MATHEMATICAL TOOLS & VECTORS

0.1 Introduction

Mathematics is the language of physics. It becomes very easier to describe, understand and apply the physical principles, if we have a good knowledge of mathematics.

For example : Newton's law of gravitation states that every body in this universe attracts every other body with a force which is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them.

This law can be expressed by a single mathematical relationship $F \propto \frac{m_1 m_2}{r^2}$ or $F = \frac{Gm_1 m_2}{r^2}$

The techniques of mathematics such as algebra, trigonometry, calculus, graph and logarithm can be used to make predictions from the basic equation.

If we are poor at grammar and vocabulary, it would be difficult for us to communicate our feelings, similarly for better understanding and expressing of physics the basic knowledge of mathematics is must.

In this introductory chapter we will learn some fundamental mathematics.

0.2 Algebra

(1) **Quadratic equation :** An equation of second degree is called a quadratic equation. Standard quadratic equation $ax^2 + bx + c = 0$

Here a is called the coefficient of x^2, b is called the coefficient of x and c is a constant term, x is the variable whose value (roots of the equation) are to be determined

Roots of the equation are : $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

This formula can be written as

$$x = \frac{-\text{Coefficient of } x \pm \sqrt{(\text{Coefficient of } x)^2 - 4(\text{Coefficient of } x^2) \times (\text{Constant term})}}{2(\text{Coefficient of } x^2)}$$

Note : ❑ If α and β be the roots of the quadratic equation then

Sum of roots $\alpha + \beta = -\frac{b}{a}$ and product of roots = $\frac{c}{a}$

***Problem* 1.** Solve the equation $10x^2 - 27x + 5 = 0$

Solution : By comparing the given equation with standard equation $a = 10$, $b = -27$, and $c = 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-27) \pm \sqrt{(-27)^2 - 4 \times 10 \times 5}}{2 \times 10} = \frac{27 \pm 23}{20}$$

$$\therefore\ x_1 = \frac{27 + 23}{20} = \frac{5}{2} \text{ and } x_2 = \frac{27 - 23}{20} = \frac{1}{5}$$

$\therefore$ Roots of the equation are $\frac{5}{2}$ and $\frac{1}{5}$.

(2) **Binomial theorem :** If n is any number positive, negative or fraction and x is any real number, such that $x < 1$ *i.e.* x lies between – 1 and + 1 then according to binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

Here 2 ! (Factorial 2) = 2 × 1, 3 ! (Factorial 3) = 3 × 2 × 1 and 4 ! (Factorial 4) = 4 × 3 × 2 × 1

Note : ❑ If $|x| << 1$ then only the first two terms are significant. It is so because the values of second and the higher order terms being very very small, can be neglected. So the expression can be written as

$$(1+x)^n = 1 + nx$$

$$(1+x)^{-n} = 1 - nx$$

$$(1-x)^n = 1 - nx$$

$$(1-x)^{-n} = 1 + nx$$

Problem 2. Evaluate $(1001)^{1/3}$ upto six places of decimal.

Solution : $(1001)^{1/3} = (1000+1)^{1/3} = 10(1+0.001)^{1/3}$

By comparing the given equation with standard equation $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

$x = 0.001$ and $n = 1/3$

$$\therefore\ 10(1+0.001)^{1/3} = 10\left[1 + \frac{1}{3}(0.001) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\times(.001)^2}{2!} + ...\right] = 10\left[1 + 0.00033 - \frac{1}{9}(0.000001) + ...\right]$$

$$= 10[1.0003301] = 10.003301 \text{ (Approx.)}$$

Problem 3. The value of acceleration due to gravity (g) at a height h above the surface of earth is given by $g' = \frac{gR^2}{(R+h)^2}$. If $h << R$ then

(a) $g' = g\left(1 - \frac{h}{R}\right)$ (b) $g' = g\left(1 - \frac{2h}{R}\right)$ (c) $g' = g\left(1 + \frac{h}{R}\right)$ (d) $g' = g\left(1 + \frac{2h}{R}\right)$

Solution : (b) $g' = g\left(\frac{R}{R+h}\right)^2 = g\left(\frac{1}{1+h/R}\right)^2 = \left(1+\frac{h}{R}\right)^{-2} = g\left[1 + (-2)\frac{h}{R} + \frac{(-2)(-3)}{2!}\left(\frac{h}{R}\right)^2 + ...\right]$

$g' = g\left(1 - \frac{2h}{R}\right)$ (if $h << R$ then by neglecting higher power of $\frac{h}{R}$.)

(3) **Arithmetic progression :** It is a sequence of numbers which are arranged in increasing order and having a constant difference between them.

Example : 1, 3, 5, 7, 9, 11, 13, …… or 2, 4, 6, 8, 10, 12, …..

In general arithmetic progression can be written as $a_0, a_1, a_2, a_3, a_4, a_5$ …….

(i) n^{th} term of arithmetic progression $a_n = a_0 + (n-1)d$

a_0 = First term, n = Number of terms, d = Common difference = $(a_1 - a_0)$ or $(a_2 - a_1)$ or $(a_3 - a_2)$

(ii) Sum of arithmetic progression $S_n = \frac{n}{2}[2a_0 + (n-1)d] = \frac{n}{2}[a_0 + a_n]$

***Problem* 4.** Find the sum of series 7 + 10 + 13 + 16 + 19 + 22 + 25

Solution : $S_n = \frac{n}{2}[a_0 + a_n] = \frac{7}{2}[7+25] = 112$ [As $n = 7$; $a_0 = 7$; $a_n = a_7 = 25$]

(4) **Geometric progression :** It is a sequence of numbers in which every term is obtained by multiplying the previous term by a constant quantity. This constant quantity is called the common ratio.

Example : 4, 8, 16, 32, 64, 128 …… or 5, 10, 20, 40, 80, …….

In general geometric progression can be written as a, ar, ar^2, ar^3, ar^4, ….

Here a = first term, r = common ratio

(i) Sum of 'n' terms of G.P. $S_n = \frac{a(1-r^n)}{1-r}$ if $r < 1$

$S_n = \frac{a(r^n - 1)}{r-1}$ if $r > 1$

(ii) Sum of infinite terms of G.P. $S_\infty = \frac{a}{1-r}$ if $r < 1$

$S_\infty = \frac{a}{r-1}$ if $r > 1$

***Problem* 5.** Find the sum of series $Q = 2q + \frac{q}{3} + \frac{q}{9} + \frac{q}{27} + ...$

Solution : Above equation can be written as $Q = q + \left[q + \frac{q}{3} + \frac{q}{9} + \frac{q}{27} + ...\right]$

By using the formula of sum of infinite terms of G.P. $Q = q + \left[\frac{q}{1-\frac{1}{3}}\right] = q + \frac{3}{2}q = \frac{5}{2}q$

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(5) **Some common formulae of algebra**

(i) $(a+b)^2 = a^2 + b^2 + 2ab$

(ii) $(a-b)^2 = a^2 + b^2 - 2ab$

(iii) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(iv) $(a+b)(a-b) = a^2 - b^2$

(v) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(vi) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

(vii) $(a + b)^2 - (a - b)^2 = 4ab$

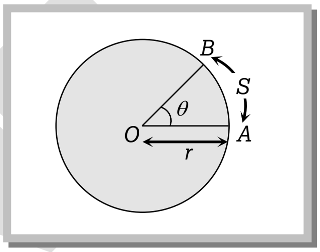
(viii) $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

(ix) $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$

(x) $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

(6) **Componendo and dividendo method :** If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

0.3 Trigonometry



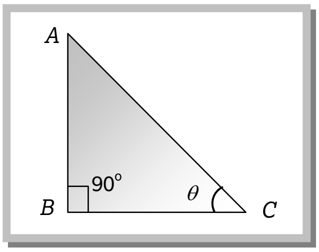
Angle $(\theta) = \frac{\text{Arc}}{\text{Radius}} = \frac{AB}{OA} = \frac{S}{r}$ (formula true for radian only)

unit of angle is radian or degree

relation between radian and degree :

$2\pi\ radian = 360^\circ$; $1\ radian = 57.3^\circ$

(1) **Trigonometric ratio :** In right angled triangle ABC, the largest side AC, which is opposite to the right angle is called hypotenuse, and if angle considered is θ, then side opposite to θ, AB, will be termed as perpendicular and BC is called the base of the triangle.



$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{AB}{AC}$ $\qquad$ $\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{AC}{AB}$

$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{BC}{AC}$ $\qquad$ $\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{AC}{BC}$

$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{AB}{BC}$ $\qquad$ $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{BC}{AB}$

(2) **Value of trigonometric ratio of standard angles**

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
sinθ	0	1/2	1/√2	√3/2	1	√3/2	1/ √2	1/2	0	– 1	0
cosθ	1	√3/2	1/√2	1/2	0	– 1/2	– 1/√2	– √3/2	– 1	0	1
tanθ	0	1/√3	1	√3	∞	– √3	– 1	– 1/√3	0	– ∞	0

(3) **Important points :**

(i) Value of $\sin\theta$ or $\cos\theta$ lies between – 1 and +1, however $\tan\theta$ and $\cot\theta$ can have any real value.

(ii) Value of $\sec\theta$ and $\text{cosec}\,\theta$ can not be numerically less than one.

(iii) $(90^\circ - \theta)$ will lie in first quadrant

$(90^\circ + \theta)$ will lie in second quadrant

$(180^\circ - \theta)$ will lie in second quadrant

$(180^\circ + \theta)$ will lie in third quadrant

$(270^\circ + \theta)$ and $(0^\circ - \theta)$ will lie in fourth quadrant.

Second quadrant (Only $\sin\theta$ and $\text{cosec}\,\theta$ are positive)	First quadrant (All *T*-ratio positive)
Third quadrant (Only $\tan\theta$ and $\cot\theta$ are positive)	Fourth quadrant (Only $\cos\theta$ and $\sec\theta$ are positive)

(4) **Fundamental trigonometrical relation**

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (ii) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ (iii) $\sec\theta = \dfrac{1}{\cos\theta}$ (iv) $\cot\theta = \dfrac{1}{\tan\theta}$

(v) $\sin^2\theta + \cos^2\theta = 1$ (vi) $\sec^2\theta - \tan^2\theta = 1$ (vii) $\text{cosec}^2\theta - \cot^2\theta = 1$

(5) ***T*-Ratios of allied angles :** The angles whose sum or difference with angle θ is zero or a multiple of 90° are called angle allied to θ.

