CLASSROOM STUDY PACKAGE

MATHEMATICS

Function

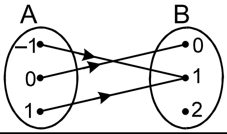
JEE EXPERT

Functions

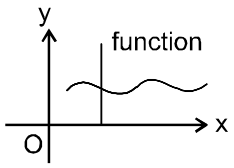
Definition :

Function is a rule (or correspondence), from a non empty set A to a non empty set B, that associates each member of A to a unique member of B. Symbolically, we write f: $A \to B$. We read it as "f is a function from A to B".

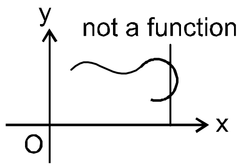
For example, let $A \equiv \{-1, 0, 1\}$ and $B \equiv \{0, 1, 2\}$.
Then $A \times B \equiv \{(-1, 0), (-1, 1), (-1, 2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$
Now, " $f : A \to B$ defined by $f(x) = x^2$ " is the function such that
$f \equiv \{(-1, 1), (0, 0), (1, 1)\}$
f can also be shown diagramatically by following mapping.



Note : Every function say $y = f(x) : A \to B$. Here x is independent variable which takes its values from A while 'y' takes its value from B. A relation will be a function if and only if
(i) x must be able to take each and every value of A and
and (ii) one value of x must be related to one and only one value of y in set B.



(a)



(b)

Graphically : If any vertical line cuts the graph at more than one point, then the graph does not represent a function.

Example # 1 : (i) Which of the following correspondences can be called a function ?

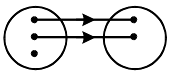
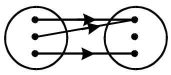
(A) $f(x) = x^3$; $\{-1, 0, 1\} \to \{0, 1, 2, 3\}$

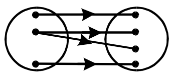
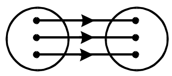
(B) $f(x) = \pm \sqrt{x}$; $\{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}$

(C) $f(x) = \sqrt{x}$; $\{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}$

(D) $f(x) = -\sqrt{x}$; $\{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}$

(ii) Which of the following pictorial diagrams represent the function

(A)  (B) 

(C)  (D) 

Solution :

(i) f(x) in (C) and (D) are functions as definition of function is satisfied. while in case of (A) the given relation is not a function, as $f(-1) \notin 2^{nd}$ set. Hence definition of function is not satisfied. While in case of (B), the given relation is not a function, as $f(1) = \pm 1$ and $f(4) = \pm 2$ i.e. element 1 as well as 4 in 1^{st} set is related with two elements of 2^{nd} set.Hence definition of function is not satisfied.

(ii) B and D. In (A) one element of domain has no image, while in (C) one element of 1st set has two images in 2nd set

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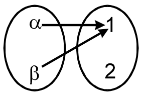
Self practice problem :

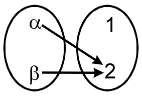
(1) Let g(x) be a function defined on [–1, 1]. If the area of the equilateral triangle with two of its vertices at (0,0) and **(**x,g(x)**)** is $\sqrt{3}/4$ sq. unit, then the function g(x) may be.

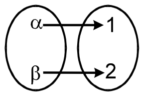
(A) g(x)= $\pm\sqrt{(1-x^2)}$ (B) g(x) = $\sqrt{(1-x^2)}$ (C) g(x) = $-\sqrt{(1-x^2)}$ (D) g(x) = $\sqrt{(1+x^2)}$

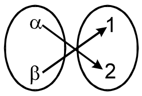
(2) Represent all possible functions defined from {α, β} to {1, 2}.

Answers : (1) B, C

(2) (i) 

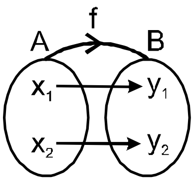
(ii) 

(iii) 

(iv) 

Domain, Co-domain and Range of a Function :

Let y = f(x) : A $\rightarrow$ B, then the set A is known as the domain of f and the set B is known as co-domain of f.



If x_1 is mapped to y_1, then y_1 is called as image of x_1 under f. Further x_1 is a pre-image of y_1 under f.
If only expression of f (x) is given (domain and co-domain are not mentioned), then domain is **complete** set of those values of x for which f (x) is real, while codomain is considered to be (– ∞, ∞) (except in inverse trigonometric functions).
Range is the complete set of values that y takes. Clearly range is a subset of Co-domain.
A function whose domain and range are both subsets of real numbers is called **a real function**.

