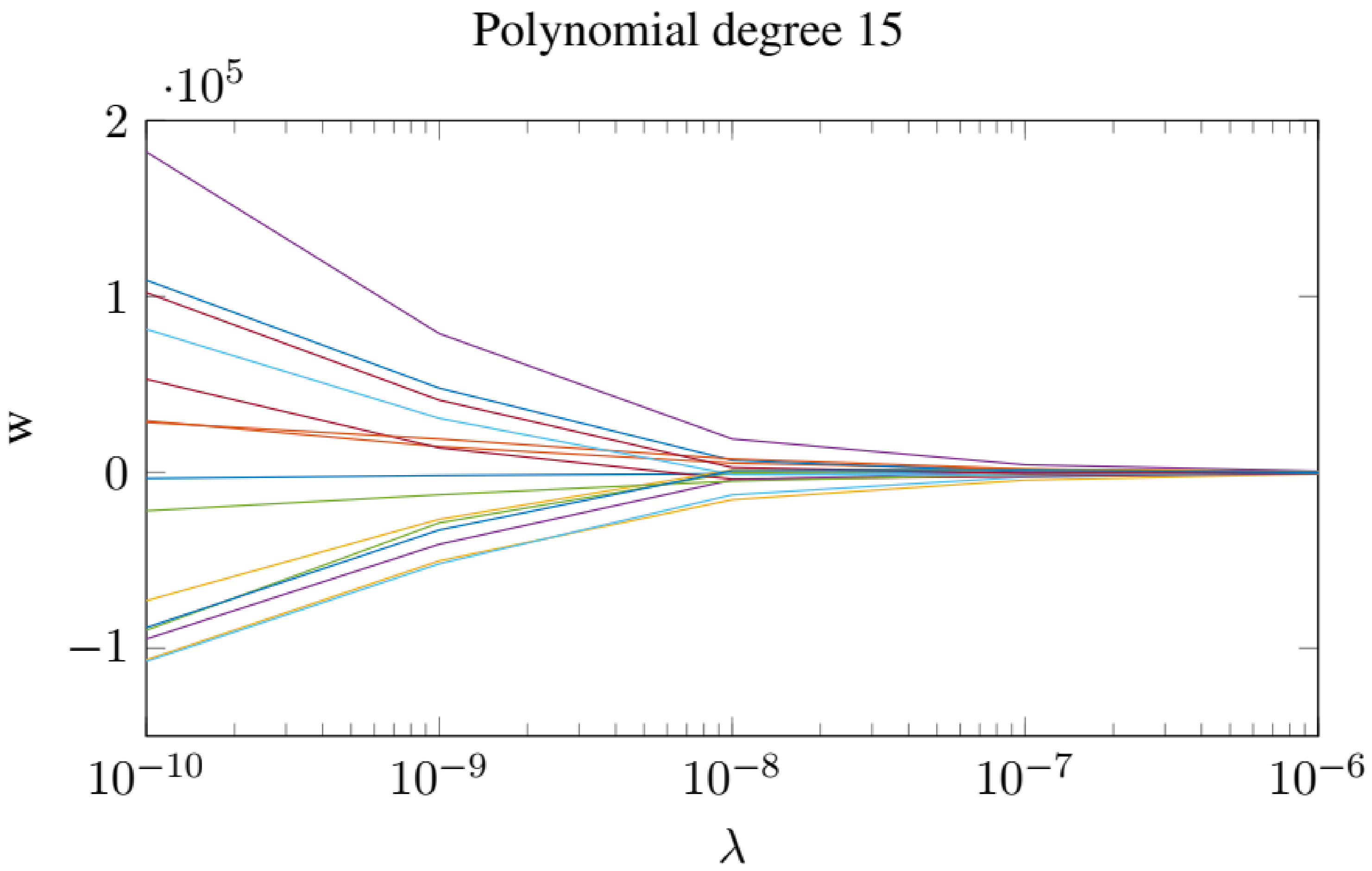


Lasso

- Another common regularization method is **lasso**:

$$L_W(\mathbf{w}) = \frac{1}{2}\|\mathbf{w}\|_1 = \frac{1}{2}\sum_{j=0}^{M-1}|w_j|$$

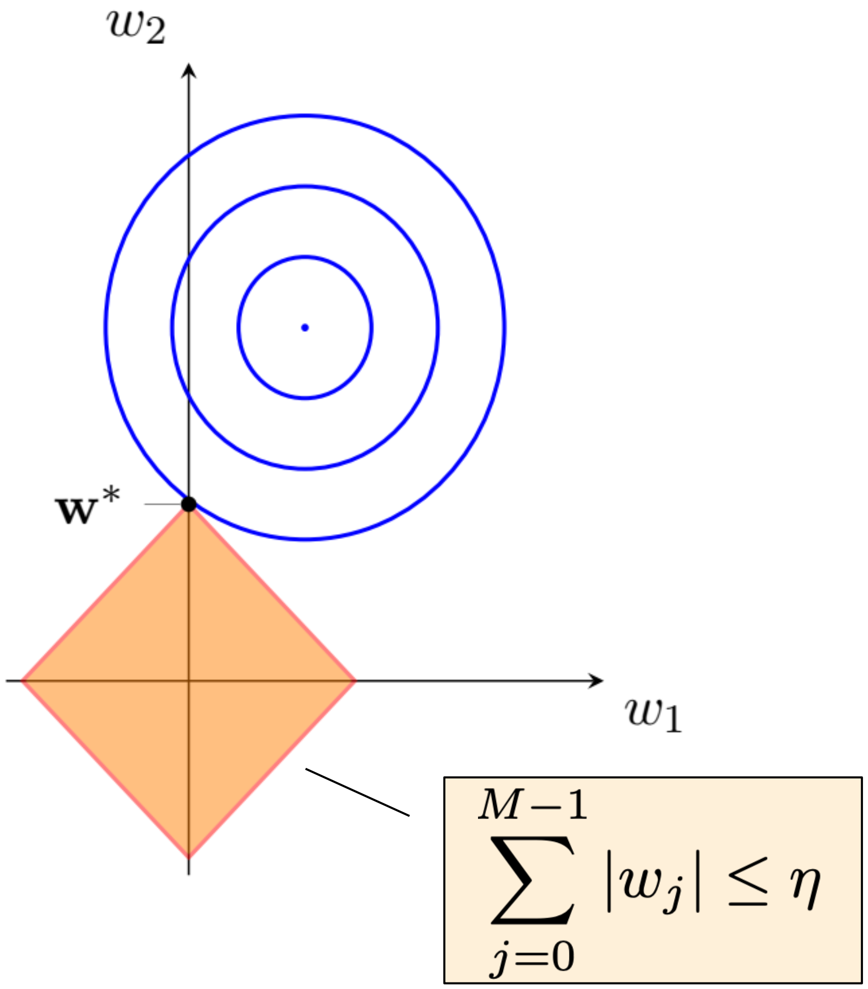
$$\Rightarrow \quad L(\mathbf{w}) = \frac{1}{2}\sum_{i=1}^{N}\left(t_i - \mathbf{w}^T\phi(\mathbf{x}_i)\right)^2 + \frac{\lambda}{2}\|\mathbf{w}\|_1$$

