

Gate ST-37.2023

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Question

- 1) Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

If $Y = \log_e X$, then $\Pr(Y < 1 | Y < 2)$ equals

Solution: Given, the probability density function of X is

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Also, $Y = \log_e X$.

Consider the cumulative distribution function(CDF) of Y ,

$$F_Y(y) = \Pr(Y \leq y) \quad (3)$$

$$= \Pr(\log_e X \leq y) \quad (4)$$

$$= \Pr(X \leq e^y) \quad (5)$$

$$= \int_1^{e^y} \frac{1}{x^2} dx \quad (6)$$

$$= 1 - \frac{1}{e^y}, y \geq 0 \quad (7)$$

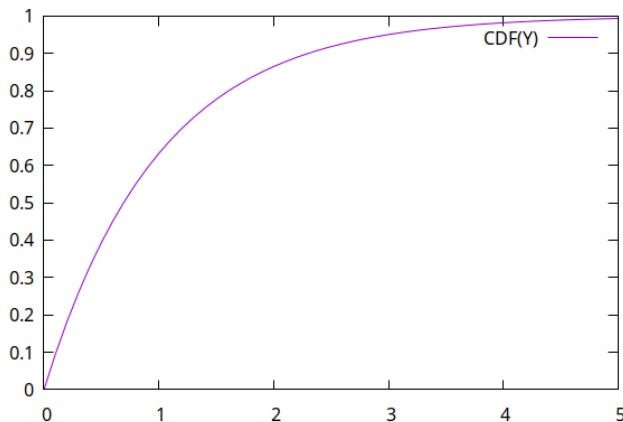


Fig. 1: CDF of Y

Now, we need to find $\Pr(Y < 1 | Y < 2)$. For that, we need to find $F_Y(1)$ and $F_Y(2)$.

Using the equation for CDF,

$$F_Y(1) = 1 - \frac{1}{e} \quad (8)$$

and

$$F_Y(2) = 1 - \frac{1}{e^2} \quad (9)$$

Now, we can find $\Pr(Y < 1 | Y < 2)$ as follows,

$$\Pr(Y < 1 | Y < 2) = \frac{\Pr(Y < 1, Y < 2)}{\Pr(Y < 2)} \quad (10)$$

$$= \frac{\Pr(Y < 1)}{\Pr(Y < 2)} \quad (11)$$

$$= \frac{F_Y(1)}{F_Y(2)} \quad (12)$$

$$= \frac{1 - \frac{1}{e}}{1 - \frac{1}{e^2}} \quad (13)$$

$$= \frac{e(e-1)}{e^2-1} \quad (14)$$

$$= \frac{e}{e+1} \quad (15)$$

- 2) C code to plot the cdf of Y .

- a) The following code generates the data for the cdf of Y .

```
#include <stdio.h>
#include <math.h>

// Define the PDF function
double pdf(double x) {
    if (x >= 1.0) {
        return 1.0 / (x * x);
    }
    return 0.0;
}

// Calculate the CDF of Y
double cdf_y(double y) {
    double sum = 0.0;
    double dx = 0.0001; // Small step size
    for numerical integration
    double x;

    for (x = 1.0; x <= exp(y); x += dx) {
        sum += pdf(x) * dx;
    }

    return sum;
}
```

```

int main() {
    FILE *dataFile = fopen("cdf_data.txt",
        "w");

    if (!dataFile) {
        printf("Error: Could not open data
            file for writing.\n");
        return 1;
    }

    double y, cdf;

    // Calculate and write the CDF data
    for (y = 0.0; y <= 5.0; y += 0.1) {
        cdf = cdf_y(y);
        fprintf(dataFile, "%.2f %.5f\n", y,
            cdf);
    }

    fclose(dataFile);

    printf("Data file 'cdf_data.txt'
        generated.\n");

    return 0;
}

```

- b) The following code plots the cdf of Y using Gnuplot.

```

set terminal pngcairo enhanced color size
    600,400
set output 'cdf_plot.png'
plot 'cdf_data.txt' using 1:2 with lines title
    'CDF(Y)'

```