(i)	$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
(ii)	$\sin(90^\circ - \theta) = \cos\theta$	$\cos(90^\circ - \theta) = \sin\theta$	$\tan(90^\circ - \theta) = \cot\theta$
(iii)	$\sin(90^\circ + \theta) = \cos\theta$	$\cos(90^\circ + \theta) = -\sin\theta$	$\tan(90^\circ + \theta) = -\cot\theta$
(iv)	$\sin(180^\circ - \theta) = \sin\theta$	$\cos(180^\circ - \theta) = -\cos\theta$	$\tan(180^\circ - \theta) = -\tan\theta$
(v)	$\sin(180^\circ + \theta) = -\sin\theta$	$\cos(180^\circ + \theta) = -\cos\theta$	$\tan(180^\circ + \theta) = \tan\theta$
(vi)	$\sin(270^\circ - \theta) = -\cos\theta$	$\cos(270^\circ - \theta) = -\sin\theta$	$\tan(270^\circ - \theta) = \cot\theta$
(vii)	$\sin(270^\circ + \theta) = -\cos\theta$	$\cos(270^\circ + \theta) = \sin\theta$	$\tan(270^\circ + \theta) = -\cot\theta$
(viii)	$\sin(360^\circ - \theta) = -\sin\theta$	$\cos(360^\circ - \theta) = \cos\theta$	$\tan(360^\circ - \theta) = -\tan\theta$
(ix)	$\sin(360^\circ + \theta) = \sin\theta$	$\cos(360^\circ + \theta) = \cos\theta$	$\tan(360^\circ + \theta) = \tan\theta$

Note :
- ❑ Angle $(2n\pi + \theta)$ lies in first quadrant, if θ in an acute angle. Similarly $(2n\pi - \theta)$ will lie in fourth quadrant. Where $n =$ 0, 1, 2, 3, 4
- ❑ Angle $(-\theta)$ is presumed always lie in fourth quadrant, whatever the value of θ.
- ❑ If parent angle is 90° or 270° then $\sin\theta$ change to $\cos\theta$, $\tan\theta$ change to $\cot\theta$ and $\sec\theta$ change to $\text{cosec}\,\theta$.
- ❑ If parent angle is 180° or 360° then no change in trigonometric function

***Problem* 6.** Find the values of (i) $\cos(-60^\circ)$ (ii) $\tan 210^\circ$ (iii) $\sin 300^\circ$ (iv) $\cos 120^\circ$ (v) $\sin(-1485^\circ)$

Solution : (i) $\cos(-60^\circ) = \cos 60^\circ = \dfrac{1}{2}$

(ii) $\tan(210^o) = \tan(180^o + 30^o) = \tan 30^o = \frac{1}{\sqrt{3}}$

(iii) $\sin(300^o) = \sin(360^o - 60^o) = -\sin 60^o = \frac{-\sqrt{3}}{2}$

(iv) $\cos(120^o) = \cos(90^o + 30^o) = -\sin 30^o = \frac{-1}{2}$

(v) $\sin(-1485^o) = -\sin(3 \times 360^o + 45^o) = -\sin 45^o = -\frac{1}{\sqrt{2}}$

(6) **Addition formulae**

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(iii) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Putting $B = A$ in these formulae, we get

(iv) $\sin 2A = 2 \sin A \cos A$

(v) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

(vi) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

***Problem* 7.** If $A = 60^o$ then value of $\sin 2A$ will be

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{2}}$

Solution : (a) $\sin 2A = 2 \sin A \cos A = 2 \sin 60 \cos 60 = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$

(7) **Difference formulae**

(i) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(ii) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(iii) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(8) **Transformation formulae**

$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$

$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$

$\cos(A-B) + \cos(A+B) = 2 \cos A \cos B$

If we put $(A+B) = C$ and $(A-B) = D$ then on adding and subtracting, we get

$A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

Putting these values in the above equation we get

(i) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(ii) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(iii) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(iv) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(9) **The sine and cosine formulae for a triangle :** In a triangle *ABC* of sides *a, b, c* and angles *A, B* and *C*, the following formulae hold good.

(i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

(ii) $a^2 = b^2 + c^2 - 2bc\cos A$

(iii) $b^2 = c^2 + a^2 - 2ca\cos B$

(iv) $c^2 = a^2 + b^2 - 2ab\cos C$

(v) Area of a triangle *ABC* = $\sqrt{S(S-a)(S-b)(S-c)}$; where, $S = (a+b+c)/3$

0.4 Logarithm

Logarithm of a number with respect to a given base is the power to which the base must be raised to represent that number.

If $a^x = N$ then $\log_a N = x$

Here *x* is called the logarithm of *N* to the base *a*.

There are two system of logarithm : Logarithm to the base 10 are called common logarithms where as logarithms to the base *e* are called natural logarithm. They are written as ln.

Conversion of natural log into common log : $\log_e x = 2.3026 \log_{10} x$

Important formulae of logarithm :

(i) $\log_a(mn) = \log_a m + \log_a n$ (Product formula)

(ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$ (Quotient formula)

(iii) $\log_a m^n = n\log_a m$ (Power formula)

(iv) $\log_a m = \log_b m \log_a b$ (Base change formula)

Note : ❑ Antilogarithm is the reverse process of logarithm *i.e.*, the number whose logarithm is *x* is called antilogarithm of *x*. If $\log n = x$ then $n =$ antilog of x

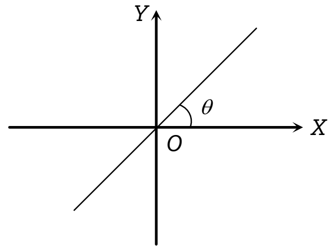
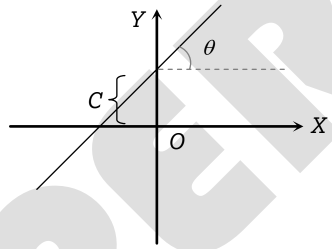
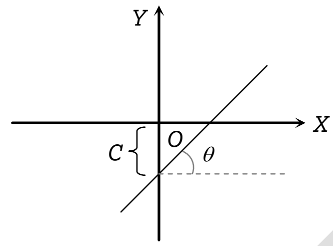
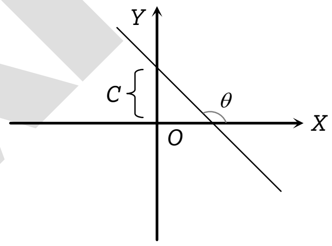
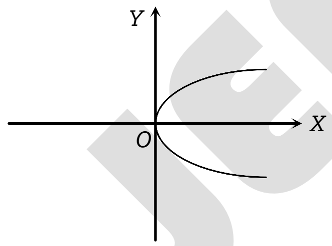
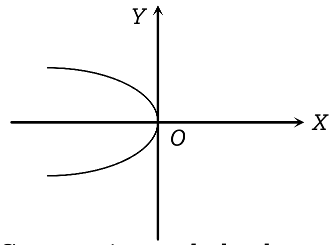
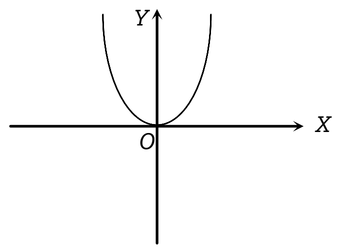
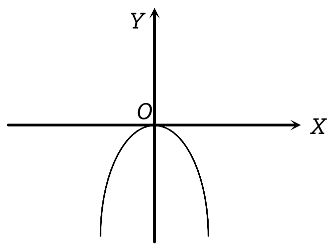
0.5 Graphs

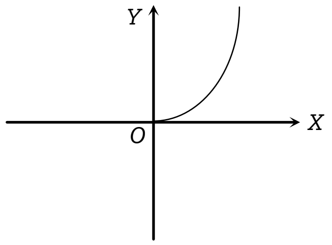
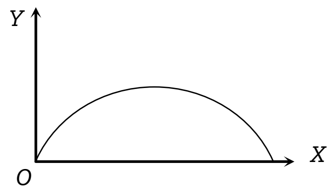
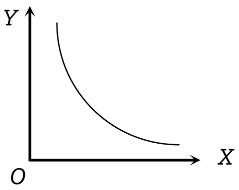
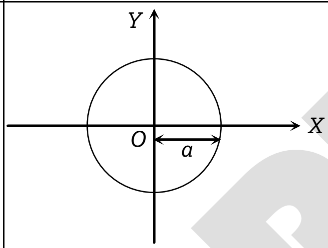
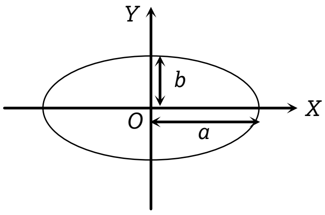
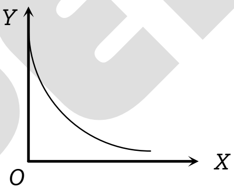
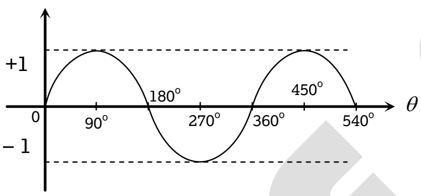
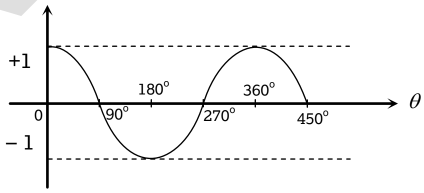
A graph is a line, straight or curved which shows the variation of one quantity *w.r.t.* other, which are interrelated with each other.

In a relation of two quantities, the quantity which is made to alter at will, is called the independent variable and the other quantity which varies as a result of this change is called the dependent variable. Conventionally, in any graph, the independent variable (*i.e.* cause) is represented along *x*-axis and dependent variable (*i.e.* effect) is represented along *y*-axis.

For example, we want to depict $V = IR$ graphically, in which R is a constant called resistance, V is the applied voltage (cause) and I (effect) is the resulting current. We will represent voltage on *x*-axis and current on *y*-axis.

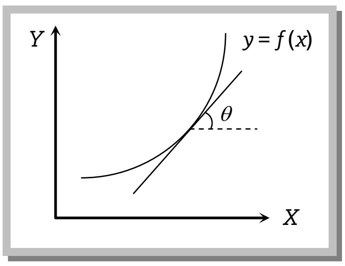
Some important graphs for various equations

 $y = mx$ $m = \tan\theta$ = slope of line with *x*-axis	 $y = mx + c$ c = Positive intercept on *y*-axis and positive slope
 $y = mx - c$ Negative intercept and positive slope	 $y = -mx + c$ Positive intercept and Negative slope
 $y^2 = kx$ Symmetric parabola about positive *X*-axis	 $y^2 = -kx$ Symmetric parabola about negative *X*-axis
 $x^2 = ky$	 $x^2 = -ky$

Symmetric parabola about positive Y-axis	Symmetric parabola about negative Y-axis
 $y = ax + bx^2$	 $y = ax - bx^2$
Asymmetric parabola	Asymmetric parabola
 $xy = \text{constant}$	 $x^2 + y^2 = a^2$
Rectangular hyperbola	Circle of radius 'a'
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	 $y = e^{-kx}$
Ellipse of semi-major axis a and semi-minor axis b.	Exponential curve
 $y = \sin\theta$	 $y = \cos\theta$
sine curve	cosine curve

0.6 Differential Calculus

The differential coefficient or derivative of variable y with respect to variable x is defined as the instantaneous rate of change of y *w.r.t.* x. It is denoted by $\frac{dy}{dx}$



Geometrically the differential coefficient of $y = f(x)$ with respect to x at any point is equal to the slope of the tangent to the curve representing $y = f(x)$ at that point

i.e. $\frac{dy}{dx} = \tan\theta$.

Note : ❑ Actually $\frac{dy}{dx}$ is a rate measurer.

❑ If $\frac{dy}{dx}$ is positive, it means y is increasing with increasing of x and vice-versa.

