1 Elements of Optics

1.1 Geometrical optics

1.1.1 Rays and ray transfer matrices

In geometrical optics, the light is described by rays. These latter are 2-dimensional vectors

$$\boldsymbol{r} = \begin{pmatrix} x \\ \theta \end{pmatrix} \tag{1.1}$$

whose first element is the distance from the optical axis, and the second is the angle between the ray and the optical axis. The propagation of light through an optical element is calculated with ray transfer matrices (see table 1.1)

$$\boldsymbol{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.2}$$

The new ray vector is calculated as the product between the matrix and the input vector

$$\boldsymbol{r}_1 = \boldsymbol{M} \cdot \boldsymbol{r}_0 = \begin{pmatrix} Ax + B\theta \\ Cx + D\theta \end{pmatrix} \tag{1.3}$$

The ray transfer matrix of a compound system is calculated as the product of the matrices of each component

$$\boldsymbol{M}_{\text{tot}} = \boldsymbol{M}_n \dots \boldsymbol{M}_2 \cdot \boldsymbol{M}_1 \tag{1.4}$$

Free space	Thin lens	Flat interface
$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$

Table 1.1: Ray transfer matrices of three common optical elements.

1.1.2 Scanning lens

A lens can be used to convert the angular displacement of a ray into a lateral displacement. Indeed, consider a ray originating from the optical axis with an angle θ. If its origin is distant f from a lens with focal length f, we find the following result

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} f\theta \\ 0 \end{pmatrix} \tag{1.5}$$

The exiting ray is laterally displaced by a quantity $f\theta$ and propagates with no angle.

1.2 Fourier optics

Maxwell's equations describe light propagation for all the components of the electric and magnetic fields

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \rho/\varepsilon & \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 & \boldsymbol{\nabla} \times \boldsymbol{B} &= \mu\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + \mu \boldsymbol{J} \end{aligned} \tag{1.6}$$

In a non-conducting medium ($\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$), they can combined to generate the wave equations

$$\nabla^2 \boldsymbol{E} = \frac{n^2}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad \nabla^2 \boldsymbol{B} = \frac{n^2}{c^2} \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \tag{1.7}$$

where $c = (\mu_0 \varepsilon_0)^{-\frac{1}{2}}$ is the speed of light in vacuum and n is the refractive index of the medium.

If the light is propagating through a homogeneous medium, the wave equations for each coordinate of the vectors are identical. Therefore, it is sufficient to solve only one scalar equation. In this case, the electromagnetic wave is effectively described by a scalar field that we identify with the symbol U. In the real world, no medium is perfectly homogeneous. Thus, a scalar description of light can be considered a good approximation only under certain conditions. The main assumption behind that theory is that the coupling between the differential equations is small, which is valid as long as the spatial inhomogeneities have a characteristic size much larger than the optical wavelength λ. Notably, different polarization components of light can still be described by the scalar theory as long as they can be treated independently.

1.2.1 Diffraction

If the scalar theory holds, we can consider a single wave equation for a scalar field U. The solution of such a differential equation is given by the *Huygens-Fresnel* integral. This latter describes how a field U_0 propagates into the U_z after a distance z

$$U_z(\boldsymbol{x}) = \frac{z}{i\lambda}\int_{\mathbb{R}^2} U_0(\boldsymbol{x}_0)\frac{\exp(ik\varrho)}{\varrho^2}\,\mathrm{d}\boldsymbol{x}_0 \tag{1.8}$$

where the position vector is $\boldsymbol{x} = (x, y)$ – the location on a plane orthogonal to the optical axis z – and ϱ is the distance between two planes, assumed to be much greater than λ, defined as

$$\varrho = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2} = z\sqrt{1 + \left(\frac{x-x_0}{z}\right)^2 + \left(\frac{y-y_0}{z}\right)^2} \tag{1.9}$$

The diffraction integral can be simplified using the proper approximation. The choice for this latter depends on the value of

$$F = \frac{D^2}{z\lambda} \tag{1.10}$$

called the *Fresnel number*. In this definition, D is the linear size of field at the starting plane.

If $F \gtrsim 1$ the diffraction takes place in the *near-field* region and the propagation is better described by the Fresnel approximation. If $F \ll 1$ the diffraction takes place in the *far-field* region. In this case the Fresnel approximation still holds, but it can be further simplified with the Fraunhofer approximation.

