Module-3.

Realization of IIR Filters.

Consider linear time-invariant system described by a difference equation of the form:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m) \quad \text{—} (1).$$

where a_k, b_m are constants with $a_0 \neq 0$.

Taking z transform of eq (1) we get

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z) \quad \text{—} (2).$$

Transfer function $H(z) = \frac{Y(z)}{X(z)}$.

$$H(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \text{—} (3).$$

Recursive and Non-Recursive Structures

Digital system can be realised in two ways.

i) Recursive Structures (having feedback) also called IIR filters. The functional realization b/w i/p and o/p sequence for recursive realization has the form

$$y(n) = F[\underbrace{y(n-1), y(n-2)}_{\text{Past o/p's}} \ldots \underbrace{x(n)}_{\text{Present i/p}}, \underbrace{x(n-1) \ldots}_{\text{past i/p}}]$$

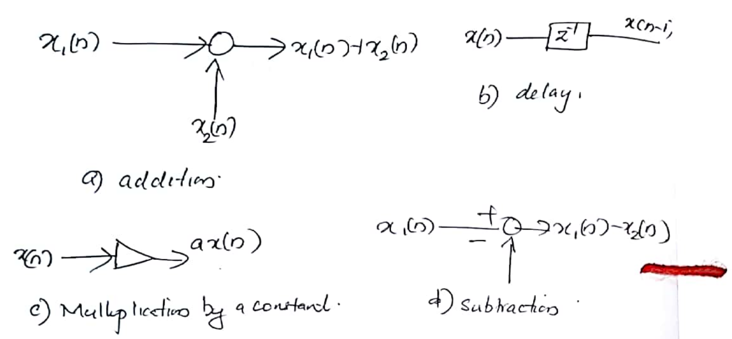
$$y(n) = y(n-3) + 8y(n-5) + x(n) + 7x(n-3).$$

ii) Non Recursive structure (No f/b).

$$\underbrace{y(n)}_{\text{Present o/p}} = F[\underbrace{x(n), x(n-1) \ldots}_{\text{Present \& Past i/ps}}]$$

They are called finite response filters

Block diagram Representation



Digital Filter structure

LTI s/m can be classified according to whether the impulse response is of finite duration or it is infinite. If the impulse response sequence is of finite duration, the system is called finite-impulse response (FIR). An infinite impulse response s/m has an impulse response of infinite duration.

Structures of IIR as follows.

1) Direct form structures. — DF–I, DF–II
2) Parallel form structures
3) Cascade form structure
4) Ladder structures

Direct form Realization of an IIR s/m:-

IIR system structures described by difference equations

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k).$$

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M) \quad — ①$$

On taking z transform of eq ①.

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) \cdots - a_N z^{-N} Y(z).$$
$$+ b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) \cdots + b_M z^{-M} X(z)$$

$$Y(z) \left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{M} b_k z^{-k} X(z). \quad — ③$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad — ④$$

eq ① can be realized by direct form I structure

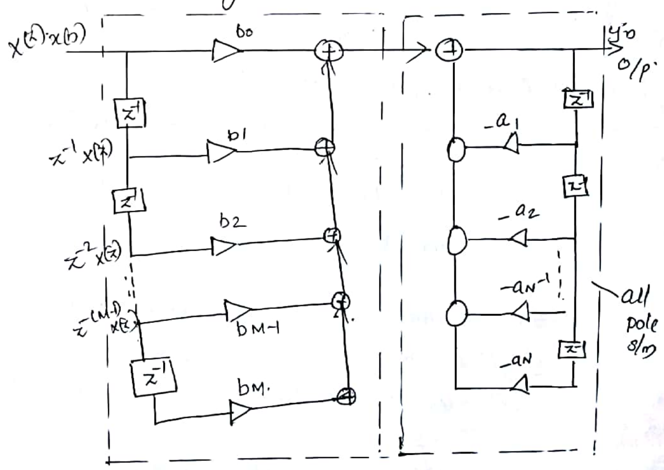
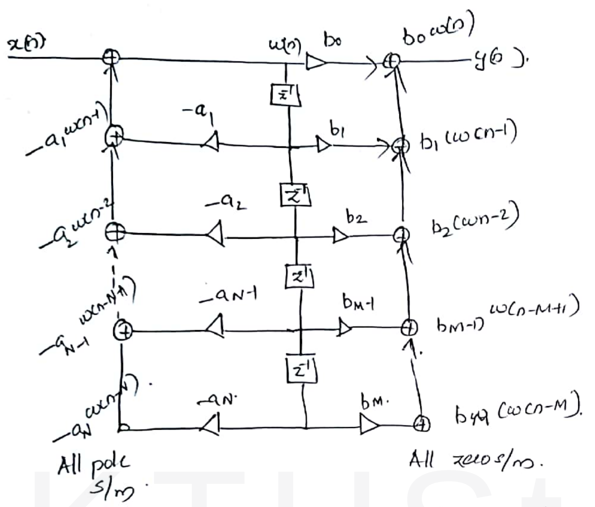


