ESE 650: Learning in Robotics
Lecture 8

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FAQ

- **Should I include training files in my submission?**
  - Yes, the most important thing in an assignment is to show your work, not the end result
- **Can I modify my code after I receive the testset to get better results for the report?**
  - Yes, I can't stop you form doing that. However, the idea of the test set and the report are that you spend time analyzing the performance and writing instead of trying to redo all the code in one day. The discussion is as important as the results.
- **How important is the speed of my code?**
  - Describing your technical approach and analyzing your performance is the most important. After than: fast-correct > slow-correct >> fast-wrong > slow-wrong.
- **How does the late day policy work?**
  - **No late days** for Projects 1 and 4. Your code and report deadlines are shifted by 24 hours for each late day you use for the other projects.

Kalman Filter (discrete time)

Motion model: $$x_{t+1} = Ax_t + Bu_t + \mathcal{N}(0, W)$$

Observation model: $$z_t = Hx_t + \mathcal{N}(0, V)$$

Prior: $$x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$$

Prediction:
$$\mu_{t+1|t} = A\mu_{t|t} + Bu_t$$
$$\Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W$$

Update:
$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - H\mu_{t+1|t})$$
$$\Sigma_{t+1|t+1} = (I - K_{t+1}H)\Sigma_{t+1|t}$$

Kalman Gain: $$K_{t+1|t} := \Sigma_{t+1|t}H^T\left(H\Sigma_{t+1|t}H^T + V\right)^{-1}$$

What is the function a(x,u)?

- **Differential drive motion model:** position $p \in \mathbb{R}^2$, orientation $\theta \in (-\pi, \pi]$, linear velocity $\mathrm{v} \in \mathbb{R}$, rotational velocity $\omega \in \mathbb{R}$

State: $x = (p, \theta)$

Control: $u = (v, \omega)$

Motion Model: $\dot{x} = a(x,u) \quad \equiv \quad \begin{cases} \dot{p} = v \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\ \dot{\theta} = \omega \end{cases}$

- **Quadrotor motion model:** position $p \in \mathbb{R}^3$, velocity $\dot{p} \in \mathbb{R}^3$, orientation $R \in SO(3)$, rotational velocity $\Omega_\mathrm{B} \in \mathbb{R}^3$, thrust $f \in \mathbb{R}$, moment $M_B \in \mathbb{R}^3$, mass $m \in \mathbb{R}_{>0}$, gravitational acceleration g, moment of inertia $J \in \mathbb{R}^{3\times 3}$, z-axis $e_3 \in \mathbb{R}^3$

State: $x = (p, \dot{p}, R, \Omega_B)$

Control: $u = (f, M_B)$

Motion Model: $\dot{x} = a(x,u) \quad \equiv \quad \begin{cases} m\ddot{p} = -mge_3 + fRe_3 \\ \dot{R} = R\hat{\Omega}_B \\ J\dot{\Omega}_B = -\Omega_B \times J\Omega_B + M_B \end{cases}$

What is the function h(x)?

- **Range sensor:** position $x \in \mathbb{R}^n$, observed point $y \in \mathbb{R}^n$, distance $z \in \mathbb{R}$

  Observation Model: $z = h(x, y) = \|x - y\|_2$

- **Bearing sensor:** position $x \in \mathbb{R}^2$, orientation $\theta \in (-\pi, \pi]$, observed point $y \in \mathbb{R}^2$, bearing $z \in (-\pi, \pi]$

  Observation Model: $z = h(x, y) = arc\tan\left(\frac{y_2 - x_2}{y_1 - x_1}\right) - \theta$

- **Camera model:** position $p \in \mathbb{R}^3$, orientation $R \in SO(3)$, intrinsic camera matrix $K \in \mathbb{R}^{2x3}$, observed point $y \in \mathbb{R}^3$, pixel $z \in \mathbb{N}^2$

  State: $x = (p, R)$

  Projection: $\pi_K(p) := \frac{1}{p_3} K \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}$

  Observation Model: $z = h(x, y) := \pi_K\left(R^T (y - p)\right)$

Moment Matching

- Let $x \sim \mathcal{N}(\mu, \Sigma)$ and $z = h(x)$. How should we approximate the joint distribution of (x, z) via a Gaussian?

