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Abstract

The project aims at studying a new kind of analytical technique for a non-linear problem called the Variational Iteration Method (VIM) is described and used to give approximate solutions for some well-known non-linear problems. It is the modification of General Lagrange Multiplier Method. In this method, the problems are initially approximated with possible unknowns. Then a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory.

The main property of the method is its flexibility and ability to solve nonlinear equations accurately and conveniently. We also study the convergence of the method for nonlinear differential equations. In VIM we get either the exact solution or numerical approximation which converges to the exact solution.

In chapter 3, the variational iteration method is applied to solve integro-differential equations. Some examples are given to illustrate the effectiveness of the method. In chapter 4, Laplace variational iteration strategy, based on the (VIM) and Laplace transform, is presented for the exact/numerical solution of linear and nonlinear differential equations.

In chapter 5, VIM is applied to different types of fractional differential equations. Also the Laplace transform is used to solve certain classes of fractional differential equations. In chapter 6, we study the applications of VIM, image restoration which is the process of recovery of images and that have been degraded by blur and noise and Perona-Malik equation provides a potential algorithm for image restoration which is solved by VIM. Another application is finding the solution of Biological Population model.

Contents

CHAPTER 1

Variational Iteration Method

1.1 Introduction

Differential equations are used in various fields as physics, chemistry, biology, mathematics and engineering. In this chapter we study a new method called Variational Iteration Method (VIM). How does this method works and converge to the solution.

1.2 VIM

Variational Iteration Method was developed by Ji-Huan-He in 1997. This method is a modification of General Lagrange Multiplier Method. VIM is a new kind of analytical technique for non-linear problems used to give approximate solutions for well known non-linear problems.In this method, the problems can be initially approximated with possible unknowns. Then a correctional functional is constructed by using General Lagrange Multiplier Method, which can be identified optimally via the variational theory.

The objective of the VIM is to make possible realistic solution of linear or non-linear simple or complex systems modelled by ODEs or PDEs. It solves Linear, Non-Linear, Homogenous, Non-Homogenous equations in bounded and unbounded domains. VIM does not depend on small parameters and is free from linearization and discretization like other methods as Homotopy Perturbation Method.

In General Lagrange multiplier method for solving non-linear problems, the main feature of the method was that the solution of a mathematical problem with linearization assumption is used as initial approximation.
For a linear operator L, non-linear operator N and analytic function g(x), consider the equation

$$Lu(x) + Nu(x) = g(x)$$

Assuming $u_0(x)$ is the solution of Lu=0,

$$u_{cor}(x_0) = u_0(x_0) + \int_0^{x_0} \lambda(Lu_0 + Nu_0 - g)dx \tag{1.1}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory and u_{cor} is the corrected solution at x_0. The above (1.1) is modified into an iteration method as following :

$$u_{n+1}(x_0) = u_n(x_0) + \int_0^{x_0} \lambda(s)\left[Lu_n(s) + N\tilde{u}_n(s) - g(s)\right] ds$$

with $u_0(x)$ as initial approximation with possible unknowns, and $\tilde{u}_n$ is considered as a restricted variation i.e. $\delta\tilde{u}_n$=0 and u_n denotes the n_{th} approximation. As x_0 is arbitrary, we can rewrite the equation as :

$$u_{n+1}(x) = u_n(x) + \int_0^{x} \lambda(s)\left[Lu_n(s) + N\tilde{u}_n(s) - g(s)\right] ds$$

The above modified method called Variational Iteration Method solves non-linear problems effectively, easily and accurately with approximations converging to accurate solutions. In this method, we obtain a series which converges fast to the exact solution.

1.3 Procedure of solving equations

Consider a general equation,

$$Lu(x) + Nu(x) = g(x)$$

where, L is linear operator, N is non linear operator and g is any analytic function.
Step 1 : For solving the equation by Variational Iteration Method, the correction functional is

$$u_{n+1}(x) = u_n(x) + \int_0^{x} \lambda(\xi)\left[Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right] d\xi \tag{1.2}$$

where, λ is Lagrange multiplier and $\tilde{u}_n$ is restricted variation i.e. $\delta\tilde{u}_n$=0.

Step 2 : λ is obtained by applying integration by parts in (1.2), taking $\delta\tilde{u}_{n+1}$=0 and comparing terms on both sides.

Step 3 : Choose initial approximation u_0, usually depending on initial conditions given in problem. And now find successive approximations u_n(x), n > 0.

step 4 : Taking lim $n \to \infty$ we get the exact solution as,

$$u(x) = \lim_{n\to\infty} u_n(x)$$

Remark 1. *We can also choose Initial approximation to be the solution of homogenous equation with given initial condition. Initial approximations can also be chosen with unknown parameters which can be found using given initial conditions in problem.*

Remark 2. *For linear problems, its exact solution can be obtained by only one iteration due to the fact that the Lagrange multiplier can be exactly identified. For nonlinear equation, the Lagrange multiplier is difficult to be identified. To overcome the difficulty, we apply restricted variations to nonlinear terms. But the restricted variations are applied only to non-linear terms, and the lesser the application of restricted variations, the faster the approximations converges to its exact solution.*

Example : 1 Now, consider an example of Ordinary Differential Equation

$$u''(t) + \omega^2 u(t) = 0$$

$$u(0) = 1,\ u'(0) = 0$$

Solution: For solving the problem by VIM, the correction functional is given by,

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left[\frac{d^2u_n(s)}{ds^2} + \omega^2\tilde{u}_n(s)\right] ds \tag{1.3}$$

For finding λ, making the above correction functional stationary using $\delta\tilde{u}_n = 0$ in above equation,

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta\int_0^t \lambda(s)\left[u_n''(s) + \omega^2\tilde{u}_n(s)\right] ds \\
&= \delta u_n(t) + \lambda(s)\delta\, u'_n(s)|_0^t - \int_0^t \lambda(s)\delta u_n'(s)ds + \int_0^t \lambda(s)\ \omega^2\ \delta\tilde{u}_n(s)\ ds \\
&= \delta u_n(t) + \lambda(s)\delta\, u'_n(s)|_{(s)=t} - \lambda'(s)\delta\, u_n(s)|_{(s)=t} + \int_0^t \lambda''(s)\delta u_n(s)ds + 0
\end{aligned}$$

So, $\delta\tilde{u}_{n+1}$=0 gives the following stationary conditions,

$$\begin{aligned}
\delta u_n(s) &: \ \lambda''(s) = 0 \\
\delta u_n(s) &: \ 1 - \lambda'(s)|_{s=t} = 0 \\
\delta u_n'(s) &: \ \lambda(s)|_{s=t} = 0
\end{aligned}$$

Solving the above equations we get,

$$\lambda(s) = (s - t)$$

Now substituting the value of $\lambda(s)$ in correction functional (1.3),

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left[\frac{d^2u_n(s)}{ds^2} + \omega^2 u_n(s)\right] ds$$

Choosing Initial approximation, $u_0(t) = 1$
Therefore, the successive approximations are,

$$u_1(t) = 1 - \frac{1}{2!}\omega^2 t^2$$
$$u_2(t) = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4$$
$$\vdots$$
$$u_n(t) = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 + ... + (-1)^n \frac{1}{(2n)!}\omega^{2n} t^{2n}$$

Therefore, the exact solution is,

$$u(t) = \lim_{n\to\infty} u_n(t)$$
$$= cos\omega t$$

1.4 Convergence Analysis

The variational iteration method changes the differential equation to a recurrence sequence of functions. The limit of that sequence is considered as the solution of the partial differential equation. Using the variational iteration method, it is possible to find the exact solution or an approximate solution of the problem. Now, our emphasis will be on the convergence of the Variational Iteration Method.

Theorem 1. *Let the equation be,*

$$Lu_n(t) + Nu_n(t) = g(t)$$

and A, defined as

$$A[u] = \int_0^t \frac{(-1)^m}{(m-1)!}(s-t)^{m-1}[Lu_n(s) + Nu_n(s) - g(s)]ds \tag{1.4}$$

be an operator from a Hilbert Space H to H. The series solution,

$$u(t) = \sum_{k=0}^{\infty} v_k(t)$$

where, $v_k(t) = A[v_0 + v_1 + v_2 + ... + v_k]$
converges if $\exists\, 0 < \gamma < 1$ *s.t.*

$$\|A[v_0 + v_1 + v_2 + ... + v_{k+1}]\| \leqslant \gamma\, \|A[v_0 + v_1 + v_2 + ... + v_k]\|$$

$(i.e. \|v_{k+1}\| \leqslant \gamma\, \|v_k\|)$.

Proof. Define the sequence of partial sums $\{S_n\}_{n=0}^{\infty}$

$$\begin{cases} S_0 = v_0 \\ S_1 = v_0 + v_1 \\ S_2 = v_0 + v_1 + v_2 \\ \quad \vdots \\ S_n = v_0 + v_1 + v_2 + \ldots + v_n \end{cases}$$

Now, we show that $\{S_n\}_{n=0}^{\infty}$ is a cauchy sequence in the Hilbert Space H. Consider,

$$\|S_{n+1} - S_n\| = \|v_{n+1}\| \leq \gamma \|v_n\| \leq \gamma^2 \|v_{n-1}\| \leq \ldots \leq \gamma^{n+1} \|v_0\|$$

For every $n, m > 0$ and $n \geq m$ we have,

$$\begin{aligned} \|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+1} - S_m)\| \\ &\leq \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + \ldots + \|(S_{m+1} - S_m)\| \\ &\leq \gamma^n \|v_0\| + \gamma^{n-1} \|v_0\| + \ldots + \gamma^{m+1} \|v_0\| \\ &= \left(\frac{1 - \gamma^{n-m}}{1 - \gamma} \right) \gamma^{m+1} \|v_0\| \end{aligned}$$

Since $0 < \gamma < 1$, we get

$$\lim_{n,m \to \infty} \|S_n - S_m\| = 0$$

Therefore, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy Sequence in the Hilbert space H and hence, the series solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ converges.

