Gate ST-37.2023

Ajay Krishnan K
EE22BTECH11003

Question Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2} & if x \geq 1 \\ 0 & otherwise. \end{cases} \tag{1}$$

If $Y = \log_e X$, then $\Pr(Y < 1 | Y < 2)$ equals

Solution:

Given, the probability density function of X is

$$f(x) = \begin{cases} \frac{1}{x^2} & if x \geq 1 \\ 0 & otherwise. \end{cases} \tag{2}$$

Also, $Y = \log_e X$.

Consider the cumulative distribution function(CDF) of Y,

$$F_Y(y) = \Pr(Y \leq y) \tag{3}$$
$$= \Pr(\log_e X \leq y) \tag{4}$$
$$= \Pr(X \leq e^y) \tag{5}$$

Now, we need to find $\Pr(Y < 1 | Y < 2)$. For that, we need to find $F_Y(1)$ and $F_Y(2)$.
Using the equation for CDF,

$$F_Y(1) = \Pr\left(X \leq e^1\right) \tag{6}$$
$$= \Pr(X \leq e) \tag{7}$$
$$= \int_1^e \frac{1}{x^2} dx \tag{8}$$
$$= 1 - \frac{1}{e} \tag{9}$$

and

$$F_Y(2) = \Pr\left(X \leq e^2\right) \tag{10}$$
$$= \int_1^{e^2} \frac{1}{x^2} dx \tag{11}$$
$$= 1 - \frac{1}{e^2} \tag{12}$$

Now, we can find $\Pr(Y < 1 | Y < 2)$ as follows,

$$\Pr(Y < 1 | Y < 2) = \frac{\Pr(Y < 1, Y < 2)}{\Pr(Y < 2)} \tag{13}$$
$$= \frac{\Pr(Y < 1)}{\Pr(Y < 2)} \tag{14}$$
$$= \frac{F_Y(1)}{F_Y(2)} \tag{15}$$
$$= \frac{1 - \frac{1}{e}}{1 - \frac{1}{e^2}} \tag{16}$$
$$= \frac{e(e-1)}{e^2 - 1} \tag{17}$$
$$= \frac{e}{e+1} \tag{18}$$