2

Image Formation Theory

2.1 Linear Space-Invariant Systems

A Linear Space-Invariant (LSI) system is a map

$$S:\ f \mapsto g \tag{2.1}$$

defined by the following properties

- **Linearity**: the output g is linear in the input f

$$g = S[f] \quad \Rightarrow \quad S[a \cdot f_1 + b \cdot f_2] = a \cdot S[f_1] + b \cdot S[f_2] = a \cdot g_1 + b \cdot g_2 \tag{2.2}$$

- **Shift-invariance**: the output of a shifted input is a shifted output

$$g(x) = S[f(x)] \quad \Rightarrow \quad g(x - x_0) = S[f(x - x_0)] \tag{2.3}$$

Any input function can be written as

$$f(x) = f(x) * \delta(x) = \int_{\mathbb{R}} f(\chi)\delta(x - \chi)\,\mathrm{d}\chi \tag{2.4}$$

Thus,

$$g(x) = \int_{\mathbb{R}} f(\chi)S[\delta(x - \chi)]\,\mathrm{d}\chi = \int_{\mathbb{R}} f(\chi)h(x - \chi)\,\mathrm{d}\chi = f(x) * h(x) \tag{2.5}$$

where $h(x)$ is the impulse response of the LSI system.

2.2 Geometrical Optics

With one or more lenses, it is possible to generate an image, namely to generate a rescaled copy of rays of light. In this section, we discuss imaging systems in terms of ray transfer matrices.

2.2.1 Single lens system

Consider a system composed of a single lens. Taking also into account the space before and after the lens, we find the following matrix

$$\begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{z_2}{f} & z_1 + z_2 - \frac{z_1 z_2}{f} \\ -\frac{1}{f} & 1 - \frac{z_1}{f} \end{pmatrix} \tag{2.6}$$

In order to have the formation of an image, the B element of the matrix has to be zero. Thus, the following condition has to be verified

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f} \tag{2.7}$$

Consequently, the A element equals $-\frac{z_2}{z_1}$ and can be interpreted as the magnification factor.

2.2.2 Two lenses system

Consider a system composed by two lenses separated by a distance d. The corresponding matrix is

$$\begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} =$$

$$= \frac{1}{f_1 f_2} \begin{pmatrix} f_1 f_2 + d z_2 - f_2 d - f_2 z_2 - f_1 z_2 & f_1 f_2 d + f_1 f_2 z_1 + f_1 f_2 z_2 - f_2 z_1 z_2 - f_1 z_1 z_2 + d z_1 z_2 - f_2 d z_1 - f_1 d z_2 \\ -f_1 - f_2 + d & f_2 f_1 + d z_1 - f_1 d - f_1 z_1 - f_2 z_1 \end{pmatrix} \tag{2.8}$$

It is easy to verify that the B element is zero if $z_1 = f_1$ and $z_2 = f_2$. Moreover, if $d = f_1 + f_2$ the C element is also zero, meaning that the system is afocal. In this case, the lateral magnification is $A = -f_2/f_1$ and the angular magnification is $D = -f_1/f_2$.

2.3 Fourier Optics

We now consider a system composed by two lenses – with focal length f_1 and f_2 – separated by a distance d. We consider the starting field at a distance z_1 from the first lens and the output field at a distance z_2 from the second lens. The propagation is calculated as follows

$$U_1 = ((((U_0 * F_{z_1}) \cdot F_{-f_1}) * F_d) \cdot F_{-f_2}) * F_{z_2} \tag{2.9}$$

The Fourier transform of the above equation is

$$\hat{U}_1 = \left(\left(\left(\left(\hat{U}_0 \cdot \hat{F}_{z_1} \right) * \hat{F}_{-f_1} \right) \cdot \hat{F}_d \right) * \hat{F}_{-f_2} \right) \cdot \hat{F}_{z_2} \tag{2.10}$$

