Set Theory

1.1.1 Definitions

A set is a well-defined class or collection of objects. By a well defined collection we mean that there exists a rule with the help of which it is possible to tell whether a given object belongs or does not belong to the given collection. The objects in sets may be anything, numbers, people, mountains, rivers etc. The objects constituting the set are called elements or members of the set.

A set is often described in the following two ways.

(1) **Roster method or Listing method :** In this method a set is described by listing elements, separated by commas, within braces {}. The set of vowels of English alphabet may be described as $\{a, e, i, o, u\}$.

The set of even natural numbers can be described as {2, 4, 6..........}. Here the dots stand for 'and so on'.

Note : ❑ The order in which the elements are written in a set makes no difference. Thus $\{a, e, i, o, u\}$ and $\{e, a, i, o, u\}$ denote the same set. Also the repetition of an element has no effect. For example, {1, 2, 3, 2} is the same set as {1, 2, 3}

(2) **Set-builder method or Rule method :** In this method, a set is described by a characterizing property $P(x)$ of its elements x. In such a case the set is described by $\{x : P(x) \text{ holds}\}$ or $\{x \mid P(x) \text{ holds}\}$, which is read as 'the set of all x such that $P(x)$ holds'. The symbol '|' or ':' is read as 'such that'.

The set E of all even natural numbers can be written as

$$E = \{x \mid x \text{ is natural number and } x = 2n \text{ for } n \in N\}$$

or $$E = \{x \mid x \in N, x = 2n, n \in N\}$$

or $$E = \{x \in N \mid x = 2n, n \in N\}$$

The set $A = \{0, 1, 4, 9, 16, ...\}$ can be written as $A = \{x^2 \mid x \in Z\}$

Note : ❑ Symbols

Symbol	Meaning
$\Rightarrow$	Implies
$\in$	Belongs to
$A \subset B$	A is a subset of B
$\Leftrightarrow$	Implies and is implied by
$\notin$	Does not belong to
s.t.	Such that
$\forall$	For every
$\exists$	There exists
Symbol	**Meaning**
iff	If and only if
&	And
$a \mid b$	a is a divisor of b
N	Set of natural numbers
I or Z	Set of integers
R	Set of real numbers
C	Set of complex numbers
Q	Set of rational numbers

Example: 1 The set of intelligent students in a class is **[AMU 1998]**

(a) A null set (b) A singleton set

(c) A finite set (d) Not a well defined collection

Solution: (d) Since, intelligency is not defined for students in a class *i.e.*, Not a well defined collection.

1.1.2 Types of Sets

(1) **Null set or Empty set:** The set which contains no element at all is called the null set. This set is sometimes also called the 'empty set' or the 'void set'. It is denoted by the symbol ϕ or {}.

A set which has at least one element is called a non-empty set.

Let $A = \{x : x^2 + 1 = 0$ and x is real)

Since there is no real number which satisfies the equation $x^2 + 1 = 0$, therefore the set A is empty set.

Note : ❑ If A and B are any two empty sets, then $x \in A$ iff $x \in B$ is satisfied because there is no element x in either A or B to which the condition may be applied. Thus $A = B$. Hence, there is only one empty set and we denote it by ϕ. Therefore, article 'the' is used before empty set.

(2) **Singleton set:** A set consisting of a single element is called a singleton set. The set {5} is a singleton set.

(3) **Finite set:** A set is called a finite set if it is either void set or its elements can be listed (counted, labelled) by natural number 1, 2, 3, … and the process of listing terminates at a certain natural number n (say).

Cardinal number of a finite set: The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by $n(A)$ or $O(A)$.

(4) **Infinite set:** A set whose elements cannot be listed by the natural numbers 1, 2, 3, …., n, for any natural number n is called an infinite set.

(5) **Equivalent set:** Two finite sets A and B are equivalent if their cardinal numbers are same *i.e.* $n(A) = n(B)$.

Example: $A = \{1, 3, 5, 7\}$; $B = \{10, 12, 14, 16\}$ are equivalent sets $[\because O(A) = O(B) = 4]$

(6) **Equal set:** Two sets A and B are said to be equal *iff* every element of A is an element of B and also every element of B is an element of A. We write "$A = B$" if the sets A and B are equal and "$A \neq B$" if the sets A and B are not equal. Symbolically, $A = B$ if $x \in A \Leftrightarrow x \in B$.

The statement given in the definition of the equality of two sets is also known as the axiom of extension.

Example: If $A = \{2, 3, 5, 6\}$ and $B = \{6, 5, 3, 2\}$. Then $A = B$, because each element of A is an element of B and vice-versa.

Note : ❑ Equal sets are always equivalent but equivalent sets may need not be equal set.

(7) **Universal set :** A set that contains all sets in a given context is called the universal set.

or

A set containing of all possible elements which occur in the discussion is called a universal set and is denoted by U.

Thus in any particular discussion, no element can exist out of universal set. It should be noted that universal set is not unique. It may differ in problem to problem.

(8) **Power set :** If S is any set, then the family of all the subsets of S is called the power set of S.

The power set of S is denoted by $P(S)$. Symbolically, $P(S) = \{T : T \subseteq S\}$. Obviously ϕ and S are both elements of $P(S)$.

Example : Let $S = \{a, b, c\}$, then $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Note : ❑ If $A = \phi$, then $P(A)$ has one element ϕ, $\therefore n[P(A)] = 1$

❑ Power set of a given set is always non-empty.

❑ If A has n elements, then $P(A)$ has 2^n elements.

❑ $P(\phi) = \{\phi\}$

$P(P(\phi)) = \{\phi, \{\phi\}\} \Rightarrow P[P(P(\phi))] = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Hence $n\{P[P(P(\phi))]\} = 4$.

(9) **Subsets (Set inclusion) :** Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.

If A is subset of B, we write $A \subseteq B$, which is read as "A is a subset of B" or "A is contained in B".

Thus, $A \subseteq B \Rightarrow a \in A \Rightarrow a \in B$.

Note : ❑ Every set is a subset of itself.

❑ The empty set is a subset of every set.

❑ The total number of subset of a finite set containing n elements is 2^n.

Proper and improper subsets: If A is a subset of B and $A \neq B$, then A is a proper subset of B. We write this as $A \subset B$.

The null set ϕ is subset of every set and every set is subset of itself, *i.e.*, $\phi \subset A$ and $A \subseteq A$ for every set A. They are called improper subsets of A. Thus every non-empty set has two improper subsets. It should be noted that ϕ has only one subset ϕ which is improper. Thus A has two improper subsets iff it is non-empty.

All other subsets of A are called its proper subsets. Thus, if $A \subset B$, $A \neq B$, $A \neq \phi$, then A is said to be proper subset of B.

Example: Let $A = \{1, 2\}$. Then A has $\phi; \{1\}, \{2\}, \{1, 2\}$ as its subsets out of which ϕ and {1, 2} are improper and {1} and {2} are proper subsets.