Theorem 2. *If the series solution*

$$u(t) = \sum_{k=0}^{\infty} v_k(t),$$

converges, then it is an exact solution of the non linear equation

$$Lu(t) + Nu(t) = g(t)$$

Proof. Suppose the series solution converges, say

$$\phi(t) = \sum_{m=0}^{\infty} v_m(t)$$

then we have,

$$\lim_{m \to \infty} v_m = 0,$$

$$\sum_{m=0}^{n}[v_{m+1}-v_m]=v_{n+1}-v_0$$

and so,

$$\sum_{m=0}^{n}[v_{m+1}-v_m]=\lim_{m\to\infty}v_m-v_0=-v_0 \tag{1.5}$$

Applying the operator Laplace operator 'L' to both sides of (1.5) then we have

$$\sum_{m=0}^{n}L[v_{m+1}-v_m]=-L[v_0]=0$$

Also, we have

$$L[v_{m+1}-v_m]=L\left[A[v_0+v_1+v_2+......+v_m]-L[A[v_0+v_1+v_2+......+v_{m-1}]]\right.$$

When $m\geq 1$, then by (1.4)

$$\begin{aligned}L[v_{m+1}-v_m]=L\Big[\int_0^t\frac{(-1)^m}{(m-1)!}(s-t)^{m-1}\Big(&L[v_0+v_1+v_2+......+v_m]\\&-L[v_0+v_1+v_2+......+v_{m-1}]+N[v_0+v_1+v_2+......+v_m]\\&-N[v_0+v_1+v_2+......+v_{m-1}]\Big)ds\Big],\qquad m\geq 1\end{aligned}$$

then,

$$L[v_{m+1}-v_m]=L[v_m]+N[v_0+v_1+v_2+......+v_m]-N[v_0+v_1+v_2+......+v_{m-1}]$$

Consequently, we have

$$\begin{aligned}\sum_{m=0}^{n}L[v_{m+1}-v_m]=&L[v_0]+N[v_0]-g(t)\\&+L[v_1]+N[v_0+v_1]-N[v_0]\\&+L[v_2]+N[v_0+v_1+v_2]-N[v_0+v_1]\\&\vdots\\&+L[v_n]+N[v_0+v_1+v_2+......+v_n]-N[v_0+v_1+v_2+......+v_{n-1}]\end{aligned}$$

Therefore,

$$\sum_{m=0}^{\infty}L[v_{m+1}-v_m]=\sum_{m=0}^{\infty}L[v_m]+\sum_{m=0}^{\infty}N[v_m]-g(t)$$

Since we have,

$$\sum_{m=0}^{\infty}L[v_{m+1}-v_m]=0$$

Then $\phi(t) = \sum_{m=0}^{\infty} v_m(t)$ is an exact solution of the given problem.

Remark 3. *A sufficient condition for convergence of VIM to the exact solution is that* $\exists\, 0 < \gamma < 1$ *s.t.*

$$\|A[v_0 + v_1 + v_2 + \ldots\ldots + v_{k+1}]\| \leqslant \gamma\, \|A[v_0 + v_1 + v_2 + \ldots\ldots + v_k]\|$$
$$i.e.\ \|v_{k+1}\| \leqslant \gamma\|v_k\|.$$

Theorem 3. *If the truncated series*

$$\sum_{k=0}^{\infty} v_k(t)$$

is used as an approximation to the solution u(t) then the maximum error $E_j(t)$ *is*

$$E_j(t) \leq \frac{1}{1-\gamma}\gamma^{j+1}\|v_0\|.$$

Proof. From Theorem 1 we have, for $n \geq m$

$$\|S_n - S_m\| \leq \frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m+1}\|v_0\|$$

Now as $n \to \infty$ then $S_n \to u(t)$ so,

$$\| u(t) - \sum_{k=0}^{m} v_k(t) \| \leq \frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m+1}\|v_0\|$$

Also, since $0 < \gamma < 1$ we have $(1 - \gamma^{n-m}) < 1$
Therefore,

$$\| u(t) - \sum_{k=0}^{m} v_k(t) \| \leq \frac{1}{1-\gamma}\gamma^{m+1}\|v_0\|$$

Remark 4. *Let us define*

$$\beta_i = \begin{cases} \dfrac{\|v_{i+1}\|}{\|v_i\|}, & if\, \|v_i\| \neq 0 \\ 0, & if\, \|v_i\| = 0 \end{cases}$$

If the first finite $\beta_i's$, *i=1,2,3.....*n_0 *are not less than 1 and* $\beta_i \leq 1$ *for* $i > n_0$, *then also the series solution converges to the exact solution i.e. the first finite terms do not affect the convergence of the series solution.*

CHAPTER 2

Solution approaches to ODE's and PDE's

2.1 Introduction

In this chapter, we will study some numerical examples of ordinary differential equations, partial differential equations and scheme of solving system of differential equations by variational iteration method.

2.2 Solving Wave Equation in unbounded Domain

Now consider the following wave equation in 1-dimension and unbounded domain

$$\begin{aligned} u_{tt} &= u_{xx}, \quad -\infty < x < \infty, \ t > 0 \\ u(x,0) &= sin\ x, \quad u_t(x,0) = 0 \end{aligned}$$

Solution:
Let the correction functional be

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s) \left[\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2} \right] ds$$

Using $\delta \tilde{u}_n = 0$ in above equation, we get Stationary conditions:

$$1 - \lambda'(s) = 0$$

$$\lambda|_{s=t} = 0$$

$$\lambda'' = 0$$

On solving , we get,

$$\lambda(s) = s - t$$

Choose, Initial approximation: $u_0(x,t) = sin\ x$
Therefore,the correction functional is

$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t) \left[\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 \tilde{u}_n(x,s)}{\partial x^2}\right] ds$

Then Successive approximations are :

$u_1(x,t) = sinx - \frac{t^2}{2!} sinx$

$u_2(x,t) = sinx - \frac{t^2}{2!} sinx + \frac{t^4}{4!} sinx$

$u_3(x,t) = sinx - \frac{t^2}{2!} sinx + \frac{t^4}{4!} sinx - \frac{t^6}{6!} sinx$

$\vdots$

$u_n(x,t) = sinx \left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + ...\right]$

Therefore the exact solution is,

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= sinx\ cost \end{aligned}$$

2.3 Comparison

Here we are comparing the exact solution and numerical approximations of a problem by Variational iteration method. Consider the following equation,

$$u''(t) - 2e^{u(t)} = 0, \quad 0 < t < 1$$

$$u(0) = 0, \quad u'(0) = 0$$

Solving by VIM correction functional is,

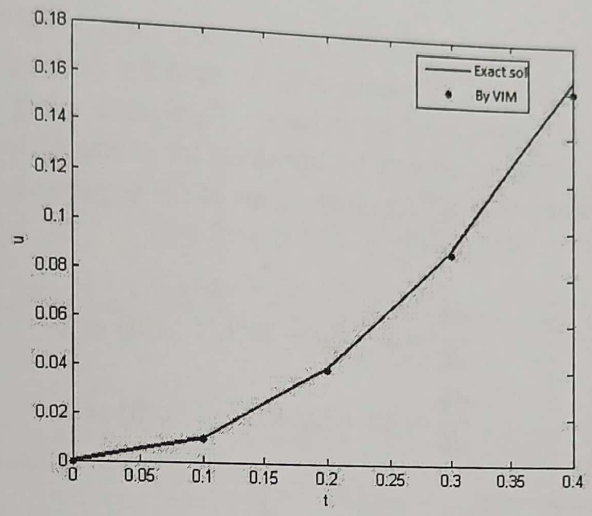
$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s) \left[\frac{d^2 u_n(s)}{ds^2} - 2e^{\tilde{u}_n(s)}\right] ds$$

By above procedure (in previous example), solving for λ.
We get, $\lambda(s) = (s-t)$
Choosing $u_0 = 0$,
Then, $u_1 = t^2$
Now we take the first approximations obtained by VIM as a solution of the given problem.
And the exact solution of the given problem is

$$u(t) = -2\ln(\cos x)$$

In the below table, comparison between the exact solution and numerical solution obtained for $u(t)$ of a 2-iterate VIM solution is shown for different points.

t	Exact solution	Approx. Sol by VIM	Absolute Error
0	0	0	0
0.1	0.010016711	0.0100	0.000016711
0.2	0.040269546	0.0400	0.000269546
0.3	0.091383312	0.0900	0.001383312
0.4	0.164458038	0.1600	0.004458038



In the above comparison, we are taking the first approximation as the solution by VIM, and we see that error is very small at starting time points, but if we calculate few more iterations by this method and calculate the error taking that to be exact solution, then error will definitely decrease. So, we conclude that VIM is very efficient in finding numerical solution of linear and non linear differential equations.

2.4 System of differential equation

Consider the m equations,

$$L_1(y_1) + N_1(y_1, y_2, \ldots, y_m) = g_1(x)$$
$$L_2(y_2) + N_2(y_1, u_2, \ldots, y_m) = g_2(x)$$
$$\vdots$$
$$L_m(y_m) + N_m(y_1, u_2, \ldots, y_m) = g_m(x)$$

where L_i is linear w.r.to y_i and N_i is non linear part of i_{th} equation.