$$\mathbb{E}[x] = \int x\phi(x; \mu, \Sigma)\, dx = \mu$$

$$Cov(x,x) = \mathbb{E}\left[xx^T\right] - \mathbb{E}[x]\mathbb{E}[x]^T = \int xx^T\phi(x;\mu,\Sigma)\,dx - \mathbb{E}[x]\mathbb{E}[x]^T = \left(\Sigma + \mu\mu^T\right) - \mu\mu^T$$

$$\mathbb{E}[z] = \mathbb{E}[h(x)] = \int h(x)\phi(x;\mu,\Sigma)\,dx = m_z$$

$$Cov(z,z) = \mathbb{E}\left[(z-m_z)(z-m_z)^T\right] = \int (h(x)-m_z)(h(x)-m_z)^T\,\phi(x;\mu,\Sigma)\,dx = S_z$$

$$Cov(x,z) = \mathbb{E}\left[(x-\mu)(z-m_z)^T\right] = \int (x-\mu)(h(x)-m_z)^T\,\phi(x;\mu,\Sigma)\,dx = C_{xz}$$

$$\begin{pmatrix} x \\ z \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ m_z \end{pmatrix}, \begin{bmatrix} \Sigma & C_{xz} \\ C_{xz}^T & S_z \end{bmatrix}\right)$$

$$x \sim \mathcal{N}(\mu, \Sigma) \qquad x \mid z \sim \mathcal{N}\left(\mu + C_{xz}S_z^{-1}(z - m_z), \Sigma - C_{xz}S_z^{-1}C_{xz}^T\right)$$

$$z \sim \mathcal{N}(m_z, S_z) \qquad z \mid x \sim \mathcal{N}\left(m_z + C_{xz}^T\Sigma^{-1}(x-\mu), S_z - C_{xz}^T\Sigma^{-1}C_{xz}\right)$$

Nonlinear KF for Additive Noise (summary)

- **Motion model:** $x_{t+1} = a(x_t, u_t) + w_t \qquad x_{t+1} \mid x_t, u_t \sim \mathcal{N}\left(a(x_t, u_t), W\right)$
- **Measurement model:** $z_t = h(x_t) + v_t \qquad z_t \mid x_t \sim \mathcal{N}\left(h(x_t), V\right)$
- **Prediction step:**

$$\mu_{t+t|t} = \int a(s, u_t)\phi\left(s; \mu_{t|t}, \Sigma_{t|t}\right) ds$$

$$\Sigma_{t+1|t} = \int a(s, u_t)\left(a(s, u_t)\right)^T \phi\left(s; \mu_{t|t}, \Sigma_{t|t}\right) ds + W - \mu_{t+t|t}\mu_{t+1|t}^T$$

- **Update step**:

$$\mu_{t+1|t+1} := \mu_{t+1|t} + K_{t+1|t}\left(z_{t+1} - m_{t+1|t}\right)$$

$$\Sigma_{t+1|t+1} := \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K_{t+1|t}$$

$$K_{t+1|t} := C_{t+1|t} S_{t+1|t}^{-1}$$

$$m_{t+1|t} := \int h(x)\phi\left(x; \mu_{t+1|t}, \Sigma_{t+1|t}\right) dx$$

$$S_{t+1|t} := \int \left(h(x) - m_{t+1|t}\right)\left(h(x) - m_{t+1|t}\right)^T \phi\left(x; \mu_{t+1|t}, \Sigma_{t+1|t}\right) dx + V$$

$$C_{t+1|t} := \int \left(x - \mu_{t+1|t}\right)\left(h(x) - m_{t+1|t}\right)^T \phi\left(x; \mu_{t+1|t}, \Sigma_{t+1|t}\right) dx$$

Extended Kalman Filter (summary)

Prior: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}\left(\mu_{t|t}, \Sigma_{t|t}\right)$

Motion Model: $x_{t+1} = a(x_t, u_t) + w_t \qquad w_t \sim \mathcal{N}(0, W)$

$$a(x_t, u_t) \approx a(\mu_{t|t}, u_t) + A_t(x_t - \mu_{t|t}) \qquad A_t := \frac{\partial a}{\partial x}\left(\mu_{t|t}, u_t\right)$$

