Unconstrained Minimization Algorithm

(1) Initialize $\boldsymbol{x}^0$, set $k := 0$.

(2) **while** *stopping condition is not satisfied at* $\boldsymbol{x}^k$

  (a) Find $\boldsymbol{x}^{k+1}$ such that $f(\boldsymbol{x}^{k+1}) < f(\boldsymbol{x}^k)$.

  (b) $k := k + 1$

endwhile

Output : $\boldsymbol{x}^* = \boldsymbol{x}^k$, a local minimum of $f(\boldsymbol{x})$.

- How to find $\boldsymbol{x}^{k+1}$ in Step 2(a) of the algorithm?
- Which *stopping condition* can be used?
- Does the algorithm converge? If yes, how fast does it converge?
- Does the convergence and its speed depend on $\boldsymbol{x}^0$?

Unconstrained Minimization Algorithm

(1) Initialize $\boldsymbol{x}^0$ and ϵ, set $k := 0$.

(2) **while** $\|\boldsymbol{g}(\boldsymbol{x}^k)\| > \epsilon$

  (a) Find $\boldsymbol{x}^{k+1}$ such that $f(\boldsymbol{x}^{k+1}) < f(\boldsymbol{x}^k)$.

  (b) $k := k + 1$

endwhile

Output : $\boldsymbol{x}^* = \boldsymbol{x}^k$, a stationary point of $f(\boldsymbol{x})$.

How to find $\boldsymbol{x}^{k+1}$ in Step 2(a)?

- Find a *descent direction* $\boldsymbol{d}^k$ for f at $\boldsymbol{x}^k$
- Take a step $\alpha^k (> 0)$ along $\boldsymbol{d}^k$ such that
  - $f(\boldsymbol{x}^{k+1}) < f(\boldsymbol{x}^k)$
  - $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \alpha^k \boldsymbol{d}^k$

Unconstrained Minimization Algorithm

(1) Initialize $\boldsymbol{x}^0$ and ϵ, set $k := 0$.

(2) **while** $\|\boldsymbol{g}(\boldsymbol{x}^k)\| > \epsilon$

  (a) Find a descent direction $\boldsymbol{d}^k$ for f at $\boldsymbol{x}^k$

  (b) Find $\alpha^k(> 0)$ along $\boldsymbol{d}^k$ such that $f(\boldsymbol{x}^k + \alpha^k\boldsymbol{d}^k) < f(\boldsymbol{x}^k)$

  (c) $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \alpha^k\boldsymbol{d}^k$

  (d) $k := k + 1$

endwhile

Output : $\boldsymbol{x}^* = \boldsymbol{x}^k$, a stationary point of $f(\boldsymbol{x})$.

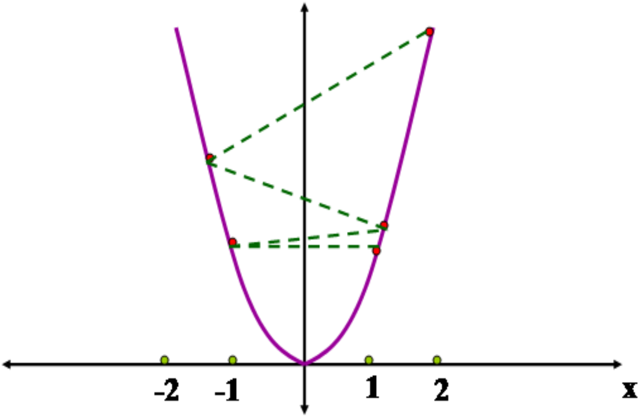
- How to determine α^k in Step 2(b)?

Step Length Determination

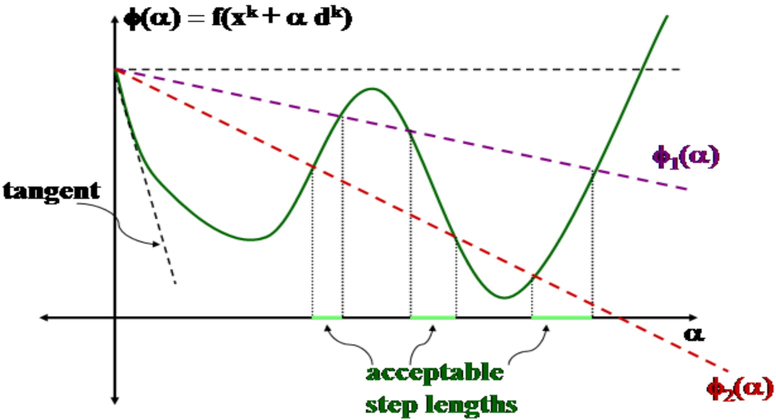
- Exact Line Search : Given a descent direction $\boldsymbol{d}^k$, determine α^k by solving the optimization problem:

$$\alpha^k = \arg\min_{\alpha > 0} \phi(\alpha) \triangleq f(\boldsymbol{x}^k + \alpha \boldsymbol{d}^k)$$

- Inexact Line Search :
  - Choice of α^k is crucial



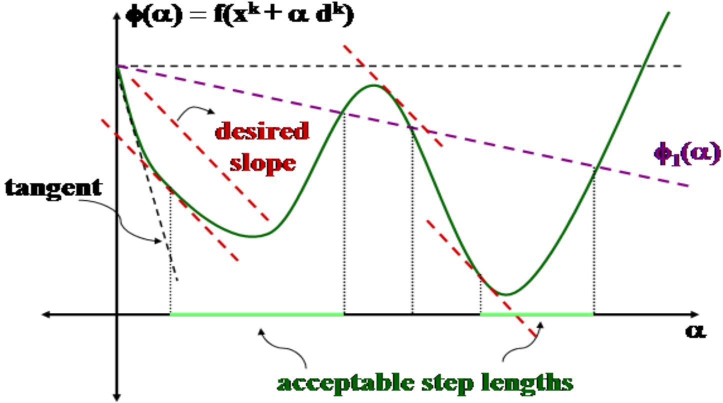
- Small decrease in function values relative to the step length



Armijo-Goldstein Conditions: Choose α^k such that

$$\phi_2(\alpha^k) \leq f(\boldsymbol{x}^k + \alpha^k \boldsymbol{d}^k) \leq \phi_1(\alpha^k)$$

Wolfe's condition ensures sufficient rate of decrease of function value in the given direction



Choose α^k such that

$$\phi'(\alpha^k) \geq c_2 \phi'(0), \quad c_2 \in (c_1, 1) \qquad \text{Wolfe's Condition}$$