Using VIM,
The correctional functionals are,

$$y_{i(n+1)} = y_{in} + \int_0^x \lambda_i \left[L_i(y_{in}(s)) + N_i(\tilde{y}_{1n}(s), ..., \tilde{y}_{mn}(s)) - g_i(s)\right] ds$$

Consider another correctional functional as

$$y_{i(n+1)} = y_{in} + \int_0^x \lambda_i \left[L_i(y_{in}(s)) + N_i(y_{1(n+1)}(s), ..., y_{(i-1)(n+1)}(s), y_{i(n)}(s).., y_{mn}(s)) - g_i(s)\right] ds \tag{2.1}$$

for i=2,....,m

In the correctional function (2.1), which is Corrected Variational Iteration Method (CVIM) updated values $y_{1(n+1)}, y_{2(n+1)}...., y_{(i-1)(n+1)}$ are used for finding $y_{i(n+1)}$. This accelerates the convergence of the system of sequences. Therefore, using just a few terms of the sequences, an accurate solution can be obtained for a larger domain of the problem. It is especially useful when computing more terms of the sequences is difficult or impossible.

Example :

$$\frac{dx}{dt} = x + 3y - 3e^2t, \quad x(0) = 1$$
$$\frac{dy}{dt} = 4x + 2y + 4e^t, \quad y(0) = 1$$

Solution :
We know the exact solution of the system is,

$$x(t) = e^t$$
$$y(t) = e^{2t}$$

Using VIM the correctional functionals are,

$$x_{n+1} = x_n - \int_0^t \left[x'_n - x_n - 3y_n + 3e^{2s}\right] ds,$$

$$y_{n+1} = y_n - \int_0^t \left[y'_n - 4x_n - 2y_n - 4e^s\right] ds$$

Choosing initial approximations for VIM, $x_0 = 1, \; y_0 = 1$
Using (2.1) the correctional functionals are,

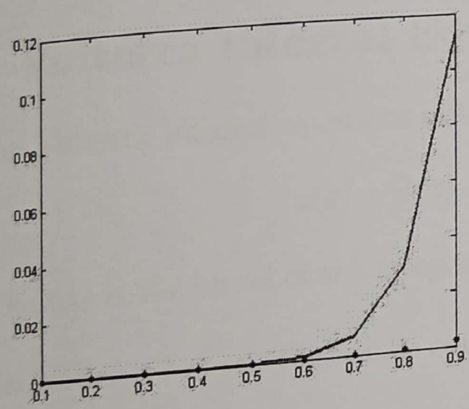
$$\overline{x}_{n+1} = \overline{x}_n - \int_0^t \left[\overline{x}'_n - \overline{x}_n - 3\overline{y}_n + 3e^{2s}\right] ds,$$

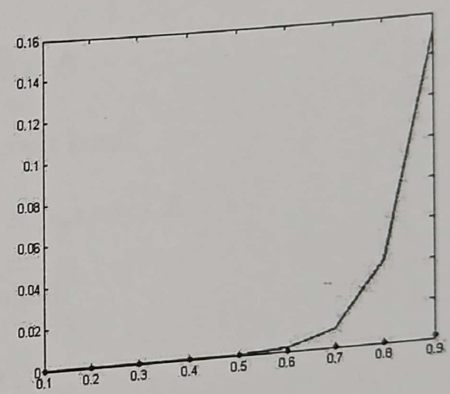
$$\overline{y}_{n+1} = \overline{y}_n - \int_0^t \left[\overline{y}'_n - 4\overline{x}_{n+1} - 2\overline{y}_n - 4e^s\right] ds$$

Choosing initial approximations for CVIM, $\overline{x}_0 = 1, \; \overline{y}_0 = 1$

In the table below, we calculate the error between the exact solution and the solutions obtained from VIM and CVIM.
(We are taking the 10^{th} approximations as the solutions of VIM and CVIM.)

t	$\lvert x(t) - x_{10}(t)\rvert$	$\lvert x(t) - \overline{x}_{10}(t)\rvert$	$\lvert y(t) - y_{10}(t)\rvert$	$\lvert y(t) - \overline{y}_{10}(t)\rvert$
0.1	0.3189×10^{-11}	0.2987×10^{-14}	0.4253×10^{-11}	0.1173×10^{-14}
0.2	0.6625×10^{-8}	0.1072×10^{-10}	0.8834×10^{-8}	0.4741×10^{-11}
0.3	0.5814×10^{-6}	0.1520×10^{-8}	0.7752×10^{-6}	0.7390×10^{-9}
0.4	0.1396×10^{-4}	0.5618×10^{-7}	0.1862×10^{-4}	0.2956×10^{-7}
0.5	0.1650×10^{-3}	0.9827×10^{-6}	0.2200×10^{-3}	0.5543×10^{-6}
0.6	0.1244×10^{-2}	0.1064×10^{-4}	0.1659×10^{-2}	0.6385×10^{-5}
0.7	0.6888×10^{-2}	0.8255×10^{-4}	0.9185×10^{-2}	0.5332×10^{-4}
0.8	0.3039×10^{-1}	0.4996×10^{-3}	0.4052×10^{-1}	0.3330×10^{-3}
0.9	0.1127	0.2498×10^{-2}	0.1503	0.1744×10^{-2}





In Fig.1 we plot the error function for $x_{10}(t), \overline{x}_{10}(t)$
In Fig.2 we plot the error function for $y_{10}(t), \overline{y}_{10}(t)$

CHAPTER 3

Integro differential Equations

3.1 Introduction

Various kind of analytical methods and approaches were used to solve Integral Equations. This chapter consists applying VIM method on different types of integral equations like Volterra integral equations, Fredholm integral equations and Integro differential equations.

3.2 Volterra Integral Equations

Consider Volterra Integral equations of the second kind

$$u(x) = f(x) + \int_a^x K(x,t)u(t)\, dt$$

where, K(x,t) is the kernel of the integral equation.

Consider Volterra Integro differential equation

Example:

$$u''(x) = 1 + xe^x - \int_0^x e^{(x-t)}u(t)dt,$$
$$u(0) = 0, u'(0) = 1$$

Solution:
Using VIM, the correction functional is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(s)\ [u''_n(s) - F(\tilde{u}_n(s))]ds$$

where,

$$F[u(x)] = 1 + xe^x - \int_0^x e^{(x-t)}u(t)dt$$

Applying stationary condition $\delta u_{n+1} = 0$ on the correctional function,
The Lagrange multiplier is, $\lambda(s) = s - x$.
So, the iteration formula is,

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\,[u_n''(s) - F(u_n(s))]ds$$

Taking initial approximation as, $u_0(x) = a + be^x$
Then from the iteration formula, on calculation we get,

$$u_1(x) = 2a + 2be^x - bxe^x + \frac{1}{2}x^2 + xe^x + \frac{1}{2}ax^2 - ae^x - 2e^x + 2 - b + x + ax$$

Applying initial conditions $u_1(0) = 0, u_1'(0) = 1$, to find the unknown parameters of initial approximations we get, a=-1, b=1.
Thus,

$$\begin{aligned} u_0(x) &= e^x - 1 \\ u_1(x) &= e^x - 1 \\ u_2(x) &= e^x - 1 \\ &\vdots \end{aligned}$$

The exact solution is,

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= e^x - 1 \end{aligned}$$

Remark 5. *The Fredholm integro differential Equation can also be solved similarly as Volterra Integro differential equation.*

3.3 Integral Equation

Now, consider the integral equation in which no initial conditions are given. So, we will use slightly different approach to VIM for choosing the initial approximations.

Example:

$$u(x) = x + \int_0^x (t-x)u(t)dt$$

Solution :
Firstly we will convert the given equation into a differential equation.
Differentiating the given equation w.r.t. x twice,

$$u'(x) = 1 - \int_0^x u(t)dt$$

$$u''(x) = -u(x)$$

Therefore,

$$u''(x) + u(x) = 0 \tag{3.1}$$

Using VIM, the correctional functional of (3.1) is,

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\ [u''(t) + u(t)]\ dt$$

Applying the stationary conditions we get, $\lambda(t) = t - x$

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\ [u''(t) + u(t)]\ dt$$

(for finding initial approximation)
Now, Suppose $v(x) = \int_0^x (t-x)u(t)dt$
Differentiating this equation w.r.t. x twice, we get

$$v'(x) = -\int_0^x u(t)dt$$

$$v''(x) = -u(x)$$

Therefore,

$$v''(x) + u(x) = 0$$

So, v(x) is a special solution of (3.1). Now, we can construct simple iteration formula,

$$u_{n+1}(x) = u_0(x) + v_n(x)$$

where $u_0(x)$ is satisfying the initial and boundary conditions.
Taking $u_0(x) = u(0) + xu'(0) = x$, we obtain the following iteration formula

$$u_{n+1}(x) = x + \int_0^x (t-x)u_n(t)dt$$

Therefore,

$$u_1(x) = x - \frac{x^3}{6}$$

$$u_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$u_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\vdots$$

Therefore, the exact solution is

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+1} \end{aligned}$$

Note : From the above analysis, we get the following iteration formula for the integral equations,

$$u_{n+1}(x) = f(x) + \int_a^x K(x,t) u_n(t)\, dt$$

Remark 6. *The Fredholm integral equations can also be solved similarly as Volterra integral equations.*

CHAPTER 4

Laplace Transform

4.1 Introduction

In this, we study a Laplace variational numerical scheme of linear and nonlinear differential equations, based on the variational iteration method (VIM) and Laplace transform. We are introducing an alternative Laplace correction functional and expressing the integral as a convolution.The results obtaining by this method confirms the simplicity, suitability, and effectiveness of this technique using only few terms of the iterative scheme.