Example # 2 : Find the domain of following functions :

(i) f(x) = $\sqrt{x^2-5}$ (ii) $\sin^{-1}(2x-1)$

Solution : (i) f(x) = $\sqrt{x^2-5}$ is real iff $x^2 - 5 \geq 0$

$\Rightarrow$ $|x| \geq \sqrt{5}$ $\Rightarrow$ $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$

$\therefore$ the domain of f is $(-\infty, -\sqrt{5}\,] \cup [\,\sqrt{5}, \infty)$

(ii) $\sin^{-1}(2x-1)$ is real iff $-1 \leq 2x - 1 \leq +1$
$\therefore$ domain is $x \in [0, 1]$

Algebraic Operations on Functions :

If f and g are real valued functions of x with domain set A and B respectively, then both f and g are defined in $A \cap B$. Now we define f + g, f – g, (f . g) and (f /g) as follows:

(i) $(f \pm g)(x) = f(x) \pm g(x)$
(ii) $(f.g)(x) = f(x).\,g(x)$
— domain in each case is $A \cap B$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B$ such that $g(x) \neq 0\}$.

Note : ☞ For domain of $\phi(x) = \{f(x)\}^{g(x)}$, conventionally, the conditions are $f(x) > 0$ and $g(x)$ must be real.

☞ For domain of $\phi(x) = {}^{f(x)}C_{g(x)}$ or $\phi(x) = {}^{f(x)}P_{g(x)}$ conventional conditions of domain are $f(x) \ge g(x)$ and $f(x) \in N$ and $g(x) \in W$

Example # 3 : Find the domain of following functions :

(i) $f(x) = \sqrt{\sin x} - \sqrt{16 - x^2}$

(ii) $f(x) = \dfrac{3}{\sqrt{4 - x^2}} \log(x^3 - x)$

(iii) $f(x) = x^{\cos^{-1} x}$

Solution : (i) $\sqrt{\sin x}$ is real iff $\sin x \ge 0 \Leftrightarrow x \in [2n\pi, 2n\pi + \pi]$, $n \in I$.

$\sqrt{16 - x^2}$ is real iff $16 - x^2 \ge 0 \Leftrightarrow -4 \le x \le 4$.

Thus the domain of the given function is $\{x : x \in [2n\pi, 2n\pi + \pi], n \in I\} \cap [-4, 4] = [-4, -\pi] \cup [0, \pi]$.

(ii) Domain of $\sqrt{4 - x^2}$ is $[-2, 2]$ but $\sqrt{4 - x^2} = 0$ for $x = \pm 2$ $\Rightarrow$ $x \in (-2, 2)$

$\log(x^3 - x)$ is defined for $x^3 - x > 0$ i.e. $x(x - 1)(x + 1) > 0$.

$\therefore$ domain of $\log(x^3 - x)$ is $(-1, 0) \cup (1, \infty)$.

Hence the domain of the given function is $\{(-1, 0) \cup (1, \infty)\} \cap (-2, 2) \equiv (-1, 0) \cup (1, 2)$.

(iii) $x > 0$ and $-1 \le x \le 1$

$\therefore$ domain is $(0, 1]$

Self practice problems :

(3) Find the domain of following functions.

(i) $f(x) = \dfrac{1}{\log(2 - x)} + \sqrt{x + 1}$ (ii) $f(x) = \sqrt{1 - x} - \sin^{-1} \dfrac{2x - 1}{3}$

Answers : (i) $[-1, 1) \cup (1, 2)$ (ii) $[-1, 1]$

Methods of determining range :

(i) **Representing x in terms of y**

If $y = f(x)$, try to express as $x = g(y)$, then domain of $g(y)$ represents possible values of y, which is range of $f(x)$.

Example # 4 : Find the range of $f(x) = \dfrac{x^2 + x + 1}{x^2 + x - 1}$

Solution : $f(x) = \dfrac{x^2 + x + 1}{x^2 + x - 1}$ $\{x^2 + x + 1$ and $x^2 + x - 1$ have no common factor$\}$

$y = \dfrac{x^2 + x + 1}{x^2 + x - 1}$

$\Rightarrow$ $yx^2 + yx - y = x^2 + x + 1$

$\Rightarrow$ $(y - 1)x^2 + (y - 1)x - y - 1 = 0$

If $y = 1$, then the above equation reduces to $-2 = 0$. Which is not true.

Further if $y \ne 1$, then $(y - 1)x^2 + (y - 1)x - y - 1 = 0$ is a quadratic and has real roots if $(y - 1)^2 - 4(y - 1)(-y - 1) \ge 0$

i.e. if $y \le -3/5$ or $y \ge 1$ but $y \ne 1$

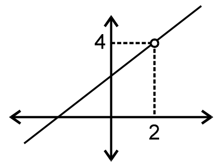
Thus the range is $(-\infty, -3/5] \cup (1, \infty)$

(ii) **Graphical Method :**
The set of y– coordinates of the graph of a function is the range.

