Last Time:

- Iterative Learning Control

Today:

- Stochastic Optimal Control

Stochastic Control:

- So far we have assumed we know the systems state perfectly.

- What happens when all we have are noisy measurements of quantities related to the state?

$$\underset{\text{measurements}}{y} = \underbrace{g(x, \underset{\text{noise}}{v})}_{\text{"measurement model"}}$$

Deterministic		Stochastic
x	$\longrightarrow$	$\underbrace{p(x \mid y)}_{\text{PDF of the state conditioned on the measurements}}$

* Stochastic Optimal Control Problem

$$\min_u E[J(x,u)]$$

- In principle, we can solve with DP

- In general, very hard

* LQG

- Special case we can solve in closed form

  Linear Dynamics

  Quadratic Costs

  Gaussian Noise

- Dynamics

$$X_{n+1} = Ax_n + Bu_k + \underbrace{w_n}_{\text{"Process noise"}}$$

$$y_n = Cx_n + \underbrace{v_n}_{\text{"measurement noise"}}$$

$$w_n \underset{\text{"Drawn from"}}{\sim} \underset{\substack{\text{Normal Distribution} \\ \text{(Gaussian)}}}{N}(0, W) \qquad v_n \sim N(\underset{\text{mean}}{0}, \underset{\text{Covariance}}{V})$$

* Multivariate Gaussian

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det(S)}} \exp\left(-\frac{1}{2}(x-\mu)^T S^{-1}(x-\mu)\right)$$

mean: $\mu = \hat{x} = E[x] \in \mathbb{R}^n$

Covariance: $S = E[(x-\mu)(x-\mu)^T] \in S^n_{++}$

$$E[f(x)] = \int_{(\text{All space})} f(x)\, p(x)\, dx$$

- Cost Function

$$J = E\left[\frac{1}{2}x_N^T Q_N x_N + \frac{1}{2}\sum_{n=0}^{N-1}(x_n^T Q x_n + u_n^T R u_n)\right]$$

- D.P. Recursion

$$V_N(x) = \frac{1}{2}E[x_N^T Q_N x_N] = \frac{1}{2}E[x_N^T P_N x_N]$$

$$V_{N-1}(x) = \min_u E\Big[\frac{1}{2}x_{N-1}^T Q x_{N-1} + \frac{1}{2}u^T R u +$$

$$\frac{1}{2}(A x_{N-1} + Bu + w_{N-1})^T P_N (A x_{N-1} + Bu + w_{N-1})\Big]$$

$$= \min_u \underbrace{E\left[\frac{1}{2}x_{N-1}^T Q x_{N-1} + \frac{1}{2}u^T R u + (Ax_{N-1} + Bu)^T P_N (Ax_{N-1} + Bu)\right]}_{\text{Standard LQR}}$$

$$+ \underbrace{E\Big[\cancel{(Ax_{N-1} + Bu)^T P_N w_{N-1}}^{\,0} + \cancel{w_{N-1}^T P_N (Ax_{N-1} + Bu)}^{\,0} + \underbrace{w_{N-1}^T P_N w_{N-1}}_{\text{Constant!}}\Big]}_{\text{Noise Terms}}$$

* Noise sample drawn at time k has nothing to do with the state (or control) at time k. X_n depends on W_{n-1} (and all previous w) but not on W_n or future W

⇒ Uncorrelated ⇒ Cross-correlation is zero

⇒ Noise term has no impact on control design!
(you just get a higher cost).

* "Certainty-Equivalence Principle"

- The optimal LQG controller is just LQR with x replaced by $E[x]$

* "Separation Principle"

- For LQG we can design an optimal feedback controller and an optimal estimator separately and then hook them together. The resulting feedback policy is optimal.

* Neither of these holds in general, but are still frequently used in practice to design sub-optimal policies.

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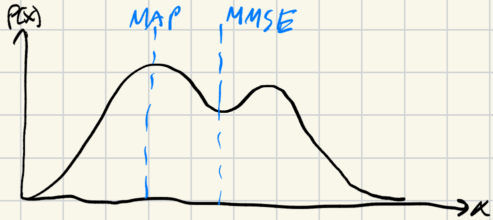
* Optimal State Estimator:

- What should I try to optimize?



- Maximum a-posteriori (MAP):

$$\arg\max(p(x))$$

- Minimum mean-squared error (MMSE) : "Least Squares"

$$\underset{\hat{x}}{\arg\min} \; E\left[(x-\hat{x})(x-\hat{x})\right]$$

* These are the same for a Gaussian!