How to compute Homography with 4 points?

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Assume homography matrix H is

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{32} \end{bmatrix} = \begin{bmatrix} \boldsymbol{h_1} \\ \boldsymbol{h_2} \\ \boldsymbol{h_3} \end{bmatrix}$$

Then if you get the right homography you should have

$$\boldsymbol{X_2} = \boldsymbol{H} * \boldsymbol{X_1} \quad (1)$$

Where $\boldsymbol{X_1} = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$, $\boldsymbol{X_2} = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$. $\boldsymbol{X_1}$ is in figure 1, $\boldsymbol{X_2}$ is its correspondence point in figure 2. Then we should be able to have

$$\boldsymbol{X_2} \times (\boldsymbol{H} * \boldsymbol{X_1}) = 0 \quad (2)$$

Why? As $\boldsymbol{X_2} = \boldsymbol{H} * \boldsymbol{X_1}$ we can write $\boldsymbol{X_2} \times (\boldsymbol{H} * \boldsymbol{X_1})$ as $\boldsymbol{X_2} \times \boldsymbol{X_2}$. Vector cross multiply itself will be 0 because

$$\boldsymbol{X_2} \times \boldsymbol{X_2} = |\boldsymbol{X_2}| * |\boldsymbol{X_2}| * \boldsymbol{sin\theta}$$

Where $\boldsymbol{\theta}$ is the angle between two vectors and it is 0 here of course. Then we can write formula (2) as

$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} \boldsymbol{h_1} \\ \boldsymbol{h_2} \\ \boldsymbol{h_3} \end{bmatrix} * \boldsymbol{X_1} \right) = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{h_1} * \boldsymbol{X_1} \\ \boldsymbol{h_2} * \boldsymbol{X_1} \\ \boldsymbol{h_3} * \boldsymbol{X_1} \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (3)$$

Then again from the property of cross product (if you have doubts here, just see wiki “cross product”, “Computing the cross product” part), vectors cross multiply each other we have results

$$\mathbf{0} = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{h_1} * \boldsymbol{X_1} \\ \boldsymbol{h_2} * \boldsymbol{X_1} \\ \boldsymbol{h_3} * \boldsymbol{X_1} \end{bmatrix} = \begin{bmatrix} y_2 * (\boldsymbol{h_3} * \boldsymbol{X_1}) - 1 * (\boldsymbol{h_2} * \boldsymbol{X_1}) \\ 1 * (\boldsymbol{h_1} * \boldsymbol{X_1}) - x_2 * (\boldsymbol{h_3} * \boldsymbol{X_1}) \\ x_2 * (\boldsymbol{h_2} * \boldsymbol{X_1}) - y_2 * (\boldsymbol{h_1} * \boldsymbol{X_1}) \end{bmatrix} (4)$$

$$\begin{bmatrix} y_2 * (\boldsymbol{h_3} * \boldsymbol{X_1}) - 1 * (\boldsymbol{h_2} * \boldsymbol{X_1}) \\ 1 * (\boldsymbol{h_1} * \boldsymbol{X_1}) - x_2 * (\boldsymbol{h_3} * \boldsymbol{X_1}) \\ x_2 * (\boldsymbol{h_2} * \boldsymbol{X_1}) - y_2 * (\boldsymbol{h_1} * \boldsymbol{X_1}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0^T} & -1 * \boldsymbol{X_1^T} & y_2 * \boldsymbol{X_1^T} \\ 1 * \boldsymbol{X_1^T} & \boldsymbol{0^T} & -x_2 * \boldsymbol{X_1^T} \\ -y_2 * \boldsymbol{X_1^T} & x_2 * \boldsymbol{X_1^T} & \boldsymbol{0^T} \end{bmatrix} * \begin{bmatrix} \boldsymbol{h_1^T} \\ \boldsymbol{h_2^T} \\ \boldsymbol{h_3^T} \end{bmatrix} (5)$$

(4), (5) is critical. For (5) you just need calculate from right side to left side to check if they equal each other). Finally, you have

$$\begin{bmatrix} \boldsymbol{0^T} & -1 * \boldsymbol{X_1^T} & y_2 * \boldsymbol{X_1^T} \\ 1 * \boldsymbol{X_1^T} & \boldsymbol{0^T} & -x_2 * \boldsymbol{X_1^T} \\ -y_2 * \boldsymbol{X_1^T} & x_2 * \boldsymbol{X_1^T} & \boldsymbol{0^T} \end{bmatrix} * \begin{bmatrix} \boldsymbol{h_1^T} \\ \boldsymbol{h_2^T} \\ \boldsymbol{h_3^T} \end{bmatrix} = \boldsymbol{0} (6)$$

