Introduction to Quantum Computing

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Multiple-Qubits States

Tensor product between matrices

- Given two matrices A (with n_a rows and m_a columns) and B (with n_b rows and m_b columns) their tensor product C is a matrix (with $n_a n_b$ rows and $m_a m_b$columns) defined as the element-by-element product between two matrices
- Example: if $A = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix}$
- their tensor product is $C = A \otimes B = \begin{bmatrix} 2 \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix} \\ 3 \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 10 & 4 \\ 3 & 12 \\ 15 & 6 \end{bmatrix}$
- Distributive property of the tensor product over addition

$$(A + B) \otimes C = A \otimes C + B \otimes C$$
$$A \otimes (B + C) = A \otimes B + A \otimes C$$

Multiple-Qubits States

- Given two qubits

$$|v_A\rangle = a_0|0\rangle + a_1|1\rangle$$
$$|v_B\rangle = b_0|0\rangle + b_1|1\rangle$$

  we wish to know their combined state
- We wish to know the probability for the two qubits to be
  - both in state $|0\rangle$,
  - or the first in state $|0\rangle$ and the second in state $|1\rangle$,
  - or the opposite,
  - or both in state $|1\rangle$
- The two qubits do not necessarily interact with each other

Multiple-Qubits States

- The state of the two qubits

$$|v_A\rangle = a_0|0\rangle + a_1|1\rangle$$
$$|v_B\rangle = b_0|0\rangle + b_1|1\rangle$$

  is described with their tensor product

The opposite is not always true!!
(a vector of 4 elements cannot always be decomposed into the tensor product of 2 qubits)

$$|v_A\rangle \otimes |v_B\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$$

- We introduce a new compact ket notation

$$|v_A v_B\rangle = |v_A\rangle|v_B\rangle = |v_A\rangle \otimes |v_B\rangle$$

Multiple-Qubits States

- We can rewrite in a different format

$$|v_A\rangle = a_0|0\rangle + a_1|1\rangle = a_0\begin{bmatrix}1\\0\end{bmatrix} + a_1\begin{bmatrix}0\\1\end{bmatrix}$$

$$|v_B\rangle = b_0|0\rangle + b_1|1\rangle = b_0\begin{bmatrix}1\\0\end{bmatrix} + b_1\begin{bmatrix}0\\1\end{bmatrix}$$

- The tensor product is

$$|v_A v_B\rangle = a_0 b_0\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} + a_0 b_1\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} + a_1 b_0\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} + a_1 b_1\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$$

Multiple-Qubits States

- We can rewrite in a different format

$$|v_A\rangle = a_0|0\rangle + a_1|1\rangle = a_0\begin{bmatrix}1\\0\end{bmatrix} + a_1\begin{bmatrix}0\\1\end{bmatrix}$$
$$|v_B\rangle = b_0|0\rangle + b_1|1\rangle = b_0\begin{bmatrix}1\\0\end{bmatrix} + b_1\begin{bmatrix}0\\1\end{bmatrix}$$

- The tensor product is

$$|v_A v_B\rangle = a_0 b_0\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} + a_0 b_1\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} + a_1 b_0\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} + a_1 b_1\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$$

$(a_0 b_0)^2$ probability of being in state $\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$

$(a_0 b_1)^2$ probability of being in state $\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$

$(a_1 b_0)^2$ probability of being in state $\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$

$(a_1 b_1)^2$ probability of being in state $\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$

Multiple-Qubits States

- We can rewrite in a different format

$$|v_A\rangle = a_0|0\rangle + a_1|1\rangle = a_0\begin{bmatrix}1\\0\end{bmatrix} + a_1\begin{bmatrix}0\\1\end{bmatrix}$$

$$|v_B\rangle = b_0|0\rangle + b_1|1\rangle = b_0\begin{bmatrix}1\\0\end{bmatrix} + b_1\begin{bmatrix}0\\1\end{bmatrix}$$

- The tensor product is

$$|v_A v_B\rangle = a_0 b_0\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} + a_0 b_1\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} + a_1 b_0\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} + a_1 b_1\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$$

- We can rewrite as

$$|v_A v_B\rangle = a_0 b_0\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + a_0 b_1\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + a_1 b_0\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + a_1 b_1\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Multiple-Qubits States

- We introduce a new notation for the basis of a two-qubit state
  - $|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}$ $\qquad |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}$
  - $|10\rangle = |1\rangle \otimes |0\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}$ $\qquad |11\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix}$
- We can rewrite the two-qubit state

$$|v_A v_B\rangle = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle$$

- or

$$|v_A v_B\rangle = c_0 |00\rangle + c_1 |01\rangle + c_2 |10\rangle + c_3 |11\rangle$$

- where coefficients c_0, c_1, c_2 and c_3 are the **amplitudes** of the multi-qubit state
- An alternative notations is

$$|v_A v_B\rangle = c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle + c_3 |3\rangle$$

Multiple-Qubits States: exercise

- Given two qubits

$$|v_A\rangle = a_0|0\rangle + a_1|1\rangle$$
$$|v_B\rangle = b_0|0\rangle + b_1|1\rangle$$