Fig: Direct form I structure of IIR filter.

Direct form II



Number of Multiplications = M+N+1

No: of addition = M+N

Memory location M or N

Difference equation for all pole filter is

$$w(n) = -\sum_{k=1}^{N} a_k\, w(n-k) + x(n) \quad \text{--- (1)}$$

Since w(n) is the i/p to all zero system.

$$y(n) = \sum_{k=0}^{M} b_k\, w(n-k) \quad \text{--- (2)}$$

The resulting structure that implement eq (1) & eq (2) is called Direct form II Realization

It is also called canonical representation.

Q Consider a causal LTI s/m with system function. Draw DF-II

$$H(z) = \frac{1+\frac{1}{4}z^{-1}}{(1-0.25z^{-1}+\frac{1}{3}z^{-2})(1+0.2z^{-1})}$$

$$H(z) = \frac{1+\frac{1}{4}z^{-1}}{1+0.2z^{-1}-0.25z^{-1}-0.050z^{-2}+\frac{1}{3}z^{-2}+0.066z^{-3}}$$

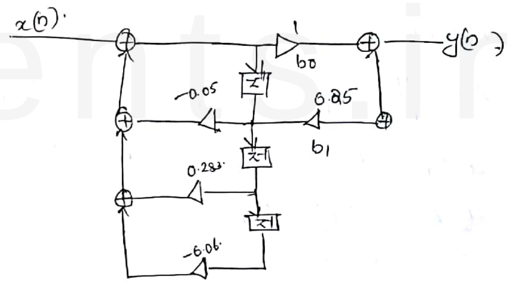
$$H(z) = \frac{1+\frac{1}{4}z^{-1}}{1-0.05z^{-1}+0.283z^{-2}+0.06z^{-3}}$$

$b_0 = 1 \quad b_1 = \frac{1}{4}$

$a_1 = -0.05$

$a_2 = 0.283$

$a_3 = 0.06$



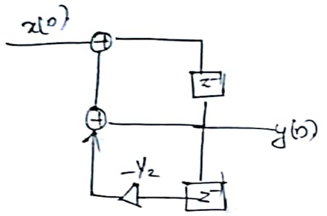
Q Draw DF-II Representation of system function

$$H(z) = \frac{z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

It is second order s/m, so memory elements = 2.

General second order system function.

$$H(z) = \frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}} \qquad \begin{aligned} b_1 &= 1 \\ a_1 &= -1 \\ a_2 &= \tfrac{1}{2} \end{aligned}$$



CASCADE Realization of IIR systems.

Let us consider a high order IIR system $(N \geq M)$ with system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} \quad \text{—— (1)}$$

H(z) can be expressed as the cascade of II^{nd} order and I^{st} order system.

$$H(z) = \prod_{k=1}^{K} H_k(z) \quad \text{where } k \text{ is a integer.}$$

II^{nd} order s/m. $H_k(z)$ has general form.

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1+a_{k1} z^{-1} + a_{k2} z^{-2}} \quad \text{—— (3)}$$

If $N > M$ then either $b_{k2} = 0$ or $b_{k1} = 0$.