Example: 2 Which of the following is the empty set

(c) $\{x : x$ is a real number and $x^2 - 9 = 0\}$ (d) $\{x : x$ is a real number and $x^2 = x + 2\}$

Solution: (b) Since $x^2 + 1 = 0$, gives $x^2 = -1 \Rightarrow x = \pm i$

$\therefore$ x is not real but x is real (given)

$\therefore$ No value of x is possible.

Example: 3 The set $A = \{x : x \in R, x^2 = 16$ and $2x = 6\}$ equals

(a) ϕ (b) [14, 3, 4] (c) [3] (d) [4]

Solution: (a) $x^2 = 16 \Rightarrow x = \pm 4$

$2x = 6 \Rightarrow x = 3$

There is no value of x which satisfies both the above equations. Thus, $A = \phi$.

Example: 4 If a set A has n elements, then the total number of subsets of A is

(a) n (b) n^2 (c) 2^n (d) $2n$

Solution: (c) Number of subsets of $A = {}^nC_0 + {}^nC_1 + \dots\dots\dots + {}^nC_n = 2^n$.

Example: 5 Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. The values of m and n are

(a) 7, 6 (b) 6, 3 (c) 5, 1 (d) 8, 7

Solution: (b) Since $2^m - 2^n = 56 = 8 \times 7 = 2^3 \times 7 \Rightarrow 2^n(2^{m-n} - 1) = 2^3 \times 7$

$\therefore$ $n = 3$ and $2^{m-n} = 8 = 2^3$

$\Rightarrow m - n = 3 \Rightarrow m - 3 = 3 \Rightarrow m = 6$

$\therefore m = 6, n = 3$.

Example: 6 The number of proper subsets of the set {1, 2, 3} is

(a) 8 (b) 7 (c) 6 (d) 5

Solution: (c) Number of proper subsets of the set {1, 2, 3) = $2^3 - 2 = 6$.

Example: 7 If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n-1) : n \in N\}$, then

(a) $X \subseteq Y$ (b) $Y \subseteq X$ (c) $X = Y$ (d) None of these

Solution: (a) Since $8^n - 7n - 1 = (7+1)^n - 7n - 1 = 7^n + {}^nC_1 7^{n-1} + {}^nC_2 7^{n-2} + ... + {}^nC_{n-1} 7 + {}^nC_n - 7n - 1$

$$= {}^nC_2 7^2 + {}^nC_3 7^3 + ... + {}^nC_n 7^n \qquad ({}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1} \text{ etc.})$$

$$= 49[{}^nC_2 + {}^nC_3(7) + ... + {}^nC_n 7^{n-2}]$$

$\therefore$ $8^n - 7n - 1$ is a multiple of 49 for $n \geq 2$.

For $n = 1$, $8^n - 7n - 1 = 8 - 7 - 1 = 0$; For $n = 2$, $8^n - 7n - 1 = 64 - 14 - 1 = 49$

$\therefore$ $8^n - 7n - 1$ is a multiple of 49 for all $n \in N$.

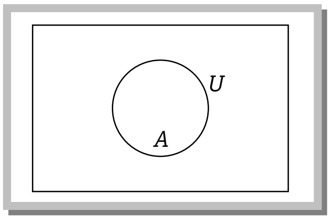
$\therefore$ X contains elements which are multiples of 49 and clearly Y contains all multiplies of 49.

$\therefore$ $X \subseteq Y$.

1.1.3 Venn-Euler Diagrams

The combination of rectangles and circles are called *Venn-Euler diagrams* or simply **Venn-diagrams**.

In venn-diagrams the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. Two disjoints sets are represented by two non-intersecting circles.

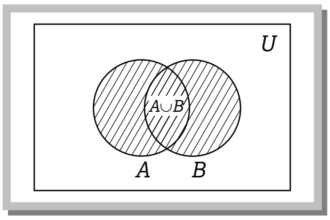


1.1.4 Operations on Sets

(1) **Union of sets :** Let A and B be two sets. The union of A and B is the set of all elements which are in set A or in B. We denote the union of A and B by $A \cup B$

which is usually read as "A union B".

symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.



It should be noted here that we take standard mathematical usage of "or". When we say that $x \in A$ or $x \in B$ we do not exclude the possibility that x is a member of both A and B.

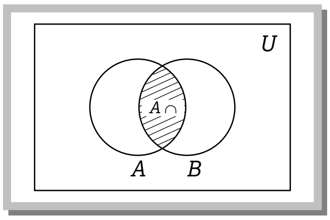
Note : ❑ If $A_1, A_2, ..., A_n$ is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or $A_1 \cup A_2 \cup A_3 ...... \cup A_n$.

(2) **Intersection of sets :** Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.

The intersection of A and B is denoted by $A \cap B$ (read as "A intersection B")

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Clearly, $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$.



In fig. the shaded region represents $A \cap B$. Evidently $A \cap B \subseteq A$, $A \cap B \subseteq B$.

Note : ❑ If $A_1, A_2, A_3 ....,\ A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or

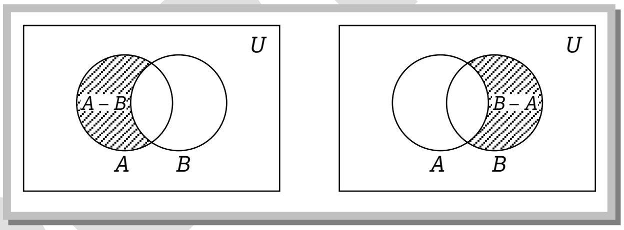
$A_1 \cap A_2 \cap A_3 \cap .... \cap A_n$.

(3) **Disjoint sets :** Two sets A and B are said to be disjoint, if $A \cap B = \phi$. If $A \cap B \neq \phi$, then A and B are said to be non-intersecting or non-overlapping sets.

In other words, if A and B have no element in common, then A and B are called disjoint sets.

Example : Sets {1, 2}; {3, 4} are disjoint sets.

(4) **Difference of sets :** Let A and B be two sets. The difference of A and B written as $A - B$, is the set of all those elements of A which do not belong to B.



Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

or $A - B = \{x \in A : x \notin B\}$

Clearly, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$. In fig. the shaded part represents $A - B$.

Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A *i.e.*

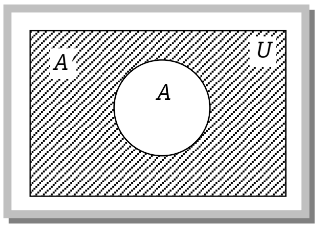
$$B - A = \{x \in B : x \notin A\}$$

Example: Consider the sets $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A - B = \{1, 2\}$; $B - A = \{4, 5\}$

As another example, $R - Q$ is the set of all irrational numbers.

(5) **Symmetric difference of two sets:** Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$. Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$

(6) **Complement of a set :** Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $C(A)$ or $U - A$ and is defined the set of all those elements of U which are not in A.



Thus, $A' = \{x \in U : x \notin A\}$.

Clearly, $x \in A' \Leftrightarrow x \notin A$

Example: Consider $U = \{1, 2, ..., 10\}$ and $A = \{1, 3, 5, 7, 9\}$.