Always remember for 1 point we should just have two formulas (one from x one from y) then if there are 3 equations only two of them work. Indeed, if you check the determinant of equation (6), it should be 0. Or you check the rank of (6), like first you do **row3 = row3+ $\boldsymbol{x_2}$*row1** then **row3 = row3 + $\boldsymbol{y_2}$*row2** you'll find **row3 = [0 0 0]**

The conclusion is any two of the formulas from (6) should work. Say we choose first 2 rows we have

$$\begin{bmatrix} \boldsymbol{0^T} & -1 * \boldsymbol{X_1^T} & y_2 * \boldsymbol{X_1^T} \\ 1 * \boldsymbol{X_1^T} & \boldsymbol{0^T} & -x_2 * \boldsymbol{X_1^T} \end{bmatrix} * \begin{bmatrix} \boldsymbol{h_1^T} \\ \boldsymbol{h_2^T} \\ \boldsymbol{h_3^T} \end{bmatrix} = \boldsymbol{0} \quad (7)$$

Write formula (7) as scalar we can have

$$\begin{bmatrix} 0 & 0 & 0 & -x_1 & -y_1 & -1 & y_2 * x_1 & y_2 * y_1 & 1 \\ x_1 & y_1 & 1 & 0 & 0 & 0 & -x_2 * x_1 & -x_2 * y_1 & -x_2 \end{bmatrix} * \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} (8)$$

Then if we have 4 points we can have 4 formulas like (8). Say we have (a_1, b_1), (a_2, b_2) be pairs, (m_1, n_1) (m_2, n_2) be pairs, (j_1, k_1) (j_2, k_2) be pairs. All of them can satisfy (8), put them together we have

$$\begin{bmatrix} 0 & 0 & 0 & -x_1 & -y_1 & -1 & y_2 * x_1 & y_2 * y_1 & 1 \\ x_1 & y_1 & 1 & 0 & 0 & 0 & -x_2 * x_1 & -x_2 * y_1 & -x_2 \\ 0 & 0 & 0 & -a_1 & -b_1 & -1 & b_2 * a_1 & b_2 * b_1 & 1 \\ a_1 & b_1 & 1 & 0 & 0 & 0 & -a_2 * a_1 & -a_2 * b_1 & -a_2 \\ 0 & 0 & 0 & -m_1 & -n_1 & -1 & n_2 * m_1 & n_2 * n_1 & 1 \\ m_1 & n_1 & 1 & 0 & 0 & 0 & -m_2 * m_1 & -m_2 * n_1 & -m_2 \\ 0 & 0 & 0 & -j_1 & -k_1 & -1 & k_2 * j_1 & k_2 * k_1 & 1 \\ j_1 & k_1 & 1 & 0 & 0 & 0 & -j_2 * j_1 & -j_2 * k_1 & -j_2 \end{bmatrix} * \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} (9)$$

Then for the 8*9 matrix do SVD and pick the last column of **V** you can get homography.

Here are some questions you may have

1: Why the 8*9 matrix is different from eqauation(9)?

As what I have said, formula (6)’s any two rows will work because it has rank 2.

2: Why some matrix is 8*8

The homography just has 8 constraints so in fact we can pick like h33 to be 1 or any number and just calculate the rest.

3: Why the third column is not 1 after I do $\boldsymbol{H} * \boldsymbol{X_1}$

Again, the H matrix can be scaled by different number, we need eliminate the scale. $\boldsymbol{H} * \boldsymbol{X_1}$ can give us

$$\begin{bmatrix} h_{11} * x_1 + h_{12} * y_1 + h_{13} \\ h_{21} * x_1 + h_{22} * y_2 + h_{23} \\ h_{31} * x_1 + h_{32} * y_2 + h_{32} \end{bmatrix}$$

How to make the last row to be 1? Just make every elements divided by third row then you can have

$$\begin{bmatrix} \frac{h_{11} * x_1 + h_{12} * y_1 + h_{13}}{h_{31} * x_1 + h_{32} * y_2 + h_{32}} \\ \frac{h_{21} * x_1 + h_{22} * y_2 + h_{23}}{h_{31} * x_1 + h_{32} * y_2 + h_{32}} \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

Do this before you calculate the residual of two points or when you map forward

or when you map backward