$B_{k2} = b_{h1} = 0$ for some k

Q. Determine the cascade realization for s/m described by equation

$$H(z) = \frac{10\,(1-\tfrac{1}{2}z^{-1})(1-\tfrac{2}{3}z^{-1})(1+2z^{-1})}{(1-\tfrac{3}{4}z^{-1})(1-\tfrac{1}{8}z^{-1})[1-(\tfrac{1}{2}+j\tfrac{1}{2})z^{-1}][1-(\tfrac{1}{2}-j\tfrac{1}{2})z^{-1}]}$$

Soln: One possible pairing of poles and zero

$$H_1(z) = \frac{(1-\tfrac{2}{3}z^{-1})}{(1-\tfrac{3}{4}z^{-1})(1-\tfrac{1}{8}z^{-1})}$$

$$H_2(z) = \frac{(1-(\tfrac{1}{2})z^{-1})(1+2z^{-1})}{[1-(\tfrac{1}{2}+j\tfrac{1}{2})z^{-1}][1-(\tfrac{1}{2}-j\tfrac{1}{2})z^{-1}]}.$$

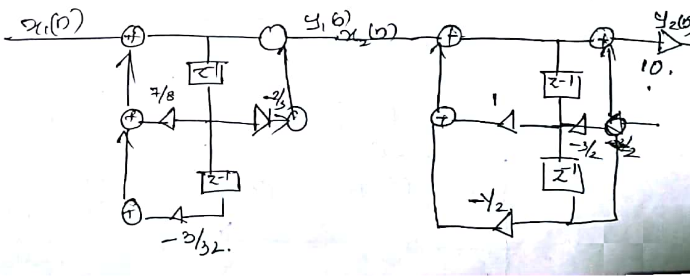
$$H(z) = 10\, H_1(z)\, H_2(z).$$

$$H_1(z) = \frac{\overset{b_0}{1} - \overset{b_1}{\tfrac{2}{3}} z^{-1}}{1-\tfrac{7}{8}z^{-1}+\tfrac{3}{32}z^{-2}} \qquad H_2(z) = \frac{\overset{b_0}{1}+\overset{b_1}{\tfrac{3}{2}}z^{-1} + \overset{b_2}{z^{-2}}}{1-\underset{a_1}{z^{-1}}+\underset{a_2}{\tfrac{1}{2}}z^{-2}}$$

General equation of second order

$$H_{sec}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1+a_1 z^{-1} + a_2 z^{-2}}.$$

Cascade structure can be drawn.



(8)

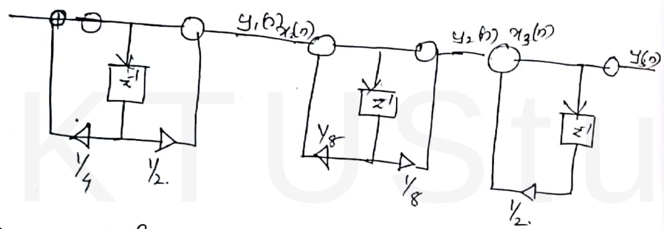
Q. using first order sections, obtain cascade realization

$$H(z) = \frac{(1+\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{8}z^{-1})}$$

using first order realization

$H(z) = H_1(z)\, H_2(z)\, H_3(z)$.

$H_1(z) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-1}}$ $\quad H_2(z) = \frac{1+\frac{1}{4}z^{-1}}{1-\frac{1}{8}z^{-1}}$ $\quad H_3 = \frac{1}{1-\frac{1}{2}z^{-1}}$



H) Parallel form Realization of IIR s/m.

A parallel form realization of an IIR system can be obtained by performing partial fraction expansion of $H(z)$. Assume $N \geq M$.

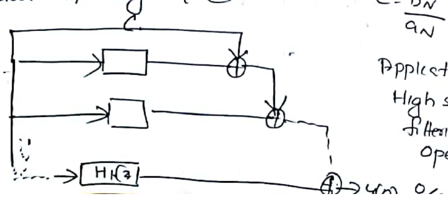
$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}} \quad \text{--- (1)}$$

$\{p_k\}$ = Poles

$\{A_k\}$ Residue

$C = \frac{b_N}{a_N}$.

Structure implied by eq (1):



Application
High speed filtering operations.

9.

Q Find the parallel form realization for discrete time linear causal s/m given by difference equation

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + \frac{1}{3}x(n-1) + x(n).$$

using 1st section module.