Measurement Model: $z_{t+1} = h(x_{t+1}) + v_{t+1} \qquad v_{t+1} \sim \mathcal{N}(0, V)$

$$h(x_{t+1}) \approx h(\mu_{t+1|t}) + H_{t+1}\left(x_{t+1} - \mu_{t+1|t}\right) \qquad H_{t+1} := \frac{\partial h}{\partial x}\left(\mu_{t+1|t}\right)$$

Prediction:

$$\mu_{t+1|t} = a(\mu_{t|t}, u_t)$$
$$\Sigma_{t+1|t} = A_t \Sigma_{t|t} A_t^T + W$$

Update:

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}\left(z_{t+1} - h(\mu_{t+1|t})\right)$$
$$\Sigma_{t+1|t+1} = (I - K_{t+1} H_{t+1})\Sigma_{t+1|t}$$

Kalman Gain: $K_{t+1|t} := \Sigma_{t+1|t} H_{t+1}^T \left(H_{t+1} \Sigma_{t+1|t} H_{t+1}^T + V\right)^{-1}$

Same as KF but using (A_t, W) and (H_t, V)

Unscented Transform

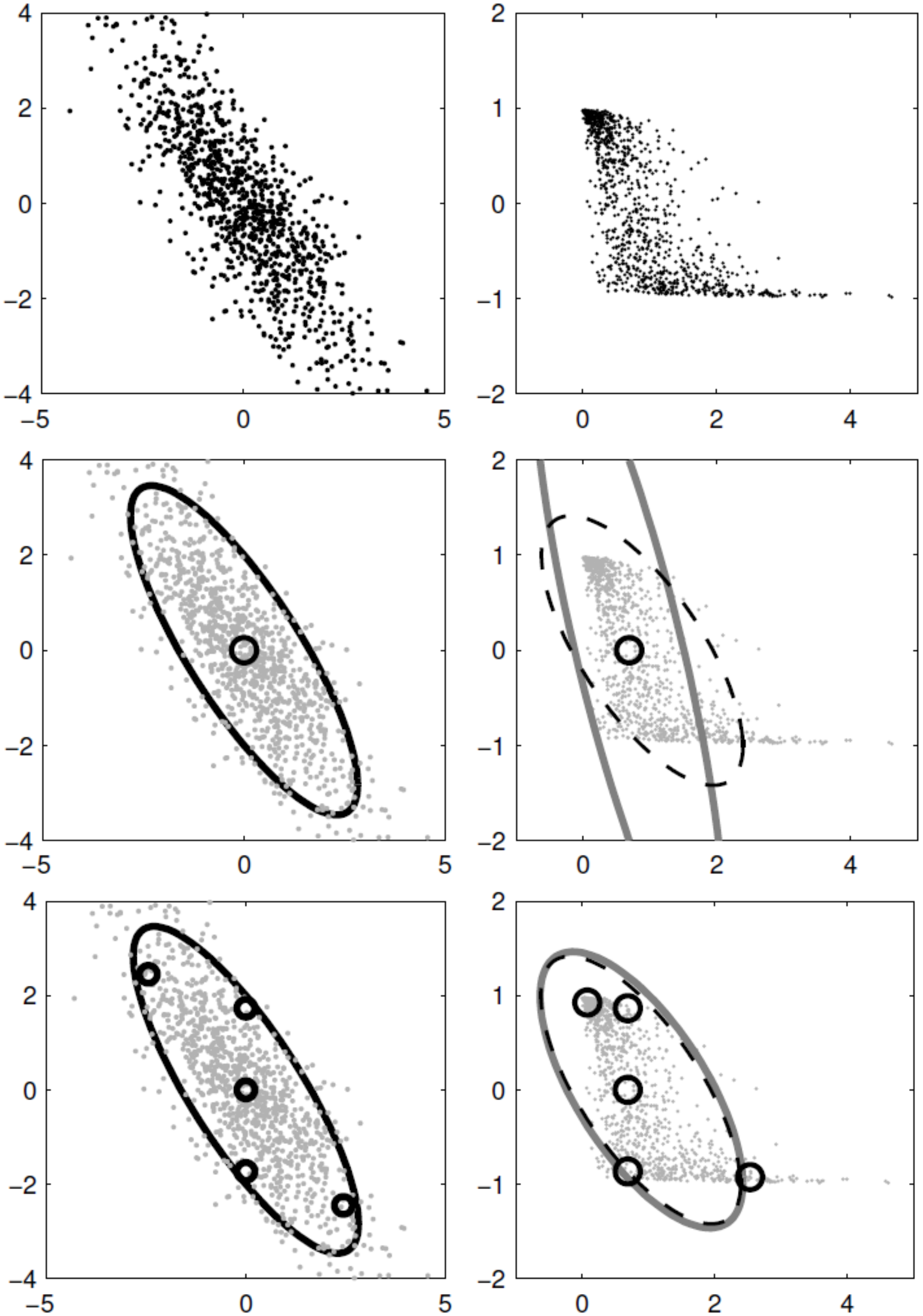
- The **unscented transform** (Julier et al., 1995, 2000) is a numerical method for approximating the joint distribution of random variables x and z defined as