4.2 A Laplace Variational Iteration Strategy

For a linear operator L, non-linear operator N and analytic function g(x), consider the equation

$$Lu(x) + Nu(x) = g(x)$$

By variational iteration method, we have the correctional functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\,[Lu_n(s) + N\tilde{u}_n(s) - g(s)]\,ds, \quad n = 0, 1, 2, ... \tag{4.1}$$

where λ, is a general Lagrange multiplier which can be identified optimally via variational theory, $\tilde{u}_n$ is considered as a restricted variation i.e. $\delta\tilde{u}_n$=0 and u_n denotes the n_{th} approximation.
We found that the general form of Lagrange multiplier is of the form,

$$\lambda = \overline{\lambda}(x - s)$$

For this form of Lagrange Multiplier the integration is the convolution hence Laplace transform is appropriate to use. Taking Laplace transform on both sides of (4.1),
So the correctional functional will be

$$\mathcal{L}[u_{n+1}(x)] = \mathcal{L}[u_n(x)] + \mathcal{L}\left[\int_0^x \overline{\lambda}(x - s)\,[Lu_n(s) + N\tilde{u}_n(s) - g(s)]\,ds\right], \quad n = 0, 1, 2, ...$$

Therefore,

$$\begin{aligned}\mathcal{L}[u_{n+1}(x)] &= \mathcal{L}[u_n(x)] + \mathcal{L}[\overline{\lambda}(x) * (Lu_n(x) + N\tilde{u}_n(x) - g(x))] \\ &= \mathcal{L}[u_n(x)] + \mathcal{L}[\overline{\lambda}(x)]\ \mathcal{L}[Lu_n(x) + N\tilde{u}_n(x) - g(x)]\end{aligned}$$

To find the optimal value of $\overline{\lambda}(x-s)$ we first take the variation w.r.to $u_n(x)$.

$$\frac{\delta}{\delta u_n}\mathcal{L}[u_{n+1}(x)] = \frac{\delta}{\delta u_n}\mathcal{L}[u_n(x)] + \frac{\delta}{\delta u_n}\mathcal{L}[\overline{\lambda}(x)]\mathcal{L}\ [Lu_n(x) + N\tilde{u}_n(x) - g(x)]$$

we have,

$$\mathcal{L}[\delta u_{n+1}(x)] = \mathcal{L}[\delta u_n(x)] + \delta\mathcal{L}[\overline{\lambda}(x)]\ \mathcal{L}[u_n(x)]$$

We assume that L is a linear differential operator with constant coefficients given by

$$L(u) = a_n u^{(n)} + a_{(n-1)}u^{(n-1)} + a_{n-2}u^{(n-2)} + \ldots + a_2 u'' + a_1 u' + a_0 u$$

where a_i's are constant coefficients and also the coefficients contain only non constants terms of the form x^k, then the Laplace variational approach is still valid.
The Laplace transform of the n_{th} derivative is given by,

$$\mathcal{L}[a_n u^n] = a_n s^n\ \mathcal{L}[u] - a_n \sum_{k=1}^{n} s^{k-1}u^{n-k}(0)$$

so, the variation w.r.to u is

$$\delta\mathcal{L}[a_n u^n] = a_n s^n\ [\delta u]$$

The other terms also follow similar arguments so the equation reduces to

$$\begin{aligned}\mathcal{L}[\delta u_{n+1}] &= \mathcal{L}[\delta u_n] + \delta\mathcal{L}[\overline{\lambda}]\ \left(\sum_{k=1}^{n} a_k s^k\right)\ \mathcal{L}[\delta u_n] \\ &= \left[1 + \delta\mathcal{L}[\overline{\lambda}]\ \left(\sum_{k=0}^{n} a_k s^k\right)\right]\ \mathcal{L}[\delta u_n]\end{aligned}$$

by $\delta u_{n+1} = 0$ we have,

$$\mathcal{L}[\overline{\lambda}] = -\frac{1}{\sum_{k=0}^{n} a_k s^k} \tag{4.2}$$

Taking inverse Laplace, (4.2), we get the optimal value of $\overline{\lambda}$. Finally, we have the iteration formula

$$\mathcal{L}[u_{n+1}(x)] = \mathcal{L}[u_n(x)] + \mathcal{L}\left[\int_0^x \overline{\lambda}(x-s)\ [Lu_n(s) + N\tilde{u}_n(s) - g(s)]\ ds\right], \quad n = 0, 1, 2, \ldots.$$

Example : Consider the following boundary value problem,

$$u'' - uu'' - \frac{1}{2} - \frac{1}{2}u'^2 = 0,$$
$$u(0) = 0, u(1) = -0.5$$

The Laplace variational iteration correction functional is as follow, for n=0,1,2,....

$$\mathcal{L}[u_{n+1}(x)] = \mathcal{L}[u_n(x)] + \mathcal{L}\left[\int_0^x \overline{\lambda}(x-s)\left(u_n''(s) - \tilde{u}_n(s)\ \tilde{u}_n''(s) - \frac{1}{2} - \frac{1}{2}(\tilde{u}')_n^2(s)\right) ds\right]$$

Hence, we have

$$\mathcal{L}[u_{n+1}(x)] = \mathcal{L}[u_n(x)] + \mathcal{L}\left[\overline{\lambda}(x) * \left(u_n''(s) - \tilde{u}_n(s)\ \tilde{u}_n''(s) - \frac{1}{2} - \frac{1}{2}\tilde{u}_n'^2(s)\right) ds\right]$$
$$= \mathcal{L}[u_n(x)] + \mathcal{L}[\overline{\lambda}(x)]\Big(s^2\mathcal{L}[u_n(x)] - su_n(0) - u_n'(0) -$$
$$\mathcal{L}[\tilde{u}_n(x)\ \tilde{u}_n''(x)] - \frac{1}{2s} - \frac{1}{2}\mathcal{L}[\tilde{u}_n'^2(s)]\Big)$$

Taking the restricted variation w.r.to $y_n(x)$

$$\frac{\delta}{\delta u_n}\mathcal{L}[u_{n+1}(x)] = \frac{\delta}{\delta u_n}\mathcal{L}[u_n(x)] + \frac{\delta}{\delta u_n}\mathcal{L}[\overline{\lambda}(x)]\Big(s^2\mathcal{L}[u_n(x)] - su_n(0)$$
$$- u_n'(0) - \mathcal{L}[\tilde{u}_n(x)\ \tilde{u}_n''(x)] - \frac{1}{2s} - \frac{1}{2}\mathcal{L}[\tilde{u}_n'^2(s)]\Big)$$

Making the correctional functional stationary,

$$\mathcal{L}[u_{n+1}] = \mathcal{L}[\delta u_n] + \mathcal{L}[\overline{\lambda}]s^2\mathcal{L}[\delta u_n] = 0$$

this implies

$$1 + s^2\mathcal{L}[\overline{\lambda}] = 0 \quad or \quad \mathcal{L}[\overline{\lambda}] = -\frac{1}{s^2}$$

therefore, $\overline{\lambda}(x) = -x$, so the iteration formula become for n=0,1,2,....

$$\mathcal{L}[u_{n+1}(x)] = \mathcal{L}[u_n(x)] + \mathcal{L}\left[\int_0^x (s-x)\left(u_n''(s) - \tilde{u}(s)\ \tilde{u}_n''(s) - \frac{1}{2} - \frac{1}{2}\tilde{u}_n'^2(s)\right) ds\right]$$

$$\mathcal{L}[u_{n+1}(x)] = \mathcal{L}[u_n(x)] + \mathcal{L}[-x]\mathcal{L}\left[u_n''(s) - \tilde{u}(s)\ \tilde{u}_n''(s) - \frac{1}{2} - \frac{1}{2}\tilde{u}_n'^2(s)\right]$$

Let, the initial condition be

$$u_0 = u(0) + u'(0)x = Ax$$

Then,

$$\begin{aligned} \mathcal{L}[y_1] &= \mathcal{L}[Ax] + \mathcal{L}[-x]\ \mathcal{L}\left[\frac{-1}{2} - \frac{1}{2}A^2\right] \\ &= \frac{A}{s^2} + \frac{1}{2s^2}\left(\frac{A^2+1}{s}\right) \end{aligned}$$

Taking inverse Laplace we get,
$u_1 = Ax + \frac{1}{4}(A^2+1)x^2$
Applying boundary condition on u_1, $u(1) = -0.5$
So, we get A= -1
Hence,

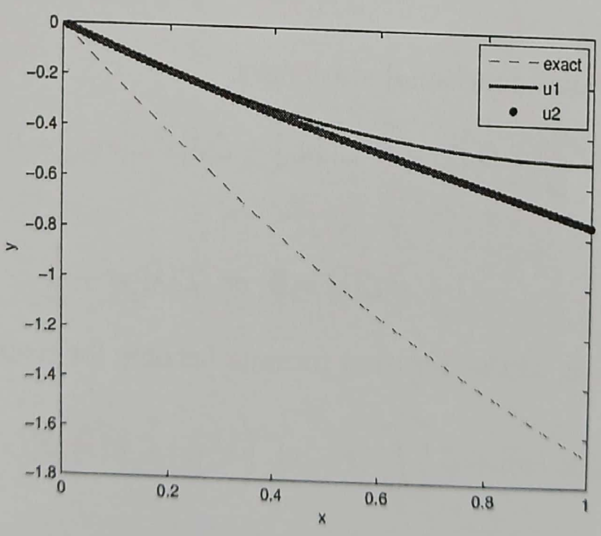
$$u_1 = -x + \frac{x^2}{2}$$

Similarly, we get

$$u_2 = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{12}$$

The exact solution of the given above equation in implicit form

$$\begin{aligned} f(x,u) = &-\sqrt{-u^2 + 0.381510869u + 0.618489131} - x + 0.5938731505 \\ &- 0.8092445655\ tan^{-1}\left(\frac{u - 0.1907554345}{\sqrt{-u^2 + 0.381510869u + 0.618489131}}\right) \end{aligned}$$



In above figure, we plot the exact solution together with the 1st, 2nd and 3rd iteration obtained by Laplace VIM.

