Propositional logic

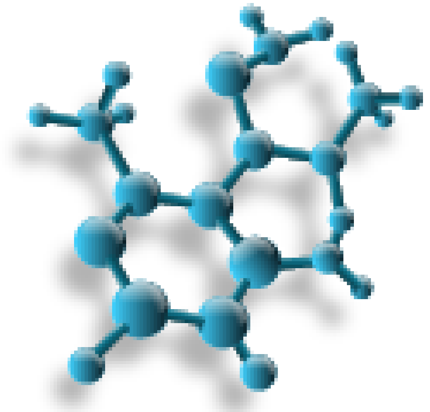
Topics of this lecture

- Propositional logic
- Definition of logic formula
- Meaning of logic formula
- Classification of logic formula
- Proof based on truth table
- Basic laws
- Clausal form/Conjunctive canonical form
- Formal proof

- 命題論理
- 論理式の定義
- 命題論理の意味
- 命題論理の種類
- 真理値表による証明
- 命題論理の法則
- 節形式/連言標準形
- 形式証明

Propositional logic

- In propositional logic, the most fundamental propositions are called **primitive propositions**.
- Primitive propositions cannot be decomposed. Propositions that can be decomposed are **compound propositions**.
- Primitive propositions can be denoted by some symbols, and these symbols are called **atomic formulas**. From atomic formulas we can construct various **logic formulas** corresponding to various compound propositions.



Examples

- Daisuke is a Japanese: P1
- Chieko is a Japanese: P2
- Daisuke and Chieko are husband and wife: P3
- Makoto is the child of Chieko: P4

- If Chieko is a Japanese **AND** Mokoto is the child of Chieko, **THEN** Mokoto is a Japanese: $P2 \wedge P4 \Rightarrow P5$
- If Daisuke and Chieko are husband and wife **AND** Makoto is the child of Chieko, **THEN** Mokoto is the child of Daisuke: $P3 \wedge P4 \Rightarrow P6$

Recursive definition of logic formula

1) An atomic formula is a logic formula.
2) If P is a logic formula, $\neg$P is also a logic formula.
3) If P and Q are logic formulas, P$\wedge$Q, P$\vee$Q, P $\Rightarrow$ Q, and P$\Leftrightarrow$Q are also logic formulas.
4) Only those defined by 1)-3) are logic formulas.

where $\neg$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are logic symbols (operators).

Well-formed formula (wff)

- Logic formulas defined in the previous page are called **well-formed formulas**.
- Similar to operators used in arithmetic calculation, logic symbols also have priorities.
- We can also use parentheses to define the priorities if the formula is ambiguous.

Logic symbols and their priorities

Symbol	Meaning	priority
¬	否定(negation, no)	1
∧	連言、論理積(conjunction, and)	2
∨	選言、論理和(disjunction, or)	3
⇒	含意(implication)	4
⇔	同値(equivalence)	5

Laws of propositional logic

- Equivalence and implication
  - $P \Leftrightarrow Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- Implication:
  - $P \Rightarrow Q = \neg P \vee Q$
- Absorption laws:
  - $P \wedge T = P \quad P \wedge F = F$
  - $P \vee T = T \quad P \vee F = P$
- Involution law （2重否定）:
  - $\neg(\neg P) = P$
- Idempotent laws （べき等律）:
  - $P \wedge P = P \quad P \vee P = P$
- Complement laws:
  - $P \wedge \neg P = F \quad P \vee \neg P = T$

- Commutative laws:
  - $P \wedge Q = Q \wedge P$, $P \vee Q = Q \vee P$
- Associative laws
  - $(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$
  - $(P \vee Q) \vee R = P \vee (Q \vee R)$
- Distributive laws
  - $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
  - $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$
- De Morgan's laws
  - $\neg(P \wedge Q) = \neg P \vee \neg Q$
  - $\neg(P \vee Q) = \neg P \wedge \neg Q$

Interpretation of logic formulas

- When we see a logic formula, we do not care about the original meaning of each proposition. Rather, we care about the logic relation between the propositions.
- The main concern is to know if a logic formula is "**true**" or "**false**" when the values of the atomic formulas contained in this formula are given.
- The process for determining the "true/false" of a logic formula based on the "true/false" of the atomic formulas is called "**interpretation**".