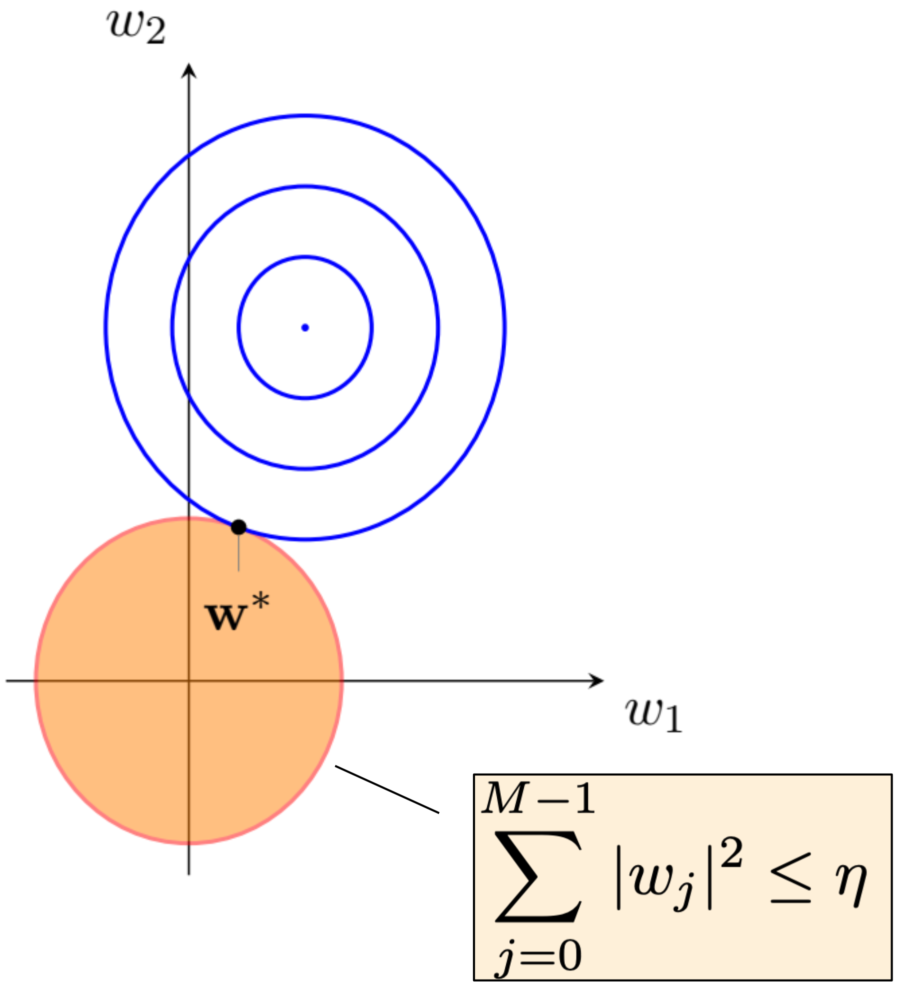
- In this case, closed-form optimization is not possible
- Nevertheless, lasso typically leads to **sparse** regression models: when regularization coefficient (λ) is large enough, some **components** of $\hat{\boldsymbol{w}}$ become **equal to zero**
- We can see regularization equivalent to minimizing $\mathcal{L}_D(w)$ subject to constraint:

$$\sum_{j=0}^{M-1}|w_j| \leq \eta$$

Lasso vs Ridge Regression

- Why lasso leads to **sparse** model? Let's visualize constraint of lasso and ridge





Least Squares and Maximum Likelihood

Maximum Likelihood (ML)

- We can deal with regression in a **probabilistic way**
  - We define a probabilistic model that maps inputs (x) to outputs (t)
  - Such probabilisic model, $f(x, w)$, will include some **unknown parameters** (w)
  - Then, we model the **likelihood**, i.e., the probability that observed data $\mathcal{D}$ is generated by a given set of **parameters** (w):

$$p(\mathcal{D}|\mathbf{w})$$

  - Finally, we can estimate **parameters** (w) by maximizing the likelihood:

$$\mathbf{w}_{ML} = \arg\max_{\mathbf{w}} p(\mathcal{D}|\mathbf{w})$$

Maximum Likelihood (ML) for linear regression

- Our probabilistic model can be defined as:

$$t = y(\mathbf{x}, \mathbf{w}) + \epsilon = \mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}) + \epsilon$$

  - we assumed a linear model for $y(x, w)$
  - we assumed $\epsilon \sim \mathcal{N}(0, \sigma^2)$

- Given a dataset $\mathcal{D}$ of N samples with inputs **X**={**x**$_1$, ..., **x**$_N$} and ouptuts **t**={t$_1$,...,t$_N$}^{T} :

$$p(\mathcal{D}|\mathbf{w}) = p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \sigma^2) = \prod_{n=1}^{N} \mathcal{N}\left(t_n|\mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}_n), \sigma^2\right)$$

Maximum Likelihood (ML) for linear regression (cont.)