Example # 5 : Find the range of $f(x) = \dfrac{x^2-4}{x-2}$

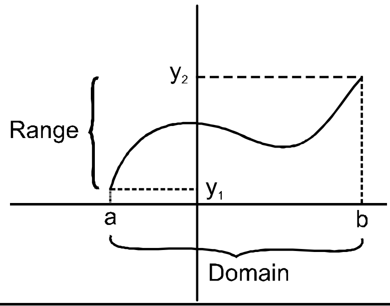
Solution : $f(x) = \dfrac{x^2-4}{x-2} = x + 2;\ x \neq 2$

$\therefore$ graph of f(x) would be

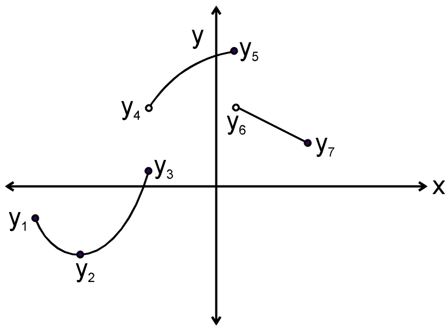


Thus the range of $f(x)$ is $R - \{4\}$

Further if f(x) happens to be continuous in its domain then range of f(x) is [min f(x), max. f(x)]. However for sectionally continuous functions, range will be union of [min f(x), max. f(x)] over all those intervals where f(x) is continuous, as shown by following example.



Example # 6 : Let graph of function y = f(x) is



Then range of above sectionally continuous function is $[y_2, y_3] \cup (y_4, y_5] \cup (y_7, y_6]$

(iii) **Using monotonocity** : Many of the functions are monotonic increasing or monotonic decreasing. In case of monotonic continuous functions the minimum and maximum values lie at end points of domain. Some of the common function which are increasing or decreasing in the interval where they are **continuous** is as under.

Monotonic increasing	Monotonic decreasing
$\log_a x,\ a > 1$	$\log_a x,\ 0 < a < 1$
e^x	e^{-x}
$\sin^{-1} x$	$\cos^{-1} x$
$\tan^{-1} x$	$\cot^{-1} x$
$\sec^{-1} x$	$\operatorname{cosec}^{-1} x$

For monotonic increasing functions in [a, b]
(i) $f'(x) \geq 0$
(ii) range is [f(a), f(b)]
for monotonic decreasing functions in [a, b]
(i) $f'(x) \leq 0$
(ii) range is [f(b), f(a)]

Example # 7 : Find the range of following functions :

(i) $y = \ell n\,(2x - x^2)$ (ii) $y = \sec^{-1}(x^2 + 3x + 1)$

Solution : (i) **Step – 1**

We have $2x - x^2 \in (-\infty, 1]$

Step – 2 Let $t = 2x - x^2$

For ℓnt to be defined accepted values are $(0, 1]$

Now, using monotonocity of $\ell n\, t$,

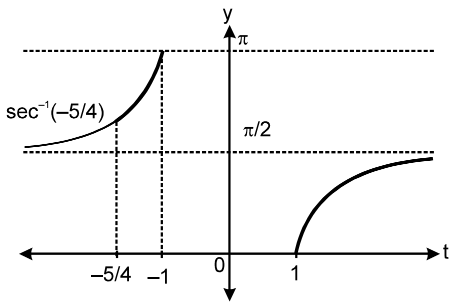
$\ell n\,(2x - x^2) \in (-\infty, 0]$

$\therefore$ range is $(-\infty, 0]$ **Ans.**

(ii) $y = \sec^{-1}(x^2 + 3x + 1)$

Let $t = x^2 + 3x + 1$ for $x \in R$, then $t \in \left[-\dfrac{5}{4}, \infty\right)$

but $y = \sec^{-1}(t)$ $\Rightarrow$ $t \in \left[-\dfrac{5}{4}, -1\right] \cup [1, \infty)$



from graph the range is $\left[0, \dfrac{\pi}{2}\right) \cup \left[\sec^{-1}\left(-\dfrac{5}{4}\right), \pi\right]$

Self practice problems :

(4) Find domain and range of following functions.

(i) $y = x^3$ (ii) $y = \dfrac{x^2 - 2x + 5}{x^2 + 2x + 5}$ (iii) $y = \dfrac{1}{\sqrt{x^2 - x}}$

(iv) $y = \cot^{-1}(2x - x^2)$ (v) $y = \ell n\left(\sin^{-1}\left(x^2 + x + \dfrac{3}{4}\right)\right)$

Answers : (i) domain R; range R (ii) domain R ; range $\left[\dfrac{3 - \sqrt{5}}{2}, \dfrac{3 + \sqrt{5}}{2}\right]$

(iii) domain R – [0, 1] ; range $(0, \infty)$ (iv) domain R ; range $\left[\dfrac{\pi}{4}, \pi\right)$

(v) domain $x \in \left[\dfrac{-2 - \sqrt{8}}{4}, \dfrac{-2 + \sqrt{8}}{4}\right]$; range $\left[\ell n\dfrac{\pi}{6}, \ell n\dfrac{\pi}{2}\right]$

Various Types of Functions :

(i) Polynomial Function :

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$, where n is a **non negative integer** and $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note : ☞ There are only two polynomial functions, satisfying the relation; f(x).f(1/x) = f(x) + f(1/x), which are $f(x) = 1 \pm x^n$

Proof : Let $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots + a_n$, then $f\left(\frac{1}{x}\right) = \frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots\ldots\ldots + a_n$.