❑ For small change Δx we use $\Delta y = \frac{dy}{dx}.\Delta x$

Example: (1) Instantaneous speed $v = \frac{ds}{dt}$

(2) Instantaneous acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

(3) Force $F = \frac{dp}{dt}$

(4) Angular velocity $\omega = \frac{d\theta}{dt}$

(5) Angular acceleration $\alpha = \frac{d\omega}{dt}$

(6) Power $P = \frac{dW}{dt}$

(7) Torque $\tau = \frac{dL}{dt}$

(1) **Fundamental formulae of differentiation :**

Function	**Differentiation**
If c is some constant	$\frac{d}{dx}(c) = 0$
If $y = cx$ where c is a constant	$\frac{dy}{dx} = \frac{d}{dx}(cx) = c\frac{dx}{dx} = c$
If $y = cu$ where c is a constant and u is a function of x	$\frac{dy}{dx} = \frac{d}{dx}(cu) = c\frac{du}{dx}$
If $y = x^n$ where n is a real number	$\frac{dy}{dx} = nx^{n-1}$
If $y = u^n$ where n is a real number and u is a function of x	$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$
If $y = u + v$ where u and v are the functions of x	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
If $y = uv$ where u and v are functions of x (product formula)	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
If $y = \frac{u}{v}$ where u and v are the functions of x (quotient formula)	$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
If $y = f(u)$ and $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
If $y = (ax + b)^n$	$\frac{dy}{dx} = n(ax + b)^{n-1} \times \frac{d}{dx}(ax + b)$

If $y = \sin x$	$\frac{dy}{dx} = \frac{d}{dx}(\sin x) = \cos x$
If $y = \cos x$	$\frac{dy}{dx} = \frac{d}{dx}(\cos x) = -\sin x$
If $y = \tan x$	$\frac{dy}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x$
If $y = \cot x$	$\frac{dy}{dx} = \frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
If $y = \sec x$	$\frac{dy}{dx} = \frac{d}{dx}(\sec x) = \tan x \sec x$
If $y = \text{cosec}\, x$	$\frac{dy}{dx} = \frac{d}{dx}(\text{cosec}\, x) = -\cot x\, \text{cosec}\, x$
If $y = \sin u$ where u is the function of x	$\frac{dy}{dx} = \frac{d}{dx}(\sin u) = \cos u \frac{d(u)}{dx}$
If $y = \cos u$ where u is the function of x	$\frac{dy}{dx} = \frac{d}{dx}(\cos u) = -\sin u \frac{d(u)}{dx}$
If $y = \tan u$ where u is the function of x	$\frac{dy}{dx} = \frac{d}{dx}(\tan u) = \sec^2 u \frac{d(u)}{dx}$
If $y = \cot u$ where u is the function of x	$\frac{dy}{dx} = \frac{d}{dx}(\cot u) = -\text{cosec}^2 u \frac{d(u)}{dx}$
If $y = \sec u$ where u is the function of x	$\frac{dy}{dx} = \frac{d}{dx}(\sec u) = \sec u \tan u \frac{d(u)}{dx}$
If $y = \text{cosec}\, u$ where u is the function of x	$\frac{dy}{dx} = \frac{d}{dx}(\text{cosec}\, u) = -\text{cosec}\, u \cot u \frac{d}{(}$
If $y = \log_a x$	$\frac{dy}{dx} = \frac{1}{x}\log_a e$

Problem 8. Differentiate the following *w.r.t x*

(i) x^3 (ii) $\sqrt{x}$ (iii) $ax^2 + bx + c$ (iv) $2x^3 - e^x$ (v) $6\log e^x - \sqrt{x} - 7$

Solution : (i) $\frac{d}{dx}(x^3) = 3x^2$

(ii) $\frac{d}{dx}(x)^{1/2} = \frac{1}{2}(x)^{\frac{1}{2}-1} = \frac{1}{2}(x)^{-1/2} = \frac{1}{2\sqrt{x}}$

(iii) $\frac{d}{dx}(ax^2 + bx + c) = a\frac{d}{dx}(x^2) + b\frac{d}{dx}(x) + \frac{d}{dx}(c) = 2ax + b$

(iv) $\frac{d}{dx}(2x^3 - e^x) = 2\frac{d}{dx}(x^3) - \frac{d}{dx}(e^x) = 6x^2 - e^x$

(v) $\frac{d}{dx}(6\log_e x - \sqrt{x} - 7) = 6\frac{d}{dx}(\log_e x) - \frac{d}{dx}(x^{1/2}) - \frac{d}{dx}(7) = \frac{6}{x} - \frac{1}{2\sqrt{x}}$

Problem 9. Differentiate the following *w.r.t. x*

(i) $\sin x + \cos x$ (ii) $\sin x + e^x$

Solution : (i) $\frac{d}{dx}(\sin x + \cos x) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x) = \cos x - \sin x$

(ii) $\frac{d}{dx}(\sin x + e^x) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) = \cos x + e^x$

Problem 10. Differentiate the following *w.r.t.* t

(i) $\sin t^2$ (ii) $e^{\sin t}$ (iii) $\sin(\omega t + \theta)$

Solution : (i) $\frac{d}{dt}(\sin t^2) = \cos t^2 \frac{d}{dt}(t^2) = 2t \cos t^2$

(ii) $\frac{d}{dt}(e^{\sin t}) = e^{\sin t} \frac{d}{dt}(\sin t) = e^{\sin t}.\cos t$

(iii) $\frac{d}{dt}[\sin(\omega t + \theta)] = \cos(\omega t + \theta).\frac{d}{dt}(\omega t + \theta) = \cos(\omega t + \theta).\omega$

Problem 11. Differentiate $\frac{x^2 + e^x}{\log x + 20}$ *w.r.t.* x

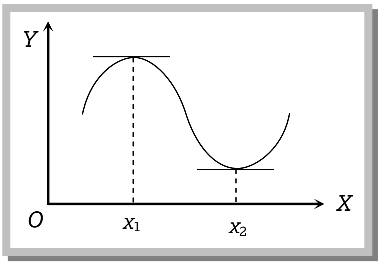
Solution : Let $y = \frac{x^2 + e^x}{\log x + 20}$.

Then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2 + e^x}{\log x + 20}\right)$

$$= \frac{(\log x + 20)\frac{d}{dx}(x^2 + e^x) - (x^2 + e^x)\frac{d}{dx}(\log x + 20)}{(\log x + 20)^2}$$

$$= \frac{(\log x + 20)(2x + e^x) - (x^2 + e^x)\left(\frac{1}{x} + 0\right)}{(\log x + 20)^2}$$

(2) **Maxima and minima :** If *a* quantity *y* depends on another quantity *x* in a manner shown in figure. It becomes maximum at x_1 and minimum at x_2.



At these points the tangent to the curve is parallel to *X*-axis and hence its slope is $\tan\theta = 0$. But the slope of the curve equals the rate of change $\frac{dy}{dx}$. Thus, at a maximum or minimum $\frac{dy}{dx} = 0$

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum. *i.e.*, $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ at a maximum.

Hence the condition of maxima : $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ (Second derivative test)

Similarly, at a minimum the slope changes from negative to positive. The slope increases at such a point and hence $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$

Hence the condition of minima : $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$. (Second derivative test)

***Problem* 12.** The height reached in time t by a particle thrown upward with a speed u is given by $h = ut - \frac{1}{2}gt^2$. Find the time taken in reaching the maximum height.

Solution : For maximum height $\frac{dh}{dt} = 0$ $\frac{d}{dt}[ut - \frac{1}{2}gt^2] = u - \frac{2gt}{2} = 0$ $\therefore t = \frac{u}{g}$

***Problem* 13.** A metal ring is being heated so that at any instant of time t in second, its area is given by

$A = 3t^2 + \frac{t}{3} + 2\, m^2$.

What will be the rate of increase of area at $t = 10$ sec.

Solution : Rate of increase of area $\frac{dA}{dt} = \frac{d}{dt}(3t^2 + \frac{t}{3} + 2) = 6t + \frac{1}{3}$

$$\left(\frac{dA}{dt}\right)_{t=10\text{ sec}} = 6 \times 10 + \frac{1}{3} = \frac{181}{3}\frac{m^2}{\text{sec}}.$$

***Problem* 14.** The radius of an air bubble is increasing at the rate of $\frac{1}{2} cm/\text{sec}$. Determine the rate of increase in its volume when the radius is 1 *cm*.

Solution : Volume of the spherical bubble $V = \frac{4}{3}\pi R^3$

Differentiating both sides *w.r.t.* time

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi R^3\right) = \frac{4}{3}\pi.3R^2.\frac{dR}{dt} = 4\pi R^2\frac{dR}{dt}$$

at $R = 1cm$, $\frac{dV}{dt} = 4\pi \times (1)^2 \times \frac{1}{2} = 2\pi\ cm^3/\text{sec}$. [Given $\frac{dR}{dt} = \frac{1}{2} cm/\text{sec}$]

***Problem* 15.** Find the angle of tangent drawn to the curve $y = 3x^2 - 7x + 5$ at the point (1, 1) with the x- axis.

Solution : $y = 3x^2 - 7x + 5$

Slope of tangent $= \frac{dy}{dx} = 6x - 7$

at (1, 1) $\frac{dy}{dx} = -1$ $\therefore \tan\theta = -1 \Rightarrow \theta = 135^o$.

0.7 Integral Calculus

The process of integration is just the reverse of differentiation. The symbol $\int$ is used to denote integration.

If $f(x)$ is the differential coefficient of function $F(x)$ with respect to x, then by integrating $f(x)$ we can get $F(x)$ again.

(1) **Fundamental formulae of integration :**

$\int x^n dx = \frac{x^{n+1}}{n+1}$, provided $n \neq -1$	$\int \sec^2 x\, dx = \tan x$
$\int dx = \int x^0 dx = \frac{x^{0+1}}{0+1} = x$	$\int \operatorname{cosec}^2 x\, dx = -\cot x$
$\int (u+v) dx = \int u\, dx + \int v\, dx$	$\int \sec x \tan x\, dx = \sec x$
$\int cu\, dx = c\int u\, dx$ where c is a constant and u is a function of x.	$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$
$\int cx^n dx = c\frac{x^{n+1}}{n+1}$	$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)\frac{d}{dx}(ax+b)} = \frac{(ax+b)^{n+1}}{a(n+1)}$
$\int x^{-1} dx = \int \frac{dx}{x} = \log_e x$	$\int \frac{a}{(ax+b)} dx = \frac{a\log_e(ax+b)}{\frac{d}{dx}(ax+b)} = \log_e(ax+b)$
$\int e^x dx = e^x$	$\int e^{ax+b} dx = \frac{e^{ax+b}}{\frac{d}{dx}(ax+b)} = \frac{e^{ax+b}}{a}$
$\int a^x dx = \frac{a^x}{\log_e a}$	$\int a^{cx+d} dx = \frac{a^{cx+d}}{\log_e a \frac{d}{dx}(cx+d)} = \frac{a^{cx+d}}{c\log_e a}$
$\int \sin x\, dx = -\cos x$	$\int \sec^2(ax+b) dx = \frac{\tan(ax+b)}{\frac{d}{dx}(ax+b)} = \frac{\tan(ax+b)}{a}$
$\int \sin nx\, dx = \frac{-\cos nx}{n}$	$\int \operatorname{cosec}^2(ax+b) dx = \frac{-\cot(ax+b)}{\frac{d}{dx}(ax+b)} = \frac{-\cot(ax+v)}{a}$
$\int \cos x\, dx = \sin x$	$\int \sec(ax+b)\tan(ax+b) dx = \frac{\sec(ax+b)}{\frac{d}{dx}(ax+b)} = \frac{\sec(ax+b)}{a}$
$\int \cos nx\, dx = \frac{\sin nx}{n}$	$\int \operatorname{cosec}(ax+b)\cot(ax+b) dx = \frac{-\operatorname{cosec}(ax+b)}{\frac{d}{dx}(ax+b)} = \frac{-\operatorname{cosec}(ax+b)}{a}$

(2) **Method of integration :** Sometimes, we come across some functions which cannot be integrated directly by using the standard integrals. In such cases, the integral of a function can be obtained by using one or more of the following methods.