CHAPTER 5

Fractional Partial Differential Equations

5.1 Introduction

The Fractional Calculus is a branch of mathematical analysis that studies the possibility of taking real and complex number powers of the differentiation and integration operator. So, fractional derivatives are derivatives of real numbers order. Basically, the fractional differential equation are generalization of differential equations.

Many phenomenons in engineering, physics, chemistry and other sciences can be described successfully using fractional calculus like nonlinear oscillations of earthquakes, the branch of physics concerned with the properties of sound, electromagnetism, electrochemistry, diffusion processes and signal processing can be modeled by fractional equations. Fractional Partial differential equations can be classified into two principal kinds:
(*i*) Space-fractional differential equation,
(*ii*) Time-fractional differential equation.

A time fractional differential equation is obtained from the classical diffusion or wave equation by replacing the first or second order time derivative by a fractional derivative of order $\alpha > 0$. Abstract Time fractional diffusion-wave equations are generalizations of classical diffusion and wave equations which are used in modeling practical phenomena of diffusion and wave in fluid flow others.An example of time fractional differential equation is Telegraph Equation:

$$D^{\alpha}u(x,t) + 2aDu(x,t) + Au(x,t) = 0, \qquad 0 < \alpha < 2$$

where, $D = d/dt$

5.2 Preliminaries

Definition : A real function f(x), $x > 0$ is said to be in the space C_{μ}, $\mu \in \mathbb{R}$, if there exist a real no $p(> \mu)$, such that $f(x) = x^{p} f_1(x)$ where, $f_1(x) \in C(0, \infty)$ and f(x) is said to be in the C_μ^m space if $f^m \in C_\mu$, where m is a natural number.

Riemann Liouville Fractional Integral : For f $\in C_\mu$, $\mu \geq -1$, the Riemanm fractional integral operator is defined as

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\zeta)^{\alpha-1} f(\zeta) d\zeta, \qquad \alpha \geq 0, \quad t \geq 0$$
$$J_t^0 f(t) = f(t)$$

Properties of Riemann Liouville fractional integral :
For f $\in C_\mu$, $\mu \geq -1$, α, $\beta \geq 0$ and $\gamma > -1$

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Riemann Liouville Fractional Derivative :
For f $\in C_\mu$, $\mu \geq -1$, the Riemanm fractional derivative operator is defined as

$$D_t^\alpha f(t) = \frac{d^n}{dt^n} D_t^{-(n-\alpha)} f(t), \qquad n-1 < \alpha \leq n, \quad t \geq 0, \; n \in \mathbb{N}$$
$$= \frac{d^n}{dt^n} J_t^{(n-\alpha)} f(t)$$

Caputo Time Fractional Derivative :
For f $\in C_\mu^n$, $\mu \geq -1$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, the Caputo sense fractional derivative operator is defined as

$$D_t^\alpha f(t) = J^{n-\alpha} D^n f(t)$$
$$= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\zeta)^{n-(\alpha+1)} f^n(\zeta) d\zeta, \qquad n-1 < \alpha \leq n, \quad t \geq 0, \; n \in \mathbb{N}$$

Remark 7. *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$D^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k!}, \qquad x > 0$$

Applications of fractional differential equations :

- To model non linear oscillations of earthquakes.
- To model fluid dynamic traffic models.
- For seepage flow in porous media.

5.3 Scheme of time fractional differential equation

Consider the equation,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = R[x]u(x,t) + q(x,t), \qquad t > 0,\ x \in \mathbb{R}$$

where, R[x] is differential operator in x, subject to below initial and boundary conditions for $0 < \alpha \le 1$,

$$u(x,0) = f(x)$$

$$u(x,t) \to 0 \quad as\ |x| \to \infty, \qquad t > 0$$

for $1 < \alpha \le 2$,

$$u(x,0) = f(x)$$

$$\frac{\partial u(x,0)}{\partial t} = g(x)$$

$$u(x,t) \to 0 \ as \ |x\,| \to \infty, \qquad t > 0$$

where, f(x), g(x) and q(x,t) are continuous functions and α is order of fractional differential equation.
Now, the correctional functional for F.D.E. according to variational method is :

$$\begin{aligned} u_{k+1}(x,t) &= u_k(x,t) + J_t^\beta\left[\lambda\left(\frac{\partial^\alpha u_k(x,\zeta)}{\partial t^\alpha} - R[x]\tilde{u}_k(x,\zeta) - q(x,\zeta)\right)\right] \\ &= u_k(x,t) + \frac{1}{\Gamma(\beta)}\int_0^t (t-\zeta)^{\beta-1}\lambda(\zeta)\left(\frac{\partial^\alpha u_k(x,\zeta)}{\partial t^\alpha} - R[x]\tilde{u}_k(x,\zeta) - q(x,\zeta)\right) d\zeta \end{aligned}$$

where J_t^β is the Riemann-Liouville Fractional Integral Operator of order β, where $\beta = \alpha - floor(\alpha)$ i.e. $\beta = \alpha + 1 - m$ with respect to the variable t.
So the correctional functional is :

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \left[\lambda(\zeta)\left(\frac{\partial^m u_k(x,\zeta)}{\partial t^m} - R[x]\tilde{u}_k(x,\zeta) - q(x,\zeta)\right)\right] d\zeta$$

Applying restricted variation on the above equation, we have

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta\int_0^t \left[\lambda(\zeta)\left(\frac{\partial^m u_k(x,\zeta)}{\partial t^m} - q(x,\zeta)\right)\right] d\zeta$$

So the lagrange multiplier is

$$\lambda = -1, \quad for \quad m = 1$$

$$\lambda = \zeta - t, \quad for \quad m = 2$$

Therefore, for m=1 $(0 < \alpha \leq 1)$ the correctional functional is

$$u_{k+1}(x,t) = u_k(x,t) - J_t^\alpha \left[\frac{\partial^\alpha u_k(x,t)}{\partial t^\alpha} - R[x]\tilde{u}_k(x,t) - q(x,t) \right]$$

for m=2 $(1 < \alpha \leq 2)$ we have the correctional functional

$$\begin{aligned} u_{k+1}(x,t) &= u_k(x,t) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\zeta)^{\alpha-2}(\zeta - t) \left(\frac{\partial^\alpha u_k(x,\zeta)}{\partial t^\alpha} - R[x]\tilde{u}_k(x,\zeta) - q(x,\zeta) \right) d\zeta \\ &= u_k(x,t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \left(\frac{\partial^\alpha u_k(x,\zeta)}{\partial t^\alpha} - R[x]\tilde{u}_k(x,\zeta) - q(x,\zeta) \right) d\zeta \end{aligned}$$

Finally, we obtain the following iteration formula

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha - 1)J_t^\alpha \left[\frac{\partial^\alpha \ u_k(x,t)}{\partial t^\alpha} - R[x]\tilde{u}_k(x,t) - q(x,t) \right]$$

Example :
We consider the one dimensional linear inhomogeneous fractional Klein-Gordan equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, \quad t > 0, \ x \in \mathbb{R}, \ 1 < \alpha \leq 2,$$

$$u(x,0) = 0, \quad u_t(x,0) = 0 \tag{5.1}$$

Solution:
By the above scheme, we have the correctional functional as

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha - 1)J_t^\alpha \left[\frac{\partial^\alpha \ u_k(x,t)}{\partial t^\alpha} - \frac{\partial^2 \ u_k(x,t)}{\partial x^2} + u - 6x^3 t - (x^3 - 6x)t^3 \right]$$

we choose initial approximation $u_0 = 0$
Thus the iterations are,

$$\begin{aligned} u_1(x,t) &= (\alpha - 1) \left[6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} \right] \\ u_2(x,t) &= 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 6(x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} \\ &\quad - (\alpha - 1)^2 \left[6(x^3 - 6x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 6(x^3 - 12x)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \right] + \ldots \\ &\ \vdots \end{aligned}$$

In the above equation, if we take α to be an integer then we get a particular Klein-Gordan Equation (ODE) for example, for $\alpha = 2$ we have the solution

$$\begin{aligned} u(x,t) =& x^3t^3 + (x^3 - 6x)\frac{6t^5}{\Gamma(6)} + 36x\frac{t^5}{\Gamma(6)} - 36x\frac{6t^7}{\Gamma(8)} \\ & - 6x^3\frac{t^5}{\Gamma(6)} - (x^3 - 6x)\frac{6t^7}{\Gamma(8)} + \dots \end{aligned}$$

In general, if we take the equation

$$(\Box + m^2)u = f(u, u', u")$$

where,

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

with initial condition : $u(x,0) = u_0(x), \quad \partial_1 u(x,0) = u_1(x)$
Taking m=2, n=3 and f=5u and the initial condition as :

$$u(x,0) = x_1x_2x_3, \quad \frac{\partial u}{\partial t}(x,0) = -x_1x_2x_3$$

The correctional functional is:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left[\frac{\partial^2 u_n}{\partial s^2}(x,s) + 4u_n(x,s) - 5u_n(x,s) - \sum_{j=1}^{3} \frac{\partial^2 \tilde{u}_n}{\partial x_j^2}(x,s) \right] ds$$

Applying the restricted variation and solving, we get the lagrange multiplier

$$\lambda(s) = -\frac{1}{2}(e^{(t-s)} - e^{(s-t)})$$

Taking the initial approximation $u_0(x,t) = x_1x_2x_3 - t(x_1x_2x_3)$
we get, $u_1(x,t) = x_1x_2x_3\ e^{-t}$
which is the exact solution.