Example (Table 3.3, p. 39)

- For example, for the formula $P \land Q \Rightarrow R$, we can interpret it using the truth table.
- Here, we have used the following fact:

$$\mathbf{P \land Q \Rightarrow R}$$
$$\mathbf{= \neg(P \land Q) \lor R}$$

P	Q	R	$P \land Q \Rightarrow R$
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

Types of logic formula

- **Tautology** (恒真式): A tautology is a logic formula that is always true regardless of the true/false of the atomic formulas. A tautology is always "**valid**".
- **Contradiction** (恒偽式): If a logic formula is always false regardless of the true/false of the atomic formulas, we say it is a contradiction. A contradiction is unsatisfiable.
- **Satisfiable** (充足可能): If there exists a set of values for the atomic formulas that makes the logic formula true, this formula is satisfiable. This set of values is called an **interpretation** of the formula.

Interpreting a formula using the truth table

P Q	$((P \Rightarrow Q) \wedge P) \Rightarrow Q$	$P \wedge \neg(\neg P \Rightarrow Q)$
T T	T	F
T F	T	F
F T	T	F
F F	T	F

To see if a given formula is a tautology or not, we can use the truth table. Although this method is not Efficient when the number of atomic formulas is large.

Example

P	Q	$\neg P$	$P \Rightarrow Q$ $\neg P \vee Q$	$(P \Rightarrow Q) \wedge P$	$\neg((P \Rightarrow Q) \wedge P)$	$((P \Rightarrow Q) \wedge P) \Rightarrow Q$ $\neg((P \Rightarrow Q) \wedge P) \vee Q$
T	T	F	T	T	F	T
T	F	F	F	F	T	T
F	T	T	T	F	T	T
F	F	T	T	F	T	T

Clausal form/conjunctive canonical form (命題論理の節形式/連言標準形)

- An atomic formula or its negation is called a **literal**, and the disjunction (logic OR) of the literals is called a **clause**.
- Suppose that Ci (i=1,2,…,n) are clauses defined by the literals Pi1,Pi2,…,Pim as follows:

$$Ci=Pi1 \vee Pi2 \vee \ldots \vee Pim$$

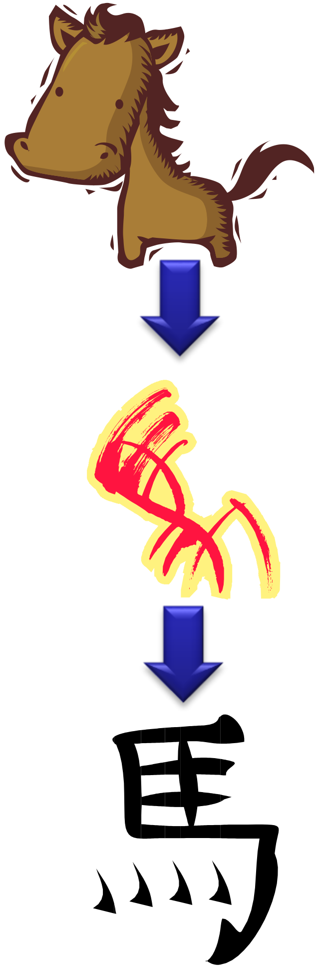
a logic formula represented by

$$C1 \wedge C2 \wedge \ldots \wedge Cn$$

is called a **clausal form** or **conjunctive canonical (normal) form**.