- To find $\mathbf{w}_{ML}$ it is convenient to maximize the log-likelihood:

$$\mathcal{N}\left(t_n|\mathbf{w}^T\boldsymbol{\phi}(\mathbf{x}_n),\sigma^2\right) = \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left\{-\frac{1}{2\sigma^2}(t_n - \mathbf{w}^T\boldsymbol{\phi}(\mathbf{x}_n))^2\right\}$$

$$\ell(\mathbf{w}) = \ln p(\mathbf{t}|\mathbf{X},\mathbf{w},\sigma^2) = \sum_{n=1}^{N} \ln p(t_n|\mathbf{x}_n,\mathbf{w},\sigma^2) = -\frac{N}{2}\ln\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}RSS(\mathbf{w})$$

- To solve the optimization problem, we equal the gradient to zero:

$$\nabla\ell(\mathbf{w}) = \sum_{n=1}^{N} t_n\boldsymbol{\phi}(\mathbf{x}_n)^T - \mathbf{w}^T\left(\sum_{n=1}^{N}\boldsymbol{\phi}(\mathbf{x}_n)\boldsymbol{\phi}(\mathbf{x}_n)^T\right) = 0$$

$$\Rightarrow \quad \mathbf{w}_{ML} = \left(\boldsymbol{\Phi}^T\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^T\mathbf{t}$$

OLS

Bayesian Linear Regression

Bayesian approach

1. We formulate our knowledge about the world in **a probabilistic way**:
   i. We define the **model** that expresses our knowledge qualitatively
   ii. Our model will have some **unknown parameters**
   iii. We capture our **assumptions** about unknown parameters with the **prior distribution** over those parameters before seeing the data
2. We observe the **data**
3. We compute the posterior probability distribution for the parameters, given observed data

$$p(parameters|data) = \frac{p(data|parameters)p(parameters)}{p(data)}$$

4. We use the posterior distribution to:
   a. Make predictions by averaging over the posterior distribution
   b. Examine/Account for uncertainty in the parameter values
   c. Make decisions by minimizing expected posterior loss

Parameters Posterior Distribution

- The **posterior distribution** for the model parameters can be found by combining the prior with the likelihood for the parameters given data:

$$p(\mathbf{w}|\mathcal{D}) = \frac{p(\mathcal{D}|\mathbf{w})p(\mathbf{w})}{p(\mathcal{D})}$$

Parameters Posterior Distribution

- The **posterior distribution** for the model parameters can be found by combining the prior with the likelihood for the parameters given data:

prior

$$p(\mathbf{w}|\mathcal{D}) = \frac{p(\mathcal{D}|\mathbf{w})p(\mathbf{w})}{p(\mathcal{D})}$$

  - $p(\mathbf{w})$ is the **prior** probability over the parameter – what we know before observing the data

Parameters Posterior Distribution

- The **posterior distribution** for the model parameters can be found by combining the prior with the likelihood for the parameters given data:

likelihood

$$p(\mathbf{w}|\mathcal{D}) = \frac{p(\mathcal{D}|\mathbf{w})p(\mathbf{w})}{p(\mathcal{D})}$$

  - $p(D|\mathbf{w})$ is the **likelihood –** the probability of observing the data (D) given some value of the parameters ($\mathbf{w}$)

Parameters Posterior Distribution

- The **posterior distribution** for the model parameters can be found by combining the prior with the likelihood for the parameters given data:

$$p(\mathbf{w}|\mathcal{D}) = \frac{p(\mathcal{D}|\mathbf{w})p(\mathbf{w})}{p(\mathcal{D})}$$

normalizing constant

$$p(\mathcal{D}) = \int p(\mathcal{D}|\mathbf{w})p(\mathbf{w})\mathrm{d}\mathbf{w}$$

  - $P(D)$ is the **marginal likelihood** and acts as **normalizing constant**

Parameters Posterior Distribution

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$$p(\mathbf{w}|\mathcal{D}) = \frac{p(\mathcal{D}|\mathbf{w})p(\mathbf{w})}{p(\mathcal{D})}$$

posterior

- $p(\mathbf{w}|D)$ is the **posterior probability** of parameters w given training data
- the most probable value of **w** given the data will be the mode of the posterior, also known as **maximum a posteriori** (MAP).

Bayesian Linear Regression

- How to model the **prior**? Assuming a Gaussian likelihood, a **conjugate prior** is the most convenient choice:

$$p(\mathbf{w}) = \mathcal{N}\left(\mathbf{w}|\mathbf{w}_0, \mathbf{S}_0\right)$$

Bayesian Linear Regression

- How to model the **prior**? Assuming a Gaussian likelihood, a **conjugate prior** is the most convenient choice:

$$p(\mathbf{w}) = \mathcal{N}\left(\mathbf{w}|\mathbf{w}_0, \mathbf{S}_0\right)$$

- As a result, the **posterior** is still Gaussian:

$$p(\mathbf{w}|\mathbf{t}, \mathbf{\Phi}, \sigma^2) \propto \mathcal{N}\left(\mathbf{w}|\mathbf{w}_0, \mathbf{S}_0\right)\mathcal{N}\left(\mathbf{t}|\mathbf{\Phi}\mathbf{w}, \sigma^2\mathbf{I}\right)$$

Bayesian Linear Regression

- How to model the **prior**? Assuming a Gaussian likelihood, a **conjugate prior** is the most convenient choice:

$$p(\mathbf{w}) = \mathcal{N}\left(\mathbf{w}|\mathbf{w}_0, \mathbf{S}_0\right)$$

- As a result, the **posterior** is still Gaussian:

$$p(\mathbf{w}|\mathbf{t}, \mathbf{\Phi}, \sigma^2) \propto \mathcal{N}\left(\mathbf{w}|\mathbf{w}_0, \mathbf{S}_0\right) \mathcal{N}\left(\mathbf{t}|\mathbf{\Phi}\mathbf{w}, \sigma^2\mathbf{I}\right)$$

$$\Rightarrow \left[\begin{array}{l} p(\mathbf{w}|\mathbf{t}, \mathbf{\Phi}, \sigma^2) = \mathcal{N}\left(\mathbf{w}|\mathbf{w}_N, \mathbf{S}_N\right) \\ \mathbf{w}_N = \mathbf{S}_N\left(\mathbf{S}_0^{-1}\mathbf{w}_0 + \dfrac{\mathbf{\Phi}^T\mathbf{t}}{\sigma^2}\right) \\ \mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \dfrac{\mathbf{\Phi}^T\mathbf{\Phi}}{\sigma^2} \end{array}\right.$$

Bayesian Linear Regression an Maximum Likelihood

- Which parameters?
  - The maximum a-posteriori (**MAP**) is the obvious choice
  - When posterior is gaussian the **MAP** is equal to the mean
- When prior is **infinitely broad** MAP is equal to ML solution:

$$\lim_{\mathbf{S}_0 \to \infty} \mathbf{w}_N = \left(\mathbf{\Phi}^T \mathbf{\Phi} \right)^{-1} \mathbf{\Phi}^T \mathbf{t}$$

$\mathbf{w}_{ML}$

$$\lim_{\mathbf{S}_0 \to \infty} \mathbf{S}_N^{-1} = \frac{\mathbf{\Phi}^T \mathbf{\Phi}}{\sigma^2}$$

$$\hat{\sigma}^2 = \frac{1}{N-M} \sum_{n=1}^{N} \left(t_n - \hat{\mathbf{w}}^T \phi(\mathbf{x}_n) \right)^2$$

  - The ML estimate of **w** has the **smallest variance** among linear unbiased estimates and the **lowest MSE** among linear **unbiased** estimates (Gauss-Markov).