(i) Integration by substitution : Those functions which cannot be integrated directly can be reduced to standard integrand by making a suitable substitution and then can be integrated by using the standard integrals. To understand the method, we take the few examples.

(ii) Integration by parts : This method of integration is based on the following rule :

Integral of a product of two functions = first function × integral of second function – integral of (differential coefficient of first function × integral of second function).

Thus, if u and v are the functions of x, then $\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\times\int v\,dx\right]dx$

Problem **16.** Integrate the following *w.r.t.* x

(i) $x^{1/2}$ (ii) $\cot^2 x$ (iii) $\frac{1}{1-\sin x}$

Solution : (i) $\int x^{1/2}dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} = \frac{2}{3}(x^{3/2})$

(ii) $\int \cot^2 x\,dx = \int(\text{cosec}^2 x - 1)dx = \int \text{cosec}^2 x\,dx - \int dx = -\cot x - x$

(iii) $\int \frac{1}{1-\sin x}dx = \int\left(\frac{1}{1-\sin x}\times\frac{1+\sin x}{1+\sin x}\right).dx = \int\frac{1+\sin x}{1-\sin^2 x}dx = \int\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}dx$

$= \int(\sec^2 x + \tan x \sec x)dx = \tan x + \sec x.$

(3) **Definite integrals :** When a function is integrated between definite limits, the integral is called definite integral. For example,

$\int_a^b f(x)dx$ is definite integral of $f(x)$ between the limits a and b and is written as $\int_a^b f(x)dx = |F(x)|_a^b = F(b) - F(a)$

Here a is called the lower limit and b is called the upper limit of integration.

Geometrically $\int_a^b f(x)dx$ equals to area of curve $F(x)$ between the limits a and b.

Problem **17.** Evaluate $\int_0^6 (2x^2+3x+5)dx$

Solution : $\int_0^6(2x^2+3x+5)dx = \int_0^6 2x^2dx + \int_0^6 3x\,dx + \int_0^6 5\,dx = \left[\frac{2x^3}{3}\right]_0^6 + \left[\frac{3x^2}{2}\right]_0^6 + [5x]_0^6 = 144+54+30 = 228.$

Problem **18.** Integrate the following

(i) $\int_0^2 \frac{1}{\sqrt{x}} dx$ (ii) $\int_0^{\pi/2} \cos x \, dx$ (iii) $\int_{r_1}^{r_2} \frac{Kq_1q_2}{r^2} .dr$ (iv) $\int_0^{\pi/4} \tan^2 x \, dx$

Solution : (i) $\int_0^2 \frac{1}{\sqrt{x}} dx = \int_0^2 x^{-1/2} dx = \left[\frac{x^{1/2}}{1/2}\right]_0^2 = \left[2x^{1/2}\right]_0^2 = 2\sqrt{2}$

(ii) $\int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} = 1$

(iii) $\int_{r_1}^{r_2} k\frac{q_1q_2}{r^2} dx = k\,q_1q_2 \int_{r_1}^{r_2} \frac{1}{r^2} dx = kq_1q_2\left(-\frac{1}{r}\right)_{r_1}^{r_2} = -kq_1q_2\left[\frac{1}{r_2} - \frac{1}{r_1}\right] = kq_1q_2\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$

(iv) $\int_0^{\pi/4} \tan^2 x \, dx = \int_o^{\pi/4} (\sec^2 x - 1) dx = [\tan x]_0^{\pi/4} - [x]_0^{\pi/4} = 1 - \frac{\pi}{4}$

0.8 General Formulae for Area and Volume

1. Area of square = (side)^2
2. Area of rectangle = length $\times$ breadth
3. Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height}$
4. Area enclosed by a circle $= \pi r^2$; where r is radius
5. Surface area of sphere $= 4\pi r^2$
6. Surface area of cube $= 6L^2$; where L is a side of cube
7. Surface area of cuboid $= 2[L \times b + b \times h + h \times L]$; where L= length, b = breadth, h = height
8. Area of curved surface of cylinder $= 2\pi rl$; where r = radius, l = length of cylinder
9. Volume of cube $= L^3$
10. Volume of cuboid $= L \times b \times h$
11. Volume of sphere $= \frac{4}{3}\pi r^3$
12. Volume of cylinder $= \pi r^2 l$
13. Volume of cone $= \frac{1}{2}\pi r^2 h$

0.9 Introduction of Vector

Physical quantities having magnitude, direction and obeying laws of vector algebra are called vectors.

Example : Displacement, velocity, acceleration, momentum, force, impulse, weight, thrust, torque, angular momentum, angular velocity *etc.*

If a physical quantity has magnitude and direction both, then it does not always imply that it is a vector. For it to be a vector the third condition of obeying laws of vector algebra has to be satisfied.

Example : The physical quantity current has both magnitude and direction but is still a scalar as it disobeys the laws of vector algebra.

0.10 Types of Vector

(1) **Equal vectors :** Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal when they have equal magnitudes and same direction.

(2) **Parallel vector :** Two vectors $\vec{A}$ and $\vec{B}$ are said to be parallel when

(i) Both have same direction.

(ii) One vector is scalar (positive) non-zero multiple of another vector.

(3) **Anti-parallel vectors :** Two vectors $\vec{A}$ and $\vec{B}$ are said to be anti-parallel when

(i) Both have opposite direction.

(ii) One vector is scalar non-zero negative multiple of another vector.

(4) **Collinear vectors :** When the vectors under consideration can share the same support or have a common support then the considered vectors are collinear.

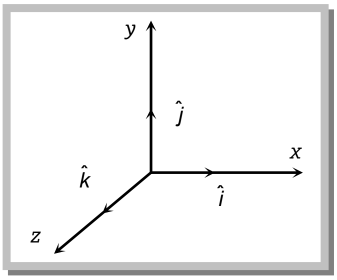
(5) **Zero vector** $(\vec{0})$**:** A vector having zero magnitude and arbitrary direction (not known to us) is a zero vector.

(6) **Unit vector :** A vector divided by its magnitude is a unit vector. Unit vector for $\vec{A}$ is $\hat{A}$ (read as A cap / A hat).

Since, $\hat{A} = \dfrac{\vec{A}}{A} \Rightarrow \vec{A} = A\hat{A}$.

Thus, we can say that unit vector gives us the direction.

(7) **Orthogonal unit vectors :** $\hat{i}, \hat{j}$ and $\hat{k}$ are called orthogonal unit vectors. These vectors must form a Right Handed Triad (It is a coordinate system such that when we Curl the fingers of right hand from *x* to *y* then we must get the direction of *z* along thumb). The

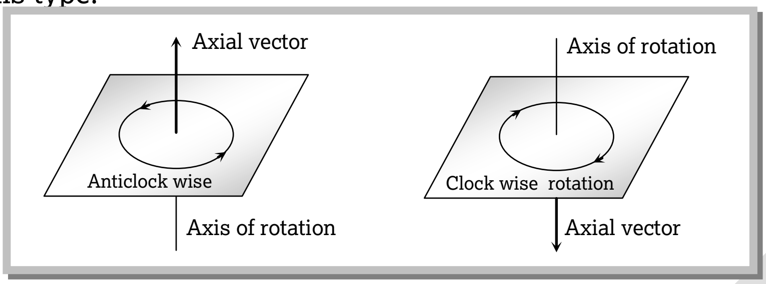


$$\hat{i} = \frac{\vec{x}}{x}, \hat{j} = \frac{\vec{y}}{y}, \hat{k} = \frac{\vec{z}}{z}$$

$$\therefore \quad \vec{x} = x\hat{i}, \ \vec{y} = y\hat{j}, \ \vec{z} = z\hat{k}$$

(8) **Polar vectors :** These have starting point or point of application . Example displacement and force *etc.*

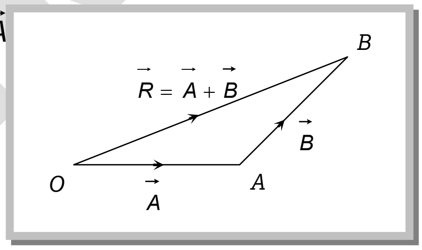
(9) **Axial Vectors :** These represent rotational effects and are always along the axis of rotation in accordance with right hand screw rule. Angular velocity, torque and angular momentum, etc., are example of physical quantities of this type.



(10) **Coplanar vector** : Three (or more) vectors are called coplanar vector if they lie in the same plane. Two (free) vectors are always coplanar.

0.11 Triangle Law of Vector Addition of Two Vectors

If two non zero vectors are represented by the two sides of a triangle taken in same order then the resultant is given by the closing side of triangle in opposite order. *i.e.* $\vec{R} = \vec{A}$

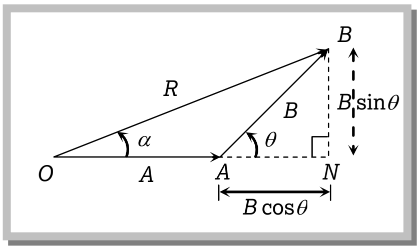


$\because\ \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

(1) **Magnitude of resultant vector**

In $\Delta\, ABN\ \cos\theta = \dfrac{AN}{B}\ \therefore\ AN = B\cos\theta$

$\sin\theta = \dfrac{BN}{B}\quad \therefore\ \ BN = B\sin\theta$



In ΔOBN, we have $OB^2 = ON^2 + BN^2$

$\Rightarrow\ R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$

$\Rightarrow\ R^2 = A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$

$\Rightarrow\ R^2 = A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$

$\Rightarrow\ R^2 = A^2 + B^2 + 2AB\cos\theta$

$\Rightarrow\ R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

(2) **Direction of resultant vectors :** If θ is angle between $\vec{A}$ and $\vec{B}$, then

$|\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

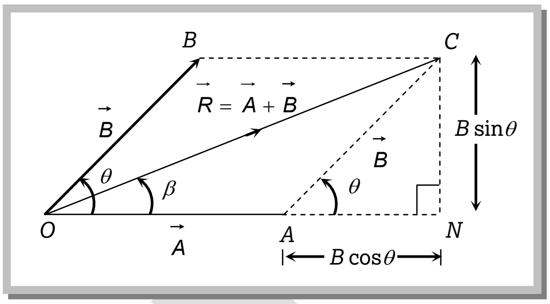
If $\vec{R}$ makes an angle α with $\vec{A}$, then in ΔOBN, then

$\tan\alpha = \dfrac{BN}{ON} = \dfrac{BN}{OA + AN}$

$\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$

0.12 Parallelogram Law of Vector Addition of Two Vectors

If two non zero vector are represented by the two adjacent sides of a parallelogram then the resultant is given by the diagonal of the parallelogram passing through the point of intersection of the two vectors.