1.2.1.1 Fresnel diffraction

Under the *paraxial approximation*, namely the assumption that diffraction angles are small with respect to the optical axis, we can expand ϱ in series

$$\varrho \sim z\left[1 + \frac{1}{2}\left(\frac{x-x_0}{z}\right)^2 + \frac{1}{2}\left(\frac{y-y_0}{z}\right)^2\right] \tag{1.11}$$

and replace it in equation 1.8. Keeping only the linear term for the denominator and up to the quadratic term for the argument of the exponential, we obtain

$$U_z(\boldsymbol{x}) = \frac{e^{ikz}}{i\lambda z}\int_{\mathbb{R}^2} U_0(\boldsymbol{x}_0)\exp\left[\frac{ik}{2z}(\boldsymbol{x}-\boldsymbol{x}_0)^2\right]\mathrm{d}\boldsymbol{x}_0 \tag{1.12}$$

This equation is known as the *Fresnel diffraction integral*, which can be seen as the convolution

$$U_z(\boldsymbol{x}) = \left[U_0(\boldsymbol{x}_0) * F_z(\boldsymbol{x}_0)\right](\boldsymbol{x}) \tag{1.13}$$

where F_z the Fresnel convolution kernel

$$F_z(x,y) = \frac{e^{ikz}}{i\lambda z} \exp\left[\frac{ik}{2z}(x^2+y^2)\right] \tag{1.14}$$

Interestingly, this kernel has a simple Fourier transform which greatly simplifies the analytical and numerical calculations of the propagation of light in free space

$$\hat{F}_z(\nu_x,\nu_y) = e^{ikz} \exp\left[-i\pi\lambda z(\nu_x^2+\nu_y^2)\right] \tag{1.15}$$

where $\boldsymbol{\nu} = (\nu_x, \nu_y)$ are the spatial frequencies.

1.2.1.2 Fraunhofer diffraction

At very large distances, the diffraction formula can be further simplified. Indeed, if we can neglect the quadratic terms of the argument of the exponential of equation 1.12, we obtain

$$U_z(\boldsymbol{x}) = \frac{e^{ikz}e^{\frac{ik}{2z}(x^2+y^2)}}{i\lambda z} \int_{\mathbb{R}^2} U_0(\boldsymbol{x}_0) \exp\left(i\frac{2\pi}{\lambda z}\boldsymbol{x}\cdot\boldsymbol{x}_0\right) \mathrm{d}\boldsymbol{x}_0 \tag{1.16}$$

This result is known as the *Fraunhofer diffraction integral*. Aside from negligible multiplicative factors, this integral is the Fourier transform of the field U_0 evaluated at spatial frequencies $\boldsymbol{\nu} = \left(\frac{x}{\lambda z}, \frac{y}{\lambda z}\right)$

$$U_z(\boldsymbol{x}) \propto \mathcal{F}\{U_0(\boldsymbol{x}_0)\}\left(\frac{\boldsymbol{x}}{\lambda z}\right) \tag{1.17}$$

1.2.2 Fourier transforming property of the lenses

The phase transformation applied by a lens to a field is

$$t_l(\boldsymbol{x}) = \exp\left(-\frac{ik}{2f}\boldsymbol{x}^2\right) \tag{1.18}$$

Notably, this transmission function has the expression of the Fresnel propagation kernel evaluated at $z = -f$, except for multiplicative constants

$$t_l(\boldsymbol{x}) \propto F_{-f}(\boldsymbol{x}) \tag{1.19}$$

Now, we calculate the effect of a lens with focal length f and a free-space propagation of the same length on a field U_0 originating from a distance z behind the lens. The result U_1 is calculated as follows

$$U_1 = [(U_0 * F_z)\cdot F_{-f}] * F_f \tag{1.20}$$

The above equation can be rewritten in frequency space by exploiting the convolution property of the Fourier transform

$$\hat{U}_1(\boldsymbol{\nu}') = \Big(\Big[\hat{U}_0(\boldsymbol{\nu}) \cdot \hat{F}_z(\boldsymbol{\nu}) \Big] * \hat{F}_{-f}(\boldsymbol{\nu}) \Big)(\boldsymbol{\nu}') \cdot \hat{F}_f(\boldsymbol{\nu}') = \\ = \int_{\mathbb{R}^2} \hat{U}_0(\boldsymbol{\nu}) \exp\left[-i\pi\lambda(z-f)\nu^2\right] \exp\left[-i2\pi\lambda f \boldsymbol{\nu}\cdot\boldsymbol{\nu}'\right] \mathrm{d}\boldsymbol{\nu} \tag{1.21}$$

