Last Time:

- Optimization with Quaternions

Today:

- LQR with Quaternions
- Quadrotor Control

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* LQR with Quaternions

- Naively linearizing a system with a quaternion state results in an uncontrollable linear system
- We'll apply our quaternion differentiation tricks to LQR to make this work.
- Given a reference $\bar{x}_n$, $\bar{u}_n$ for a discrete-time system:

$$\bar{x}_{n+1} + \delta x_{n+1} = f(\bar{x}_n + \delta x_n, \bar{u}_n + \delta u_n)$$

$$\approx f(\bar{x}_n, \bar{u}_n) + \underbrace{A_n}_{\frac{\partial f}{\partial x}} \delta x_n + \underbrace{B_n}_{\frac{\partial f}{\partial u}} \delta u_n$$

- For the quaternion part of the state, we apply the attitude Jacobian to convert $\delta q \rightarrow \phi \in \mathbb{R}^3$

$$x = \begin{bmatrix} r \\ q \\ \sigma \\ \dot{r} \\ \omega \\ \dot{\sigma} \end{bmatrix} \quad \begin{matrix} x[1:3] \\ x[4:7] \\ x[8:n] \\ \vdots \end{matrix}$$

$$\underbrace{\begin{bmatrix} \delta x_{n}[1:3] \\ \phi_{n+1} \\ \delta x[8:n] \end{bmatrix}}_{\delta \tilde{x}_{n+1}} \simeq \underbrace{\begin{bmatrix} I & & 0 \\ & G(\bar{q}_{n+1}) & \\ 0 & & I \end{bmatrix}^{T}}_{E(\bar{x}_{n+1})} A_n \underbrace{\begin{bmatrix} I & & 0 \\ & G(\bar{q}_n) & \\ 0 & & I \end{bmatrix}}_{E(\bar{x}_n)} \begin{bmatrix} \delta x_n[1:3] \\ \phi_n \\ \delta x[8:n] \end{bmatrix}$$

$$+ E(\bar{x}_{n+1})^{T} B_n \, \delta u_n$$

- Once we have these "reduced" Jacobians $\tilde{A}_n$, $\tilde{B}_n$:

$$\tilde{A}_n = E(\bar{x}_{n+1})^{T} A_n E(\bar{x}_n), \quad \tilde{B}_n = E(\bar{x}_{n+1})^{T} B_n$$

we compute the LQR controller as usual

- When we run the controller, we calculate $\delta \tilde{x}$ before multiplying by K:

given x_n, $\delta \tilde{x}_n = \begin{bmatrix} x_n[1:3] - \bar{x}_n[1:3] \\ \phi(L(\bar{q}_n)^{T} q_n) \\ x_n[8:n] - \bar{x}_n[8:n] \end{bmatrix}$

whatever 3-parameter representation you like

$$u_n = \bar{u}_n - K_n \delta \tilde{x}_n$$

* Computing error/delta rotations!

- many possible conventions

- We will write it as rotation from body frame B to the reference/desired body frame R:

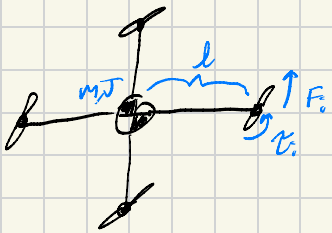
$$\Rightarrow \underbrace{{}^{N}Q^{B}}_{Q} = {}^{N}Q^{R} \, {}^{R}Q^{B} \quad \Rightarrow \quad ({}^{N}Q^{R})^{-1} \, {}^{N}Q^{B} = {}^{R}Q^{B}$$

$$Q = \bar{Q} \, \delta Q \quad \Rightarrow \quad \delta Q = \bar{Q}^{T} Q$$

- Using quaternions

$$\delta q = \bar{q}^{+} * q = L(q)^T q$$

* 3D Quadrotor



$$F_i = k_T u_i \quad , \quad u \in \mathbb{R}^4$$
$$\tau_i = k_\tau u_i$$

- State :

$$x = \begin{bmatrix} {}^N r \in \mathbb{R}^3 & \text{position} \\ {}^N_B q \in \mathbb{H} & \text{attitude} \\ {}^B v \in \mathbb{R}^3 & \text{linear velocity in } \mathcal{B} \text{ frame} \\ {}^B \omega \in \mathbb{R}^3 & \text{angular velocity} \end{bmatrix}$$

- Kinematics

$${}^N \dot{r} = {}^N v = Q\, {}^B v$$

$$\dot{q} = \frac{1}{2} q * \hat{\omega} = \frac{1}{2} L(q) H\, {}^B\omega = \frac{1}{2} G(q)\, {}^B\omega$$

- Translation Dynamics :

$$m\, {}^N \dot{v} = {}^N F \quad \leftarrow \text{total force}$$

↑ need to rotate into $\mathcal{B}$ frame

$${}^N v = Q\, {}^B v \Rightarrow {}^N \dot{v} = \underbrace{\dot{Q}}_{Q\hat{\omega}}\, {}^B v + Q\, {}^B \dot{v} = Q \hat{\omega}\, {}^B v + Q\, {}^B \dot{v}$$

$$\Rightarrow {}^B \dot{v} = Q^T\, {}^N \dot{v} - {}^B\omega \times {}^B v$$

$Q^T\, {}^N \dot{v}$: rotate into $\mathcal{B}$ frame

${}^B\omega \times {}^B v$: extra rotating-frame term

$$\Rightarrow {}^{B}\dot{V} = \frac{1}{m}{}^{B}F - {}^{B}\omega \times {}^{B}V$$

$${}^{B}F = Q^{T}\begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_T & k_T & k_T & k_T \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

- Rotation Dynamics

$$J\,{}^{B}\dot{\omega} + {}^{B}\omega \times J\,{}^{B}\omega = {}^{B}\tau$$ ← total torque

J: Inertia Matrix

Euler's Equation

$${}^{B}\tau = \begin{bmatrix} l k_T (u_2 - u_4) \\ l k_T (u_3 - u_1) \\ k_\tau (u_1 - u_2 + u_3 - u_4) \end{bmatrix}$$