(1) **Magnitude**

Since, $R^2 = ON^2 + CN^2$

$\Rightarrow R^2 = (OA + AN)^2 + CN^2$

$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$

$\therefore \quad R = |\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

Special cases : $R = A + B$ when $\theta = 0^\circ$

$R = A - B$ when $\theta = 180^\circ$

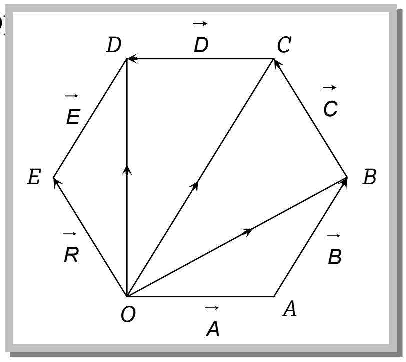
$R = \sqrt{A^2 + B^2}$ when $\theta = 90^\circ$

(2) **Direction**

$$\tan\beta = \frac{CN}{ON} = \frac{B\sin\theta}{A + B\cos\theta}$$

0.13 Polygon Law of Vector Addition

If a number of non zero vectors are represented by the $(n - 1)$ sides of an n-sided polygon then the resultant is given by the closing side or the n^{th} side of the polygon taken in op



$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$

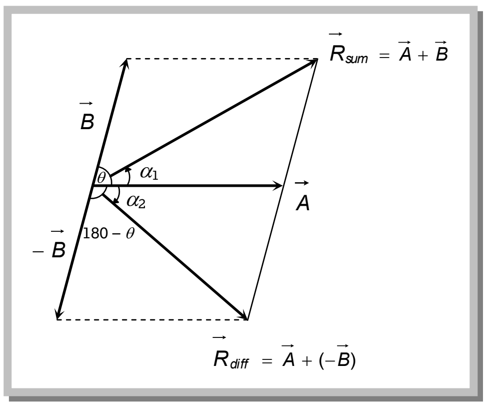
$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{OE}$

Note : ❑ Resultant of two unequal vectors can not be zero.

❑ Resultant of three co-planar vectors may or may not be zero

❑ Resultant of three non co- planar vectors can not be zero.

0.14 Subtraction of Vectors



Since, $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ and $|\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\Rightarrow \qquad |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos(180^\circ - \theta)}$

Since, $\cos(180 - \theta) = -\cos\theta$

$\Rightarrow \qquad |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

$$\tan\alpha_1 = \frac{B\sin\theta}{A + B\cos\theta}$$

and $$\tan\alpha_2 = \frac{B\sin(180 - \theta)}{A + B\cos(180 - \theta)}$$

But $\sin(180 - \theta) = \sin\theta$ and $\cos(180 - \theta) = -\cos\theta$

$\Rightarrow \quad \tan\alpha_2 = \dfrac{B\sin\theta}{A - B\cos\theta}$

Sample problem based on addition and subtraction of vectors

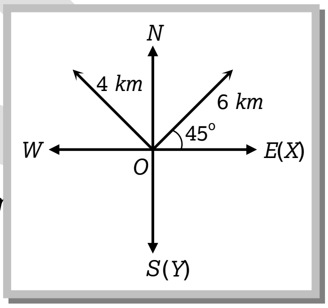
Problem 19. A car travels 6 *km* towards north at an angle of 45° to the east and then travels distance of 4 *km* towards north at an angle of 135° to the east. How far is the point from the starting point. What angle does the straight line joining its initial and final position makes with the east

(a) $\sqrt{50}\ km$ and $\tan^{-1}(5)$ (b) 10 *km* and $\tan^{-1}(\sqrt{5})$

(c) $\sqrt{52}\ km$ and $\tan^{-1}(5)$ (d) $\sqrt{52}\ km$ and $\tan^{-1}(\sqrt{5})$

Solution : (c) Net movement along *x*-direction $S_x = (6-4)\cos 45°\ \hat{i} = 2\times\frac{1}{\sqrt{2}} = \sqrt{2}\ km$

Net movement along *y*-direction $S_y = (6+4)\sin 45°\ \hat{j} = 10\times\frac{1}{\sqrt{2}} = 5\sqrt{2}\ km$



Net movement from starting point $|\vec{s}| = \sqrt{s_x^2 + s_y^2} = \sqrt{(\sqrt{2})^2 + (5\sqrt{2})^2} = \sqrt{52}\ km$

Angle which makes with the east direction $\tan\theta = \dfrac{\text{Y – component}}{\text{X – component}} = \dfrac{5\sqrt{2}}{\sqrt{2}}$

$\therefore\ \theta = \tan^{-1}(5)$

Problem 20. There are two force vectors, one of 5 *N* and other of 12 *N* at what angle the two vectors be added to get resultant vector of 17 *N*, 7 *N* and 13 *N* respectively

(a) 0°, 180° and 90° (b) 0°, 90° and 180° (c) 0°, 90° and 90° (d) 180°, 0° and 90°

Solution : (a) For 17 *N* both the vector should be parallel *i.e.* angle between them should be zero.

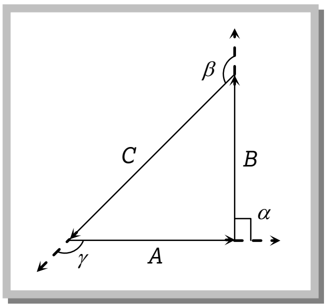
For 7 *N* both the vectors should be antiparallel *i.e.* angle between them should be 180°

For 13 *N* both the vectors should be perpendicular to each other *i.e.* angle between them should be 90°

Problem 21. Given that $\vec{A} + \vec{B} + \vec{C} = 0$ out of three vectors two are equal in magnitude and the magnitude of third vector is $\sqrt{2}$ times that of either of the two having equal magnitude. Then the angles between vectors are given by

(a) 30°, 60°, 90° (b) 45°, 45°, 90° (c) 45°, 60°, 90° (d) 90°, 135°, 135°

Solution : (d) From polygon law, three vectors having summation zero should form a closed polygon. (Triangle) since the two vectors are having same magnitude and the third vector is $\sqrt{2}$ times that of either of two having equal magnitude. *i.e.* the triangle should be right angled triangle



Angle between *A* and *B*, $\alpha = 90º$

Angle between *B* and *C*, $\beta = 135º$

Angle between *A* and *C*, $\gamma = 135º$

Problem 22. If $\vec{A} = 4\hat{i} - 3\hat{j}$ and $\vec{B} = 6\hat{i} + 8\hat{j}$ then magnitude and direction of $\vec{A} + \vec{B}$ will be

(a) $5, \tan^{-1}(3/4)$ (b) $5\sqrt{5}, \tan^{-1}(1/2)$ (c) $10, \tan^{-1}(5)$ (d) $25, \tan^{-1}(3/4)$

Solution : (b) $\vec{A} + \vec{B} = 4\hat{i} - 3\hat{j} + 6\hat{i} + 8\hat{j} = 10\hat{i} + 5\hat{j}$

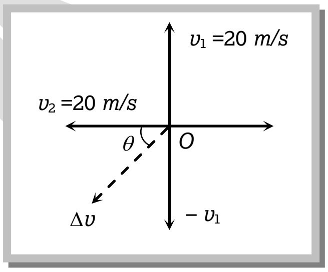
$|\vec{A} + \vec{B}| = \sqrt{(10)^2 + (5)^2} = 5\sqrt{5}$

$\tan\theta = \frac{5}{10} = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$

Problem 23. A truck travelling due north at 20 *m/s* turns west and travels at the same speed. The change in its velocity be

(a) 40 *m/s N–W* (b) $20\sqrt{2}$ *m/s N–W* (c) 40 *m/s S–W* (d) $20\sqrt{2}$ *m/s S–W*

Solution : (d) From fig.



$\vec{v}_1 = 20\hat{j}$ and $\vec{v}_2 = -20\hat{i}$

$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1 = -20(\hat{i} + \hat{j})$

$|\Delta\vec{v}| = 20\sqrt{2}$ and direction $\theta = \tan^{-1}(1) = 45°$ *i.e. S–W*

Problem 24. If the sum of two unit vectors is a unit vector, then magnitude of difference is

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $1/\sqrt{2}$ (d) $\sqrt{5}$

Solution : (b) Let $\hat{n}_1$ and $\hat{n}_2$ are the two unit vectors, then the sum is

$\vec{n}_s = \hat{n}_1 + \hat{n}_2$ or $n_s^2 = n_1^2 + n_2^2 + 2n_1n_2\cos\theta = 1 + 1 + 2\cos\theta$

Since it is given that n_s is also a unit vector, therefore $1 = 1 + 1 + 2\cos\theta$

or $\cos\theta = -\frac{1}{2}$ or $\theta = 120°$

Now the difference vector is $n_d = n_1 - n_2$ or $n_d^2 = n_1^2 + n_2^2 - 2n_1n_2\cos\theta = 1 + 1 - 2\cos(120°)$

$\therefore n_d^2 = 2 - 2(-1/2) = 2 + 1 = 3 \quad \Rightarrow n_d = \sqrt{3}$

Problem 25. The sum of the magnitudes of two forces acting at point is 18 and the magnitude of their resultant is 12. If the resultant is at 90° with the force of smaller magnitude, what are the, magnitudes of forces

(a) 12, 5 (b) 14, 4 (c) 5, 13 (d) 10, 8

Solution : (c) Let *P* be the smaller force and *Q* be the greater force then according to problem –

$P + Q = 18$ ......(i)

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = 12$ ....(ii)

$\tan\phi = \frac{Q\sin\theta}{P + Q\cos\theta} = \tan 90 = \infty$

$\therefore\ P + Q\cos\theta = 0$ ....(iii)

By solving (i), (ii) and (iii) we will get $P = 5$, and $Q = 13$

Problem 26. Two forces $F_1 = 1\,N$ and $F_2 = 2\,N$ act along the lines $x = 0$ and $y = 0$ respectively. Then the resultant of forces would be

(a) $\hat{i} + 2\hat{j}$ (b) $\hat{i} + \hat{j}$ (c) $3\hat{i} + 2\hat{j}$ (d) $2\hat{i} + \hat{j}$

Solution : (d) $x = 0$ means y-axis $\Rightarrow \vec{F}_1 = \hat{j}$

$y = 0$ means x-axis $\Rightarrow \vec{F}_2 = 2\hat{i}$ so resultant $\vec{F} = \vec{F}_1 + \vec{F}_2 = 2\hat{i} + \hat{j}$

Problem 27. Let $\vec{A} = 2\hat{i} + \hat{j}, B = 3\hat{j} - \hat{k}$ and $\vec{C} = 6\hat{i} - 2\hat{k}$ value of $\vec{A} - 2\vec{B} + 3\vec{C}$ would be

(a) $20\hat{i} + 5\hat{j} + 4\hat{k}$ (b) $20\hat{i} - 5\hat{j} - 4\hat{k}$ (c) $4\hat{i} + 5\hat{j} + 20\hat{k}$ (d) $5\hat{i} + 4\hat{j} + 10\hat{k}$

Solution : (b) $\vec{A} - 2\vec{B} + 3\vec{C} = (2\hat{i} + \hat{j}) - 2(3\hat{j} - \hat{k}) + 3(6\hat{i} - 2\hat{k})$