Transforming back into real space, we obtain

$$U_1(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{U}_0(\boldsymbol{\nu}) \exp\left[-i\pi\lambda(z-f)\nu^2\right] \underbrace{\int_{\mathbb{R}^2} \exp\left[-i2\pi\lambda f \boldsymbol{\nu}\cdot\boldsymbol{\nu}'\right] \exp\left[i2\pi\boldsymbol{x}\cdot\boldsymbol{\nu}'\right] \mathrm{d}\boldsymbol{\nu}'}_{\delta(\boldsymbol{x}-\boldsymbol{\nu}\lambda f)} \mathrm{d}\boldsymbol{\nu} = \\ = \hat{U}_0\left(\frac{\boldsymbol{x}}{\lambda f}\right) \exp\left[\frac{ik}{2f}\left(1-\frac{z}{f}\right)\boldsymbol{x}^2\right] \tag{1.22}$$

Interestingly, the field at the focal plane of the lens is the Fourier transform of the input field evaluated at spatial frequencies $\boldsymbol{\nu} = (\frac{x}{\lambda f}, \frac{y}{\lambda f})$, aside from a phase factor. when the distance between the input and the lens is matching the focal length, i.e. $z = f$, the phase factor equals to 1 and the result is exactly the Fourier transform. In other words, the field at the focal plane can be calculated as as Fraunhofer diffraction evaluated at $z = f$.

1.3 Vectorial Optics

1.3.1 Polarization

The solution of Maxwell's equations in vacuum (or in a homogeneous insulating material) can be written as a superposition of the following plane waves:

$$\boldsymbol{E}(z,t) = \begin{pmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y} \end{pmatrix} e^{i(kz-\omega t)} \tag{1.23}$$

where the amplitude term is described by a 2-vector called phasor. The polarization state of light depends on the relative amplitude and phase of the two components on the phasor.

1.3.1.1 Jones formalism

The normalized phasor is called *Jones vector*, which can be equivalently written in the linear or circular basis (see Table 1.2). Namely, a generic Jones vector $|P\rangle$ is

$$|P\rangle = c_H |H\rangle + c_V |V\rangle = \tag{1.24}$$
$$= c_R |R\rangle + c_L |L\rangle \tag{1.25}$$

where the coefficients c are complex numbers. Thus, the polarized electric field is

Linear basis		Diagonal basis		Circular basis	
$\|H\rangle$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\|D\rangle$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\|R\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$
$\|V\rangle$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\|A\rangle$	$\begin{pmatrix}1\\-1\end{pmatrix}$	$\|L\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

Table 1.2: The most common bases used to represent a Jones vector with the corresponding Ket notation.

$$|E\rangle = E_0 e^{ikz - i\omega t} |P\rangle \tag{1.26}$$

Anisotropic optical elements can act either on the amplitude or the phase of a polarization state. Elements acting only on the amplitude are called *polarizers* and those acting only on the phase are called *phase retarders* or *wave plates*. Such elements are described by the 2×2 Jones matrices of Table 1.3. Notably, Jones Matrices are unitary and with determinant 1. In other words, they are members of the SU(2) group.

Linear Polarizer	Phase Retarder
$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$\begin{pmatrix}e^{ik\delta} & 0\\0 & 1\end{pmatrix}$

Table 1.3: Jones matrices of optical elements aligned horizontally.

Optical elements act differently depending on the orientation of their optical axis. The matrix J of a rotated optical element can be calculated as follows

$$J(\theta) = R(-\theta) J R(\theta) \tag{1.27}$$

where the rotation matrix is

$$R(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{1.28}$$

Two wave plates are of particular interest for their practical applications. The matrix of a half-wave plate can be obtained by assuming that the thickness of the retarder is $\delta = \lambda/2$. In this case, the phase difference of the two polarization components is π. Similarly, the matrix of a quarter wave-plate can be obtained assuming $\delta = \lambda/4$. The corresponding phase difference is $\pi/2$.

With some algebra, we can prove that a half-wave plate rotated by 45° rotates a linearly polarization state by 90°.

$$J_{\lambda/2}(45^\circ)|H\rangle = |V\rangle \qquad J_{\lambda/2}(45^\circ)|V\rangle = |H\rangle \tag{1.29}$$

Similarly, a quarter-wave plate rotated by $45°$ converts a linearly polarization state into a circularly polarization state.

$$J_{\lambda/4}(45°)\,|H\rangle = |R\rangle \qquad J_{\lambda/4}(45°)\,|V\rangle = |L\rangle \tag{1.30}$$