Soln Taking z transform

$$Y(z) = \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + \frac{1}{3}X(z)z^{-1} + X(z)$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = \frac{1}{3}X(z)z^{-1} + X(z)$$

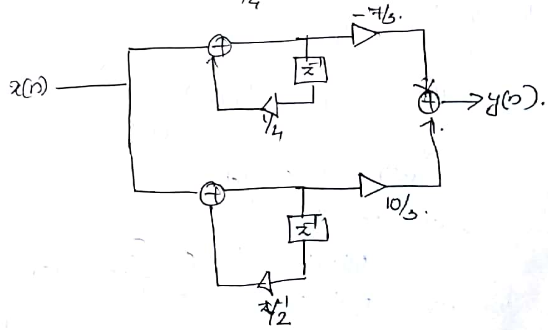
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \frac{z^2+\frac{1}{3}z}{z^2-\frac{3}{4}z+\frac{1}{8}}.$$

$$\frac{H(z)}{z} = \frac{z+\frac{1}{3}}{z^2-\frac{3}{4}z+\frac{1}{8}} = \frac{z+\frac{1}{3}}{(z-\frac{1}{4})(z-\frac{1}{2})} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{2}}$$

$A = -7/3 \qquad B = 10/3$.

$$\frac{H(z)}{z} = \frac{-7/3}{z-\frac{1}{4}} + \frac{10/3}{z-\frac{1}{2}}$$

$$H(z) = \frac{\overset{b_0}{-\frac{7}{3}}z}{z-\frac{1}{4}} + \frac{\overset{b_{01}}{\frac{10}{3}}z}{1-\frac{1}{4}z^{-1}}$$



Ladder structure:

A continued fraction expansion of $H(z)$ is given by

$$H(z) = \alpha_0 + \underbrace{\left[\cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{\alpha_1 + \cfrac{1}{B_2 z^{-1} + \cdots + \cdots 1/\alpha_n}}} \right]}_{H_1(z)} \quad \text{--- (1)}$$

Now eq (1) can be written

$$H(z) = \alpha_0 + H_1(z)$$

$$\frac{Y(z)}{X(z)} = \alpha_0 + H_1(z)$$

$$Y(z) = \alpha_0 X(z) + H_1(z) X(z). \quad \text{--- (2)}$$

Eq (2) can be drawn as

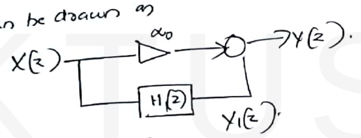


fig a.

$$H_1(z) = \frac{Y_1(z)}{X_1(z)}. \quad \text{--- (3)}$$

from eq (2)

$$H(z) = \alpha_0 + 1/B_1 z^{-1} + 1/R_1(z)$$

$$H_1(z) = 1/B_1 z^{-1} + 1/R_1(z). \quad \text{--- (4)}$$

$$\therefore \frac{Y_1(z)}{X_1(z)} = \frac{1}{B_1 z^{-1} + 1/R_1(z)} \quad \text{--- (5)}$$

$$Y_1(z) = \frac{X_1(z) R_1(z)}{R_1(z) B_1 z^{-1} + 1}$$

$$X(z) = B_1 z^{-1} Y_1(z) + 1/R_1(z)\, Y_1(z) \quad \text{--- (6)}$$

$$X(z) R_1(z) = B_1 z^{-1} Y_1(z) + R_1(z) 1 Y_1(z) \quad \text{--- (7)}$$

$$Y_1(z) = \left[\frac{X(z) - B_L z^{-1} Y_1(z)}{X_1(z)} \right] R_1(z) \quad \text{--- 8.}$$

$$R_1(z) = Y_1(z)/X_1(z).$$

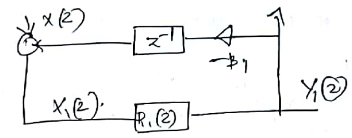
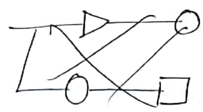


fig (b)

$H_1(z)$ can be replaced by fig (b) in fig a.



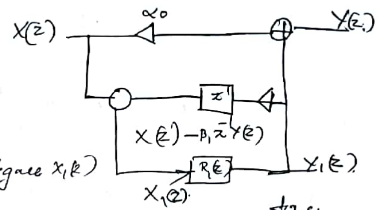


fig c.

Similarly we can obtain the figure $X_1(z)$ and $Y_1(z)$.