Example : Consider, the linear fourth-order fractional integro-differential equation:

$$D^\alpha u(x) = x(1+e^x) + 3e^x + u(x) - \int_0^x u(t)dt$$

$$0 < x < 1, \ 3 < \alpha \le 4$$

$$u(0) = 0, \quad u''(0) = 2, \quad u(1) = 1 + e, \quad u''(1) = 3e$$

Solution : The exact solution for $\alpha = 4$,

$$u(x) = x(1 + xe^x)$$

Now, according to VIM the correction functional is :

$$u_{n+1}(x) = u_n(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6} J^\alpha \left(D^\alpha u_n(x) - x(1+e^x) - 3e^x + u(x) + \int_0^x u_n(t)dt \right)$$

Taking initial approximation $u_0(x) = 1 + Ax + x^2 + B\frac{x^3}{6}$,
where A and B are unknown parameters to be determined using given boundary conditions.
Now,

$$u_1(x) = u_0(x) - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6} J^\alpha \left(D^\alpha u_0(x) - 3 - 5x - \frac{5}{2}x^2 - x^3 - \frac{x^4}{6} - u_0(x) + \int_0^x u_n(t)dt \right)$$

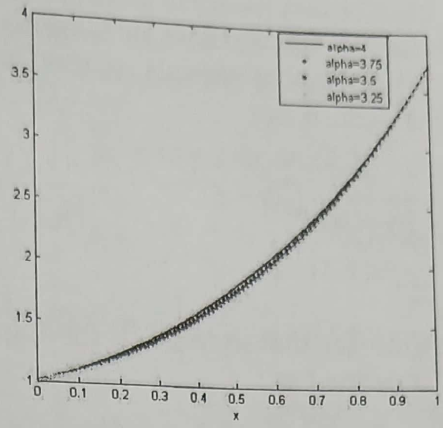
$$= 1 + Ax + x^2 + B\frac{x^3}{6} - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6} x^\alpha \left[-\frac{4}{\Gamma(\alpha+1)} - \frac{(4+A)x}{\Gamma(\alpha+2)} + \frac{(A-7)x^2}{\Gamma(\alpha+3)} - \frac{(4+B)x^3}{\Gamma(\alpha+4)} + \frac{(B-4)x^4}{\Gamma(\alpha+5)} \right]$$

We plot the graph for different values of α.

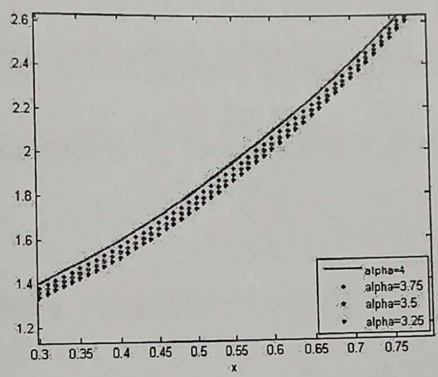
Table 5.1: Values of A and B for different values of α

	$\alpha = 3.25$	$\alpha = 3.5$	$\alpha = 3.75$
A	0.74031475165214	0.81642134845857	0.90761047783198
B	5.40426563043794	4.54507997600139	3.71105498995859

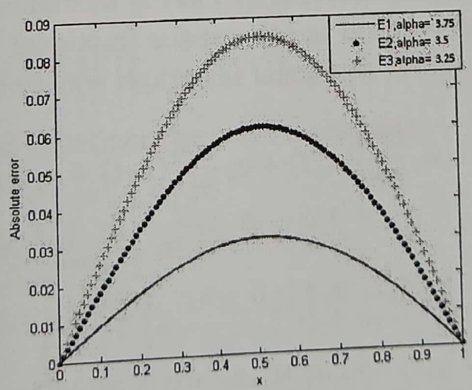
Graph for exact solution and different values of α.



We cant get difference between the lines from the above graph. After zooming the above graph we have the following graph for the different values of α.



The error for different values of α is shown by the graph below.



5.4 Scheme of space fractional differential equations

Space and Time Fractional Derivatives in Caputo Sense :
For n to be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as :

$$D_t^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} \frac{\partial^n u(x,\xi)}{\partial \xi^n} d\xi, & \text{if } n-1 < \alpha < n \\ \frac{\partial^n}{\partial t^n} u(x,t), & \text{if } \alpha = n \end{cases}$$

For m to be the smallest integer that exceeds β, the Caputo space fractional derivative operator of order $\beta > 0$ is defined as :

$$D_x^{\beta}u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^x (x-\xi)^{m-\beta-1} \frac{\partial^m u(\xi,t)}{\partial \xi^m} d\xi, & \text{if } m-1 < \beta < m \\ \frac{\partial^m}{\partial x^m} u(x,t), & \text{if } \beta = m \end{cases}$$

Laplace transform of fractional order derivative :
The laplace transform of fractional order derivative is defined as :

$$L[D_x^{\alpha} f(x)] = s^{\alpha} L\left[f(x)\right] - \sum_{k=0}^{n-1} s^{\alpha-k-1}[f^{(k)}(x)]_{x=0}, \quad n-1 < \alpha \leq n$$

Mittag-Leffler function:
The Mittag-Leffler function with two parameters is defined by

$$E_{\alpha,\beta}(u) = \sum_{n=0}^{\infty} \frac{u^n}{\Gamma(n\alpha+\beta)}, \quad \alpha, \beta, u \in C, Re(\alpha), Re(\beta) > 0$$

Laplace Variational Iteration Method(LVIM):

Laplace Variational Iteration Method($LVIM$) is a combined form of Laplace transform and variational iteration method to solve space fractional telegraph equations. Now, we will apply LVIM on a general fractional telegraph equation :

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}}(x,t)] = a_1 \frac{\partial^{\beta} u}{\partial t^{\beta}}(x,t) + a_2 \frac{\partial^{\gamma} u}{\partial t^{\gamma}}(x,t) + a_3 u(x,t) + f(x,t) \tag{5.2}$$

where, $1 < \alpha, \beta \leq 2$, $0 < \gamma \leq 1$, $x, t \geq 0$

$u(0,t) = h(t)$, $u_x(0,t) = g(t)$ and a_1, a_2, a_3 are constants.
The new approach of LVIM is as follows :
Step(i): We first remove the fractional derivative operator of order α w.r.t. x by first

applying laplace and then inverse laplace transform.
Step(ii): Now differentiate the equation obtained (without any fractional derivative w.r.t. x) w.r.t x partially. Then, using variational iteration correctional functional we find the lagrange multiplier, and simply calculate the iterations choosing initial approximation $u_0(x,t)$.
Apply laplace transform on both sides in equation (5.2),

$$L\left[\frac{\partial^\alpha u}{\partial x^\alpha}(x,t)\right] = L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t) + f(x,t)\right]$$

$$s^\alpha U(s,t) - s^{\alpha-1}u(0,t) - s^{\alpha-2}u_x(0,t) = L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t) + f(x,t)\right]$$

$$U(s,t) = \frac{1}{s^\alpha}\left[s^{\alpha-1}u(0,t) + s^{\alpha-2}u_x(0,t) + L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t) + f(x,t)\right]\right]$$

$$U(s,t) = \frac{1}{s}u(0,t) + \frac{1}{s^2}u_x(0,t) + \frac{1}{s^\alpha}\left[L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t) + f(x,t)\right]\right]$$

$$U(s,t) = \frac{1}{s}h(t) + \frac{1}{s^2}g(t) + \frac{1}{s^\alpha}\left[L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t) + f(x,t)\right]\right]$$

By taking the inverse laplace on both sides of above equation we get,

$$L^{-1}[U(s,t)] = L^{-1}\left[\frac{1}{s}h(t) + \frac{1}{s^2}g(t) + \frac{1}{s^\alpha}\left[L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t) + f(x,t)\right]\right]\right]$$

$$u(x,t) = h(t) + xg(t) + L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t) + f(x,t)\right]\right]\right]$$

Now the fractional derivative w.r.t. x is removed, and the dependent variable $u(x,t)$ in the left hand side is free of derivatives. So differentiating w.r.t. x, to get

$$u_x(x,t) = g(t) + \frac{\partial}{\partial x}\left[L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t) + f(x,t)\right]\right]\right]\right]$$

$$u_x(x,t) = g(t) + \frac{\partial}{\partial x}\left[L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[a_1\frac{\partial^\beta u}{\partial t^\beta}(x,t) + a_2\frac{\partial^\gamma u}{\partial t^\gamma}(x,t) + a_3 u(x,t)\right]\right]\right]\right]$$
$$+ \frac{\partial}{\partial x}\left[L^{-1}\left[\frac{1}{s^\alpha}L[f(x,t)]\right]\right]$$

Therefore, the correctional functional by VIM method is,

$$\begin{aligned} u_{n+1}(x,t) =& u_n(x,t) + \int_0^x \lambda\left[\frac{\partial u_n}{\partial \xi}(\xi,t) - g(t) - \right. \\ & \frac{\partial}{\partial \xi}\left[L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[a_1\frac{\partial^\beta \tilde{u}_n}{\partial t^\beta}(\xi,t) + a_2\frac{\partial^\gamma \tilde{u}_n}{\partial t^\gamma}(\xi,t) + a_3\tilde{u}_n(\xi,t)\right]\right]\right]\right] \\ & \left. + \frac{\partial}{\partial \xi}\left[L^{-1}\left[\frac{1}{s^\alpha}L[f(\xi,t)]\right]\right]\right] d\xi \end{aligned}$$

Applying restricted variation,

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^x \lambda \left[\frac{\partial u_n}{\partial \xi}(\xi,t)\right] d\xi$$

The lagrange multiplier is found to be : $\lambda(\xi) = -1$
So, the iteration formula for n=0,1,2,... is

$$\begin{aligned} u_{n+1}(x,t) =& u_n(x,t) - \int_0^x \left[\frac{\partial u_n}{\partial \xi}(\xi,t) - g(t) - \right. \\ & \frac{\partial}{\partial \xi}\left[L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[a_1 \frac{\partial^\beta \tilde{u}_n}{\partial t^\beta}(\xi,t) + a_2 \frac{\partial^\gamma \tilde{u}_n}{\partial t^\gamma}(\xi,t) + a_3 \tilde{u}_n(\xi,t)\right]\right]\right]\right] \\ & \left. + \frac{\partial}{\partial \xi}\left[L^{-1}\left[\frac{1}{s^\alpha} L[f(\xi,t)]\right]\right]\right] d\xi \end{aligned}$$

And, we can choose the initial approximation to be :

$$\begin{aligned} u_0(x,t) &= u(0,t) + x u_x(x,t) \\ &= h(t) + x g(t) \end{aligned}$$

So, the exact solution is calculated as : $u(x,t) = \lim_{n\to\infty} u_n(x,t)$

Examples :

Example 1.