Conversion of a logic formula to normal form

- Any logic formula can be converted to a clausal form as follows.
  – Remove the equivalence and implication symbols:
    - $P \Leftrightarrow Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
    - $P \Rightarrow Q = \neg P \vee Q$
  – Put the negation symbol just before the literals:
    - $\neg(\neg P) = P$
    - $\neg(P \wedge Q) = \neg P \vee \neg Q$
    - $\neg(P \vee Q) = \neg P \wedge \neg Q$
  – Adopt the distributive law:
    - $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$



Example 3.2 (pp. 42-43)

$$
\begin{aligned}
&\neg P \Rightarrow ((Q \Rightarrow \neg R) \wedge \neg(R \Rightarrow \neg Q)) \\
&\quad = P \vee ((\neg Q \vee \neg R) \wedge \neg(\neg R \vee \neg Q)) \\
&\quad = P \vee ((\neg Q \vee \neg R) \wedge (R \wedge Q)) \\
&\quad = (P \vee (\neg Q \vee \neg R)) \wedge (P \vee (R \wedge Q)) \\
&\quad = (P \vee \neg Q \vee \neg R) \wedge ((P \vee R) \wedge (P \vee Q)) \\
&\quad = (P \vee \neg Q \vee \neg R) \wedge (P \vee R) \wedge (P \vee Q)
\end{aligned}
$$

Similar to the arithmetic formula (a+b)x(c-d)

Reasoning with logic formulas

- We can conduct inference or reasoning formally based on propositional logic formulas.
- By “formally” here we mean that reasoning can be conducted based on symbols, without considering the physical meanings of the atomic formulas.
- Reasoning formally enables a computing machine to make decisions faster than human.

Reasoning with logic formulas

- There are two types of reasoning (inference). The first one is **deductive reasoning**, which makes a decision for a given observation based on existing knowledge.
- The other one is **inductive reasoning**, which produces knowledge from various observations.
- Here, we consider only deductive reasoning. We will study inductive reasoning in the context of machine learning.

Rules for reasoning (inference)

- Reasoning is the process that derives a conclusion from the premise.
- In formal reasoning, the premise is given as a set of logic formulas, and the conclusion is also a logic formula. That is

(前提) P1,P2,…,Pn ⇒ Q（結論）

肯定式(modus ponens)	否定式(modus tollens)	三段論法(syllogism)
$P \Rightarrow Q$ P ――――― Q	$P \Rightarrow Q$ $\neg Q$ ――――― $\neg P$	$P \Rightarrow Q$ $Q \Rightarrow R$ ――――― $P \Rightarrow R$

Well-know rules for reasoning

Logical consequence (論理的帰結)

- If for any interpretation that makes the premise formulas P1,P2,…,Pn true, the conclusion Q is also true, we say Q is the **logical consequence** of P1,P2,…, Pn.
- In this case, we say the reasoning process that derives Q from {P1,P2,…,Pn} is **sound**（健全）.

Logical consequence (論理的帰結)

Theorem: For the logic formulas P1,P2,…,Pn and Q, Q is the logical consequence of P1, P2, …, Pn if and only if

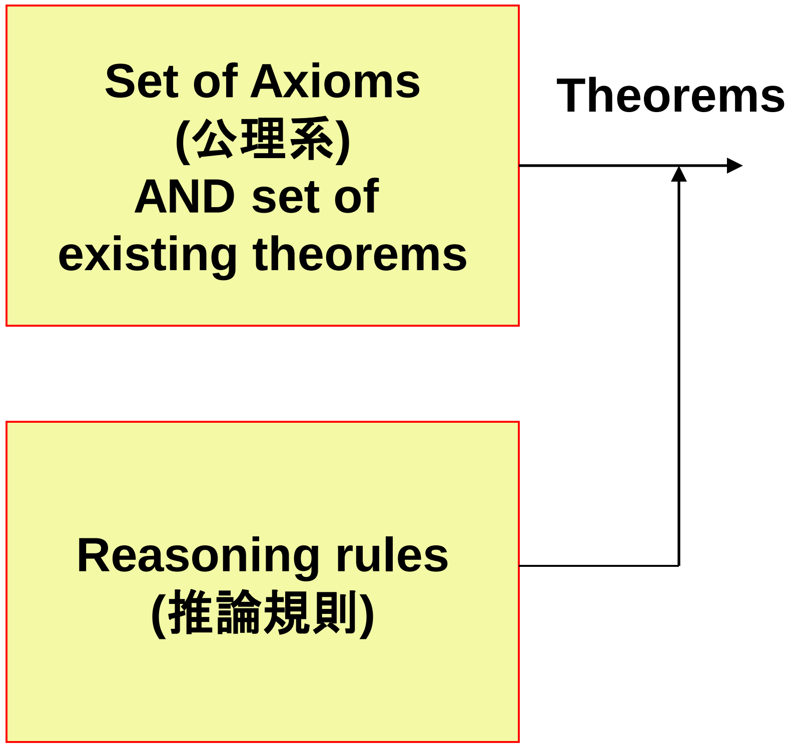
$$(P1 \wedge P2 \wedge \cdots \wedge Pn) \Rightarrow Q \quad or \quad \{P1, P2, \dots, Pn\} \Rightarrow Q$$