Since the relation holds for many values of x,

$\therefore$ Comparing the coefficients of x^n, we get $a_0 a_n = a_0$ $\Rightarrow$ $a_n = 1$

Similary comparing the coefficients of x^{n-1}, we get $a_0 a_{n-1} + a_1 a_n = a_1$ $\Rightarrow a_{n-1} = 0$, like wise a_{n-2}, ..., a_1 are all zero.

Comparing the constant terms, we get $a_0^2 + a_1^2 + \ldots\ldots + a_n^2 = 2a_n^2$ $\Rightarrow$ $a_0 = \pm 1$

(ii) Algebraic Function :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form, $P_0(x)\, y^n + P_1(x)\, y^{n-1} + \ldots\ldots + P_{n-1}(x)\, y + P_n(x) = 0$, where n is a positive integer and $P_0(x)$, $P_1(x)$....... are polynomials in x. e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note : ☞ All polynomial functions are algebraic but not the converse.

☞ A function that is not algebraic is called **Transcendental Function**.

(iii) Rational Function :

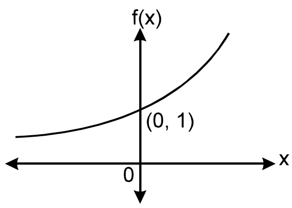
A rational function is a function of the form, $y = f(x) = \frac{g(x)}{h(x)}$, where g (x) and h (x), $h(x) \neq 0$ are polynomials.

(iv) Exponential Function :

A function $f(x) = a^x = e^{x \ln a}$ ($a > 0$, $a \neq 1$, $x \in R$) is called an exponential function. Graph of exponential function can be as follows :

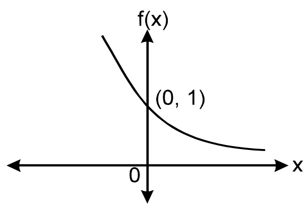
Case - I

For a > 1



Case - II

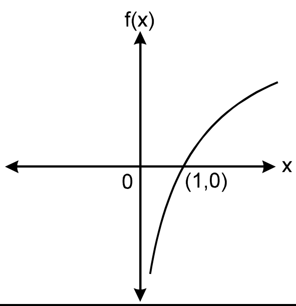
For 0 < a < 1



(v) Logarithmic Function : $f(x) = \log_a x$ is called logarithmic function, where $a > 0$ and $a \neq 1$ and $x > 0$. Its graph can be as follows

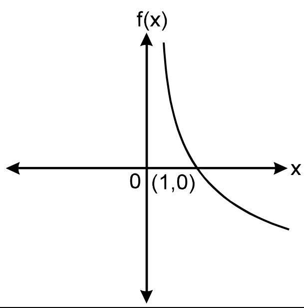
Case- I

For a > 1



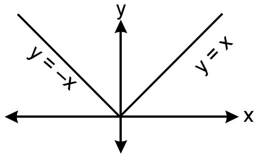
Case- II

For 0 < a < 1



(vi) Absolute Value Function / Modulus Function :

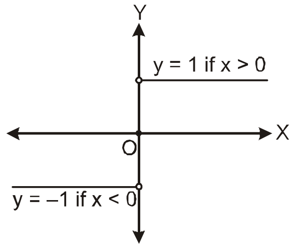
The symbol of modulus function is f (x) = $|x|$ and is defined as: y = $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.



y = |x|

(vii) Signum Function : (Also known as sgn(x))

A function f (x) = sgn (x) is defined as follows :



y = sgn x

$$f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as sgn x = $\begin{cases} \frac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$

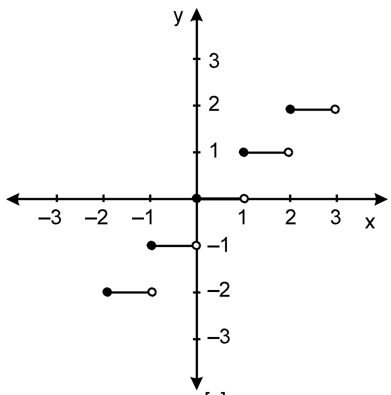
Note : sgn f(x) = $\begin{cases} \frac{|f(x)|}{f(x)}; & f(x) \neq 0 \\ 0; & f(x) = 0 \end{cases}$

(viii) Greatest Integer Function or Step Function :

The function y = f (x) = [x] is called the greatest integer function, where [x] equals to the greatest integer less than or equal to x. For example :

for $-1 \leq x < 0$; [x] = – 1 ; for $0 \leq x < 1$; [x] = 0

for $1 \leq x < 2$; [x] = 1 ; for $2 \leq x < 3$; [x] = 2 and so on.



y = [x]

Properties of greatest integer function :

(a) $x - 1 < [x] \leq x$

(b) If m is an integer, then $[x \pm m] = [x] \pm m$.