Then $A' = \{2, 4, 6, 8, 10\}$

Example: 8 Given the sets $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, then $A \cup (B \cap C)$ is

(a) {3} (b) {1, 2, 3, 4} (c) {1, 2, 4, 5} (d) {1, 2, 3, 4, 5, 6}

Solution: (b) $B \cap C = \{4\}$, $\therefore$ $A \cup (B \cap C) = \{1, 2, 3, 4\}$.

Example: 9 If $A \subseteq B$, then $A \cup B$ is equal to

(a) A (b) $B \cap A$ (c) B (d) None of these

Solution: (c) Since $A \subseteq B \Rightarrow A \cup B = B$.

Example: 10 If A and B are any two sets, then $A \cup (A \cap B)$ is equal to

(a) A (b) B (c) A^c (d) B^c

Solution: (a) $A \cap B \subseteq A$. Hence $A \cup (A \cap B) = A$.

Example: 11 If A and B are two given sets, then $A \cap (A \cap B)^c$ is equal to

(a) A (b) B (c) ϕ (d) $A \cap B^c$

Solution: (d) $A \cap (A \cap B)^c = A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) = \phi \cup (A \cap B^c) = A \cap B^c$.

Example: 12 If $N_a = \{an : n \in N\}$, then $N_3 \cap N_4 =$

(a) N_7 (b) N_{12} (c) N_3 (d) N_4

Solution: (b) $N_3 \cap N_4 = \{3, 6, 9, 12, 15 ......\} \cap \{4, 8, 12, 16, 20, ...\}$

$= \{12, 24, 36......\} = N_{12}$

Trick: $N_3 \cap N_4 = N_{12}$ [$\because$ 3, 4 are relatively prime numbers]

Example: 13 If $aN = \{ax : x \in N\}$ and $bN \cap cN = dN$, where b, $c \in N$ are relatively prime, then

(a) $d = bc$ (b) $c = bd$ (c) $b = cd$ (d) None of these

Solution: (a) $bN =$ the set of positive integral multiples of b, cN = the set of positive integral multiples of c.

$\therefore$ $bN \cap cN$ = the set of positive integral multiples of bc = $b \subset N$ [$\because b, c$ are prime]

$\therefore$ $d = bc$.

Example: 14 If the sets A and B are defined as

$A = \{(x, y) : y = \frac{1}{x}, 0 \neq x \in R\}$

$B = \{(x, y) : y = -x, x \in R\}$, then

(a) $A \cap B = A$ (b) $A \cap B = B$ (c) $A \cap B = \phi$ (d) None of these

Solution: (c) Since $y = \frac{1}{x}, y = -x$ meet when $-x = \frac{1}{x} \Rightarrow x^2 = -1$, which does not give any real value of x

Hence $A \cap B = \phi$.

Example: 15 Let $A = [x : x \in R, |x| < 1]$; $B = [x : x \in R, |x - 1| \geq 1]$ and $A \cup B = R - D$, then the set D is

(a) $[x : 1 < x \leq 2]$ (b) $[x : 1 \leq x < 2]$ (c) $[x : 1 \leq x \leq 2]$ (d) None of these

Solution: (b) $A = [x : x \in R, -1 < x < 1]$

$B = [x : x \in R : x - 1 \leq -1$ or $x - 1 \geq 1] = [x : x \in R : x \leq 0$ or $x \geq 2]$

$\therefore$ $A \cup B = R - D$

Where $D = [x : x \in R, 1 \leq x < 2]$

Example: 16 If the sets A and B are defined as

$A = \{(x, y) : y = e^x, x \in R\}$

$B = \{(x, y) : y = x, x \in R\}$, then

(a) $B \subseteq A$ (b) $A \subseteq B$ (c) $A \cap B = \phi$ (d) $A \cup B = A$

Solution: (c) Since, $y = e^x$ and $y = x$ do not meet for any $x \in R$

$\therefore$ $A \cap B = \phi$.

Example: 17 If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n - 1) : n \in N\}$, then $X \cup Y$ is equal to

(a) X (b) Y (c) N (d) None of these

Solution: (b) Since, $4^n - 3n - 1 = (3 + 1)^n - 3n - 1 = 3^n + {}^nC_1 3^{n-1} + {}^nC_2 3^{n-2} + ... + {}^nC_{n-1} 3 + {}^nC_n - 3n - 1$

$= {}^nC_2 3^2 + {}^nC_3 .3^3 + ... + {}^nC_n 3^n$ $\quad ({}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}$ etc.)

$= 9[{}^nC_2 + {}^nC_3(3) + ... + {}^nC_n 3^{n-1}]$

$\therefore$ $4^n - 3n - 1$ is a multiple of 9 for $n \ge 2$.

For $n = 1$, $4^n - 3n - 1 = 4 - 3 - 1 = 0$, For $n = 2$, $4^n - 3n - 1 = 16 - 6 - 1 = 9$

$\therefore$ $4^n - 3n - 1$ is a multiple of 9 for all $n \in N$

$\therefore$ X contains elements which are multiples of 9 and clearly Y contains all multiples of 9.

$\therefore$ $X \subseteq Y$, $\therefore$ $X \cup Y = Y$.

1.1.5 Some Important Results on Number of Elements in Sets

If A, B and C are finite sets and U be the finite universal set, then

(1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(2) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.

(3) $n(A - B) = n(A) - n(A \cap B)$ *i.e.* $n(A - B) + n(A \cap B) = n(A)$

(4) $n(A \Delta B)$ = Number of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ $\quad$ [$\because$ $(A - B)$ and $(B - A)$ are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B) = n(A) + n(B) - 2n(A \cap B)$

(5) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(6) n (Number of elements in exactly two of the sets A, B, C) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(7) n(Number of elements in exactly one of the sets A, B, C) $= n(A) + n(B) + n(C)$

$- 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(8) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

(9) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Example: 18 Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$

(a) 3 (b) 6 (c) 9 (d) 18

Solution: (b) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - n(A \cap B)$

Since maximum number of elements in $A \cap B = 3$

$\therefore$ Minimum number of elements in $A \cup B = 9 - 3 = 6$.

Example: 19 If A and B are two sets such that $n(A) = 70$, $n(B) = 60$ and $n(A \cup B) = 110$, then $n(A \cap B)$ is equal to

(a) 240 (b) 50 (c) 40 (d) 20

Solution: (d) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow 110 = 70 + 60 - n(A \cap B)$

$\therefore$ $n(A \cap B) = 130 - 110 = 20$.