$$z = h(x) \qquad x \sim \mathcal{N}(\mu, \Sigma)$$

- **Idea**: deterministically choose a fixed number of **sigma points** that capture the mean and covariance of the original distribution of x exactly

- Instead of linearizing the function, propagate the sigma points through the nonlinear function and then estimate the mean and covariance of the transformed variable from them

- This resembles Monte Carlo approximation but the sigma points are selected **deterministically**



Applying a nonlinear transformation to the random variable on the left results in a new random variable (right)

EKF: Linearization-based approximation to the transformation formed by calculating the curvature at the mean (solid) and true covariance (dashed).

UKF: Unscented transform approximation to the transformation formed by propagating sigma points through the nonlinearity and estimating the covariance from them (solid) and true covariance (dashed).

- The **unscented transform** Gaussian approximation to the joint distribution of x and a transformed random variable $z = h(x, v)$ where $x \sim \mathcal{N}(\mu, \Sigma)$ and $v \sim \mathcal{N}(0, V)$ is:

$$\begin{pmatrix} x \\ z \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ m_U \end{pmatrix}, \begin{bmatrix} \Sigma & C_U \\ C_U^T & S_U \end{bmatrix}\right)$$

- Form a set of $\mathbf{2n+1}$ sigma points using the i-th columns of the square root $\sqrt{\Sigma}$ of the covariance $\Sigma = \left[\sqrt{\Sigma}\right]\left[\sqrt{\Sigma}\right]^T$ (**note**: $\sqrt{\Sigma}$ is lower-triangular and can be obtained, e.g., via **Cholesky factorization**) as follows:

$$\mathcal{X}^{(0)} = \mu$$

$$\mathcal{X}^{(i)} = \mu \pm \sqrt{n+\lambda}\left[\sqrt{\Sigma}\right]_i \qquad i = 1, \ldots, n$$

$$\lambda := \alpha^2(n+k) - n \qquad (\alpha, k): \text{ determine sigma points spread, e.g., } \alpha = \sqrt{2}, k = 0$$

- Estimate the mean and covariance of $z = h(x)$ from the transformed sigma points:

$$m_U = \sum_{i=0}^{2n} W_i^{(m)} h\left(\mathcal{X}^{(i)}\right) \qquad W_0^{(m)} = \frac{\lambda}{n+\lambda} \qquad W_i^{(m)} = \frac{1}{2(n+\lambda)} \qquad i = 1, \ldots, 2n$$

$$S_U = \sum_{i=0}^{2n} W_i^{(c)} \left(h\left(\mathcal{X}^{(i)}\right) - m_U\right)\left(h\left(\mathcal{X}^{(i)}\right) - m_U\right)^T + V \qquad W_0^{(c)} = \frac{\lambda}{n+\lambda} + \left(1 - \alpha^2 + \beta\right)$$

$$C_U = \sum_{i=0}^{2n} W_i^{(c)} \left(\mathcal{X}^{(i)} - \mu\right)\left(h\left(\mathcal{X}^{(i)}\right) - m_U\right)^T \qquad W_i^{(c)} = \frac{1}{2(n+\lambda)} \qquad i = 1, \ldots, 2n$$

Unscented Kalman Filter (summary)