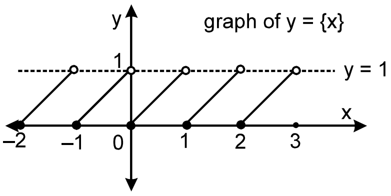
(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

(d) $[x] + [-x] = \begin{cases} 0, & \text{if x is an integer} \\ -1, & \text{if x is not an integer} \end{cases}$

(ix) Fractional Part Function:

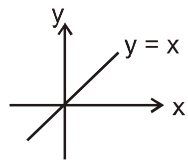
It is defined as, y = {x} = x – [x], where [.] denotes greatest integer function.

e.g. the fractional part of the number 2.1 is 2.1 – 2 = 0.1 and {–3.7} = 0.3.
The period of this function is 1 and graph of this function is as shown.



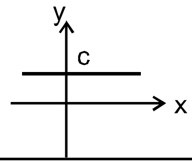
(x) Identity function :

The function f : A → A defined by, f(x) = x, ∀ x ∈ A is called the identity function on A and is denoted by I_A. It is easy to observe
that identity function is a bijection.



(xi) Constant function :

A function f : A → B is said to be a constant function, if every element of A has the same f image in B. Thus f : A → B; f(x) = c, ∀ x ∈ A, c ∈ B is a constant function.



Example # 8 : (i) Let {x} and [x] denote the fractional and integral part of a real number x respectively. Solve 4{x} = x + [x]

(ii) Draw graph of f(x) = sgn (ℓn x)

Solution : (i) As x = [x] + {x}

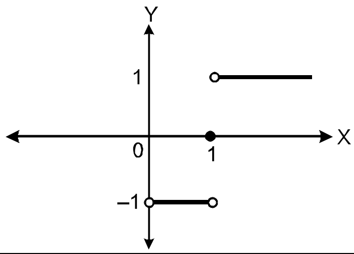
∴ Given equation ⇒ 4{x} = [x] + {x} + [x] ⇒ $\{x\} = \dfrac{2\,[x]}{3}$

As [x] is always an integer and {x} ∈ [0, 1), possible values are

[x]	{x}	x = [x] + {x}
0	0	0
1	$\frac{2}{3}$	$\frac{5}{3}$

∴ There are two solution of given equation x = 0 and x = $\frac{5}{3}$

(ii)



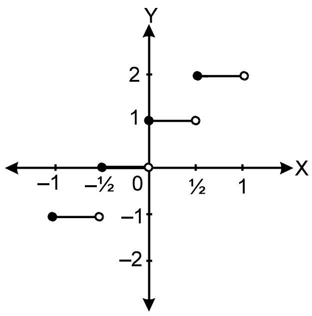
Self practice problems :

(5) If f : R → R satisfying the conditions f(0) = 1, f(1) = 2 and f(x + 2) = 2f (x) + f(x + 1), then find f (6).

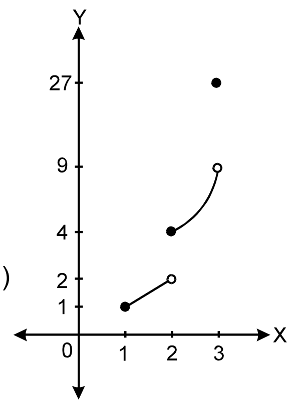
(6) Draw the graph of following functions, where [.] denotes greatest integer function
(i) y = [2 x] + 1 (ii) $y = x^{[x]}$, 1 ≤ x ≤ 3 (iii) y = sgn (x^2 – x)

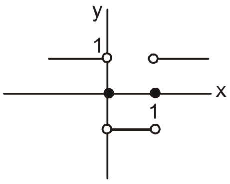
Answers : (5) 64

(6) (i)



(ii)



(iii) 

Equal or Identical Functions :

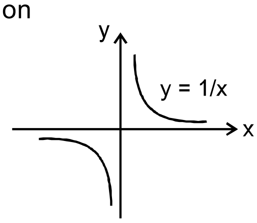
Two functions f and g are said to be identical (or equal) iff :

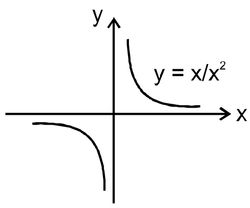
(i) The domain of f $\equiv$ the domain of g.

(ii) f(x) = g(x), for every x belonging to their common domain.

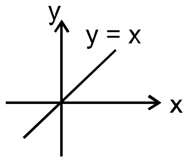
e.g. $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ are identical functions.

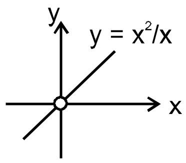
Clearly the graphs of f(x) and g(x) are exactly same





But f(x) = x and $g(x) = \frac{x^2}{x}$ are not identical functions.





Clearly the graphs of f(x) and g(x) are different at x = 0.

Example # 9 : Examine whether following pair of functions are identical or not ?