Example: 20 Let $n(U) = 700, n(A) = 200, n(B) = 300$ and $n(A \cap B) = 100$, then $n(A^c \cap B^c) =$

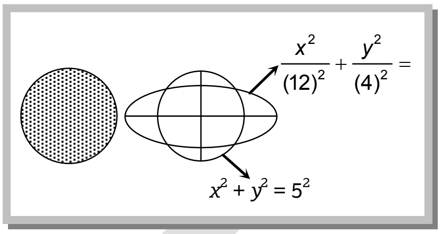
(a) 400 (b) 600 (c) 300 (d) 200

Solution: (c) $n(A^c \cap B^c) = n[(A \cup B)^c] = n(U) - n(A \cup B) = n(U) - [n(A) + n(B) - n(A \cap B)] = 700 - [200 + 300 - 100] = 300.$

Example: 21 If $A = [(x, y) : x^2 + y^2 = 25]$ and $B = [(x, y) : x^2 + 9y^2 = 144]$, then $A \cap B$ contains

(a) One point (b) Three points (c) Two points (d) Four points

Solution: (d) A = Set of all values (x, y) : $x^2 + y^2 = 25 = 5^2$



$B = \frac{x^2}{144} + \frac{y^2}{16} = 1$ *i.e.,* $\frac{x^2}{(12)^2} + \frac{y^2}{(4)^2} = 1$.

Clearly, $A \cap B$ consists of four points.

Example: 22 In a town of 10,000 families it was found that 40% family buy newspaper A, 20% buy newspaper B and 10% families buy newspaper C, 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three newspapers, then number of families which buy A only is

(a) 3100 (b) 3300 (c) 2900 (d) 1400

Solution: (b) $n(A)$ = 40% of 10,000 = 4,000

$n(B)$ = 20% of 10,000 = 2,000

$n(C)$ = 10% of 10,000 = 1,000

$n(A \cap B)$ = 5% of 10,000 = 500, $n(B \cap C)$ = 3% of 10,000 = 300

$n(C \cap A)$ = 4% of 10,000 = 400, $n(A \cap B \cap C)$ = 2% of 10,000 = 200

We want to find $n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$

$= n(A) - n[A \cap (B \cup C)] = n(A) - n[(A \cap B) \cup (A \cap C)] = n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$

$= 4000 - [500 + 400 - 200] = 4000 - 700 = 3300.$

Example: 23 In a city 20 percent of the population travels by car, 50 percent travels by bus and 10 percent travels by both car and bus. Then persons travelling by car or bus is

(a) 80 percent (b) 40 percent (c) 60 percent (d) 70 percent

Solution: (c) $n(C) = 20$, $n(B) = 50$, $n(C \cap B) = 10$

Now, $n(C \cup B) = n(C) + n(B) - n(C \cap B) = 20 + 50 - 10 = 60.$

Hence, required number of persons = 60%.

Example: 24 Suppose $A_1, A_2, A_3, \ldots\ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2, \ldots\ldots, B_n$ are n sets each with 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each elements of S belongs to exactly 10 of the $A_i^{'}s$ and exactly 9 of the $B_j^{'}s$. Then n is equal to

(a) 15 (b) 3 (c) 45 (d) None of these

Solution: (c) $O(S) = O\left(\bigcup_{i=1}^{30} A_i\right) = \frac{1}{10}(5 \times 30) = 15$

Since, element in the union S belongs to 10 of $A_i^{'}s$

Also, $O(S) = O\left(\bigcup_{j=1}^{n} B_j\right) = \frac{3n}{9} = \frac{n}{3}$, $\therefore \frac{n}{3} = 15 \Rightarrow n = 45$.

Example: 25 In a class of 55 students, the number of students studying different subjects are 23 in Mathematics, 24 in Physics, 19 in Chemistry, 12 in Mathematics and Physics, 9 in Mathematics and Chemistry, 7 in Physics and Chemistry and 4 in all the three subjects. The number of students who have taken exactly one subject is

(a) 6 (b) 9 (c) 7 (d) All of these

Solution: (d) $n(M) = 23$, $n(P) = 24$, $n(C) = 19$

$n(M \cap P) = 12$, $n(M \cap C) = 9$, $n(P \cap C) = 7$

$n(M \cap P \cap C) = 4$

We have to find $n(M \cap P' \cap C')$, $n(P \cap M' \cap C')$, $n(C \cap M' \cap P')$

Now $n(M \cap P' \cap C') = n[M \cap (P \cup C)']$

$= n(M) - n(M \cap (P \cup C)) = n(M) - n[(M \cap P) \cup (M \cap C)]$

$= n(M) - n(M \cap P) - n(M \cap C) + n(M \cap P \cap C) = 23 - 12 - 9 + 4 = 27 - 21 = 6$

$n(P \cap M' \cap C') = n[P \cap (M \cup C)']$

$= n(P) - n[P \cap (M \cup C)] = n(P) - n[(P \cap M) \cup (P \cap C)] = n(P) - n(P \cap M) - n(P \cap C) + n(P \cap M \cap C)$

$= 24 - 12 - 7 + 4 = 9$

$n(C \cap M' \cap P') = n(C) - n(C \cap P) - n(C \cap M) + n(C \cap P \cap M) = 19 - 7 - 9 + 4 = 23 - 16 = 7$

Hence (d) is the correct answer.

1.1.6 Laws of Algebra of Sets

(1) **Idempotent laws :** For any set A, we have

(i) $A \cup A = A$ (ii) $A \cap A = A$

(2) **Identity laws :** For any set A, we have

(i) $A \cup \phi = A$ (ii) $A \cap U = A$

i.e. ϕ and U are identity elements for union and intersection respectively.

(3) **Commutative laws :** For any two sets A and B, we have

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$ (iii) $A \Delta B = B \Delta A$

i.e. union, intersection and symmetric difference of two sets are commutative.

(iv) $A - B \neq B - A$ (iv) $A \times B \neq B \times A$

i.e., difference and cartesian product of two sets are not commutative

(4) **Associative laws :** If A, B and C are any three sets, then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$ (iii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

i.e., union, intersection and symmetric difference of two sets are associative.

(iv) $(A - B) - C \neq A - (B - C)$ (v) $(A \times B) \times C \neq A \times (B \times C)$

i.e., difference and cartesian product of two sets are not associative.

(5) **Distributive law :** If A, B and C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union respectively.

(iii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (iv) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (v) $A \times (B - C) = (A \times B) - (A \times C)$

(6) **De-Morgan's law :** If A and B are any two sets, then

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(iii) $A - (B \cup C) = (A - B) \cap (A - C)$ (iv) $A - (B \cap C) = (A - B) \cup (A - C)$

Note : ❑ **Theorem 1:** If A and B are any two sets, then

(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$ (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(viii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

❑ **Theorem 2 :** If A, B and C are any three sets, then

(i) $A - (B \cap C) = (A - B) \cup (A - C)$ (ii) $A - (B \cup C) = (A - B) \cap (A - C)$

(iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (iv) $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$

Example: 26 If A, B and C are any three sets, then $A \times (B \cap C)$ is equal to

(a) $(A \times B) \cup (A \times C)$ (b) $(A \times B) \cap (A \times C)$ (c) $(A \cup B) \times (A \cup C)$ (d) $(A \cap B) \times (A \cap C)$

Solution: (b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$. It is distributive law.

Example: 27 If A, B and C are any three sets, then $A \times (B \cup C)$ is equal to

(a) $(A \times B) \cup (A \times C)$ (b) $(A \cup B) \times (A \cup C)$ (c) $(A \times B) \cap (A \times C)$ (d) None of these

Solution: (a) It is distributive law.