Explicitly, it is written as

$$\hat{U}_1(\boldsymbol{\nu}'') = \int_{\mathbb{R}^4} \hat{U}_0(\boldsymbol{\nu}) \exp\left[-i\pi\lambda \Big(z_1 \boldsymbol{\nu}^2 - f_1(\boldsymbol{\nu}' - \boldsymbol{\nu})^2 + d\boldsymbol{\nu}'^2 - f_2(\boldsymbol{\nu}'' - \boldsymbol{\nu}')^2 + z_2 \boldsymbol{\nu}''^2 \Big) \right] \mathrm{d}\boldsymbol{\nu}\, \mathrm{d}\boldsymbol{\nu}' \tag{2.11}$$

Assuming $z_1 = f_1$ and $z_2 = f_2$, we have

$$\hat{U}_1(\boldsymbol{\nu}'') = \int_{\mathbb{R}^4} \hat{U}_0(\boldsymbol{\nu}) e^{-i\pi\lambda(d-f_1-f_2)\nu'^2} e^{-i2\pi\lambda f_1 \nu\nu'} e^{-i2\pi\lambda f_2 \nu'\nu''}\, \mathrm{d}\nu\, \mathrm{d}\nu' \tag{2.12}$$

Transforming back to the real space, we obtain

$$\begin{aligned} U_1(\boldsymbol{x}) &= \int_{\mathbb{R}^4} \hat{U}_0(\boldsymbol{\nu}) e^{-i\pi\lambda(d-f_1-f_2)\nu'^2} e^{-i2\pi\lambda f_1 \boldsymbol{\nu}\cdot\boldsymbol{\nu}'} \underbrace{\int_{\mathbb{R}^2} e^{-i2\pi\lambda f_2 \boldsymbol{\nu}'\cdot\boldsymbol{\nu}''} e^{i2\pi \boldsymbol{x}\cdot\boldsymbol{\nu}''}\, \mathrm{d}\boldsymbol{\nu}''}_{\delta(\boldsymbol{x}-\boldsymbol{\nu}'\lambda f_2)}\, \mathrm{d}\boldsymbol{\nu}\, \mathrm{d}\boldsymbol{\nu}' = \\ &= e^{-i\pi\lambda(d-f_1-f_2)\frac{r^2}{\lambda^2 f_2^2}} \int_{\mathbb{R}^2} \hat{U}_0(\boldsymbol{\nu}) e^{-i2\pi\boldsymbol{\nu}\cdot\boldsymbol{x} f_1/f_2}\, \mathrm{d}\boldsymbol{\nu} = \\ &= U_0\left(-\frac{f_1}{f_2}\boldsymbol{x} \right) e^{-i\pi\lambda(d-f_1-f_2)\frac{r^2}{\lambda^2 f_2^2}} \end{aligned} \tag{2.13}$$

This result shows that under imaging conditions (i.e. $z_1 = f_1$ and $z_2 = f_2$) the amplitude of the output field U_1 is a copy of the amplitude of the initial field U_0, but inverted and rescaled by the magnification factor $M = \frac{f_2}{f_1}$. This implies that the light intensity at the two planes is identical, thus at z_2 there is an *image* of the plane at z_1. If the distance between the two lenses is equal to $d = f_1 + f_2$, then the two fields are identical both in amplitude and in phase. In this case the two planes are said to be *optically conjugated*.

2.3.1 Impulse response of an imaging system

We now consider the effect of the finite size of the lenses. The pupil function describes the limited aperture of a lens, and it is defined as follows

$$P(r) = \begin{cases} 1 & \text{if } r \le R \\ 0 & \text{if } r > R \end{cases} \tag{2.14}$$

where $r = \sqrt{x^2 + y^2}$ and R is the radius of the lens. We now calculate the propagation of a point-like source $U_0(\boldsymbol{x}) = \delta(\boldsymbol{x})$ through a two-lenses imaging system. As shown before, the value of d has no effect on the intensity at the image plane. Therefore, we choose $d = 0$ for the sake of simplicity. The output field is

$$H = ((\delta * F_{z_1}) \cdot P \cdot F_{-f_1} \cdot F_{-f_2}) * F_{z_2} \tag{2.15}$$