(i) $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$

(ii) $f(x) = \sin^2x + \cos^2x$ and $g(x) = \sec^2x - \tan^2x$

Solution : (i) No, as domain of f(x) is R – {1} while domain of g(x) is R

(ii) No, as domain are not same. Domain of f(x) is R while that of g(x) is $R - \left\{(2n+1)\frac{\pi}{2}; n \in I\right\}$

Self practice problems

(7) Examine whether the following pair of functions are identical or not :

(i) f(x) = sgn (x) and $g(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(ii) $f(x) = \sin^{-1}x + \cos^{-1}x$ and $g(x) = \frac{\pi}{2}$

Answers : (i) Yes (ii) No

Composite Function :

Let f: $X \to Y_1$ and g: $Y_2 \to Z$ be two functions and D is the set of values of x such that if $x \in X$, then $f(x) \in Y_2$. If $D \not\equiv \phi$, then the function h defined on D by h(x) = g{f(x)} is called composite function of g and f and is denoted by gof. It is also called function of a function.

Note : ☞ Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If D = X, then $f(X) \subseteq Y_2$.

Pictorially gof(x) can be viewed as under $\longrightarrow \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$

Note that gof(x) exists only for those x when range of f(x) is a subset of domain of g(x).

Properties of Composite Functions :

(a) In general gof $\neq$ fog (i.e. not commutative)

(b) The composition of functions are associative i.e. if three functions f, g, h are such that fo (goh) and (fog) oh are defined, then fo (goh) = (fog) oh.

Example # 10 : Describe fog and gof wherever is possible for the following functions

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$ (ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Solution : (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[1, \infty)$.

<u>For gof(x)</u>

Since range of f is a subset of domain of g,

$\therefore$ domain of gof is $[-3, \infty)$ {equal to the domain of f }

gof (x) = g{f(x)} = g ($\sqrt{x+3}$) = 1 + (x+3) = x + 4. Range of gof is $[1, \infty)$.

<u>For fog(x)</u>

since range of g is a subset of domain of f,

$\therefore$ domain of fog is R {equal to the domain of g}

fog (x) = f{g(x)}= f(1+ x^2) = $\sqrt{x^2+4}$ Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[-1, \infty)$.

<u>For gof(x)</u>

Since range of f is a subset of the domain of g,

$\therefore$ domain of gof is $[0, \infty)$ and g{f(x)}= g($\sqrt{x}$) = x – 1. Range of gof is $[-1, \infty)$

<u>For fog(x)</u>

Since range of g is not a subset of the domain of f

i.e. $[-1, \infty) \not\subset [0, \infty)$

$\therefore$ fog is not defined on whole of the domain of g.

Domain of fog is {$x \in R$, the domain of g : $g(x) \in [0, \infty)$, the domain of f}.

Thus the domain of fog is D = {$x \in R$: $0 \leq g(x) < \infty$}

i.e. D = { $x \in R$: $0 \leq x^2 - 1$}= { $x \in R$: $x \leq -1$ or $x \geq 1$ }= $(-\infty, -1] \cup [1, \infty)$

fog (x) = f{g(x)} = f(x^2–1) = $\sqrt{x^2-1}$ Its range is $[0, \infty)$.

Example # 11 : Let $f(x) = e^x$; $R^+ \to R$ and $g(x) = \sin^{-1} x$; $[-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Find domain and range of fog(x)

Solution : Domain of f(x) : $(0, \infty)$ Range of g(x) : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

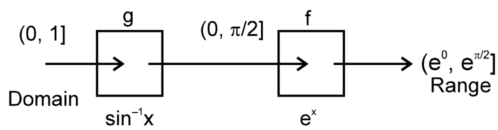
values in range of g(x) which are accepted by f(x) are $\left(0, \frac{\pi}{2}\right]$

$\Rightarrow$ $0 < g(x) \leq \frac{\pi}{2}$ $\Rightarrow$ $0 < \sin^{-1}x \leq \frac{\pi}{2}$ $\Rightarrow$ $0 < x \leq 1$

Hence domain of fog(x) is $x \in (0, 1]$

Therefore Domain : (0, 1]

Range : $(1, e^{\pi/2}]$



Example # 12 : Composition of piecewise defined functions :

If $\quad f(x) = |\,|x-3|-2\,| \qquad 0 \le x \le 4$

$\quad g(x) = 4 - |2-x| \qquad -1 \le x \le 3$

then find fog(x) and draw rough sketch of fog(x).

Solution : $f(x) = |\,|x-3|-2| \qquad 0 \le x \le 4$

$$= \begin{cases} |x-1| & 0 \le x < 3 \\ |x-5| & 3 \le x \le 4 \end{cases} = \begin{cases} 1-x & 0 \le x < 1 \\ x-1 & 1 \le x < 3 \\ 5-x & 3 \le x \le 4 \end{cases}$$