- and their state

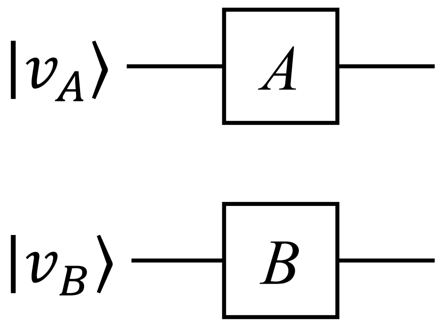
$$|v_A v_B\rangle = a_0 b_0|00\rangle + a_0 b_1|01\rangle + a_1 b_0|10\rangle + a_1 b_1|11\rangle$$

- show that the amplitudes normalize to 1

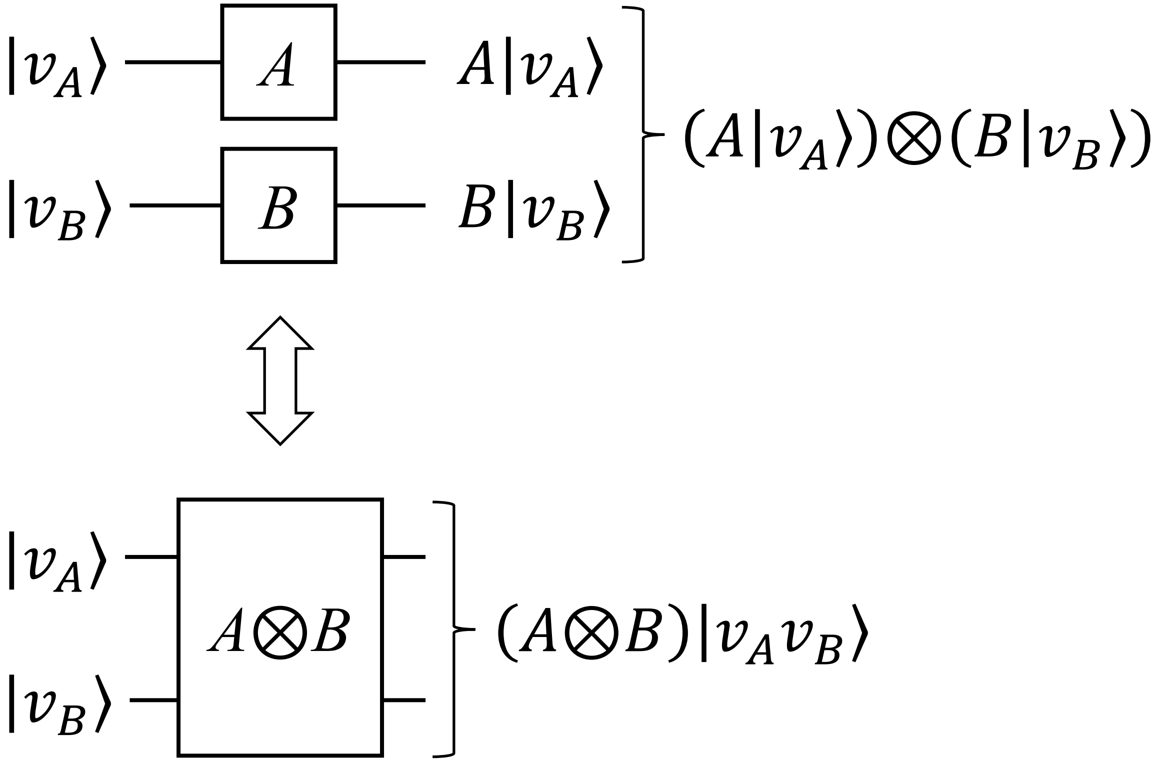
- We can write the sum of the square of the amplitudes as

$$a_0^2 b_0^2 + a_0^2 b_1^2 + a_1^2 b_0^2 + a_1^2 b_1^2 = a_0^2\left(b_0^2 + b_1^2\right) + a_1^2\left(b_0^2 + b_1^2\right) = a_0^2 + a_1^2 = 1$$

Multiple-Qubits Circuits (Parallel Gates)