Bayesian Linear Regression and Regularization

- What about regularization?
  - When $\mathbf{w}_0$=0 and $\mathbf{S}_0=\tau^2\mathbf{I}$, we have:

$$\ln p(\mathbf{w}|\mathbf{t}) = -\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(t_i - \mathbf{w}^T\phi(\mathbf{x}_i)\right)^2 - \frac{1}{2\tau^2}\|\mathbf{w}\|_2^2$$

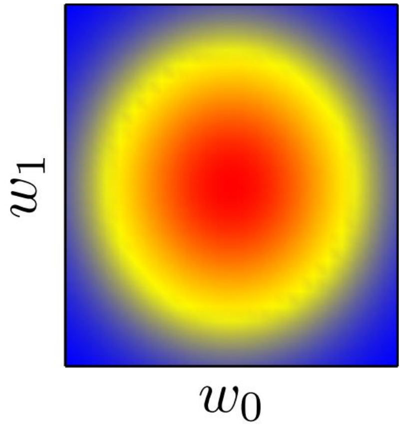
  - In this case, **MAP** ($\mathbf{w}_N$) is equivalent to the solution of ridge regression ($\hat{\mathbf{w}}_{ridge}$) with $\lambda = \frac{\sigma^2}{\tau^2}$

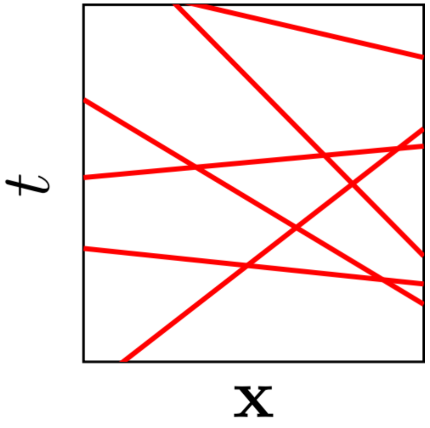
Bayesian Linear Regression: sequential learning

- How to exploit the Bayesian approach for sequential learning?
  - We compute **posterior** with initial data
  - When additional data is available, the **posterior** becomes the **prior**

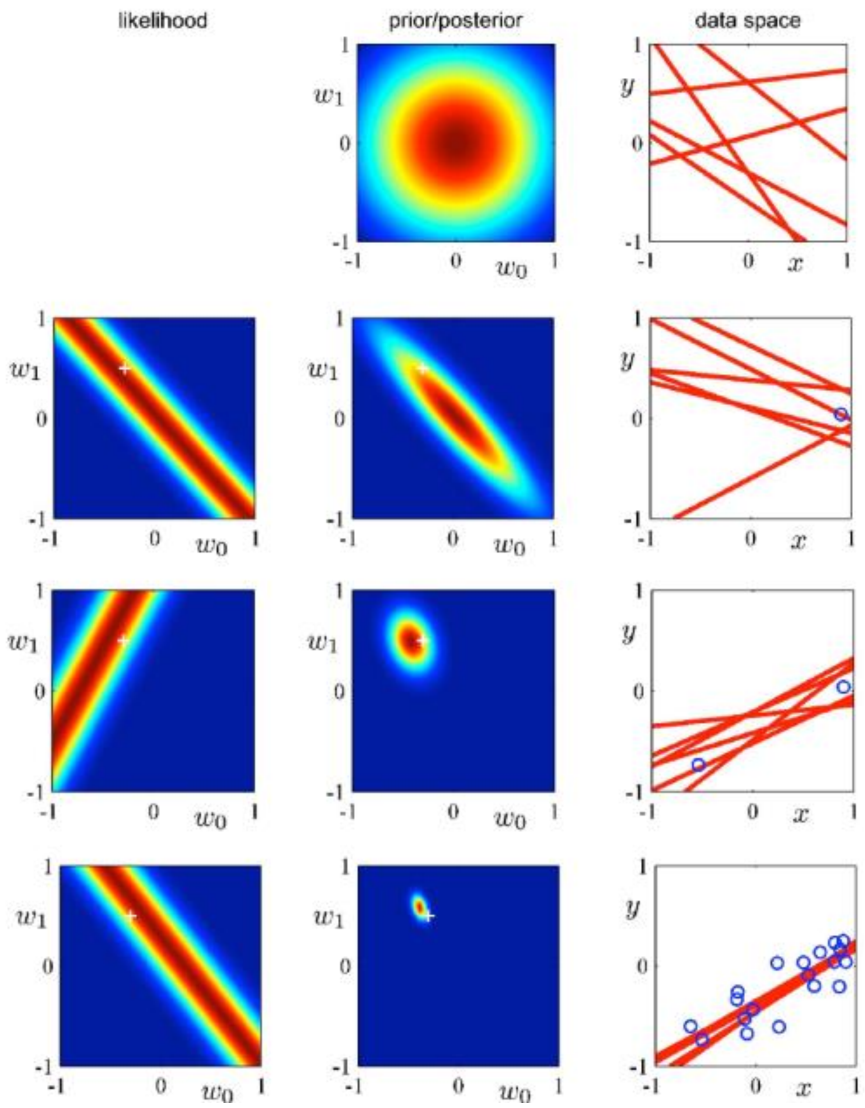
Bayesian Linear Regression: an example

- How to exploit the Bayesian approach for sequential learning?
  - We compute **posterior** with initial data
  - When additional data is available, the **posterior** becomes the **prior**
- Let see an example
  - Let assume data is generated as $t(x) = -0.3 + 0.5x + \varepsilon$
  - Let assume that $x \sim U(-1,1)$ and $\varepsilon \sim \mathcal{N}(0,0.04)$
  - Let use as model: $y(x, \mathbf{w}) = w_0 + w_1 x$
  - Let assume as prior: $p(\mathbf{w}) = \mathcal{N}(\mathbf{w_0}, \tau^2 \mathbf{I})$ with $\tau^2 = 0.5$ and $\mathbf{w_0} = [0,0]^T$