$g(x) = 4 - |2-x| \qquad -1 \le x \le 3$

$$= \begin{cases} 4-(2-x) & -1 \le x < 2 \\ 4-(x-2) & 2 \le x \le 3 \end{cases} = \begin{cases} 2+x & -1 \le x < 2 \\ 6-x & 2 \le x \le 3 \end{cases}$$

$$\therefore \quad fog\,(x) = \begin{cases} 1-g(x) & 0 \le g(x) < 1 \\ g(x)-1 & 1 \le g(x) < 3 \\ 5-g(x) & 3 \le g(x) \le 4 \end{cases} = \begin{cases} 1-(2+x) & 0 \le 2+x < 1 & \text{and} & -1 \le x < 2 \\ 2+x-1 & 1 \le 2+x < 3 & \text{and} & -1 \le x < 2 \\ 5-(2+x) & 3 \le 2+x \le 4 & \text{and} & -1 \le x < 2 \\ 1-6+x & 0 \le 6-x < 1 & \text{and} & 2 \le x \le 3 \\ 6-x-1 & 1 \le 6-x < 3 & \text{and} & 2 \le x \le 3 \\ 5-6+x & 3 \le 6-x \le 4 & \text{and} & 2 \le x \le 3 \end{cases}$$

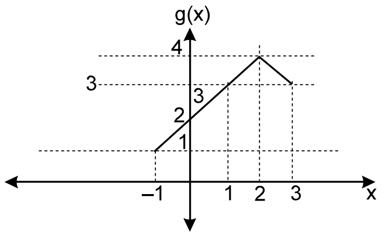
$$= \begin{cases} -1-x & -2 \le x < -1 & \text{and} & -1 \le x < 2 \\ 1+x & -1 \le x < 1 & \text{and} & -1 \le x < 2 \\ 3-x & 1 \le x \le 2 & \text{and} & -1 \le x < 2 \\ x-5 & -6 \le -x < -5 & \text{and} & 2 \le x \le 3 \\ 5-x & -5 \le -x < -3 & \text{and} & 2 \le x \le 3 \\ x-1 & -3 \le -x \le -2 & \text{and} & 2 \le x \le 3 \end{cases} = \begin{cases} -1-x & -2 \le x < -1 & \text{and} & -1 \le x < 2 \\ 1+x & -1 \le x < 1 & \text{and} & -1 \le x < 2 \\ 3-x & 1 \le x \le 2 & \text{and} & -1 \le x < 2 \\ x-5 & 5 < x \le 6 & \text{and} & 2 \le x \le 3 \\ 5-x & 3 < x \le 5 & \text{and} & 2 \le x \le 3 \\ x-1 & 2 \le x \le 3 & \text{and} & 2 \le x \le 3 \end{cases}$$

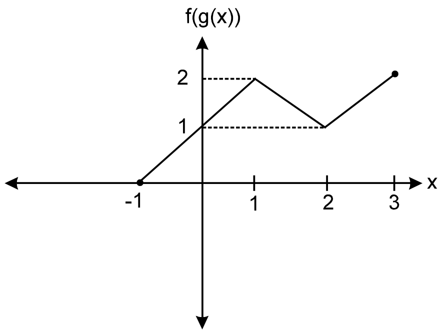
$$= \begin{cases} 1+x & -1 \le x < 1 \\ 3-x & 1 \le x < 2 \\ x-1 & 2 \le x \le 3 \end{cases}$$

Alternate method for finding fog

$$g(x) = \begin{cases} 2+x & -1 \le x < 2 \\ 6-x & 2 \le x \le 3 \end{cases}$$

graph of g(x) is





$$\therefore \quad fog(x) = \begin{cases} 1-g(x) & 0 \le g(x) < 1 \\ g(x)-1 & 1 \le g(x) < 3 \\ 5-g(x) & 3 \le g(x) \le 4 \end{cases}$$

$$= \begin{cases} 1-g(x) & \text{for no value} \\ g(x)-1 & -1 \le x < 1 \\ 5-g(x) & 1 \le x \le 3 \end{cases} = \begin{cases} 2+x-1 & -1 \le x < 1 \\ 5-(2+x) & 1 \le x < 2 \\ 5-(6-x) & 2 \le x \le 3 \end{cases} = \begin{cases} x+1 & -1 \le x < 1 \\ 3-x & 1 \le x < 2 \\ x-1 & 2 \le x \le 3 \end{cases}$$

Self practice problems

(8) Define fog(x) and gof(x). Also find their domain and range.

(i) $f(x) = [x]$, $g(x) = \sin x$ (ii) $f(x) = \tan x$, $x \in (-\pi/2, \pi/2)$; $g(x) = \sqrt{1-x^2}$

(9) Let $f(x) = e^x : R^+ \to R$ and $g(x) = x^2 - x : R \to R$. Find domain and range of fog (x) and gof (x)

Answers :

(8) (i) gof = $\sin [x]$ domain : R range $\{\sin a : a \in I\}$
fog = $[\sin x]$ domain : R range : $\{-1, 0, 1\}$

(ii) gof $\equiv \sqrt{1-\tan^2 x}$, domain : $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ range : $[0, 1]$

fog $\equiv \tan \sqrt{1-x^2}$ domain : $[-1, 1]$ range $[0, \tan 1]$

(9) fog (x) gof (x)
Domain : $(-\infty, 0) \cup (1, \infty)$ Domain : $(0, \infty)$
Range : $(1, \infty)$ Range : $(0, \infty)$

