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Ajay Krishnan K
EE22BTECH11003

Question

Fig. 1: CDF of Y

1) Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2} & if\, x \geq 1 \\ 0 & otherwise. \end{cases} \quad (1)$$

If $Y = \log_e X$, then $\Pr(Y < 1 | Y < 2)$ equals
Solution: Given, the probability density function of X is

$$f(x) = \begin{cases} \frac{1}{x^2} & if\, x \geq 1 \\ 0 & otherwise. \end{cases} \quad (2)$$

Also, $Y = \log_e X$.
Consider the cumulative distribution function(CDF) of X,

$$F_X(x) = \Pr(X \leq x) \quad (3)$$
$$= \int_1^x \frac{1}{x^2} dx \quad (4)$$
$$= 1 - \frac{1}{x}, x \geq 1 \quad (5)$$

Now, we need to find the CDF of Y.

$$F_Y(y) = \Pr(Y \leq y) \quad (6)$$
$$= \Pr(\log_e X \leq y) \quad (7)$$
$$= \Pr(X \leq e^y) \quad (8)$$
$$= F_X(e^y) \quad (9)$$
$$= 1 - \frac{1}{e^y}, y \geq 0 \quad (10)$$

Now, we need to find $\Pr(Y < 1 | Y < 2)$. For that, we need to find $F_Y(1)$ and $F_Y(2)$.
Using the equation for CDF,

$$F_Y(1) = 1 - \frac{1}{e} \quad (11)$$

and

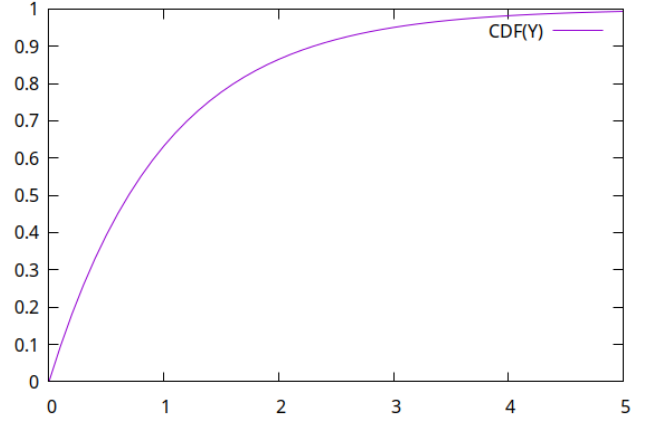$$F_Y(2) = 1 - \frac{1}{e^2} \quad (12)$$

Now, we can find $\Pr(Y < 1 | Y < 2)$ as follows,

$$\Pr(Y < 1 | Y < 2) = \frac{\Pr(Y < 1, Y < 2)}{\Pr(Y < 2)} \quad (13)$$
$$= \frac{\Pr(Y < 1)}{\Pr(Y < 2)} \quad (14)$$
$$= \frac{F_Y(1)}{F_Y(2)} \quad (15)$$
$$= \frac{1 - \frac{1}{e}}{1 - \frac{1}{e^2}} \quad (16)$$
$$= \frac{e(e-1)}{e^2 - 1} \quad (17)$$
$$= \frac{e}{e+1} \quad (18)$$

2) Steps to plot the cdf of Y.

a) Generate X from uniform distribution.
b) Use X to generate $Y = \log_e X$.
c) Calculate the CDF of Y.
d) Store the values of CDF in data file.
e) Plot the CDF using GNUPlot.