Example: 28 If A, B and C are any three sets, then $A - (B \cup C)$ is equal to

(a) $(A - B) \cup (A - C)$ (b) $(A - B) \cap (A - C)$ (c) $(A - B) \cup C$ (d) $(A - B) \cap C$

Solution: (b) It is De' Morgan law.

Example: 29 If $A = [x : x$ is a multiple of 3] and $B = [x : x$ is a multiple of 5], then $A - B$ is ($\overline{A}$ means complement of A)

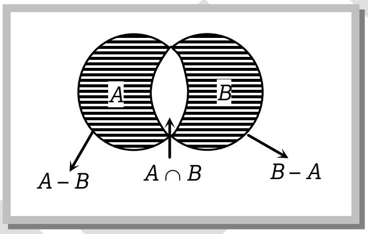
(a) $\overline{A} \cap B$ (b) $A \cap \overline{B}$ (c) $\overline{A} \cap \overline{B}$ (d) $\overline{A \cap B}$

Solution: (b) $A - B = A \cap B^c = A \cap \overline{B}$.

Example: 30 If A, B and C are non-empty sets, then $(A - B) \cup (B - A)$ equals

(a) $(A \cup B) - B$ (b) $A - (A \cap B)$ (c) $(A \cup B) - (A \cap B)$ (d) $(A \cap B) \cup (A \cup B)$

Solution: (c) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.



1.1.7 Cartesian Product of Sets

Cartesian product of sets : Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.

Thus, $A \times B = [(a, b) : a \in A$ and $b \in B]$

If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

Example : Let $A = \{a, b, c\}$ and $B = \{p, q\}$.

Then $A \times B = \{(a, p), (a, q), (b, p), (b, q), (c, p), (c, q)\}$

Also $B \times A = \{(p, a), (p, b), (p, c), (q, a), (q, b), (q, c)\}$

Important theorems on cartesian product of sets :

Theorem 1 : For any three sets A, B, C

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Theorem 2 : For any three sets A, B, C

$$A \times (B - C) = (A \times B) - (A \times C)$$

Theorem 3 : If A and B are any two non-empty sets, then

$$A \times B = B \times A \Leftrightarrow A = B$$

Theorem 4 : If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

Theorem 5 : If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set C.

Theorem 6 : If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$

Theorem 7 : For any sets A, B, C, D

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

Theorem 8 : For any three sets A, B, C

(i) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$ (ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

Theorem 9 : Let A and B two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

Example: 31 If $A = \{0, 1\}$, and $B = \{1, 0\}$, then $A \times B$ is equal to

(a) $\{0, 1, 1, 0\}$ (b) $\{(0, 1), (1, 0)\}$ (c) $\{0, 0\}$ (d) $\{(0,1),(0,0),(1,1),(1,0)\}$

Solution: (d) By the definition of cartesian product of sets

Clearly, $A \times B = \{(0, 1), (0, 0), (1, 1), (1, 0)\}$.

Example: 32 If $A = \{2, 4, 5\}$, $B = \{7, 8, 9\}$, then $n(A \times B)$ is equal to

(a) 6 (b) 9 (c) 3 (d) 0

Solution: (b) $A \times B = \{(2, 7), (2, 8), (2, 9), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9)\}$

$n(A \times B) = n(A) \,.\, n(B) = 3 \times 3 = 9$.

Example: 33 If the set A has p elements, B has q elements, then the number of elements in $A \times B$ is

(a) $p + q$ (b) $p + q + 1$ (c) pq (d) p^2

Solution: (c) $n(A \times B) = pq$.

Example: 34 If $A = \{a, b\}$, $B = \{c, d\}$, $C = \{d, e\}$, then $\{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$ is equal to

(a) $A \cap (B \cup C)$ (b) $A \cup (B \cap C)$ (c) $A \times (B \cup C)$ (d) $A \times (B \cap C)$

Solution: (c) $B \cup C = \{c, d\} \cup \{d, e\} = \{c, d, e\}$

$\therefore\ A \times (B \cup C) = \{a, b\} \times \{c, d, e\} = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$.

Example: 35 If $A = \{x : x^2 - 5x + 6 = 0\}$, $B = \{2, 4\}$, $C = \{4, 5\}$, then $A \times (B \cap C)$ is

(a) $\{(2, 4), (3, 4)\}$ (b) $\{(4, 2), (4, 3)\}$ (c) $\{(2, 4), (3, 4), (4, 4)\}$ (d) $\{(2,2), (3,3), (4,4), (5,5)\}$

Solution: (a) Clearly, $A = \{2, 3\}$, $B = \{2, 4\}$, $C = \{4, 5\}$

$B \cap C = \{4\}$

$\therefore\ A \times (B \cap C) = \{(2, 4), (3, 4)\}$.

Relations

1.2.1 Definition

Let A and B be two non-empty sets, then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a, b) \in R$. Then we say that a is related to b by the relation R and write it as aRb. If $(a, b) \in R$, we write it as aRb.

Example: Let $A = \{1, 2, 5, 8, 9\}$, $B = \{1, 3\}$ we set a relation from A to B as: $a\ R\ b$ iff $a \le b$; $a \in A, b \in B$. Then $R = \{(1, 1)\}, (1, 3), (2, 3)\} \subset A \times B$

(1) **Total number of relations :** Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subset of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

(2) **Domain and range of a relation :** Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

(3) **Relation on a set :** Let A be a non-void set. Then, a relation from A to itself *i.e.* a subset of $A \times A$ is called a relation on set A.

Example: 1 Let $A = \{1, 2, 3\}$. The total number of distinct relations that can be defined over A is

(a) 2^9 (b) 6 (c) 8 (d) None of these

Solution: (a) $n(A \times A) = n(A).n(A) = 3^2 = 9$

So, the total number of subsets of $A \times A$ is 2^9 and a subset of $A \times A$ is a relation over the set A.

Example: 2 Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 5, 7, 9\}$. Which of the following is/are relations from X to Y

(a) $R_1 = \{(x, y) | y = 2 + x, x \in X, y \in Y\}$ (b) $R_2 = \{(1,1), (2,1), (3,3), (4,3), (5,5)\}$

(c) $R_3 = \{(1,1), (1,3)(3,5), (3,7), (5,7)\}$ (d) $R_4 = \{(1,3), (2,5), (2,4), (7,9)\}$

Solution: (a,b,c) R_4 is not a relation from X to Y, because $(7, 9) \in R_4$ but $(7, 9) \notin X \times Y$.

Example: 3 Given two finite sets A and B such that $n(A) = 2$, $n(B) = 3$. Then total number of relations from A to B is

(a) 4 (b) 8 (c) 64 (d) None of these

Solution: (c) Here $n(A \times B) = 2 \times 3 = 6$

Since every subset of $A \times B$ defines a relation from A to B, number of relation from A to B is equal to number of subsets of $A \times B = 2^6 = 64$, which is given in (c).