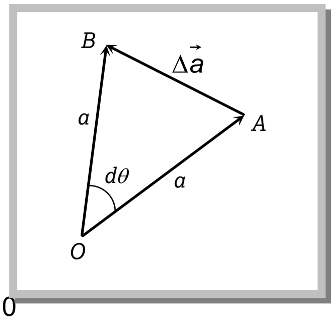
$= 2\hat{i} + \hat{j} - 6\hat{j} + 2\hat{k} + 18\hat{i} - 6\hat{k}$

$= 20\hat{i} - 5\hat{j} - 4\hat{k}$

Problem 28. A vector $\vec{a}$ is turned without a change in its length through a small angle $d\theta$. The value of $|\Delta\vec{a}|$ and Δa are respectively

(a) $0, a\,d\theta$ (b) $a\,d\theta, 0$ (c) 0, 0 (d) None of these

Solution : (b) From the figure $|\overrightarrow{OA}| = a$ and $|\overrightarrow{OB}| = a$



Also from triangle rule $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \Delta\vec{a} \Rightarrow |\Delta\vec{a}| = AB$

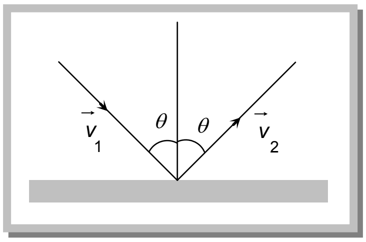
Using angle $= \dfrac{\text{arc}}{\text{radius}} \Rightarrow AB = a\,.\,d\theta$

So $|\Delta\vec{a}| = a\,d\theta$

Δa means change in magnitude of vector *i.e.* $|\overrightarrow{OB}| - |\overrightarrow{OA}| \Rightarrow a - a = 0$

So $\Delta a = 0$

Problem 29. An object of *m kg* with speed of *v m/s* strikes a wall at an angle θ and rebounds at the same speed and same angle. The magnitude of the change in momentum of the object will be

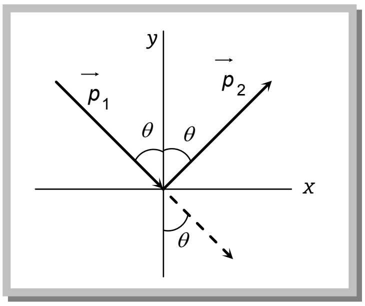


(a) $2mv\cos\theta$ (b) $2mv\sin\theta$ (c) 0 (d) $2mv$

Solution : (a) $\vec{P}_1 = mv\sin\theta\,\hat{i} - mv\cos\theta\,\hat{j}$ and $\vec{P}_2 = mv\sin\theta\,\hat{i} + mv\cos\theta\,\hat{j}$

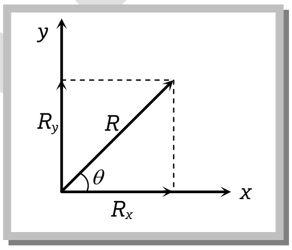
So change in momentum $\overrightarrow{\Delta P} = \vec{P}_2 - \vec{P}_1 = 2mv\cos\theta\,\hat{j}$

$|\Delta\vec{P}| = 2mv\cos\theta$



0.15 Resolution of Vector Into Components

Consider a vector $\vec{R}$ in *x-y* plane as shown in fig. If we draw orthogonal vectors $\vec{R}_x$ and $\vec{R}_y$ along *x* and *y* axes respectively, by law of vector addition, $\vec{R} = \vec{R}_x + \vec{R}_y$



Now as for any vector $\vec{A} = A\hat{n}$ so, $\vec{R}_x = \hat{i}R_x$ and $\vec{R}_y = \hat{j}R_y$

so $\vec{R} = \hat{i}R_x + \hat{j}R_y$ .....(i)

But from fig $R_x = R\cos\theta$ .....(ii)

and $R_y = R\sin\theta$ .....(iii)

Since R and θ are usually known, Equation (ii) and (iii) give the magnitude of the components of $\vec{R}$ along x and *y*-axes respectively.

Here it is worthy to note once a vector is resolved into its components, the components themselves can be used to specify the vector as –

(1) The magnitude of the vector $\vec{R}$ is obtained by squaring and adding equation (ii) and (iii), *i.e.*

$$R = \sqrt{R_x^2 + R_y^2}$$

(2) The direction of the vector $\vec{R}$ is obtained by dividing equation (iii) by (ii), *i.e.*

$$\tan\theta = (R_y / R_x) \quad \text{or} \quad \theta = \tan^{-1}(R_y / R_x)$$

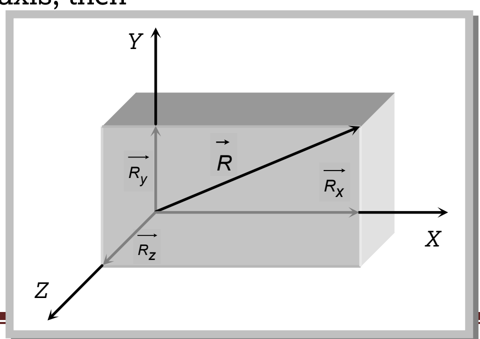
0.16 Rectangular Components of 3-D Vector

$\vec{R} = \vec{R}_x + \vec{R}_y + \vec{R}_z$ or $\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$

If $\vec{R}$ makes an angle α with *x* axis, β with *y* axis and γ with *z* axis, then

$$\Rightarrow \cos\alpha = \frac{R_x}{R} = \frac{R_x}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = l$$

$$\Rightarrow \cos\beta = \frac{R_y}{R} = \frac{R_y}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = m$$



$$\Rightarrow \cos\gamma = \frac{R_z}{R} = \frac{R_z}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = n$$

where l, m, n are called Direction Cosines of the vector $\vec{R}$

$$l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{R_x^2 + R_y^2 + R_z^2}{R_x^2 + R_y^2 + R_z^2} = 1$$

Note :
- ❑ When a point P have coordinate (x, y, z) then its position vector $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$
- ❑ When a particle moves from point (x_1, y_1, z_1) to (x_2, y_2, z_2) then its displacement vector

$$\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Sample problem based on representation and resolution of vector

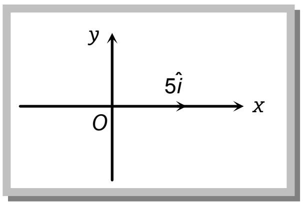
Problem 30. If a particle moves 5 m in +x- direction. The displacement of the particle will be

(a) $5\,j$ (b) $5\,i$ (c) $-5\,j$ (d) $5\,k$

Solution : (b) Magnitude of vector = 5

Unit vector in +x direction is $\hat{i}$

So displacement = $5\,\hat{i}$



Problem 31. Position of a particle in a rectangular-co-ordinate system is (3, 2, 5). Then its position vector will be

(a) $3\hat{i} + 5\hat{j} + 2\hat{k}$ (b) $3\hat{i} + 2\hat{j} + 5\hat{k}$ (c) $5\hat{i} + 3\hat{j} + 2\hat{k}$ (d) None of these

Solution : (b) If a point have coordinate (x, y, z) then its position vector $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

Problem 32. If a particle moves from point P (2,3,5) to point Q (3,4,5). Its displacement vector be

(a) $\hat{i} + \hat{j} + 10\hat{k}$ (b) $\hat{i} + \hat{j} + 5\hat{k}$ (c) $\hat{i} + \hat{j}$ (d) $2\hat{i} + 4\hat{j} + 6\hat{k}$

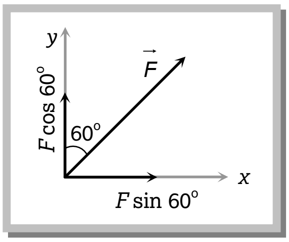
Solution : (c) Displacement vector $\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k} = (3-2)\hat{i} + (4-3)\hat{j} + (5-5)\hat{k} = \hat{i} + \hat{j}$

Problem 33. A force of 5 N acts on a particle along a direction making an angle of 60° with vertical. Its vertical component be

(a) 10 N (b) 3 N (c) 4 N (d) 5.2 N

Solution : (d) The component of force in vertical direction will be $F\cos\theta = F\cos 60^\circ$

$$= 5 \times \frac{1}{2} = 2.5\,N$$



Problem 34. If $A = 3\hat{i} + 4\hat{j}$ and $B = 7\hat{i} + 24\hat{j}$, the vector having the same magnitude as B and parallel to A is

(a) $5\hat{i}+20\hat{j}$ (b) $15\hat{i}+10\hat{j}$ (c) $20\hat{i}+15\hat{j}$ (d) $15\hat{i}+20\hat{j}$

Solution : (d) $|B|=\sqrt{7^2+(24)^2}=\sqrt{625}=25$

Unit vector in the direction of A will be $\hat{A}=\frac{3\hat{i}+4\hat{j}}{5}$

So required vector $=25\left(\frac{3\hat{i}+4\hat{j}}{5}\right)=15\hat{i}+20\hat{j}$

***Problem* 35.** Vector $\vec{A}$ makes equal angles with *x*, *y* and *z* axis. Value of its components (in terms of magnitude of $\vec{A}$) will be

(a) $\frac{A}{\sqrt{3}}$ (b) $\frac{A}{\sqrt{2}}$ (c) $\sqrt{3}A$ (d) $\frac{\sqrt{3}}{A}$

Solution : (a) Let the components of $\vec{A}$ makes angles α,β and γ with *x*, *y* and *z* axis respectively then $\alpha=\beta=\gamma$

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \Rightarrow 3\cos^2\alpha=1 \Rightarrow \cos\alpha=\frac{1}{\sqrt{3}}$

$\therefore A_x=A_y=A_z=A\cos\alpha=\frac{A}{\sqrt{3}}$

***Problem* 36.** If $\vec{A}=2\hat{i}+4\hat{j}-5\hat{k}$ the direction of cosines of the vector $\vec{A}$ are

(a) $\frac{2}{\sqrt{45}},\frac{4}{\sqrt{45}}$ and $\frac{-5}{\sqrt{45}}$ (b) $\frac{1}{\sqrt{45}},\frac{2}{\sqrt{45}}$ and $\frac{3}{\sqrt{45}}$ (c) $\frac{4}{\sqrt{45}},0$ and $\frac{4}{\sqrt{45}}$ (d) $\frac{3}{\sqrt{45}},\frac{2}{\sqrt{45}}$ and $\frac{5}{\sqrt{45}}$

Solution : (a) $|\vec{A}|=\sqrt{(2)^2+(4)^2+(-5)^2}=\sqrt{45}$

$\therefore \cos\alpha=\frac{2}{\sqrt{45}}, \quad \cos\beta=\frac{4}{\sqrt{45}}, \quad \cos\gamma=\frac{-5}{\sqrt{45}}$

***Problem* 37.** The vector that must be added to the vector $\hat{i}-3\hat{j}+2\hat{k}$ and $3\hat{i}+6\hat{j}-7\hat{k}$ so that the resultant vector is a unit vector along the *y*-axis is

(a) $4\hat{i}+2\hat{j}+5\hat{k}$ (b) $-4\hat{i}-2\hat{j}+5\hat{k}$ (c) $3\hat{i}+4\hat{j}+5\hat{k}$ (d) Null vector

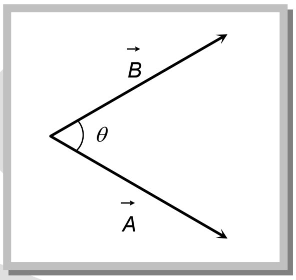
Solution : (b) Unit vector along *y* axis $=\hat{j}$ so the required vector $=\hat{j}-[(\hat{i}-3\hat{j}+2\hat{k})+(3\hat{i}+6\hat{j}-7\hat{k})]=-4\hat{i}-2\hat{j}+5\hat{k}$

0.17 Scalar Product of Two Vectors

(1) **Definition :** The scalar product (or dot product) of two vectors is defined as the product of the magnitude of two vectors with cosine of angle between them.

Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having angle θ between them, then their scalar product written as $\vec{A}.\vec{B}$ is defined as $\vec{A}.\vec{B} = AB\cos\theta$

(2) **Properties :** (i) It is always a scalar which is positive if angle between the vectors is acute (*i.e.*, < 90°) and negative if angle between them is obtuse (*i.e.* 90°<θ< 180°).



(ii) It is commutative, *i.e.* $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

(iii) It is distributive, *i.e.* $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

(iv) As by definition $\vec{A}.\vec{B} = AB\cos\theta$

The angle between the vectors $\theta = \cos^{-1}\left[\dfrac{\vec{A}.\vec{B}}{AB}\right]$

(v) Scalar product of two vectors will be maximum when $\cos\theta = \max = 1$, *i.e.* $\theta = 0^\circ$, *i.e.*, vectors are parallel

$$(\vec{A}.\vec{B})_{\max} = AB$$

(vi) Scalar product of two vectors will be minimum when $|\cos\theta| = \min = 0$, *i.e.* $\theta = 90^\circ$

$$(\vec{A}.\vec{B})_{\min} = 0$$

i.e., if the scalar product of two nonzero vectors vanishes the vectors are orthogonal.

(vii) The scalar product of a vector by itself is termed as self dot product and is given by $(\vec{A})^2 = \vec{A}.\vec{A} = AA\cos\theta = A^2$

i.e., $A = \sqrt{\vec{A}.\vec{A}}$

(viii) In case of unit vector $\hat{n}$

$$\hat{n}.\hat{n} = 1\times 1\times\cos 0 = 1 \quad \text{so } \hat{n}.\hat{n} = \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$$

(ix) In case of orthogonal unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 1\times 1\cos 90 = 0$

(x) In terms of components $\vec{A}.\vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = [A_xB_x + A_yB_y + A_zB_z]$

(3) **Example :** (i) Work W: In physics for constant force work is defined as, $W = Fs\cos\theta$ ....(i)

But by definition of scalar product of two vectors, $\vec{F}.\vec{s} = Fs\cos\theta$ ....(ii)

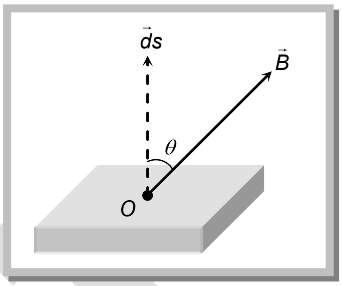
So from eq^n (i) and (ii) $W = \vec{F}.\vec{s}$ *i.e.* work is the scalar product of force with displacement.

(ii) Power P:

As $W = \vec{F}.\vec{s}$ or $\frac{dW}{dt} = \vec{F}.\frac{d\vec{s}}{dt}$ [As $\vec{F}$ is constant]

or $P = \vec{F}.\vec{v}$ *i.e.*, power is the scalar product of force with velocity. $\left[\text{As} \frac{dW}{dt} = P \text{ and } \frac{d\vec{s}}{dt} = \vec{v}\right]$

(iii) Magnetic Flux ϕ:



Magnetic flux through an area is given by $d\phi = B\,ds\cos\theta$ ......(i)

But by definition of scalar product $\vec{B}.d\vec{s} = Bds\cos\theta$ ......(ii)

So from eq^n (i) and (ii) we have

$d\phi = \vec{B}.d\vec{s}$ or $\phi = \int \vec{B}.d\vec{s}$

(iv) Potential energy of a dipole U: If an electric dipole of moment $\vec{p}$ is situated in an electric field $\vec{E}$ or a magnetic dipole of moment $\vec{M}$ in a field of induction $\vec{B}$, the potential energy of the dipole is given by :

$U_E = -\vec{p}.\vec{E}$ and $U_B = -\vec{M}.\vec{B}$

Sample problem based on dot product

Problem 38. $\vec{A} = 2\hat{i} + 4\hat{j} + 4\hat{k}$ and $\vec{B} = 4\hat{i} + 2\hat{j} - 4\hat{k}$ are two vectors. The angle between them will be

(a) 0° (b) 45° (c) 60° (d) 90°

Solution : (d) $\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}|.|\vec{B}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{A}|.|\vec{B}|} = \frac{2\times4 + 4\times2 - 4\times4}{|\vec{A}|.|\vec{B}|} = 0$

$\therefore\ \theta = \cos^{-1}(0°) \Rightarrow \theta = 90°$

Problem 39. If two vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $-4\hat{i} - 6\hat{j} - \lambda\hat{k}$ are parallel to each other then value of λ be

(a) 0 (b) 2 (c) 3 (d) 4

Solution : (c) Let $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{B} = -4\hat{i} - 6\hat{j} + \lambda\hat{k}$

$\vec{A}$ and $\vec{B}$ are parallel to each other $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ *i.e.* $\frac{2}{-4} = \frac{3}{-6} = \frac{-1}{\lambda} \Rightarrow \lambda = 2$.

Problem 40. In above example if vectors are perpendicular to each other then value of λ be

(a) 25 (b) 26 (c) – 26 (d) – 25

Solution : (c) If $\vec{A}$ and $\vec{B}$ are perpendicular to each other then $\vec{A}.\vec{B} = 0 \Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

So, $2(-4) + 3(-6) + (-1)(\lambda) = 0 \Rightarrow \lambda = -26$

Problem 41. If $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{B} = -\hat{i} + 3\hat{j} + 4\hat{k}$ then projection of $\vec{A}$ on $\vec{B}$ will be

(a) $\frac{3}{\sqrt{13}}$ (b) $\frac{3}{\sqrt{26}}$ (c) $\sqrt{\frac{3}{26}}$ (d) $\sqrt{\frac{3}{13}}$

Solution : (b) $|\vec{A}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4+9+1} = \sqrt{14}$

$|\vec{B}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{1+9+16} = \sqrt{26}$

$\vec{A}.\vec{B} = 2(-1) + 3\times 3 + (-1)(4) = 3$

The projection of $\vec{A}$ on $\vec{B}$ $= \dfrac{\vec{A}.\vec{B}}{|\vec{B}|} = \dfrac{3}{\sqrt{26}}$

Problem 42. A body, acted upon by a force of 50 *N* is displaced through a distance 10 *meter* in a direction making an angle of 60° with the force. The work done by the force be

(a) 200 *J* (b) 100 *J* (c) 300 (d) 250 *J*

Solution : (d) $W = \vec{F}.\vec{S} = FS\cos\theta = 50\times 10\times\cos 60^\circ = 50\times 10\times\frac{1}{2} = 250\ J.$

Problem 43. A particle moves from position $3\hat{i}+2\hat{j}-6\hat{k}$ to $14\hat{i}+13\hat{j}+9\hat{k}$ due to a uniform force of $4\hat{i}+\hat{j}+3\hat{k}\,N$. If the displacement in meters then work done will be

(a) 100 *J* (b) 200 *J* (c) 300 *J* (d) 250 *J*

Solution : (a) $S = \vec{r}_2 - \vec{r}_1$

$W = \vec{F}.\vec{S} = (4\hat{i}+\hat{j}+3\hat{k}).(11\hat{i}+11\hat{j}+15\hat{k}) = (4\times 11 + 1\times 11 + 3\times 15) = 100\ J.$

Problem 44. If for two vector $\vec{A}$ and $\vec{B}$, sum $(\vec{A}+\vec{B})$ is perpendicular to the difference $(\vec{A}-\vec{B})$. The ratio of their magnitude is

(a) 1 (b) 2 (c) 3 (d) None of these

Solution : (a) $(\vec{A}+\vec{B})$ is perpendicular to $(\vec{A}-\vec{B})$. Thus

$(\vec{A}+\vec{B}).(\vec{A}-\vec{B}) = 0$ or $A^2 + \vec{B}.\vec{A} - \vec{A}.\vec{B} - B^2 = 0$

Because of commutative property of dot product $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

$\therefore\ A^2 - B^2 = 0$ or $A = B$

Thus the ratio of magnitudes $A/B = 1$

Problem 45. A force $\vec{F} = -K(y\hat{i} + x\hat{j})$ (where *K* is a positive constant) acts on a particle moving in the *x-y* plane. Starting from the origin, the particle is taken along the positive *x*- axis to the point (*a*, 0) and then parallel to the *y*-axis to the point (*a*, *a*). The total work done by the forces $\vec{F}$ on the particle is

(a) $-2Ka^2$ (b) $2Ka^2$ (c) $-Ka^2$ (d) Ka^2

Solution : (c) For motion of the particle form (0, 0) to (*a*, 0)

$\vec{F} = -K(0\hat{i} + a\hat{j}) \Rightarrow \vec{F} = -Ka\hat{j}$

Displacement $\vec{r} = (a\hat{i} + 0\hat{j}) - (0\hat{i} + 0\hat{j}) = a\hat{i}$

So work done from (0, 0) to (*a*, 0) is given by $W = \vec{F}.\vec{r} = -Ka\hat{j}.a\hat{i} = 0$

For motion (*a*, 0) to (*a*, *a*)

$\vec{F} = -K(a\hat{i} + a\hat{j})$ and displacement $\vec{r} = (a\hat{i} + a\hat{j}) - (a\hat{i} + 0\hat{j}) = a\hat{j}$

So work done from (*a*, 0) to (*a*, *a*) $W = \vec{F}.\vec{r} = -K(a\hat{i} + a\hat{j}).a\hat{j} = -Ka^2$

So total work done $= -Ka^2$

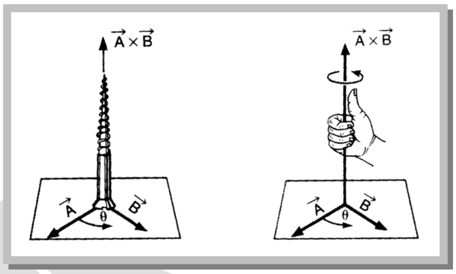
0.18 Vector Product of Two Vector

(1) **Definition** : The vector product or cross product of two vectors is defined as a vector having a magnitude equal to the product of the magnitudes of two vectors with the sine of angle between them, and direction perpendicular to the plane containing the two vectors in accordance with right hand screw rule.

$$\vec{C} = \vec{A} \times \vec{B}$$

Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product written as $\vec{A} \times \vec{B}$ is a vector $\vec{C}$ defined by

$$\vec{C} = \vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$$



The direction of $\vec{A} \times \vec{B}$, *i.e.* $\vec{C}$ is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and in the sense of advance of a right handed screw rotated from $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the smaller angle between them. Thus, if a right handed screw whose axis is perpendicular to the plane framed by $\vec{A}$ and $\vec{B}$ is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A} \times \vec{B}$ *i.e.* $\vec{C}$

(2) **Properties :**

(i) Vector product of any two vectors is always a vector perpendicular to the plane containing these two vectors, *i.e.*, orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.

