Continuous-Time Dynamics

- The most generic/general for a smooth system:

"state" $\in \mathbb{R}^n$

$$\dot{x} = f(x, u)$$

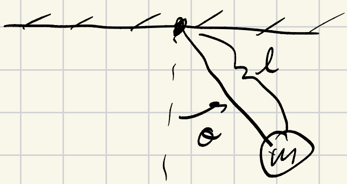
time derivative of state — "dynamics" — "input" $\in \mathbb{R}^m$

"configuration"/"pose" (not always a vector)

- For a mechanical system: $x = \begin{bmatrix} q \\ v \end{bmatrix}$

"velocity"

- Example (pendulum):



$$m\ell^2\ddot{\theta} + mg\ell\sin(\theta) = \tau$$

$$q = \theta, \quad v = \dot{\theta}, \quad u = \tau$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\theta) + \frac{1}{m\ell^2}u \end{bmatrix}}_{f(x,u)}$$

$q \in S^1$ (circle), $\quad x \in S^1 \times \mathbb{R}$ (cylinder)

* Control-Affine Systems

$$\dot{x} = \underbrace{f_0(x)}_{\text{"drift"}} + \underbrace{B(x)}_{\text{"input Jacobian"}} u$$

- most systems can be put in this form
- pendulum:

$$f_0(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\theta) \end{bmatrix}, \quad B(x) = \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix}$$

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* Manipulator Dynamics

$$\underbrace{M(q)}_{\text{mass matrix}} \dot{v} + \underbrace{C(q,v)}_{\substack{\text{"Dynamic bias"} \\ \text{(Coriolis + Gravity)}}} = \underbrace{B(q)}_{\text{Input Jacobian}} u + F \quad \text{"external forces"}$$

$$\underbrace{\dot{q} = G(q) v}_{\text{"velocity kinematics"}} \qquad \dot{x} = f(x,u) = \begin{bmatrix} G(q) v \\ M^{-1}(q) (B(q) u + F - C) \end{bmatrix}$$

- Pendulum:

$$M(q) = m\ell^2, \quad C(q,v) = mg\ell\sin(\theta), \quad B = I, \quad G = I$$

- All mechanical systems can be written this way
- This is just a way of re-writing the Euler-Lagrange equation for:

$$L = \underbrace{\tfrac{1}{2} v^T M(q) v}_{\text{Kinetic Energy}} - \underbrace{U(q)}_{\text{Potential Energy}}$$

* Linear Systems:

$$\dot{x} = A(t)x + B(t)u$$

- Called "time invariant" if $A(t) = A$, $B(t) = B$
- Called "time varying" otherwise
- Super important in control
- We often approximate nonlinear systems with linear ones:

$$\dot{x} = f(x, u) \Rightarrow A = \frac{\partial f}{\partial x}, \quad B = \frac{\partial f}{\partial u}$$