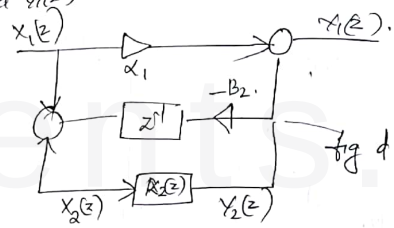
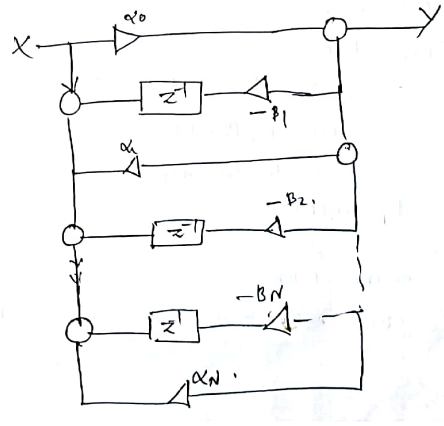


fig d.

Thus by continued fraction expansion method resultant structure will be obtain for transfer fn. $H(z)$.



Lattice Structure of an IIR system

Let us consider an all pole-system with system function

$$H(z) = \frac{1}{1+\sum_{k=1}^{N} a_N z^{-k}} = \frac{1}{A_N(z)} \quad \text{—(1)}$$

The difference equation for this IIR s/m will be.

$$y(n) = -\sum_{k=1}^{N} a_N(k)\, y(n-k) + x(n) \quad \text{—(2)}$$

$$x(n) = y(n) + \sum_{k=1}^{N} a_N(k)\, y(n-k) \quad \text{—(3)}$$

For N=1 we have

$$x(n) = y(n) + a_1(k)\, y(n-1) \quad \text{—(4)}$$

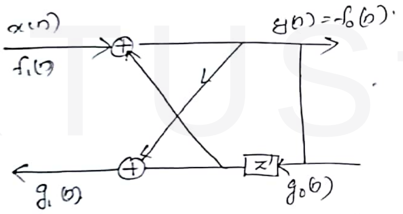


fig. single stage all pole lattice filter.

$x(n) = f_1(n)$

$y(n) = f_0(n)$

$= f_1(n) - k_1 g_0(n-1) = x(n) - k_1 y(n-1)$.

$g_1(n) = k_1 f_0(n) + g_0(n-1)$ —(5)

$\quad = k_1 y(n) + y(n-1)$.

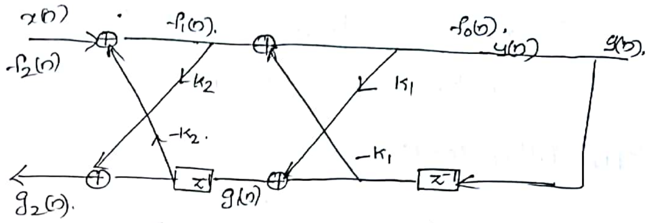
$k_1 = a_1(1)$.

(2) N=2

$x(n) = f_2(n)$.

$y(n) = x(n) - a_2(1)\, y(n-1) - a_2(2)\, y(n-2)$ —(6).

This o/p can be obtained from a two stage lattice filter.



Two stage all pole lattice filter.

$f_2(n) = x(n)$

$f_1(n) = f_2(n) - k_2 g_1(n-1)$

$g_2(n) = k_2 f_1(n) + g_1(n-1)$.

$f_0(n) = f_1(n) - k_1 g_0(n-1)$

$g_1(n) = k_1 f_0(n) + g_0(n-1)$.

$y(n) = f_0(n) = g(n)$

$= f_1(n) - k_1 g_0(n-1) = f_2(n) - k_2 g_1(n-1) - k_1 g_0(n-1)$.

$= f_2(n) - k_2 [k_1 f_0(n-1) + g_0(n-2)] - k_1 g_0(n-1)$

$= f_2(n) - k_2 [k_1 f_0(n-1) + g_0(n-2)] - k_1 g_0(n-1)$

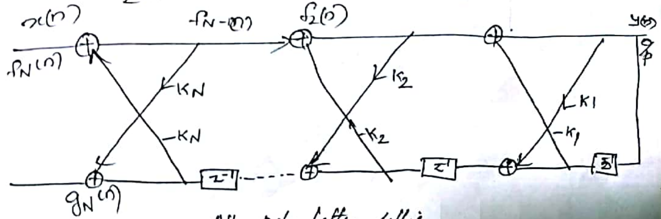
$y(n) = x(n) - k_1(1+k_2)\, y(n-1) - k_2 y(n-2)$.