Consider the space fractional homogenous equation

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) \qquad x,t \geq 0, 1 < \alpha \leq 2 \tag{5.3}$$

$$u(0,t) = e^{-t}, \quad u_x(0,t) = e^{-t}$$

Solution:

Applying Laplace transform w.r.to x to the given equation on both sides,

$$s^\alpha U(s,t) - s^{\alpha-1}u(0,t) - s^{\alpha-2}u_x(0,t) = \mathcal{L}\left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t)\right]$$

So,

$$U(s,t) = \frac{1}{s}e^{-t} + \frac{1}{s^2}e^{-t} + \frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t)\right]$$

Now taking inverse Laplace transform we get,

$$u(x,t) = e^{-t} + xe^{-t} + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t)\right]\right] \tag{5.4}$$

Differentiating equation (5.4) with respect to x

$$\frac{\partial u}{\partial x} = e^{-t} + \frac{\partial}{\partial x}\left[\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t)\right]\right]\right]$$

Applying VIM we obtain the following correctional functional

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(\xi)\Big[\frac{\partial u_n(\xi,t)}{\partial \xi} - e^{-t} - \frac{\partial}{\partial \xi}\Big[\mathcal{L}^{-1}\Big[\frac{1}{s^{\alpha}}\mathcal{L}\Big[\frac{\partial^2 \tilde{u}_n(\xi,t)}{\partial t^2} \\ \frac{\partial \tilde{u}_n(\xi,t)}{\partial t} + \tilde{u}_n(\xi,t)\Big]\Big]\Big]\Big]\, d\xi$$

Applying restricted variation and Integration by parts we have the value of Lagrange multiplier $\lambda(\xi) = -1$

So, the iteration formula is,

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x \left[\frac{\partial u_n(\xi,t)}{\partial \xi} - e^{-t} - \frac{\partial}{\partial \xi}\left[\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\dot{\mathcal{L}}\left[\frac{\partial^2 \tilde{u}_n(\xi,t)}{\partial t^2} \right.\right.\right.\right. \\ \left.\left.\left.\left. + \frac{\partial \tilde{u}_n(\xi,t)}{\partial t} + \tilde{u}_n(\xi,t)\right]\right]\right]\right] d\xi$$

By the above procedure we choose the initial approximation
$u_0(x,t) = u(0,t) + xu_x(0,t) = e^{-t} + xe^{-t}$
Then,

$$u_1(x,t) = e^{-t} + xe^{-t} + \frac{e^{-t}x^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^{-t}x^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$u_2(x,t) = e^{-t} + xe^{-t} + \frac{e^{-t}x^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^{-t}x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{e^{-t}x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{e^{-t}x^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$u_3(x,t) = e^{-t} + xe^{-t} + \frac{e^{-t}x^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^{-t}x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{e^{-t}x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{e^{-t}x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{e^{-t}x^{3\alpha}}{\Gamma(3\alpha+1)} \\ + \frac{e^{-t}x^{3\alpha+1}}{\Gamma(3\alpha+2)}$$

$$\vdots$$

Similarly we get

$$u_n(x,t) = e^{-t}\left[\sum_{k=1}^{n}\left(\frac{x^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{x^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)\right]$$

So the general solution is

$$\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\
&= e^{-t}\left[\sum_{k=0}^{\infty}\left(\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}+\frac{x^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)\right] \\
&= e^{-t}\left(E_{\alpha,1}(x^\alpha)+xE_{\alpha,2}(x^\alpha)\right)
\end{aligned}$$

For $\alpha = 2$,we have

$$\begin{aligned}
u(x,t) &= e^{-t}\left(E_{2,1}(x^2)+xE_{2,2}(x^2)\right) \\
&= e^{-t}\left[\sum_{k=0}^{\infty}\left(\frac{x^{2k}}{\Gamma(2k+1)}+\frac{x^{2k+1}}{\Gamma(2k+2)}\right)\right] \\
&= e^{-t}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right) \\
&= e^{x-t}
\end{aligned}$$

Example 2: Consider nonhomogenous fractional equation

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + u(x,t) - x^2 - t^2 - 2t \qquad x,t \geq 0,$$

$$1 < \alpha,\beta \leq 2,\ \frac{2}{3} < \gamma \leq 1$$

$$u(0,t) = t^2,\ \ u_x(0,t) = 0,\ 2 < \beta+\gamma \leq 3$$

Solution:
Applying Laplace transform w.r.to x to the given equation on both sides,

$$s^\alpha U(s,t) - s^{\alpha-1}u(0,t) - s^{\alpha-2}u_x(0,t) = \mathcal{L}\left[\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + u(x,t) - x^2 - t^2 - 2t\right]$$

So,

$$U(s,t) = \frac{t^2}{s} + \frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + u(x,t)\right] - \frac{2}{s^{3+\alpha}} - \frac{t^2}{s^{\alpha+1}} - \frac{2t}{s^{\alpha+1}}$$

Now taking inverse Laplace transform we get,

$$u(x,t) = t^2 + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + u(x,t)\right]\right] - \frac{2x^{2+\alpha}}{\Gamma(3+\alpha)} - t^2\frac{x^\alpha}{\Gamma(\alpha+1)} - 2t\frac{x^\alpha}{\Gamma(\alpha+1} \tag{5.5}$$

Differentiating equation (5.5) with respect to x,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left[\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^{\beta}u(x,t)}{\partial t^{\beta}}+\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}}+u(x,t)\right]\right]\right]-\frac{2x^{\alpha+1}}{\Gamma(2+\alpha)}-t^{2}\frac{x^{\alpha-1}}{\Gamma(\alpha)}-2t\frac{x^{\alpha-1}}{\Gamma(\alpha)}$$

Applying VIM, we obtain the following correctional functional

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^x\lambda(\xi)\left[\frac{\partial u_n(\xi,t)}{\partial\xi}-\frac{\partial}{\partial\xi}\left[\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^{\beta}\tilde{u}_n(x,t)}{\partial t^{\beta}}+\frac{\partial^{\gamma}\tilde{u}_n(x,t)}{\partial t^{\gamma}}+\tilde{u}_n(x,t)\right]\right]\right]+\frac{2\xi^{\alpha+1}}{\Gamma(2+\alpha)}+t^{2}\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}+2t\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\right]d\xi$$

After applying restricted variation and Integration by parts, we have the value of Lagrange multiplier $\lambda(\xi)=-1$

So, the iteration formula is,

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^x\left[\frac{\partial u_n(\xi,t)}{\partial\xi}-\frac{\partial}{\partial\xi}\left[\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^{\beta}\tilde{u}_n(\xi,t)}{\partial t^{\beta}}+\frac{\partial^{\gamma}\tilde{u}_n(\xi,t)}{\partial t^{\gamma}}+\tilde{u}_n(\xi,t)\right]\right]\right]+\frac{2\xi^{\alpha+1}}{\Gamma(2+\alpha)}+t^{2}\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}+2t\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\right]d\xi$$

By the above procedure we choose the initial approximation,

$u_0(x,t)=u(0,t)+xu_x(0,t)=t^2$

Then we find out fractional derivative with respect to t using the formula

$$D_t^{\beta}(u_0=t^2)=\frac{\Gamma(3)}{\Gamma(3-\beta)}t^{2-\beta},\qquad 2-1<\beta\leq 2$$
$$D_t^{\gamma}(u_0=t^2)=\frac{\Gamma(3)}{\Gamma(3-\gamma)}t^{2-\gamma},\qquad \frac{2}{3}<\gamma\leq 1$$

$$u_1(x,t)=t^2-\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{2t^{2-\beta}}{\Gamma(3-\beta)}+\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)}-2t\right)$$

$$u_2(x,t)=t^2-\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{2t^{2-\beta}}{\Gamma(3-\beta)}+\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)}-2t\right)-\frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\left(\frac{2t^{2-2\gamma}}{\Gamma(3-2\gamma)}-\frac{2t^{1-\gamma}}{\Gamma(2-\gamma)}+\frac{2t^{2-\beta}}{\Gamma(3-\beta)}+\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)}-2t\right)$$

$\vdots$

Similarly we get $u_n(x,t)$

$$u_n(x,t) = t^2 - 2x^2\sum_{k=1}^{n}\frac{(x^\alpha)^k}{\Gamma(k\alpha+3)} + \sum_{k=1}^{n}\frac{(x^\alpha)^k}{\Gamma(k\alpha+1)}\left(\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + \frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} - 2t\right)$$
$$+\sum_{k=2}^{n}\frac{(k-1)(x^\alpha)^k}{\Gamma(k\alpha+1)}\left(\frac{2t^{2-2\gamma}}{\Gamma(3-2\gamma)} - \frac{2t^{1-\gamma}}{\Gamma(2-\gamma)}\right) \qquad ,n \geq 2$$

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= t^2 + x^2(1 - 2E_{\alpha,3}(x^\alpha)) + (E_{\alpha,1}(x^\alpha) - 1)\left[\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + \frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} - 2t\right] \\ &+ (1 + E_{\alpha,1}^2(x^\alpha) - 2E_{\alpha,1}(x^\alpha))\left[\frac{2t^{2-2\gamma}}{\Gamma(3-2\gamma)} - \frac{2t^{1-\gamma}}{\Gamma(2-\gamma)}\right] \end{aligned}$$

For $\alpha = \beta = 2$ *and* $\gamma = 1$ we have the solution

$$\begin{aligned} u(x,t) &= t^2 + x^2 - 2 - 2x^2E_{2,3}(x^2) + 2E_{2,1}(x^2) + 0 \\ &= t^2 + x^2 \end{aligned}$$

This is the solution of particular values of α,β,γ.