Explicitly

$$H(\boldsymbol{x}) = \int_{\mathbb{R}^2} P(\boldsymbol{x}') \exp\left[\frac{ik}{2}\left(\frac{1}{nz_1} - \frac{1}{f_1} - \frac{1}{f_2} + \frac{1}{z_2}\right)r'^2\right] \exp\left[-\frac{ik}{z_2}\boldsymbol{x}\cdot\boldsymbol{x}'\right] \mathrm{d}\boldsymbol{x}' \tag{2.16}$$

Where we neglected pure multiplicative phase factors. The imaging condition implies

$$\frac{1}{nz_1} + \frac{1}{z_2} - \frac{1}{f_1} - \frac{1}{f_2} = 0 \tag{2.17}$$

Therefore, we impose $z_2 = f_2$ and $z_1 = f_1/n + z$. Using a McLaurin expansion, we get $\frac{1}{nz_1} \sim \frac{1}{f_1}\left(1 - n\frac{z}{f_1}\right)$. By substituting these values, we get

$$H(\boldsymbol{x}) = \int_{\mathbb{R}^2} P(r') \exp\left[-\frac{ik}{2}\frac{nzr'^2}{f_1^2}\right] \exp\left[-\frac{ik}{f_2}\boldsymbol{x}\cdot\boldsymbol{x}'\right] \mathrm{d}\boldsymbol{x}' \tag{2.18}$$

That is the Fourier transform of a circularly symmetric function. Therefore, we can rewrite the integral as a zero-order Hankel transform

$$H(r) = \int_0^R \exp\left(-\frac{ik}{2}\frac{nzr'^2}{f_1^2}\right) J_0\left(\frac{k}{f_2}rr'\right) r'\,\mathrm{d}r' \tag{2.19}$$

Changing the variable r' with $\rho = r'/R$ and defining the numerical aperture of the first lens as $\mathrm{NA} = nR/f_1$ we finally obtain

$$H(r,z) = \int_0^1 \exp\left(-\frac{ik}{2}\frac{\mathrm{NA}^2}{n}\rho^2 z\right) J_0\left(\frac{k\mathrm{NA}}{M}\rho r\right) \rho\,\mathrm{d}\rho \tag{2.20}$$

where we neglected pure multiplicative factors and used the definition of the magnification as $M = nf_2/f_1$.

2.3.2 Coherence of light

The finite temporal coherence of light can be described by random phase shifts of the electromagnetic wave

$$E_1(t) = \sum_{n=-\infty}^{+\infty} \exp(i\omega t + i\phi_n)\Pi\left(\frac{t}{T} - \frac{n}{2}\right) \tag{2.21}$$

Now consider the same electric field, time-shifted by τ

$$E_2(t-\tau) = \sum_{m=-\infty}^{+\infty} \exp(i\omega t - i\omega\tau + i\phi_m)\Pi\left(\frac{t-\tau}{T} - \frac{m}{2}\right) \tag{2.22}$$

The total intensity of the sum of the two fields is calculated as

$$|E_1 + E_2|^2 = |E_1|^2 + |E_2|^2 + E_1^* E_2 + E_1 E_2^* \tag{2.23}$$

The first two terms are proportional to the intensity of each field, the last two terms describe the interference between the two fields.

$$E_1^* E_2 = \exp(-i\omega\tau) \sum_{m,n} \exp(i\phi_n - i\phi_m) \Pi\left(\frac{t}{T} - \frac{n}{2}\right) \Pi\left(\frac{t-\tau}{T} - \frac{m}{2}\right) \tag{2.24}$$

given a fixed τ, the product of the two rectangular functions is either 0 or 1, depending on the value of $n - m$. We now consider only the couple (n, m) such as this product is equal to 1. Therefore,

$$\sum_{m,n} \exp(i\phi_n - i\phi_m) \approx \delta_{m,n} \tag{2.25}$$

Therefore the interference term is not zero only if $m = n$ and

$$\Pi\left(\frac{t}{T}\right) \Pi\left(\frac{t-\tau}{T}\right) > 0 \tag{2.26}$$

which implies $|\tau| < T$. Indeed, T is the coherence time and defines the maximum delay beyond which the interference term can be neglected.