$g_2(n) = k_2 y(n) + k_1(1+k_2) y(n-1) + y(n-2)$.

on comparing above equations

$a_2(0) = 1$

$a_2(1) = k_1(1+k_2)$, $a_2(2) = k_2$



All pole lattice filter

$f_N(n) = x(n)$

$f_{m-1}(n) = f_m(n) - k_m g_{m-1}(n-1) \quad m = N, N-\ldots$

$g_m(n-1) = k_m f_{m-1}(n-1) + g_{m-1}(n-2) \quad m = N, N-1 \ldots 1$

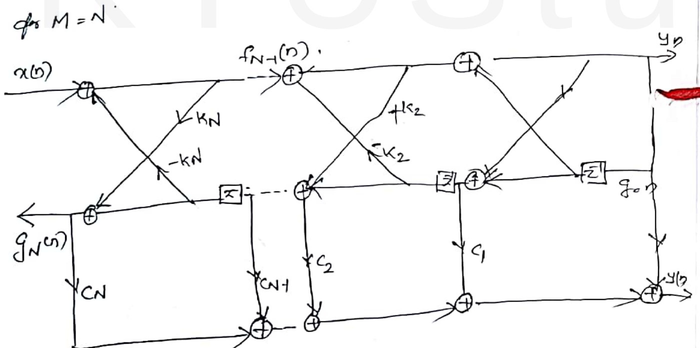
The lattice ladder-structure

Let us consider an IIR filter with following system T.F

$$H(z) = \frac{B_M(z)}{A_N(z)} = \frac{\sum_{k=0}^{N} b_M(k) z^{-k}}{1 + \sum_{k=1}^{N} a_N k\, z^{-k}} \quad -① $$

where $N \geq M$.

A lattice structure in eq ① may be constructed by first realizing an all pole lattice coefficients k_m where $1 \leq m \leq N$ for the denominator $A_N(z)$ and then adding ladder part as shown in figure for $M = N$.



Lattice ladder structure of pole zero IIR S/m.

Structures for Realization of FIR S/ms.

In general FIR S/m is described by difference equation

$$y(n) = \sum_{k=0}^{N-1} b_k\, x(n-k). \quad — ①$$

On taking z transform of eq ① we get

$$Y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z) \quad — ②$$

$$Y(z)/X(z) = H(z) = \sum_{k=0}^{N-1} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{N-1} z^{-(N-1)} \quad — ③$$

H(z) is transfer function of FIR S/m.

$H(z) = Z\,[h(n)]$ where $h(n)$ is impulse response of FIR System. Let us replace the index n by k.

$$H(z) = Z\,[h(k)] = \sum_{k=0}^{N-1} h(k)\, z^{-k} = h(0) + h(1) z^{-1} + h(2) z^{-2} + \ldots + \ldots h(N-1) z^{-(N-1)} \quad — ④$$

On comparing equation ③ and ④.

we get $b_k = h(k)$ for $k = 0, 1, 2 \ldots (N-1)$

These eqns can be used to construct the blockdiagram of system delays, adder and multipliers.

The different types for realizing FIR system are:

1) Direct form realization
2) Cascade Realization
3) Linear phase Realization

Direct Form Realization

The direct form can be obtained from z domain equation governing FIR system. The general z domain equation governing a FIR S/m is given by eq ②.

$$Y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z).$$

$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + b_3 z^{-3} X(z) + \ldots$$
$$\ldots + b_{N-2} z^{-(N-2)} X(z) + b_{N-1} z^{-(N-1)} X(z) \quad \text{--- (6)}$$

The direct form structure is draw using eq (6).