Bayesian Linear Regression: an example (2)



Step 0: 0 samples observed

Step 1: 1 samples observed

Step 2: 2 samples observed

Step 20: 20 samples observed

Predictive Distribution for Bayesian Regression

- With our assumptions, we can compute the **predictive distribution:**

$$p(t|\mathbf{x},\mathcal{D},\sigma^2) = \int p(t|\mathbf{x},\mathbf{w},\sigma^2)p(\mathbf{w}|\mathbf{w}_N,\mathbf{S}_N)\mathrm{d}\mathbf{w} = \int \mathcal{N}\left(t|\mathbf{w}^T\boldsymbol{\phi}(\mathbf{x}),\sigma^2\right)\mathcal{N}\left(\mathbf{w}|\mathbf{w}_N,\mathbf{S}_N\right)\mathrm{d}\mathbf{w}$$

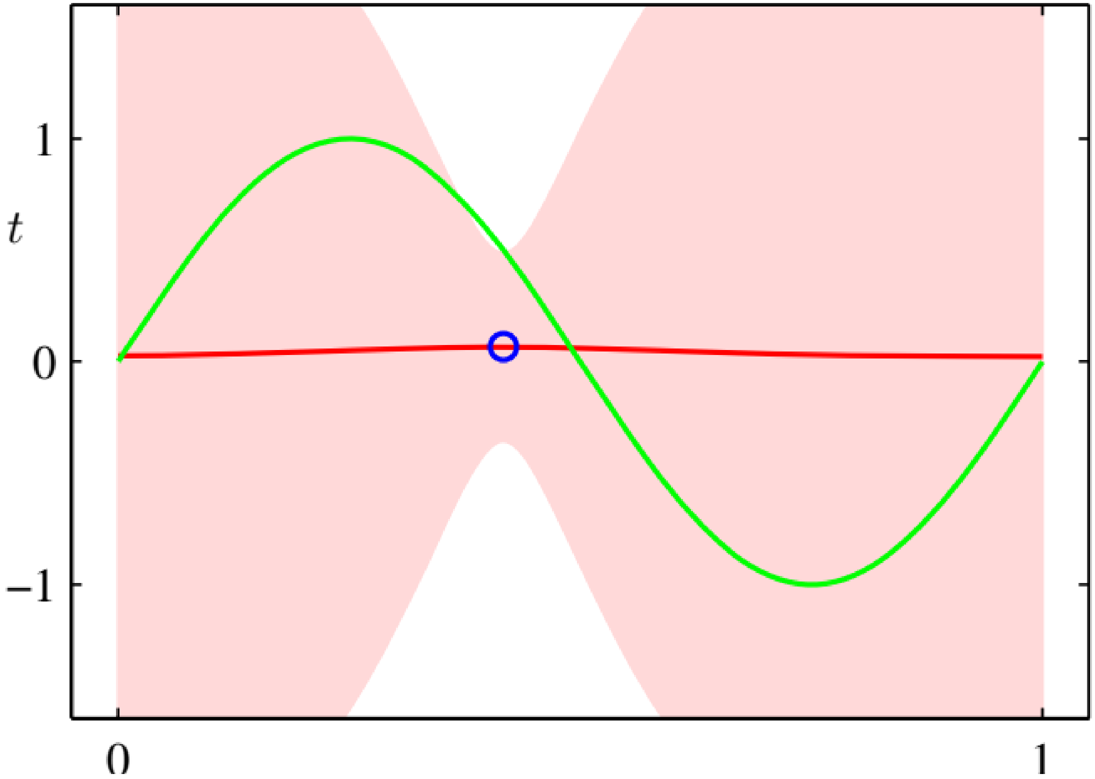
$$p(t|\mathbf{x},\mathcal{D},\sigma^2) = \mathcal{N}\left(t|\mathbf{w}_N{}^T\boldsymbol{\phi}(\mathbf{x}),\sigma_N^2(\mathbf{x})\right)$$

$$\Rightarrow \quad \sigma_N^2(\mathbf{x}) = \underbrace{\sigma^2}_{\text{data}} + \underbrace{\boldsymbol{\phi}(\mathbf{x})^T\mathbf{S}_N\boldsymbol{\phi}(\mathbf{x})}_{\text{parameters}}$$

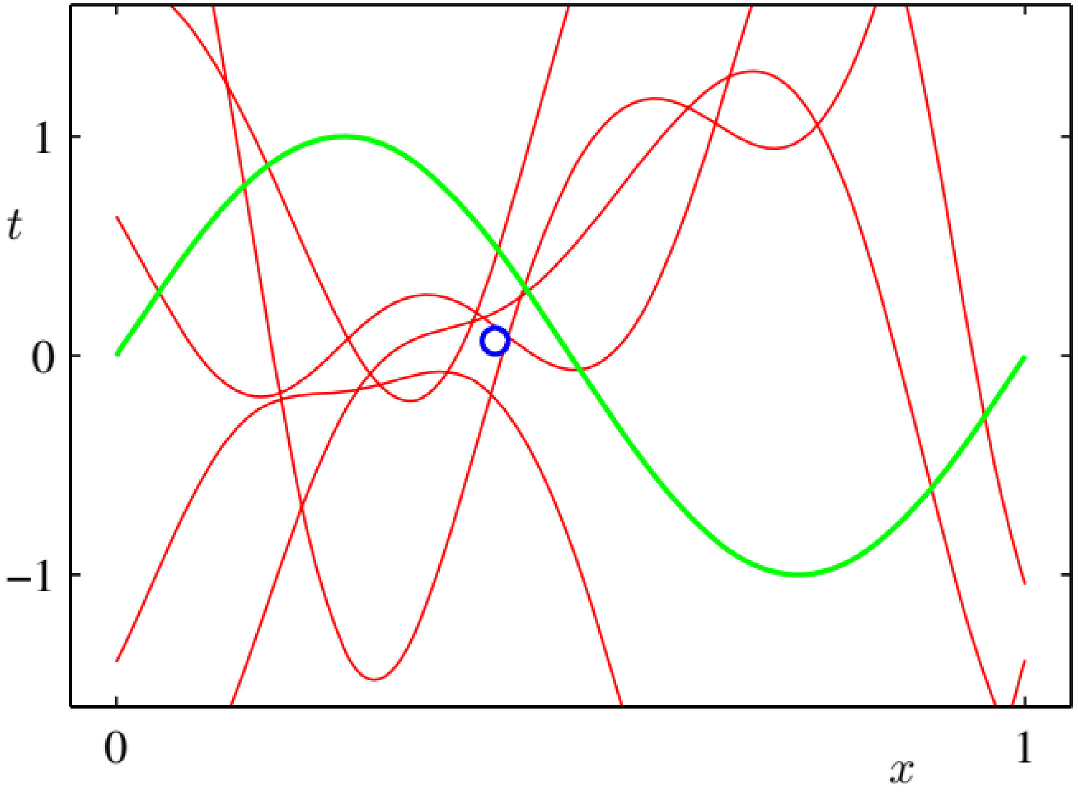
  - when $N \to \infty$ the uncertainty associated to parameters (second term) goes to zero and the variance of predictive distribution depends only on variance of data (σ^2)