The value τ defines the visibility of the interference. Indeed, the total signal collected in an ideally infinite amount of time from the interference terms is

$$\begin{aligned} &\int_{\mathbb{R}} \left[E_1^*(t) E_2(t-\tau) + E_1(t) E_2^*(t-\tau)\right] \mathrm{d}t = 2\cos(\omega\tau) \int_{\mathbb{R}} \Pi\left(\frac{t}{T}\right) \Pi\left(\frac{t-\tau}{T}\right) \mathrm{d}t = \\ &= (T - |\tau|) 2\cos(\omega\tau) \end{aligned} \tag{2.27}$$

Thus, the visibility of the interference $\cos(\omega t)$ decreases linearly with $|\tau|$. This linear behaviour is a consequence of this simplified model which uses rectangular coherence windows. A more realistic model would still predict a visibility monotonically decreasing with τ, but with a non-linear trend. For perfectly incoherent light $T \to 0$ and the interference signal can be seen as $\delta(\tau)$.

The same reasoning which led to the results of this section can be applied to space to describe spatial coherence.

2.3.3 Incoherent imaging

$$\begin{aligned} i(\boldsymbol{x}) &= \int_0^T \left|\left[O(\boldsymbol{x}', t - \tau(\boldsymbol{x}')) * H(\boldsymbol{x}')\right](\boldsymbol{x})\right|^2 \mathrm{d}t = \\ &= \int_0^T \int_{\mathbb{R}^6} O(\boldsymbol{x}', t - \tau(\boldsymbol{x}')) O^*(\boldsymbol{x}'', t - \tau(\boldsymbol{x}'')) H(\boldsymbol{x} - \boldsymbol{x}') H^*(\boldsymbol{x} - \boldsymbol{x}'') \,\mathrm{d}\boldsymbol{x}' \,\mathrm{d}\boldsymbol{x}'' \,\mathrm{d}t \end{aligned} \tag{2.28}$$

where $\tau(\boldsymbol{x})$ is the time the light needs to reach the detector starting from the coordinate $\boldsymbol{x}$ of the object plane. Since H is a sharply peaked function, the product $H(\boldsymbol{x}-\boldsymbol{x}')H^*(\boldsymbol{x}-\boldsymbol{x}'')$ is non zero only for $\boldsymbol{x}' \approx \boldsymbol{x}''$. Therefore, we can approximate $\tau(\boldsymbol{x}') \approx \tau(\boldsymbol{x}'')$ and neglect both time delays. Thus, the time integral is just the correlation function at zero delay. For perfectly incoherent light it becomes

$$\int_0^T O(\boldsymbol{x}', t) O^*(\boldsymbol{x}'', t)\,\mathrm{d}t = T|O(\boldsymbol{x}')|^2 \delta(\boldsymbol{x}' - \boldsymbol{x}'') \tag{2.29}$$

Therefore, the image formed with incoherent light is

$$i(\boldsymbol{x}) = T \int_{\mathbb{R}^3} |O(\boldsymbol{x}')|^2 |H(\boldsymbol{x}-\boldsymbol{x}')|^2\,\mathrm{d}x' \propto [o * h](\boldsymbol{x}) \tag{2.30}$$

where

$$h(\boldsymbol{x}) = |H(\boldsymbol{x})|^2 \tag{2.31}$$

is the intensity Point Spred Function (PSF) and

$$o(\boldsymbol{x}) = |O(\boldsymbol{x})|^2 \tag{2.32}$$

is the distribution of light-emitters in the object plane.