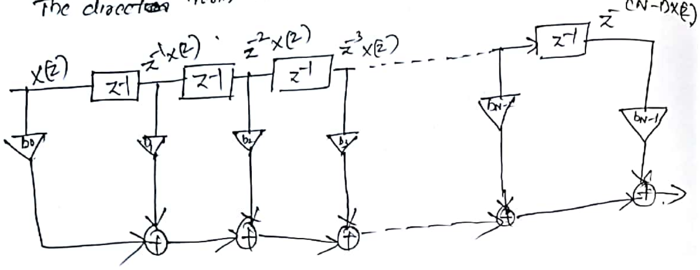


fig: Direct from structure of FIR s/m.

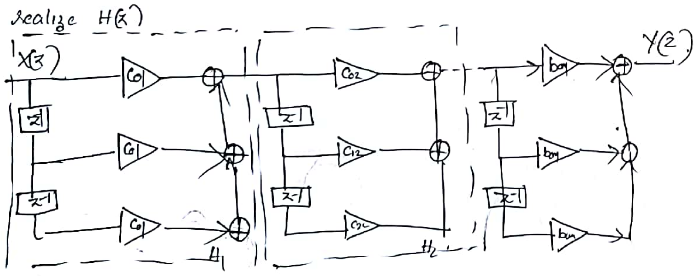
2. CASCADE REALIZATION of FIR S/M.

The transfer function of FIR system eq (3) and (4) is $(N-1)^{th}$ order polynomial in z. This polynomial can be factorized into second order factor. (when N is odd).

and the transfer function can be expressed

when N is odd, $H(z) = \sum_{k=0}^{N-1} b_k z^{-k} = \prod_{k=1}^{\frac{N-1}{2}} C_{01} + c$ --- (7)

when N is odd then (N-1) will be even so H(z) will have $N-1/2$ second order factors. Each second order factor of eq (7) can be realized in direct from and all second order system are connected in cascade to realize H(z)



Analog to Digital Transformation

Design of IIR filters from analog filters

There are several methods that can be used to design digital filter having an infinite duration unit sample response. The techniques are based on converting an analog filter into digital filter.

Properties

1. The $j\Omega$ axis in the s plane should map into unit circle in z plane. Thus there will be a direct relationship b/t two frequency variables in two domains.

2. The left half plane of s plane should map into inside the unit circle in z plane. Thus a stable analog filter will be converted to a stable digital filter.

The four most widely used method for digitising the analog filter into digital filter include

1. Approximation of Derivatives

One of the simplest method of digitising an analog filter into a digital filter is to approximate the differential equation by an equivalent difference equation

For the derivative $\frac{dy(t)}{dt}$ at time $t = nT$.

we substitute backward difference $\frac{y(nT) - y(nT-T)}{T}$

Thus $$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T} \quad \text{--- (1)}$$
$$= y(n) - y(n-1)/T.$$

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T represents sampling interval $y(n) = y(nT)$.

We know Laplace transform of $\frac{dy(t)}{dt} = SY(s)$

$$y(t) \rightarrow \boxed{H(s)=s} \rightarrow dy(t)/dt$$

z transform of $\frac{y(n)-y(n-1)}{T}$ is $\frac{(1-z^{-1})Y(z)}{T}$

which can be represented as shown in fig

$$y(t) \rightarrow \boxed{1-z^{-1}/T} \rightarrow \frac{y(n)-y(n-1)}{T}$$

$$S = \frac{1-z^{-1}}{T} \quad \text{—— (2)}$$

Consequently the s/m function for digital IIR filter obtained as a result of approximation of derivative by finite difference is

$$H(z) = H(s)\Big|_{s=\frac{1-z^{-1}}{T}}$$

from the relation $s = \frac{1-z^{-1}}{T}$

$$z = \frac{1}{1-sT} = \frac{1}{1-j\Omega T} = \frac{1+j\Omega T}{1+\Omega^2T^2} \quad \text{—— (3)}$$

$$= \frac{1}{1+\Omega^2T^2} + \frac{j\Omega T}{1+\Omega^2T^2}$$

$$= x+jy$$

$$x^2+y^2 = x.$$

Design of IIR filter using Impulse Invariance Technique.

In impulse invariance method the IIR filter is designed such that unit impulse response $h(n)$ of digital filter is the sampled version of impulse response of analog filter.