Prior: $x_t \mid z_{0:t}, u_{0:t-1} \sim \mathcal{N}\left(\mu_{t|t}, \Sigma_{t|t}\right)$

Motion Model: $x_{t+1} = a(x_t, u_t) + w_t \qquad w_t \sim \mathcal{N}(0, W)$

Measurement Model: $z_{t+1} = h(x_{t+1}) + v_{t+1} \qquad v_{t+1} \sim \mathcal{N}(0, V)$

Prediction:

$$\mu_{t+1|t} = \sum_{i=0}^{2n} W_i^{(m)} a\left(\mathcal{X}_{t|t}^{(i)}, u_t\right) \qquad \mathcal{X}_{t|t}^{(0)} = \mu_{t|t} \qquad \mathcal{X}_{t|t}^{(i)} = \mu_{t|t} \pm \sqrt{n+\lambda}\left[\sqrt{\Sigma_{t|t}}\right]_i$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2n} W_i^{(c)} \left(a\left(\mathcal{X}_{t|t}^{(i)}, u_t\right) - \mu_{t+1|t}\right)\left(a\left(\mathcal{X}_{t|t}^{(i)}, u_t\right) - \mu_{t+1|t}\right)^T + W$$

Update:

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t}\left(z_{t+1} - m_{t+1|t}\right)$$

$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K_{t+1|t}^T$$

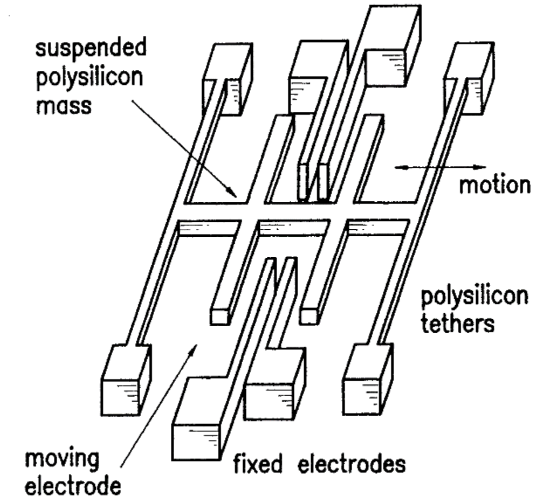
Kalman Gain: $K_{t+1|t} := C_{t+1|t} S_{t+1|t}^{-1}$

$$m_{t+1|t} = \sum_{i=0}^{2n} W_i^{(m)} h\left(\mathcal{X}_{t+1|t}^{(i)}\right) \qquad \mathcal{X}_{t+1|t}^{(0)} = \mu_{t+1|t} \qquad \mathcal{X}_{t+1|t}^{(i)} = \mu_{t+1|t} \pm \sqrt{n+\lambda}\left[\sqrt{\Sigma_{t+1|t}}\right]_i$$

$$S_{t+1|t} = \sum_{i=0}^{2n} W_i^{(c)} \left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)\left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)^T + V$$

$$C_{t+1|t} = \sum_{i=0}^{2n} W_i^{(c)} \left(\mathcal{X}_{t+1|t}^{(i)} - \mu_{t+1|t}\right)\left(h\left(\mathcal{X}_{t+1|t}^{(i)}\right) - m_{t+1|t}\right)^T$$