Predictive Distribution: an example

- Approximating a sinusoidal dataset with a linear model w/ 9 Gaussian basis functions: 1 sample observed



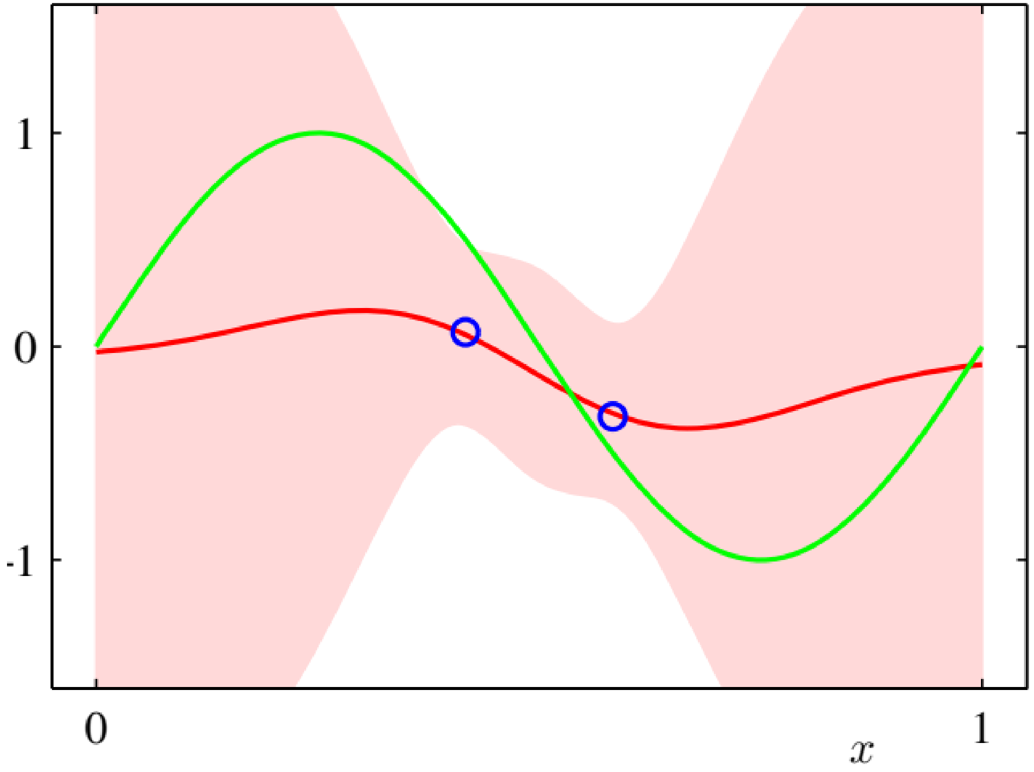
Predictive Distribution



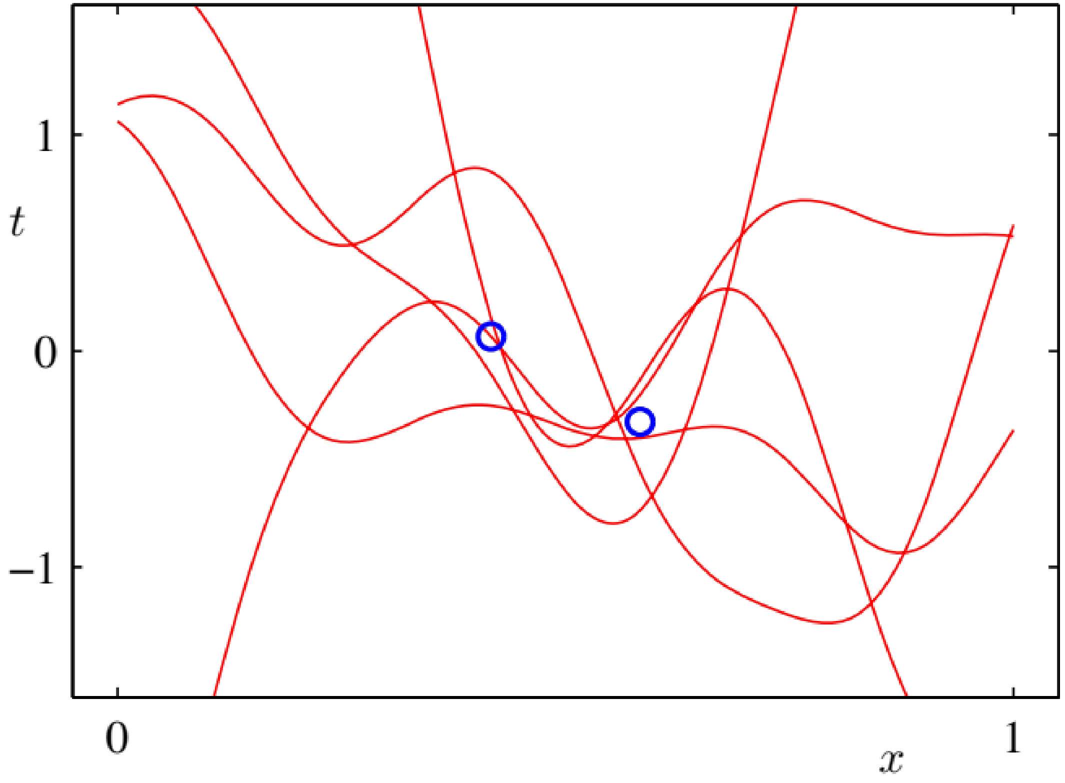
Sampled Models

Predictive Distribution: an example

- Approximating a sinusoidal dataset with a linear model w/ 9 Gaussian basis functions: 2 samples observed