1.3.1.2 Stokes formalism

The Jones formalism is about the fields, while the Stokes formalism is about the intensity. A Stokes vector has four entries

$$\boldsymbol{s} = (s_0, s_1, s_2, s_3)^T \tag{1.31}$$

which are the Stokes parameters, calculated as

$$s_0 = |\langle E|E\rangle|^2 \tag{1.32}$$
$$s_1 = |\langle E|H\rangle|^2 - |\langle E|V\rangle|^2 \tag{1.33}$$
$$s_2 = |\langle E|D\rangle|^2 - |\langle E|A\rangle|^2 \tag{1.34}$$
$$s_3 = |\langle E|R\rangle|^2 - |\langle E|L\rangle|^2 \tag{1.35}$$

In other words, the first component describes the total intensity of light, while the other components describe the excess of linear, diagonal, and circular polarization, respectively. Importantly, the stokes parameters are not linearly independent. Namely, the following relation applies

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 \tag{1.36}$$

If normalized with respect to s_0, the last three elements of the Stokes vector can be written as

$$s_1 = \cos(2\psi)\cos(2\chi) \tag{1.37}$$
$$s_2 = \sin(2\psi)\cos(2\chi) \tag{1.38}$$
$$s_3 = \sin(2\chi) \tag{1.39}$$

where ψ is the azimuthal angle and χ is the ellipticity angle. Namely, they parameterize in spherical coordinates the surface of a unit sphere, known as *Poincaré sphere*. A point on the surface of the Poincaré sphere uniquely defines a polarization state.

Optical elements can be described as matrices also using Stokes formalism. In this case, they become 4×4 matrices and are known as *Mueller* matrices M. They can be obtained from the corresponding Jones Matrices J using the following transformation

$$M = A(J \otimes J^*)A^{-1} \tag{1.40}$$

where $\otimes$ denotes the Kronecker product and

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix} \tag{1.41}$$

Two examples are shown in Table 1.4. Note that Mueller matrices have always real entries.

Linear Polarizer	Phase retarder
$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos k\delta & \sin k\delta \\ 0 & 0 & -\sin k\delta & \cos k\delta \end{pmatrix}$

Table 1.4: Mueller matrices of optical elements aligned horizontally.

Once again, we can find the matrix of a rotated component applying the following transformation

$$M(\theta) = R(-\theta) M R(\theta) \tag{1.42}$$

where, in this case, the rotation matrix is

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\theta) & \sin(2\theta) & 0 \\ 0 & -\sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.43}$$

1.4 Quantum optics

The Hamiltonian of a harmonic oscillator is

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{q}^2}{2} \tag{1.44}$$

where $\hat{p}$ and $\hat{q}$ are, respectively, the momentum and position operators, which follow the canonical commutator relation

$$[\hat{q}, \hat{p}] = \hbar \tag{1.45}$$

They are conveniently rewritten into the following dimensionless operators

$$\hat{X} = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{q} \tag{1.46}$$

$$\hat{Y} = \left(\frac{1}{2m\hbar\omega}\right)^{1/2} \hat{p} \tag{1.47}$$

known as the *field quadrature operators*. The Hamiltonian can be rewritten using the quadrature operators as

$$\hat{\mathcal{H}} = \hbar\omega\left(\hat{X} + \hat{Y}\right) \tag{1.48}$$

The quadrature operators can be identified with the electric and magnetic part of an oscillating electromagnetic field. Equivalently, they can be seen as the in-phase and the in-quadrature part of the electric field. Namely, the operator representing the electric field with a specific polarization and wavelength is

$$\hat{E}(\boldsymbol{r}, t) = \left(\frac{2\hbar\omega}{\varepsilon_0 V}\right)^{1/2}\left(\hat{X}\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) + \hat{Y}\sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})\right) \tag{1.49}$$

We now define the following operators

$$\hat{a} = \hat{X} + i\hat{Y} \tag{1.50}$$

$$\hat{a}^\dagger = \hat{X} - i\hat{Y} \tag{1.51}$$

known, respectively, as the *destruction* and *creation* operators. Using those operators, the electric field can be rewritten as

$$\hat{E}(\boldsymbol{r}, t) = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}\left(\hat{a}e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} + \hat{a}^\dagger e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\right) \tag{1.52}$$

Namely, the destruction and creation operators are proportional, respectively, to the positive and negative frequency part of the electric field. The Hamiltonian is reshaped as

$$\hat{\mathcal{H}} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \tag{1.53}$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator. The eigenstates $|n\rangle$ are called number states. The corresponding Eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{1.54}$$

Namely, the energy levels are discrete with spacing $\hbar\omega$.

Each longitudinal mode and polarization mode of an electromagnetic field is described by the Hamiltonian 1.53 whose eigenstates are known as *photons*.