is **deductively valid**.

An argument form is deductively valid if and only if it is impossible that its conclusion is false when its premises are true.

Formal proof (形式的証明)

- Theorem proof is a special case of formal reasoning.
- Starting from a set of valid formulas called axioms, we can derive valid formula called theorems based on reasoning rules.
- The number of axioms is often small, but the number of theorems can be very large.
- The proved theorems can also be used to derive new theorems based on the reasoning rules.



Definition of formal proof

- The **proof** of a logic formula Bn is a finite sequence of logic formulas B1,B2, …,Bn satisfying
  – Bi (1<=i<=n) is an axiom, or
  – Bi (1<=i<=n) is a formula derived from Bj and Bk (1<=j,k<i) based on a reasoning rule.
- A logic formula B is **provable** if a proof defined above exists. A provable formula B is denoted by

$$\vdash B$$

- Provable logic formulas are called **theorems**.

Example 3.4 pp. 45-46

Axiom set:

- A1: $P \Rightarrow (Q \Rightarrow P)$
- A2: $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow$
  $((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$
- A3: $(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P)$

Reasoning rule:

If $P \Rightarrow Q$ AND P
Then Q

Theorem to prove: $P \Rightarrow P$

- P1= $P \Rightarrow ((P \Rightarrow P) \Rightarrow P)$ (from A1)
- P2= $(P \Rightarrow ((P \Rightarrow P) \Rightarrow P)) \Rightarrow$
  $((P \Rightarrow (P \Rightarrow P)) \Rightarrow (P \Rightarrow P))$ (from A2)
- P3= $(P \Rightarrow (P \Rightarrow P)) \Rightarrow (P \Rightarrow P)$
  (derived from P1 and P2)
- P4= $P \Rightarrow (P \Rightarrow P)$ (from A1)
- P5= $P \Rightarrow P$
  (derived from P3 and P4)

{P1, P2, P3, P4} is the proof of P5.

Homework for lecture 7 (1)

(submit the answers in the exercise class)

1. Show that the following logic formulas are valid using the truth table:
   - $P \Rightarrow P$
   - $((P \Rightarrow Q) \land P) \Rightarrow Q$

2. Show that Q is the logic consequence of $\{P \Rightarrow Q, P\}$ (This is the problem 3.3 in the textbook, p. 45).

Homework for lecture 7 (2)

1. Download the skeleton file from the web page of this course, and complete the program.
2. The completed program should be able to print out the truth table of a given logic formula.
3. Try to find the logic formula for the truth table given in the right table, confirm its correctness using your program, and write your results in `summary_07.txt`.

x	y	z	output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Quizzes of today

- The most fundamental proposition is called “P___________ proposition”.
- A primitive proposition, when denoted by a symbol, is an “A_______ formula”.
- Write the double negation (Involution) law:
- Write the distributive law:

- Atomic formula and its negation is called a “L_______”.
- The disjunction of several literals is called a “C________”.
- Any logic formula can be converted to a Clausal _________________.
- To conduct formal proof, it is necessary to have a set of “A________” and the reasoning rules.
- A provable formula B is denoted by