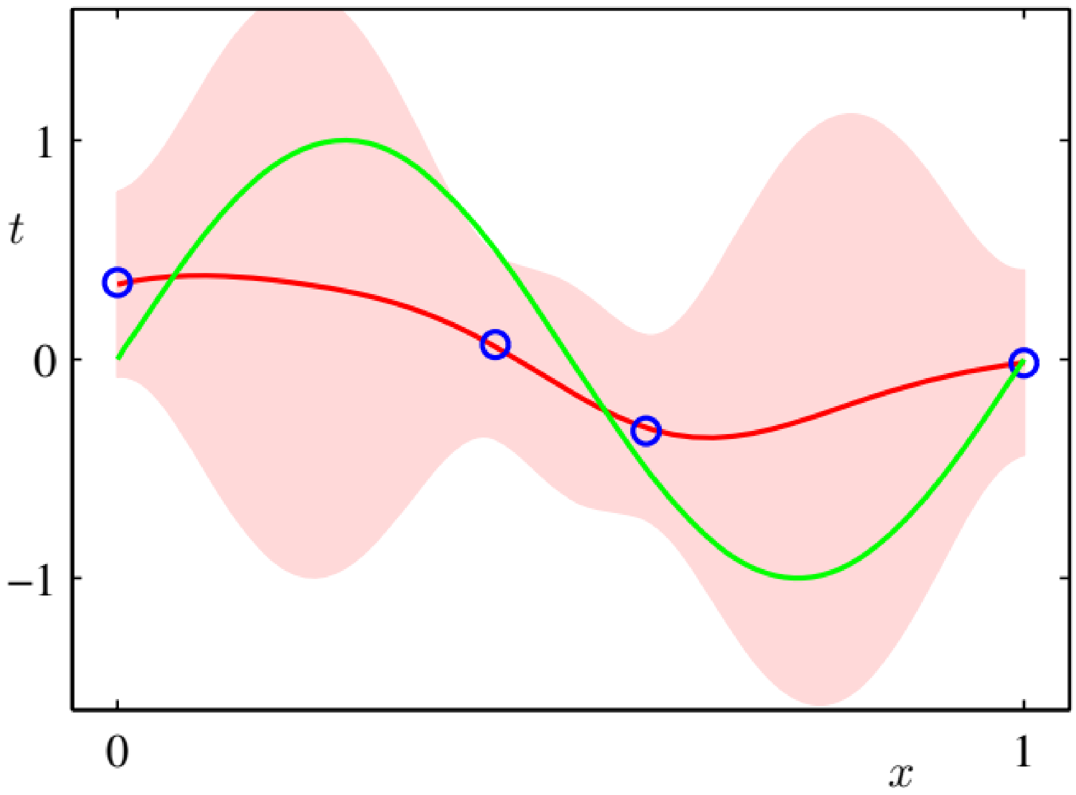
Predictive Distribution



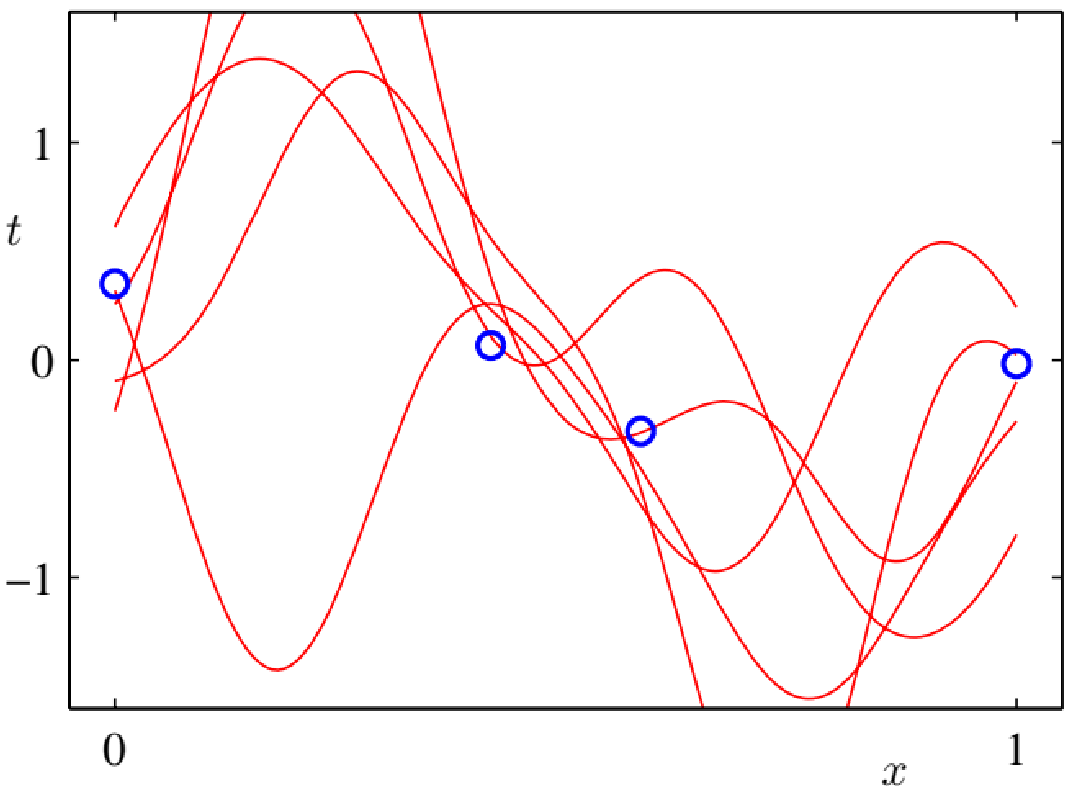
Sampled Models

Predictive Distribution: an example

- Approximating a sinusoidal dataset with a linear model w/ 9 Gaussian basis functions: 4 samples observed