Example: 4 The relation R defined on the set of natural numbers as $\{(a, b) : a$ differs from b by 3$\}$, is given by

(a) {(1, 4, (2, 5), (3, 6),.....} (b) {(4, 1), (5, 2), (6, 3),.....} (c) {(1, 3), (2, 6), (3, 9),..} (d) None of these

Solution: (b) $R = \{(a,b) : a,b \in N, a-b=3\} = \{((n+3),n) : n \in N\} = \{(4,1),(5,2),(6,3).....\}$

1.2.2 Inverse Relation

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b,a) : (a,b) \in R\}$

Clearly $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$. Also, Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1})

Example: Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $R = \{(a, 1), (a, 3), (b, 3), (c, 3)\}$.

Then, (i) $R^{-1} = \{(1, a), (3, a), (3, b), (3, c)\}$

(ii) Dom $(R) = \{a, b, c\}$ = Range (R^{-1})

(iii) Range $(R) = \{1, 3\}$ = Dom (R^{-1})

Example: 5 Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. A relation $R : A \to B$ is defined by $R = \{(1, 3), (1, 5), (2, 1)\}$. Then R^{-1} is defined by

(a) {(1,2), (3,1), (1,3), (1,5)} (b) {(1, 2), (3, 1), (2, 1)} (c) {(1, 2), (5, 1), (3, 1)} (d) None of these

Solution: (c) $(x, y) \in R \Leftrightarrow (y, x) \in R^{-1}$, $\therefore$ $R^{-1} = \{(3,1),(5,1),(1,2)\}$.

Example: 6 The relation R is defined on the set of natural numbers as $\{(a, b) : a = 2b\}$. Then R^{-1} is given by

(a) {(2, 1), (4, 2), (6, 3).....} (b) {(1, 2), (2, 4), (3, 6)....} (c) R^{-1} is not defined (d) None of these

Solution: (b) $R = \{(2, 1), (4, 2), (6, 3),......\}$ So, $R^{-1} = \{(1, 2), (2, 4), (3, 6),.....\}$.

1.2.3 Types of Relations

(1) **Reflexive relation :** A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Example: Let $A = \{1, 2, 3\}$ and $R = \{(1, 1); (1, 3)\}$

Then R is not reflexive since $3 \in A$ but $(3, 3) \notin R$

Note :
- The identity relation on a non-void set A is always reflexive relation on A. However, a reflexive relation on A is not necessarily the identity relation on A.
- The universal relation on a non-void set A is reflexive.

(2) **Symmetric relation :** A relation R on a set A is said to be a symmetric relation *iff*

$$(a, b) \in R \Rightarrow (b, a) \in R \text{ for all } a, b \in A$$

i.e. $$aRb \Rightarrow bRa \text{ for all } a, b \in A.$$

it should be noted that R is symmetric iff $R^{-1} = R$

Note :
- The identity and the universal relations on a non-void set are symmetric relations.
- A relation R on a set A is not a symmetric relation if there are at least two elements $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.
- A reflexive relation on a set A is not necessarily symmetric.

(3) **Anti-symmetric relation :** Let A be any set. A relation R on set A is said to be an anti-symmetric relation *iff* $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

Thus, if $a \neq b$ then a may be related to b or b may be related to a, but never both.

Example: Let N be the set of natural numbers. A relation $R \subseteq N \times N$ is defined by xRy iff x divides y(*i.e.*, x/y).

Then $x R y,\ y R x \Rightarrow x$ divides y, y divides $x \Rightarrow x = y$

Note : ❑ The identity relation on a set A is an anti-symmetric relation.

❑ The universal relation on a set A containing at least two elements is not anti-symmetric, because if $a \neq b$ are in A, then a is related to b and b is related to a under the universal relation will imply that $a = b$ but $a \neq b$.

❑ The set $\{(a, a) : a \in A\} = D$ is called the diagonal line of $A \times A$. Then "the relation R in A is antisymmetric iff $R \cap R^{-1} \subseteq D$".

(4) **Transitive relation :** Let A be any set. A relation R on set A is said to be a transitive relation *iff*

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$ *i.e.*, aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

In other words, if a is related to b, b is related to c, then a is related to c.

Transitivity fails only when there exists a, b, c such that $a R b$, $b R c$ but $a \not R c$.

Example: Consider the set $A = \{1, 2, 3\}$ and the relations

$R_1 = \{(1, 2), (1, 3)\}$; $R_2 = \{(1, 2)\}$; $R_3 = \{(1, 1)\}$; $R_4 = \{(1, 2), (2, 1), (1, 1)\}$

Then R_1, R_2, R_3 are transitive while R_4 is not transitive since in $R_4, (2, 1) \in R_4; (1, 2) \in R_4$ but $(2, 2) \notin R_4$.

Note : ❑ The identity and the universal relations on a non-void sets are transitive.

❑ The relation 'is congruent to' on the set T of all triangles in a plane is a transitive relation.

(5) **Identity relation :** Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only. Every identity relation will be reflexive, symmetric and transitive.

Example: On the set = $\{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A .

Note : ❑ It is interesting to note that every identity relation is reflexive but every reflexive relation need not be an identity relation.

Also, identity relation is reflexive, symmetric and transitive.

(6) **Equivalence relation :** A relation R on a set A is said to be an equivalence relation on A *iff*

(i) It is reflexive *i.e.* $(a, a) \in R$ for all $a \in A$

(ii) It is symmetric *i.e.* $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$

(iii) It is transitive *i.e.* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Note : ❑ **Congruence modulo (m) :** Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a-b$ is divisible by m and we write $a \equiv b$ (mod m).

Thus $a \equiv b$ (mod m) $\Leftrightarrow a-b$ is divisible by m. For example, $18 \equiv 3$ (mod 5) because 18 – 3 = 15 which is divisible by 5. Similarly, $3 \equiv 13$ (mod 2) because 3 – 13 = –10 which is divisible by 2. But 25 $\neq$ 2 (mod 4) because 4 is not a divisor of 25 – 3 = 22.

The relation "Congruence modulo m" is an equivalence relation.

Important Tips

- *If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.*
- *The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.*
- *The inverse of an equivalence relation is an equivalence relation.*

1.2.4 Equivalence Classes of an Equivalence Relation

Let R be equivalence relation in $A(\neq \phi)$. Let $a \in A$. Then the equivalence class of a, denoted by $[a]$ or $\{\bar{a}\}$ is defined as the set of all those points of A which are related to a under the relation R. Thus $[a] = \{x \in A : x\ R\ a\}$.

It is easy to see that

(1) $b \in [a] \Rightarrow a \in [b]$ (2) $b \in [a] \Rightarrow [a] = [b]$ (3) Two equivalence classes are either disjoint or identical.

As an example we consider a very important equivalence relation $x \equiv y (\text{mod } n)$ iff n divides $(x-y)$, n is a fixed positive integer. Consider $n = 5$. Then

$[0] = \{x : x \equiv 0 \text{ (mod 5)}\} = \{5p : p \in Z\} = \{0, \pm 5, \pm 10, \pm 15, ...\}$

$[1] = \{x : x \equiv 1 (\text{mod } 5)\} = \{x : x - 1 = 5k, k \in Z\} = \{5k + 1 : k \in Z\} = \{1, 6, 11, ..., -4, -9, ...\}$.