MEMS Strapdown IMU



Surface Micromachined Accelerometer

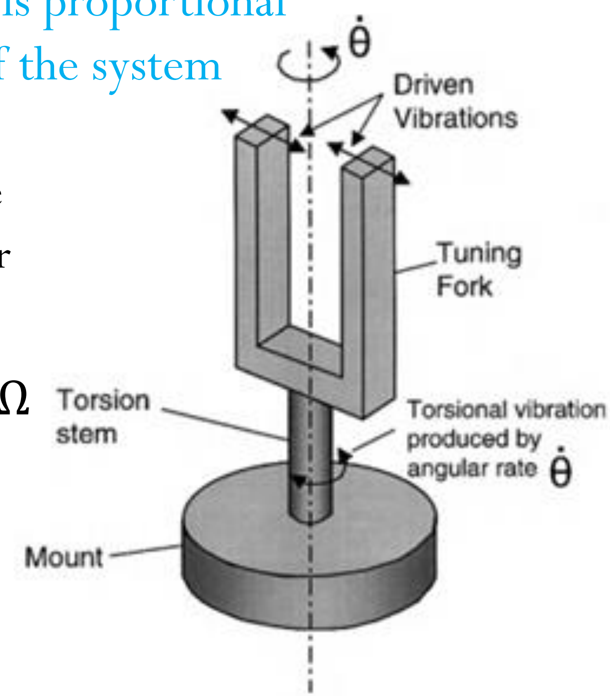
- **MEMS:** micro-electro-mechanical system
- **IMU**: inertial measurement unit – consist of a triaxial accelerometer (measures linear acceleration) and a triaxial gyroscope (measures angular velocity)
- **Strapdown**: the IMU and the object/vehicle inertial frames are joined together/identical
- **Accelerometer**: a mass m on a spring with spring constant k

$$F = ma = kd \quad \Rightarrow \quad d = \frac{ma}{k}$$

Spring displacement is proportional to the acceleration of the system

  - VLSI Fabrication: the displacement of a metal plate with mass m is measured with respect to another plate using capacitance
  - Used for car airbags (if the acceleration goes above $2g$, the car is hitting something!)

- **Gyroscope**: uses Coriolis force to detect rotational velocity Ω $(\dot{R} = R\widehat{\Omega})$. The rotational velocity changes the mechanical resonance of a tuning fork
- **Homework**: Read Kraft's paper on quaternion-based UKF (**Canvas**)



Project 2 Overview

- **State:** $x_t \coloneqq (q_t, \Omega_t)$, orientation q_t, rotational velocity $\Omega_t \in \mathbb{R}^3$
- **Control**: $u_t = q_\Delta \coloneqq \left[\cos\left(\frac{\|y_t\|\Delta t}{2}\right), \frac{y_t}{\|y_t\|}\sin\left(\frac{\|y_t\|\Delta t}{2}\right)\right]$, gyroscope measurement y_t in deg/sec during a time interval Δt
- **Noise**: $w_t = \begin{pmatrix} w_t^q \\ w_t^\Omega \end{pmatrix} \sim \mathcal{N}(0, W)$ with covariance $W \in S_{\geq 0}^{6\times 6}$ symmetric positive-semidefinite matrices
- **Motion model**: $$x_{t+1} = a(x_t, u_t, w_t) \coloneqq \begin{pmatrix} q_t q_{w_t^q} q_\Delta \\ \Omega_t + w_t^\Omega \end{pmatrix}$$
- **Observation model**: global frame gravitational acceleration g ("down"), global frame magnetic field b ("north")

$$x_{t+1} = a(x_t, u_t, w_t) \coloneqq \begin{pmatrix} q_t q_{w_t^q} q_\Delta \\ \Omega_t + w_t^\Omega \end{pmatrix}$$

- **Bias:** use stationary portions of the training data to estimate the bias
- **Scale**: set to datasheet value
- Correct both by comparing to Vicon ground truth

Gaussian Mixture Filter

Motion model: $x_{t+1} = Ax_t + Bu_t + \mathcal{N}(0,W)$

Observation model: $z_t = Hx_t + \mathcal{N}(0,V)$

Prior: $x_t \mid z_{0:t}, u_{0:t-1} \sim p_{t|t}(x_t) := \sum_k a_{t|t}^{(k)} \phi\left(x_t; \mu_{t|t}^{(k)}, \Sigma_{t|t}^{(k)}\right)$

Prediction:

$$p_{t+1|t}(x) = \int p_a(x \mid s, u_t)\, p_{t|t}(s)\, ds = \int \phi(x; As + Bu_t, W) \sum_k a_{t|t}^{(k)} \phi\left(s; \mu_{t|t}^{(k)}, \Sigma_{t|t}^{(k)}\right) ds$$

$$= \sum_k a_{t|t}^{(k)} \int \phi(x; As + Bu_t, W)\, \phi\left(s; \mu_{t|t}^{(k)}, \Sigma_{t|t}^{(k)}\right) ds$$

$$= \sum_k a_{t|t}^{(k)} \phi\left(x; A\mu_{t|t}^{(k)} + Bu_t, A\Sigma_{t|t}^{(k)} A^T + W\right)$$