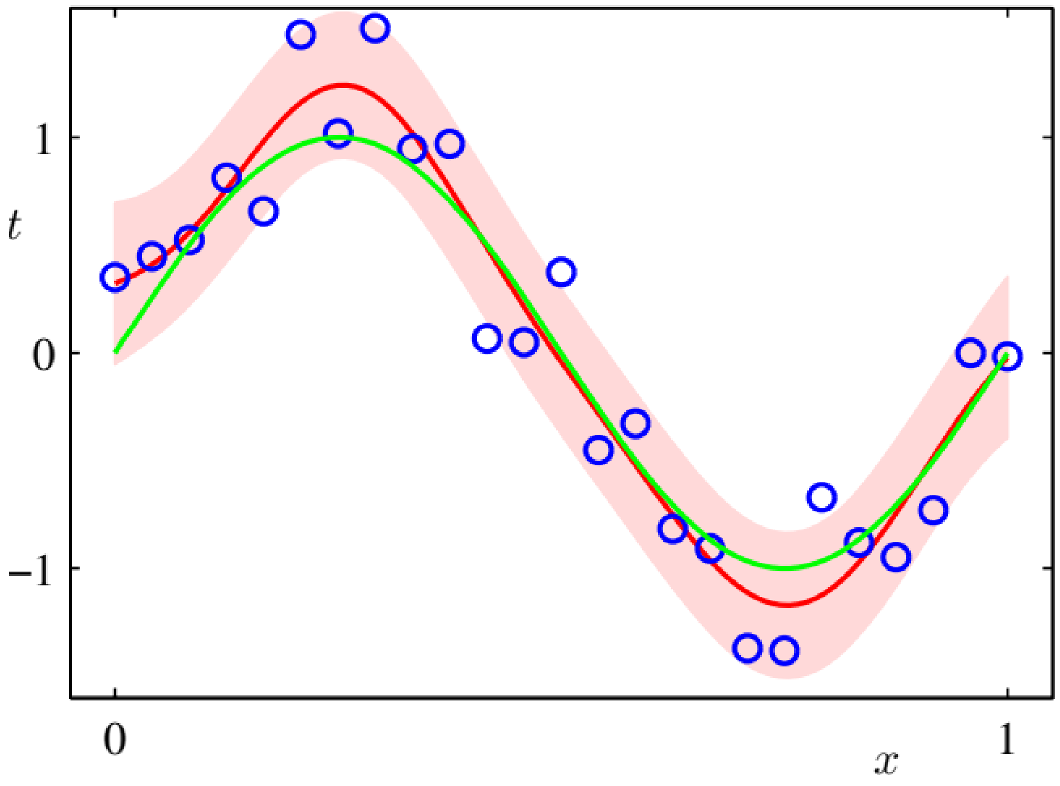
Predictive Distribution



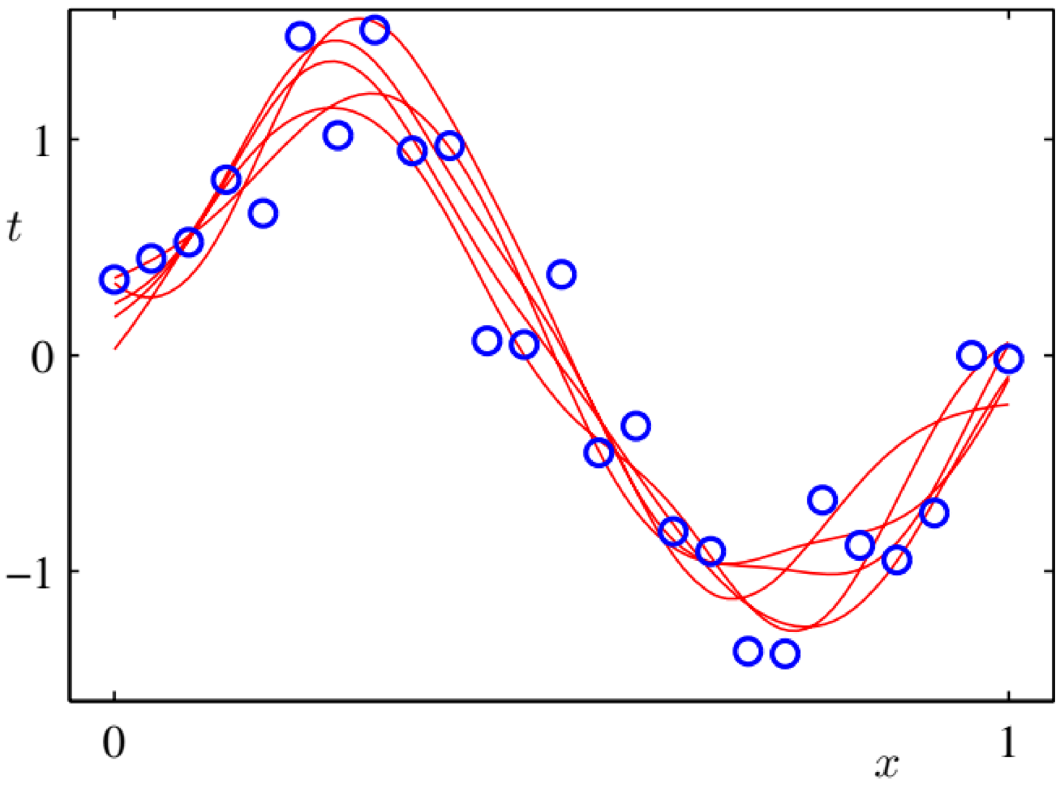
Sampled Models

Predictive Distribution: an example

- Approximating a sinusoidal dataset with a linear model w/ 9 Gaussian basis functions: 25 samples observed



Predictive Distribution



Sampled Models

Challenges and Limitation of Linear Regression

Challenges

- Modeling Challenges
  - Our model should fit well all the functions (we think) are **likely**
  - The prior should not give zero or small probabilities to possible values, but at the same time also avoid spreading out the probability (**uninformative**)
- Computational Challenges
  - **Analytical integration** is possible only using **conjugate priors** and works for **simple models**
  - **Approximated** approaches are instead to be used in the more general case, such as Gaussian (Laplace) approximation, Monte Carlo integration, and Variational approximation

Limitation of Fixed Basis Functions

- Linear models with fixed basis functions have several advantages
  - Allow closed-form solution
  - Tractable **Bayesian treatment**
  - Can model non-linear relationship with proper basis function
- However, they have also several limitations
  - Basis functions are not *adaptive* with respect to the training data
  - These models suffer of the **curse of dimensionality**