2.3.4 Lateral and axial resolution

Using equation 2.20 with object plane coordinates, the intensity PSF is

$$h(r, z) = \left| \int_0^1 \exp\left(-\frac{ik}{2}\frac{\mathrm{NA}^2}{n}\rho^2 z\right) J_0(k\mathrm{NA}\rho r)\rho\,\mathrm{d}\rho \right|^2 \tag{2.33}$$

In perfect focus condition ($z = 0$), this equation becomes

$$h(r, 0) = \left| \int_0^1 J_0(k\mathrm{NA}r\rho)\rho\,\mathrm{d}\rho \right|^2 = \left| \int_0^{k\mathrm{NA}r} \frac{J_0(x)x}{(k\mathrm{NA}r)^2}\,\mathrm{d}x \right|^2 = \left| \frac{J_1(k\mathrm{NA}r)}{k\mathrm{NA}r} \right|^2 \tag{2.34}$$

where we used the property of Bessel functions $\frac{\mathrm{d}}{\mathrm{d}x}[J_\nu(x)x^\nu] = J_{\nu-1}(x)x^\nu$. The first zero of $J_1(x)$ is at $x_0 \approx 3.8317$. Solving the equation $k\mathrm{NA}r = x_0$ for r, we obtain the distance between the peak of the PSF and its first minimum

$$r_{\mathsf{min}} = 0.61\frac{\lambda}{\mathrm{NA}} \tag{2.35}$$

This is the minimum lateral distance resolvable by a standard imaging system, according to Rayleigh's criterion.

Along the optical axis ($r = 0$) the intensity profile is

$$h(0,z) = \left| \int_0^1 \exp\left(-\frac{ik}{2}\frac{\mathrm{NA}^2}{n}\rho^2 z\right)\rho\,\mathrm{d}\rho \right|^2 = \left| \frac{n}{\mathrm{NA}^2 kz}\left[\exp\left(-\frac{ik}{2}\frac{\mathrm{NA}^2}{n}z\right) - 1\right] \right|^2 = $$
$$= \left(\frac{2n}{kz\mathrm{NA}^2}\right)^2 \sin^2\left(\frac{kz\mathrm{NA}^2}{4n}\right) \tag{2.36}$$

The first zero of the cardinal sine function $\frac{\sin(x)}{x}$ is at $x_0 = \pi$. Solving the equation $\frac{kz\mathrm{NA}^2}{4n} = x_0$ for z, we obtain the distance between the peak of the axial PSF and its first minimum

$$z_{\mathsf{min}} = \frac{2\lambda n}{\mathrm{NA}^2} \tag{2.37}$$

This is the minimum axial distance resolvable by a standard imaging system, according to Rayleigh's criterion.

2.4 Frequency analysis of imaging systems

A linear imaging system acts as a low-pass filter on the signal emitted from a sample. The spatial frequencies that can be collected are defined by the Optical Transfer Function (OTF), which is the Fourier Transform of the impulse response (PSF).

From equation 2.18, we see that in focus ($z = 0$) the field PSF is the Fourier transform of the pupil function. Thus, the OTF for coherent imaging is

$$\mathrm{OTF} = \mathcal{F}\{H(\boldsymbol{x})\} = P\left(\frac{\boldsymbol{x}'}{\lambda f}\right) \tag{2.38}$$

The OTF for incoherent imaging is

$$\mathrm{OTF} = \mathcal{F}\{h(\boldsymbol{x})\} = P\left(\frac{\boldsymbol{x}'}{\lambda f}\right) * P\left(-\frac{\boldsymbol{x}'}{\lambda f}\right) \tag{2.39}$$

If P is a rectangular function with length $2R$, then the incoherent OTF is a triangular function with cut-off frequency

$$\nu_o = \frac{2R}{\lambda f} = \frac{2\mathrm{NA}}{\lambda} \tag{2.40}$$

where $\mathrm{NA} = f/R$.