Update:

$$p_{t+1|t+1}(x) = \frac{p_h(z_{t+1} \mid x)\, p_{t+1|t}(x)}{p(z_{t+1} \mid z_{0:t}, u_{0:t})} = \frac{\phi(z_{t+1}; Hx, V) \sum_k a_{t+1|t}^{(k)} \phi\left(x; \mu_{t+1|t}^{(k)}, \Sigma_{t+1|t}^{(k)}\right)}{\int \phi(z_{t+1}; Hs, V) \sum_j a_{t+1|t}^{(j)} \phi\left(s; \mu_{t+1|t}^{(j)}, \Sigma_{t+1|t}^{(j)}\right) ds}$$

$$= \sum_k \left(\frac{a_{t+1|t}^{(k)} \phi(z_{t+1}; Hx, V)\, \phi\left(x; \mu_{t+1|t}^{(k)}, \Sigma_{t+1|t}^{(k)}\right)}{\sum_j a_{t+1|t}^{(j)} \phi\left(z_{t+1}; H\mu_{t+1|t}^{(j)}, H\Sigma_{t+1|t}^{(j)} H^T + V\right)} \times \frac{\phi\left(z_{t+1}; H\mu_{t+1|t}^{(k)}, H\Sigma_{t+1|t}^{(k)} H^T + V\right)}{\phi\left(z_{t+1}; H\mu_{t+1|t}^{(k)}, H\Sigma_{t+1|t}^{(k)} H^T + V\right)} \right)$$

$$= \sum_k \left[\frac{a_{t+1|t}^{(k)} \phi\left(z_{t+1}; H\mu_{t+1|t}^{(k)}, H\Sigma_{t+1|t}^{(k)} H^T + V\right)}{\sum_j a_{t+1|t}^{(j)} \phi\left(z_{t+1}; H\mu_{t+1|t}^{(j)}, H\Sigma_{t+1|t}^{(j)} H^T + V\right)} \right] \phi\left(x; \mu_{t+1|t} + K_{t+1|t}(z_{t+1} - H\mu_{t+1|t}), (I - K_{t+1}H)\Sigma_{t+1|t}\right)$$

Kalman Gain: $K_{t+1|t} := \Sigma_{t+1|t} H^T \left(H\Sigma_{t+1|t} H^T + V\right)^{-1}$

Gaussian Mixture Filter

pdf: $x_t \mid z_{0:t}, u_{0:t-1} \sim p_{t|t}(x_t) := \sum_k a_{t|t}^{(k)} \phi\left(x_t; \mu_{t|t}^{(k)}, \Sigma_{t|t}^{(k)}\right)$

Mean: $\mu_{t|t} := \mathbb{E}\left[x_t \mid z_{0:t}, u_{0:t-1}\right] = \int x p_{t|t}(x) dx = \sum_k a_{t|t}^{(k)} \mu_{t|t}^{(k)}$

Covariance: $\Sigma_{t|t} := \mathbb{E}\left[x_t x_t^T \mid z_{0:t}, u_{0:t-1}\right] - \mu_{t|t}\mu_{t|t}^T = \int x x^T p_{t|t}(x) dx - \mu_{t|t}\mu_{t|t}^T = \sum_k a_{t|t}^{(k)} \left(\Sigma_{t|t}^{(k)} + \mu_{t|t}^{(k)} \left(\mu_{t|t}^{(k)}\right)^T\right) - \mu_{t|t}\mu_{t|t}^T$

- The GMF is just **a bank of Kalman filters**; sometimes called **Gaussian Sum filter**
- If the motion or observation models are nonlinear, we can apply the EKF or UKF tricks to get a nonlinear GMF
- Additional operations are needed when strong nonlinearities are present in the motion or observation models:
  - **Refinement**: introduces additional components to reduce the linearization error
  - **Pruning**: approximate the overall distribution with a smaller number of components (e.g., using KL divergence as a measure of accuracy)
- More details:

Marco Huber, "Nonlinear Gaussian Filtering: Theory, Algorithms, and Applications," KIT Scientific Publishing, 2015.

Simo Särkkä, "Bayesian Filtering and Smoothing," Cambridge University Press, 2013.