One can easily see that there are only 5 distinct equivalence classes *viz.* [0], [1], [2], [3] and [4], when n = 5.

Example: 7 Given the relation R = {(1, 2), (2, 3)} on the set A = {1, 2, 3}, the minimum number of ordered pairs which when added to R make it an equivalence relation is

(a) 5 (b) 6 (c) 7 (d) 8

Solution: (c) R is reflexive if it contains (1, 1), (2, 2), (3, 3)

$\because (1, 2) \in R, (2, 3) \in R$

$\therefore$ R is symmetric if (2, 1), (3, 2) $\in R$. Now, $R = \{(1,1), (2,2), (3,3), (2,1), (3,2), (2,3), (1,2)\}$

R will be transitive if (3, 1); (1, 3) $\in R$. Thus, R becomes an equivalence relation by adding (1, 1) (2, 2) (3, 3) (2, 1) (3,2) (1, 3) (3, 1). Hence, the total number of ordered pairs is 7.

Example: 8 The relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)} on set A = {1, 2, 3} is

(a) Reflexive but not symmetric (b) Reflexive but not transitive

(c) Symmetric and Transitive (d) Neither symmetric nor transitive

Solution: (a) Since (1, 1); (2, 2); (3, 3) $\in R$ therefore R is reflexive. (1, 2) $\in R$ but (2, 1) $\notin R$, therefore R is not symmetric. It can be easily seen that R is transitive.

Example: 9 Let R be the relation on the set R of all real numbers defined by $a\,R\,b$ iff $|a-b| \le 1$. Then R is

(a) Reflexive and Symmetric (b) Symmetric only (c) Transitive only (d) Anti-symmetric only

Solution: (a) $|a-a| = 0 < 1$ $\therefore a\,R\,a \;\forall\, a \in R$

$\therefore$ R is reflexive, Again $a\,R\,b \Rightarrow |a-b| \le 1 \Rightarrow |b-a| \le 1 \Rightarrow bRa$

$\therefore$ R is symmetric, Again $1R\frac{1}{2}$ and $\frac{1}{2}R1$ but $\frac{1}{2} \ne 1$

$\therefore$ R is not anti-symmetric

Further, 1 R 2 and 2 R 3 but $1 \not R 3$

$[\because |1-3| = 2 > 1]$

$\therefore$ R is not transitive.

Example: 10 The relation "less than" in the set of natural numbers is

(a) Only symmetric (b) Only transitive (c) Only reflexive (d) Equivalence relation

Solution: (b) Since $x < y, y < z \Rightarrow x < z \;\forall\, x, y, z \in N$

$\therefore$ $xRy, yRz \Rightarrow xRz$, $\therefore$ Relation is transitive, $\therefore$ $x < y$ does not give $y < x$, $\therefore$ Relation is not symmetric.

Since $x < x$ does not hold, hence relation is not reflexive.

Example: 11 With reference to a universal set, the inclusion of a subset in another, is relation, which is

(a) Symmetric only (b) Equivalence relation (c) Reflexive only (d) None of these

Solution: (d) Since $A \subseteq A$ $\therefore$ relation '$\subseteq$' is reflexive.

Since $A \subseteq B,\;\; B \subseteq C \Rightarrow A \subseteq C$

$\therefore$ relation '$\subseteq$' is transitive.

But $A \subseteq B, \not\Rightarrow B \subseteq A$, $\therefore$ Relation is not symmetric.

Example: 12 Let $A = \{2, 4, 6, 8\}$. A relation R on A is defined by $R = \{(2,4), (4,2), (4,6), (6,4)\}$. Then R is

(a) Anti-symmetric (b) Reflexive (c) Symmetric (d) Transitive

Solution: (c) Given $A = \{2, 4, 6, 8\}$

$R = \{(2, 4)(4, 2)\;(4, 6)\;(6, 4)\}$

$(a, b) \in R \Rightarrow (b, a) \in R$ and also $R^{-1} = R$. Hence R is symmetric.

Example: 13 Let $P = \{(x, y) \mid x^2 + y^2 = 1, x, y \in R\}$. Then P is

(a) Reflexive (b) Symmetric (c) Transitive (d) Anti-symmetric

Solution: (b) Obviously, the relation is not reflexive and transitive but it is symmetric, because $x^2 + y^2 = 1 \Rightarrow y^2 + x^2 = 1$.

Example: 14 Let R be a relation on the set N of natural numbers defined by $nRm \Leftrightarrow n$ is a factor of m (*i.e.*, $n|m$). Then R is

(a) Reflexive and symmetric (b) Transitive and symmetric

(c) Equivalence (d) Reflexive, transitive but not symmetric

Solution: (d) Since $n \mid n$ for all $n \in N$, therefore R is reflexive. Since $2 \mid 6$ but $6 \nmid 2$, therefore R is not symmetric.

Let $n\,R\,m$ and $m\,R\,p \Rightarrow n|m$ and $m|p \Rightarrow n|p \Rightarrow nRp$. So R is transitive.

Example: 15 Let R be an equivalence relation on a finite set A having n elements. Then the number of ordered pairs in R is

(a) Less than n (b) Greater than or equal to n (c) Less than or equal to n (d) None of these

Solution: (b) Since R is an equivalence relation on set A, therefore $(a, a) \in R$ for all $a \in A$. Hence, R has at least n ordered pairs.

Example: 16 Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by (a, b) R (c, d) if $ad(b+c) = bc(a+d)$, then R is

(a) Symmetric only (b) Reflexive only (c) Transitive only (d) An equivalence relation

Solution: (d) For $(a, b), (c, d) \in N \times N$

$(a,b)R(c,d) \Rightarrow ad(b+c) = bc(a+d)$

Reflexive: Since $ab(b+a) = ba(a+b) \forall ab \in N$,

$\therefore$ $(a,b)R(a,b)$, $\therefore$ R is reflexive.

Symmetric: For $(a,b),(c,d) \in N \times N$, let $(a,b)R(c,d)$

$\therefore$ $ad(b+c) = bc(a+d) \Rightarrow bc(a+d) = ad(b+c) \Rightarrow cb(d+a) = da(c+b) \Rightarrow (c,d)R(a,b)$

$\therefore$ R is symmetric

Transitive: For $(a,b),(c,d),(e,f) \in N \times N$, Let $(a,b)R(c,d),(c,d)R(e,f)$

$\therefore$ $ad(b+c) = bc(a+d)$, $cf(d+e) = de(c+f)$

$\Rightarrow adb + adc = bca + bcd$ .....(i) and $cfd + cfe = dec + def$ ....(ii)

(i) × ef + (ii) × ab gives, $adbef + adcef + cfdab + cfeab = bcaef + bcdef + decab + defab$

$\Rightarrow adcf(b+e) = bcde(a+f) \Rightarrow af(b+e) = be(a+f) \Rightarrow (a,b)R(e,f)$. $\therefore$ R is transitive. Hence R is an equivalence relation.