(ii) Vector product of two vectors is not commutative, *i.e.*, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ [but $= -\vec{B} \times \vec{A}$]

Here it is worthy to note that

$$|\vec{A} \times \vec{B}| = |\vec{B} \times \vec{A}| = AB\sin\theta$$

i.e., in case of vector $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ magnitudes are equal but directions are opposite.

(iii) The vector product is distributive when the order of the vectors is strictly maintained, *i.e.*

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iv) As by definition of vector product of two vectors $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$

So $\quad |\vec{A} \times \vec{B}| = AB\sin\theta \quad$ *i.e.*, $\quad \theta = \sin^{-1}\left[\dfrac{|\vec{A} \times \vec{B}|}{|\vec{A}||\vec{B}|}\right]$

(v) The vector product of two vectors will be maximum when $\sin\theta = \max = 1$, *i.e.*, $\theta = 90^o$

$$[\vec{A} \times \vec{B}]_{\max} = AB\,\hat{n}$$

i.e., vector product is maximum if the vectors are orthogonal.

(vi) The vector product of two non- zero vectors will be minimum when $|\sin\theta| =$ minimum $= 0$, *i.e.*, $\theta = 0^o$ or 180^o

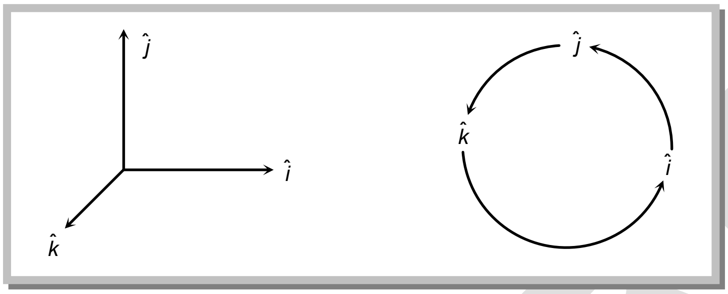
$$[\vec{A} \times \vec{B}]_{\min} = 0$$

i.e. if the vector product of two non-zero vectors vanishes, the vectors are collinear.

(vii) The self cross product, *i.e.*, product of a vector by itself vanishes, *i.e.*, is null vector $\vec{A} \times \vec{A} = AA \sin 0^o \hat{n} = \vec{0}$

(viii) In case of unit vector $\hat{n} \times \hat{n} = \vec{0}$ so that $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

(ix) In case of orthogonal unit vectors, $\hat{i}, \hat{j}, \hat{k}$ in accordance with right hand screw rule :



$\hat{i} \times \hat{j} = \hat{k}$, $\quad \hat{j} \times \hat{k} = \hat{i}$ $\quad$ and $\quad \hat{k} \times \hat{i} = \hat{j}$

And as cross product is not commutative,

$\hat{j} \times \hat{i} = -\hat{k}$ $\quad \hat{k} \times \hat{j} = -\hat{i}$ $\quad$ and $\quad \hat{i} \times \hat{k} = -\hat{j}$

(x) In terms of components $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x)$

(3) **Example :** Since vector product of two vectors is a vector, vector physical quantities (particularly representing rotational effects) like torque, angular momentum, velocity and force on a moving charge in a magnetic field and can be expressed as the vector product of two vectors. It is well – established in physics that :

(i) Torque $\vec{\tau} = \vec{r} \times \vec{F}$

(ii) Angular momentum $\vec{L} = \vec{r} \times \vec{p}$

(iii) Velocity $\vec{v} = \vec{\omega} \times \vec{r}$

(iv) Force on a charged particle q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by $\vec{F} = q(\vec{v} \times \vec{B})$

(v) Torque on a dipole in a field $\vec{\tau_E} = \vec{p} \times \vec{E}$ and $\vec{\tau_B} = \vec{M} \times \vec{B}$

Sample problem based on vector product

***Problem* 46.** If $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$ then value of $|\vec{A} \times \vec{B}|$ will be

(a) $8\sqrt{2}$ (b) $8\sqrt{3}$ (c) $8\sqrt{5}$ (d) $5\sqrt{8}$

Solution : (b) $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = (1\times 4 - 2\times -2)\hat{i} + (2\times 2 - 4\times 3)\hat{j} + (3\times -2 - 1\times 2)\hat{k} = 8\hat{i} - 8\hat{j} - 8\hat{k}$

$\therefore$ Magnitude of $\vec{A}\times\vec{B} = |\vec{A}\times\vec{B}| = \sqrt{(8)^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$

***Problem* 47.** In above example a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ will be

(a) $+\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$ (b) $-\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$ (c) Both (a) and (b) (d) None of these

Solution : (c) $\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{8\hat{i} - 8\hat{j} - 8\hat{k}}{8\sqrt{3}} = \frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$

There are two unit vectors perpendicular to both $\vec{A}$ and $\vec{B}$ they are $\hat{n} = \pm\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$

***Problem* 48.** The vectors from origin to the points *A* and *B* are $\vec{A} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} + \hat{j} - 2\hat{k}$ respectively. The area of the triangle *OAB* be

(a) $\frac{5}{2}\sqrt{17}$ *sq.unit* (b) $\frac{2}{5}\sqrt{17}$ *sq.unit* (c) $\frac{3}{5}\sqrt{17}$ *sq.unit* (d) $\frac{5}{3}\sqrt{17}$ *sq.unit*

Solution : (a) Given $\overrightarrow{OA} = \vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\overrightarrow{OB} = \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\therefore (\vec{a}\times\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix} = (12 - 2)\hat{i} + (4 + 6)\hat{j} + (3 + 12)\hat{k}$

$= 10\hat{i} + 10\hat{j} + 15\hat{k} \Rightarrow |\vec{a}\times\vec{b}| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}$

Area of $\Delta OAB = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{5\sqrt{17}}{2}$ *sq.unit.*

***Problem* 49.** The angle between the vectors $\vec{A}$ and $\vec{B}$ is θ. The value of the triple product $\vec{A}.(\vec{B}\times\vec{A})$ is

(a) A^2B (b) Zero (c) $A^2B\sin\theta$ (d) $A^2B\cos\theta$

Solution : (b) Let $\vec{A}.(\vec{B}\times\vec{A}) = \vec{A}.\vec{C}$

Here $\vec{C} = \vec{B}\times\vec{A}$ Which is perpendicular to both vector $\vec{A}$ and $\vec{B}$ $\therefore \vec{A}.\vec{C} = 0$

***Problem* 50.** The torque of the force $\vec{F} = (2\hat{i} - 3\hat{j} + 4\hat{k})N$ acting at the point $\vec{r} = (3\hat{i} + 2\hat{j} + 3\hat{k})\,m$ about the origin be

(a) $6\hat{i} - 6\hat{j} + 12\hat{k}$ (b) $17\hat{i} - 6\hat{j} - 13\hat{k}$ (c) $-6\hat{i} + 6\hat{j} - 12\hat{k}$ (d) $-17\hat{i} + 6\hat{j} + 13\hat{k}$

Solution : (b) $\vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix} = [(2\times 4) - (3\times -3)]\hat{i} + [(2\times 3) - (3\times 4)]\hat{j} + [(3\times -3) - (2\times 2)]\hat{k} = 17\hat{i} - 6\hat{j} - 13\hat{k}$

***Problem* 51.** If $\vec{A}\times\vec{B} = \vec{C}$, then which of the following statements is wrong

(a) $\vec{C} \perp \vec{A}$ (b) $\vec{C} \perp \vec{B}$ (c) $\vec{C} \perp (\vec{A} + \vec{B})$ (d) $\vec{C} \perp (\vec{A} \times \vec{B})$

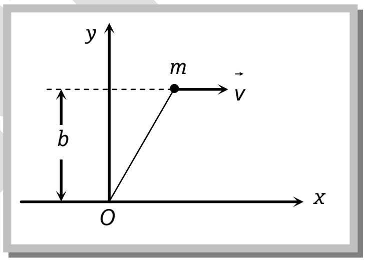
Solution : (d) From the property of vector product, we notice that $\vec{C}$ must be perpendicular to the plane formed by vector $\vec{A}$ and $\vec{B}$. Thus $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ and $(\vec{A} + \vec{B})$ vector also must lie in the plane formed by vector $\vec{A}$ and $\vec{B}$. Thus $\vec{C}$ must be perpendicular to $(\vec{A} + \vec{B})$ also but the cross product $(\vec{A} \times \vec{B})$ gives a vector $\vec{C}$ which can not be perpendicular to itself. Thus the last statement is wrong.

***Problem* 52.** If a particle of mass m is moving with constant velocity v parallel to x-axis in x-y plane as shown in fig. Its angular momentum with respect to origin at any time t will be

(a) $mvb\,\hat{k}$ (b) $-mvb\,\hat{k}$ (c) $mvb\,\hat{i}$ (d) $mv\,\hat{i}$

Solution : (b) We know that, Angular momentum

$\vec{L} = \vec{r} \times \vec{p}$ in terms of component becomes $\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$



As motion is in x-y plane ($z = 0$ and $P_z = 0$), so $\vec{L} = \vec{k}(xp_y - yp_x)$

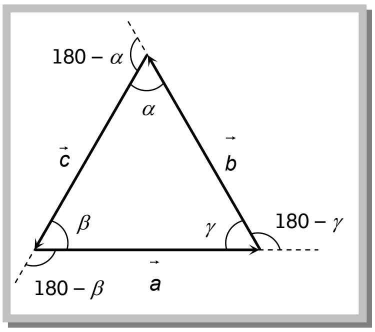
Here $x = vt$, $y = b$, $p_x = mv$ and $p_y = 0$

$\therefore\ \vec{L} = \vec{k}[vt \times 0 - b\,mv] = -mvb\,\hat{k}$

0.19 Lami's Theorem

In any ΔABC with sides $\vec{a}, \vec{b}, \vec{c}$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$



i.e., for any triangle the ratio of the sine of the angle containing the side to the length of the side is a constant.

For a triangle whose three sides are in the same order we establish the Lami's theorem in the following manner. For the triangle shown

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$ [All three sides are taken in order] .....(i)

$\Rightarrow\ \vec{a} + \vec{b} = -\vec{c}$ .....(ii)

Pre-multiplying both sides by $\vec{a}$

$\vec{a} \times (\vec{a} + \vec{b}) = -\vec{a} \times \vec{c} \ \Rightarrow\ \vec{0} + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c} \ \Rightarrow\ \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ .....(iii)

Pre-multiplying both sides of (ii) by $\vec{b}$

$\vec{b} \times (\vec{a} + \vec{b}) = -\vec{b} \times \vec{c} \ \Rightarrow\ \vec{b} \times \vec{a} + \vec{b} \times \vec{b} = -\vec{b} \times \vec{c} \ \Rightarrow\ -\vec{a} \times \vec{b} = -\vec{b} \times \vec{c} \Rightarrow \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ .....(iv)

From (iii) and (iv), we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Taking magnitude, we get $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

$\Rightarrow\ ab\sin(180 - \gamma) = bc\sin(180 - \alpha) = ca\sin(180 - \beta)$

$\Rightarrow \ ab\sin\gamma = bc\sin\alpha = ca\sin\beta$

Dividing through out by *abc*, we have $\Rightarrow \ \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$