CHAPTER 6

Applications of VIM

6.1 Image Restoration

6.1.1 Introduction

In Image restoration, we study to repair error distortions in an image, caused during the image creation process. Also, it is the operation of taking a corrupted noisy image and estimating the original clean image. Corruption may come in forms such as motion blur, noise and camera misfocus. The process of image restoration is referred as image deblurring or deconvolution.

6.1.2 Perona Malik Equation

The Perona Malik Equation was proposed in 1990 by Pietro Perona and Jitendra Malik which is a parabolic partial differential equation. So, on their names it is called Perona Malik Equation (P-M Equation). This equation could be applied as a new mathematical tool with great capabilities to image restoring for purposes like smoothing, restoration, segmentation, filtering or detecting edges by convolving the original image.

It is commonly believed that the Perona-Malik equation provides a potential algorithm for image segmentation, noise removing, edge detection, and image enhancement. The basic idea behind the PeronaŪMalik algorithm is to evolve an original image $u_0(x,y)$, under an edge controlled diffusion operator.

The original form of P-M equation is,

$$\frac{\partial u(x,y,t)}{\partial t} = \nabla.(c(x,y,t)\nabla u(x,y,t))$$

$$u(x,y,0) = u_0(x,y)$$

where, c(x,y,t) is diffusion factor and has two forms

$$c(x,y,t) = \frac{u}{\left(1+\frac{|\nabla u|^2}{k^2}\right)}$$

and

$$c(x, y, t) = \exp\Big(-\frac{|\nabla u|^2}{k^2}\Big)$$

Here, $u_0(x, y)$ is the original image which is to be restored and u(x,y,t), the solution obtained is the restored image.
We put our PDE in the VIM algorithm to obtain the solution.
Assume, the following non-linear and modified form of P-M Equation with given initial condition

$$\begin{aligned} u_t =& \frac{1}{u_x^2 + u_y^2}(u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}) \\ u(x, y, 0) =& \sqrt{x^2 + y^2} - 1 \end{aligned}$$

For solving this equation by VIM, correctional functional is:

$$\begin{aligned} u_{n+1}(x, y, t) =& u_n(x, y, t) \\ &+ \int_0^t \lambda(s)\Bigg[\frac{du_n(x, y, s)}{ds} - \frac{1}{\Big(\frac{\partial u_n}{\partial x}\Big)^2 + \Big(\frac{\partial u_n}{\partial y}\Big)^2} \\ &\Bigg[\Big(\frac{\partial u_n}{\partial y}\Big)^2 \frac{\partial^2 u_n}{\partial x^2} - 2\frac{\partial u_n}{\partial x}\frac{\partial u_n}{\partial y}\frac{\partial^2 u_n}{\partial x \partial y} + \Big(\frac{\partial u_n}{\partial x}\Big)^2 \frac{\partial^2 u_n}{\partial y^2}\Bigg]\Bigg] ds \end{aligned}$$

To make the correctional functional stationary w.r.to u_n, we have

$$\begin{aligned} \delta u_{n+1}(x, y, t) =& \delta u_n(x, y, t) + \\ &\delta \int_0^t \lambda(s) \Bigg[\frac{du_n(x, y, s)}{ds} - \frac{1}{\Big(\frac{\partial \tilde{u_n}}{\partial x}\Big)^2 + \Big(\frac{\partial \tilde{u_n}}{\partial y}\Big)^2} \\ &\Bigg[\Big(\frac{\partial \tilde{u_n}}{\partial y}\Big)^2 \frac{\partial^2 \tilde{u_n}}{\partial x^2} - 2\frac{\partial \tilde{u_n}}{\partial x}\frac{\partial \tilde{u_n}}{\partial y}\frac{\partial^2 \tilde{u_n}}{\partial x \partial y} + \Big(\frac{\partial \tilde{u_n}}{\partial x}\Big)^2 \frac{\partial^2 \tilde{u_n}}{\partial y^2}\Bigg]\Bigg] ds \\ =& \delta u_n(x, y, t) + \int_0^t \lambda(s)\delta\Big(\frac{\partial u_n}{\partial s}\Big) ds \end{aligned}$$

Using $\delta u_{n+1} = 0$, and applying integration by parts,

$$\delta u_n + \lambda(s)\delta u_n|_{s=t} - \int_o^t \lambda'(s)\, \delta u_n\, ds = 0$$

We obtain these equation,

$\lambda'(s) = 0, \quad 1 + \lambda(s)|_{s=t} = 0$

On solving we have,

$$\lambda(s) = -1$$

Now substituting the value of $\lambda(s)$ in correction functional,

$$\begin{aligned} u_{n+1}(x,y,t) =& u_n(x,y,t) \\ &+ \int_0^t (-1)\left[\frac{du_n(x,y,s)}{ds} - \frac{1}{\left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial y}\right)^2} \right. \\ &\left. \left[\left(\frac{\partial u_n}{\partial y}\right)^2 \frac{\partial^2 u_n}{\partial x^2} - 2\frac{\partial u_n}{\partial x}\frac{\partial u_n}{\partial y}\frac{\partial^2 u_n}{\partial x \partial y} + \left(\frac{\partial u_n}{\partial x}\right)^2 \frac{\partial^2 u_n}{\partial y^2}\right]\right] ds \end{aligned}$$

Choosing initial approximation,

$$u_0(x,y) = \sqrt{x^2+y^2} - 1$$

Therefore, the successive approximations are

$$u_1(x,y,t) = \frac{x^2+y^2+t-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$u_2(x,y,t) = \frac{2x^4+2x^2t-2x^2\sqrt{x^2+y^2}+4x^2y^2+2y^4+2y^2t-t^2-2y^2\sqrt{x^2+y^2}}{2(x^2+y^2)^{3/2}}$$

$\vdots$

u(x,y,t)$= \lim_{n\to\infty} u_n(x,y,t)$

Assume the solution of the given problem as an iteration $u_n(x,y,t)$ for some n. Therefore, for different values of t we get the result as the restored image after time t.

6.2 Biological Model

6.2.1 Introduction

In this chapter, we will solve the degenerate parabolic equation in the spatial diffusion of biological populations. Here, considering degenerate parabolic equation in biological population model :

$$\rho_t = \Phi(\rho_{xx}) + \Phi(\rho_{yy}) + \sigma(\rho), \qquad t \geq 0, \quad x,\ y\ \in \Re$$

with given initial condition $\rho(x,y,0)$, where ρ denotes the population density and σ represents the population supply due to births and deaths.

6.2.2 Population model of animals

In animals, movements are made generally either by mature animals or by young animals just reaching maturity moving out to establish breeding territory of their own. It is supposed that they will be directed towards nearby vacant territory. In this model therefore, movement will take place almost exclusively down the population density gradient, and will be much more rapid at high population densities than at low ones. To model the animal population model we take, $\Phi(\rho) = \rho^2$, and so the equation is :

$$\rho_t = \rho^2_{xx} + \rho^2_{yy} + \sigma(\rho), \qquad t \geq 0, \quad x,\ y \in \Re$$

Considering, general form of $\sigma(\rho) = h\rho^\alpha(1 - r\rho^\beta)$.
The general equation is :

$$\rho_t = \rho^2_{xx} + \rho^2_{yy} + h\rho^\alpha(1 - r\rho^\beta)$$

where α, β, h and r are real numbers.

Example :

$$\rho_t = \rho^2_{xx} + \rho^2_{yy} - \rho(1 + \frac{8}{9}\rho)$$

with initial condition, $\rho(x, y, 0) = \exp{(\frac{x+y}{3})}$

Solution :
The correctional functional according to VIM is :

$$\rho_{n+1}(x,y,t) = \rho_n(x,y,t) + \int_0^t \lambda\Big[(\rho_n(x,y,s))_s - (\tilde{\rho}^2_n(x,y,s))_{xx} - (\tilde{\rho}^2_n(x,y,s))_{yy} + \rho_n(x,y,s) + \frac{8}{9}(\tilde{\rho}^2_n(x,y,s))\Big] ds$$

Applying the restricted variation, we get

$$\delta\rho_n \ : \ 1 + \lambda(t) = 0$$

$$\delta\rho_n \ : \ \lambda' - \lambda = 0$$

On solving, we get $\lambda(s) = -\exp{(s-t)}$.

Choosing $\rho_0(x,y,t) = \rho(x,y,0) = \exp{(\frac{x+y}{3})}$
So,

$$\rho_1(x,y,t) = \exp(\frac{x+y}{3} - t)$$

and this is the exact solution.

Conclusion

From the above analysis, we have concluded that correction functional can be easily constructed by a general Lagrange multiplier, and the multiplier can be optimally identified by variational theory. The application of restricted variations in correction functional makes it much easier to determine the multiplier.
The initial approximation can be freely selected with unknown constants, which can be determined via various methods.
The first-order approximations are of extreme accuracy . Moreover, it converges to the exact solution. This technique is a very powerful tool for solving various differential equation.

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