$|v_C\rangle = ?$



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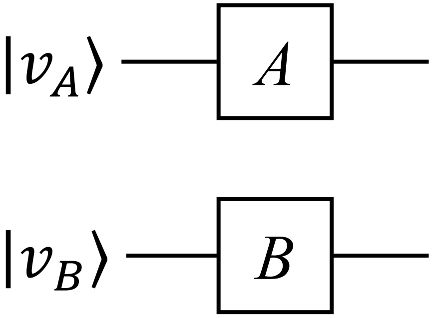
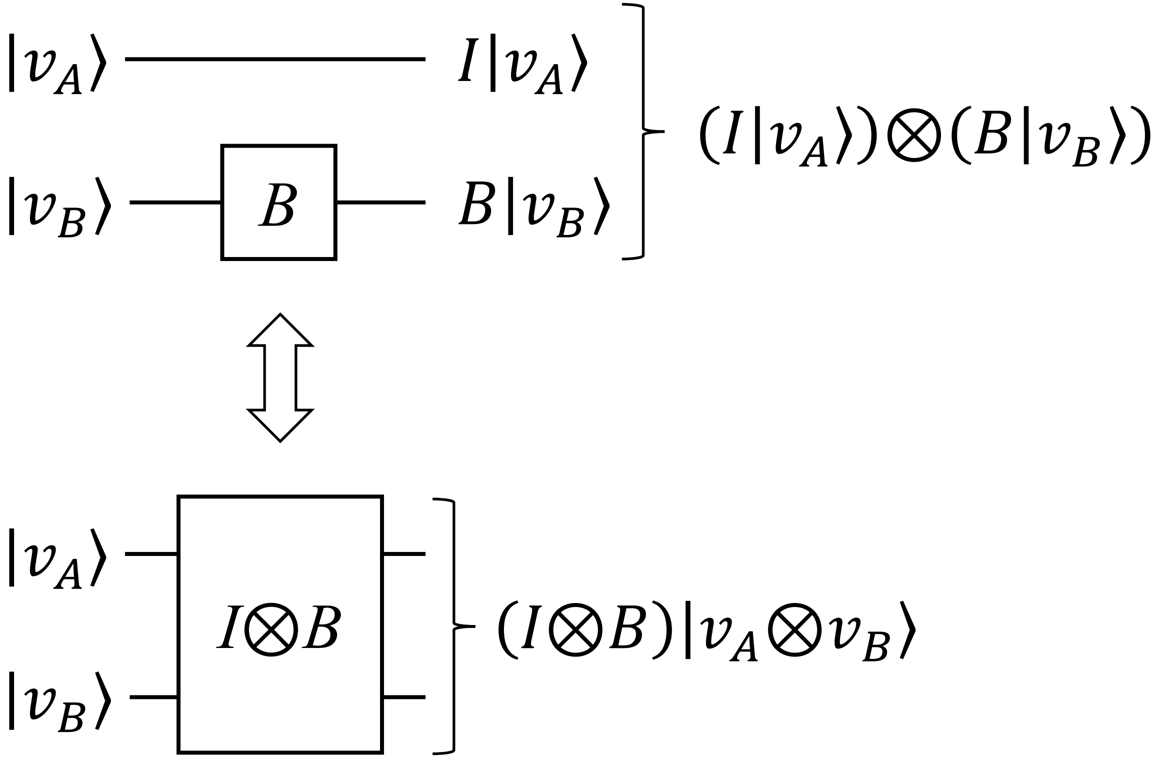


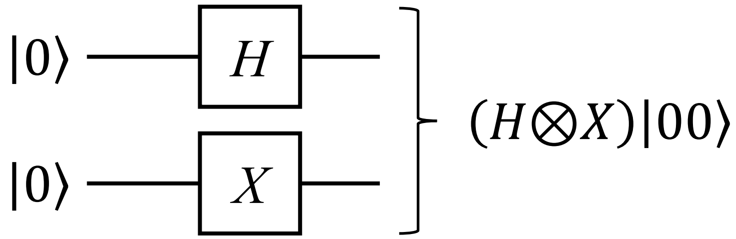
$|v_C\rangle = ?$



Parallel Gates: Example 1

$$H \otimes X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & -1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$



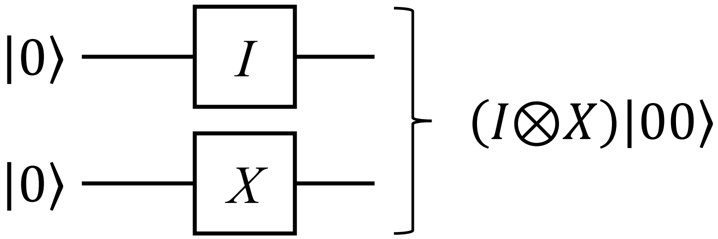
$$(H \otimes X)|00\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}} (|01\rangle + |11\rangle)$$

Parallel Gates: Example 2

$$I \otimes X = \begin{bmatrix} 1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(I \otimes X)|00\rangle = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

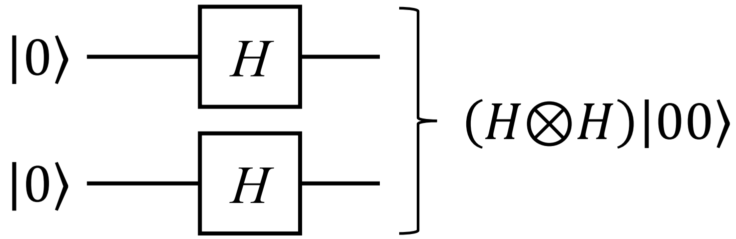


Parallel Gates: the $H^{\otimes 2}$ Hadamard transform

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & -\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$(H \otimes H)|(|0\rangle \otimes |0\rangle)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$



- This circuit performs the $H^{\otimes 2}$ **Hadamard transform** on two qubits
- Similarly, we can define the $H^{\otimes n}$ Hadamard transform on n qubits
- Hadamard transform places the state in a "*uniform*" superposition across all qubits

Thanks

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