Example: 17 For real numbers x and y, we write $xRy \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is

(a) Reflexive (b) Symmetric (c) Transitive (d) None of these

Solution: (a) For any $x \in R$, we have $x - x + \sqrt{2} = \sqrt{2}$ an irrational number.

$\Rightarrow xRx$ for all x. So, R is reflexive.

R is not symmetric, because $\sqrt{2}R1$ but $1 \not R \sqrt{2}$, R is not transitive also because $\sqrt{2}\,R\,1$ and $1R2\sqrt{2}$ but $\sqrt{2} \not R 2\sqrt{2}$.

Example: 18 Let X be a family of sets and R be a relation on X defined by 'A is disjoint from B'. Then R is

(a) Reflexive (b) Symmetric (c) Anti-symmetric (d) Transitive

Solution: (b) Clearly, the relation is symmetric but it is neither reflexive nor transitive.

Example: 19 Let R and S be two non-void relations on a set A. Which of the following statements is false

(a) R and S are transitive $\Rightarrow R \cup S$ is transitive (b) R and S are transitive $\Rightarrow R \cap S$ is transitive

(c) R and S are symmetric $\Rightarrow R \cup S$ is symmetric (d) R and S are reflexive $\Rightarrow R \cap S$ is reflexive

Solution: (a) Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2)\}$, $S = \{(2, 2)\ (2, 3)\}$ be transitive relations on A.

Then $R \cup S = \{(1, 1); (1, 2); (2, 2); (2, 3)\}$

Obviously, $R \cup S$ is not transitive. Since $(1, 2) \in R \cup S$ and $(2,3) \in R \cup S$ but $(1, 3) \notin R \cup S$.

Example: 20 The solution set of $8x \equiv 6 (\text{mod } 14), x \in Z$, are

(a) $[8] \cup [6]$ (b) $[8] \cup [14]$ (c) $[6] \cup [13]$ (d) $[8] \cup [6] \cup [13]$

Solution: (c) $8x - 6 = 14P (P \in Z) \Rightarrow x = \frac{1}{8}[14P + 6], x \in Z$

$\Rightarrow x = \frac{1}{4}(7P+3) \Rightarrow x = 6, 13, 20, 27, 34, 41, 48, \ldots\ldots.$

$\therefore$ Solution set = {6, 20, 34, 48,.....} $\cup$ {13, 27, 41, ...} = [6] $\cup$ [13].

Where [6], [13] are equivalence classes of 6 and 13 respectively.

1.2.5 Composition of Relations

Let R and S be two relations from sets A to B and B to C respectively. Then we can define a relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\ b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation is called the composition of R and S.

For example, if $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $C = \{p, q, r, s\}$ be three sets such that $R = \{(1, a), (2, c), (1, c), (2, d)\}$ is a relation from A to B and $S = \{(a, s), (b, q), (c, r)\}$ is a relation from B to C. Then SoR is a relation from A to C given by $SoR = \{(1, s)\ (2, r)\ (1, r)\}$

In this case RoS does not exist.

In general $RoS \neq SoR$. Also $(SoR)^{-1} = R^{-1}oS^{-1}$.

Example: 21 If R is a relation from a set A to a set B and S is a relation from B to a set C, then the relation SoR

(a) Is from A to C (b) Is from C to A (c) Does not exist (d) None of these

Solution: (a) It is obvious.

Example: 22 If $R \subset A \times B$ and $S \subset B \times C$ be two relations, then $(SoR)^{-1} =$

(a) $S^{-1}oR^{-1}$ (b) $R^{-1}oS^{-1}$ (c) SoR (d) RoS

Solution: (b) It is obvious.

Example: 23 If R be a relation < from $A = \{1,2, 3, 4\}$ to $B = \{1, 3, 5\}$ *i.e.,* $(a, b) \in R \Leftrightarrow a < b$, then RoR^{-1} is

(a) {(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)}

(b) {(3, 1) (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)}

(c) {(3, 3), (3, 5), (5, 3), (5, 5)}

(d) {(3, 3) (3, 4), (4, 5)}

Solution: (c) We have, $R = \{(1, 3); (1, 5); (2, 3); (2, 5); (3, 5); (4, 5)\}$

$R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2); (5, 3); (5, 4)\}$

Hence $RoR^{-1} = \{(3, 3); (3, 5); (5, 3); (5, 5)\}$

Example: 24 Let a relation R be defined by $R = \{(4, 5); (1, 4); (4, 6); (7, 6); (3, 7)\}$ then $R^{-1}oR$ is

(a) {(1, 1), (4, 4), (4, 7), (7, 4), (7, 7), (3, 3)} (b) {(1, 1), (4, 4), (7, 7), (3, 3)}

(c) {(1, 5), (1, 6), (3, 6)} (d) None of these

Solution: (a) We first find R^{-1}, we have $R^{-1} = \{(5,4);(4,1);(6,4);(6,7);(7,3)\}$ we now obtain the elements of $R^{-1}oR$ we first pick the element of R and then of R^{-1}. Since $(4,5) \in R$ and $(5,4) \in R^{-1}$, we have $(4,4) \in R^{-1}oR$

Similarly, $(1,4) \in R, (4,1) \in R^{-1} \Rightarrow (1,1) \in R^{-1}oR$

$(4,6) \in R, (6,4) \in R^{-1} \Rightarrow (4,4) \in R^{-1}oR$, $(4,6) \in R, (6,7) \in R^{-1} \Rightarrow (4,7) \in R^{-1}oR$

$(7,6) \in R, (6,4) \in R^{-1} \Rightarrow (7,4) \in R^{-1}oR$, $(7,6) \in R, (6,7) \in R^{-1} \Rightarrow (7,7) \in R^{-1}oR$

$(3,7) \in R, (7,3) \in R^{-1} \Rightarrow (3,3) \in R^{-1}oR$, Hence $R^{-1}oR = \{(1, 1); (4, 4); (4, 7); (7, 4), (7, 7); (3, 3)\}$.

1.2.6 Axiomatic Definitions of the Set of Natural Numbers (Peano's Axioms)

The set N of natural numbers ($N = \{1, 2, 3, 4......\}$) is a set satisfying the following axioms (known as peano's axioms)

(1) N is not empty.

(2) There exist an injective (one-one) map $S : N \to N$ given by $S(n) = n^+$, where n^+ is the immediate successor of n in N *i.e.*, $n + 1 = n^+$.

(3) The successor mapping S is not surjective (onto).

(4) If $M \subseteq N$ such that,

(i) M contains an element which is not the successor of any element in N, and

(ii) $m \in M \Rightarrow m^+ \in M$, then $M = N$

This is called the axiom of induction. We denote the unique element which is not the successor of any element is 1. Also, we get $1^+ = 2, 2^+ = 3$.

Note : ❑ Addition in N is defined as,

$$n + 1 = n^+$$

$$n + m^+ = (n + m)^+$$

❑ Multiplication in N is defined by,

$$n.1 = n$$

$$n.m^+ = n.m + n$$