The z-transform of an infinite impulse response is given by

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n) e^{-sTn} \quad \text{—— (1)}$$

Let us consider the mapping of points from s plane to z plane implied by the relation

$$z = e^{sT}$$

If we substitute $s=\sigma+j\Omega$ and express the complex variable z in polar form $z = re^{j\omega}$.

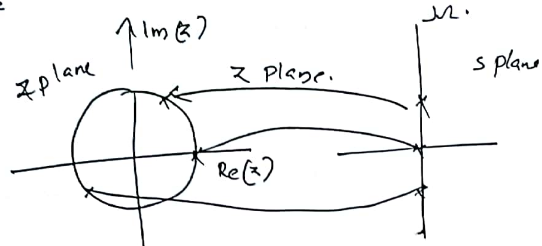
$$re^{j\omega} = e^{(\sigma+j\Omega)t}$$

$$= e^{\sigma t} e^{j\Omega t} \quad \text{—— (3)}$$

$$r = e^{\sigma T} \quad \omega = \Omega T$$

The first term in eq (3) $e^{\sigma t}$ has a magnitude of $e^{\sigma t}$ and an angle of 0 as real number.
The second term $e^{j\Omega t}$ has unity magnitude and an angle of ΩT. Thus our analog pole is mapped to a place in z plane of magnitude $e^{\sigma t}$ and angle ΩT.
The real part of analog pole determines the radius of z plane pole and imaginary part of analog pole dictates the angle of digital pole.

1st case

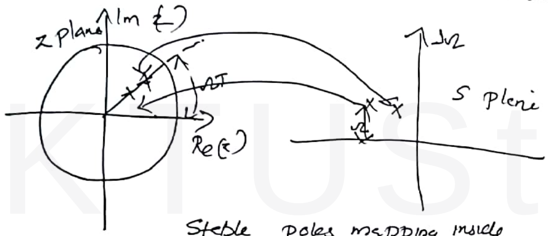


jΩ axis mapping to unit circle

2nd case

Poles in left half of s plane where $\sigma < 0$

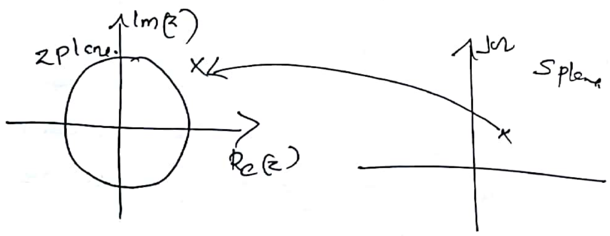
These poles map inside unit circle $r = e^{\sigma T} < 1$ for $\sigma < 0$.



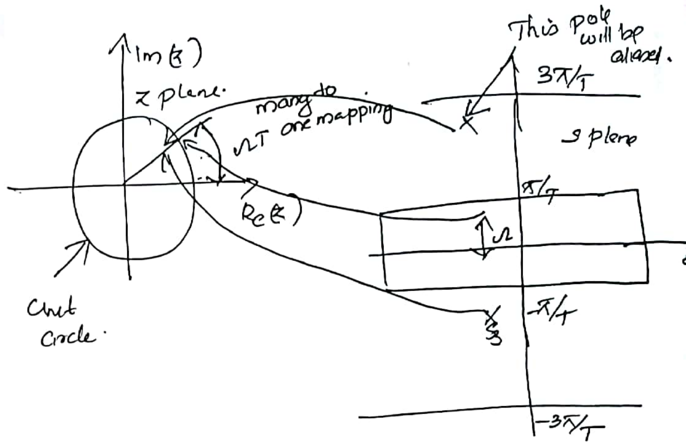
Stable poles mapping inside unit circle.

Case 3 All poles in right half of s plane to digital poles outside the unit circle

$r = e^{\sigma T} > 1$ for $\sigma > 0$.



unstable poles mapping outside unit circle.



Steps to design a digital filter using Impulse Invariant method.

1. For the given specifications, find $H_a(s)$, the transfer function of an analog filter.
2. Select the sampling rate of digital filter, T seconds per sample.
3. Express the analog filter transfer function as sum of single pole filter.

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k}$$

4) Compute the z transform of digital filter using formula

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_k T} z^{-1}}$$

for high sampling rate

$$H(z) = \sum_{k=1}^{N} \frac{T C_k}{1-e^{P_k T} z^{-1}}$$