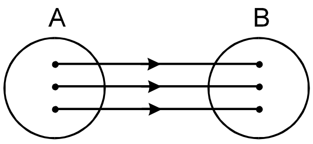
Classification of Functions :

Functions can be classified as "One – One Function (Injective Mapping)" and "Many – One Function":

One - One Function :

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \ne x_2 \Leftrightarrow f(x_1) \ne f(x_2)$.
Diagrammatically an injective mapping can be shown as



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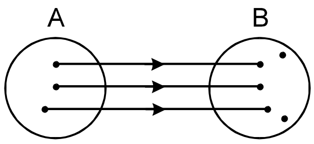
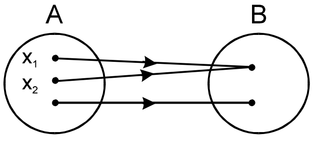


Many - One function :

A function $f : A \to B$ is said to be a many one function if there exist at least two or more elements of A having the same f image in B.

Thus $f : A \to B$ is many one iff there exist atleast two elements $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$ but $x_1 \ne x_2$.

Diagrammatically a many one mapping can be shown as



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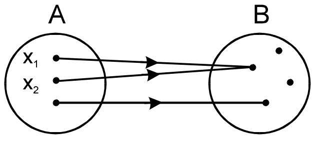


Note : ☞ If a function is one–one, it cannot be many–one and vice versa.

Methods of determining whether a given function is ONE-ONE or MANY-ONE :

(a) If $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, equate $f(x_1)$ and $f(x_2)$ and if it implies that $x_1 = x_2$, then and only then function is ONE-ONE otherwise MANY-ONE.

(b) If there exists a straight line parallel to x-axis, which cuts the graph of the function atleast at two points, then the function is MANY-ONE, otherwise ONE-ONE.

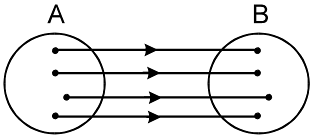
(c) If either $f'(x) \geq 0$, $\forall x \in$ domain or $f'(x) \leq 0$ $\forall x \in$ domain, where equality can hold at discrete point(s) only i.e. strictly monotonic, then function is ONE-ONE, otherwise MANY-ONE.

Note : If f and g both are one-one, then gof and fog would also be one-one (if they exist). Functions can also be classified as "Onto function (Surjective mapping)" and "Into function":

Onto function :

If the function $f : A \to B$ is such that each element in B (co–domain) must have atleast one pre–image in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall b \in B$, there exists some $a \in A$ such that f (a) = b.

Diagrammatically surjective mapping can be shown as



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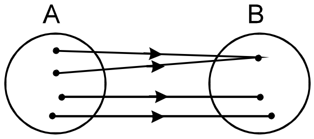
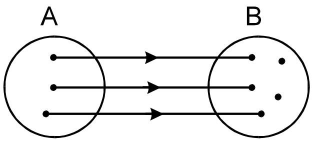


Into function :

If $f : A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function can be shown as



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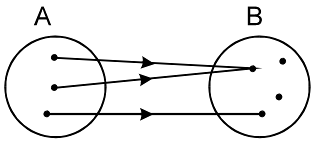
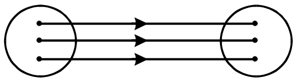


Note : (i) If range $\equiv$ co–domain, then f(x) is onto, otherwise into

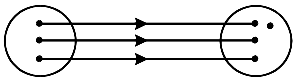
(ii) If a function is onto, it cannot be into and vice versa.

A function can be one of these four types:

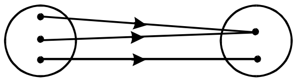
(a) one–one onto (injective and surjective)



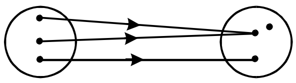
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note : (i) If f is both injective and surjective, then it is called a **bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains 'n' distinct elements, then the number of different functions defined from $A \to A$ is n^n and out of which n! are one one.

(iii) If f and g both are onto, then gof or fog may or may not be onto.

(iv) The composite of two bijections is a bijection iff f and g are two bijections such that gof is defined, then gof is also a bijection only **when co-domain of f is equal to the domain of g**.

Example#13 : (i) Find whether $f(x) = x + \cos x$ is one-one.

(ii) Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$ for $f : R \to R$ is ONTO or INTO

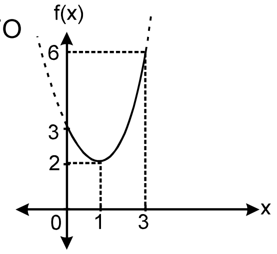
(iii) $f(x) = x^2 - 2x + 3$; $[0, 3] \to A$. Find whether f(x) is injective or not. Also find the set A, if f(x) is surjective.

Solution : (i) The domain of f(x) is R. $f'(x) = 1 - \sin x$.

$\therefore$ $f'(x) \geq 0\ \forall\ x \in$ complete domain and equality holds at discrete points only

$\therefore$ f(x) is strictly increasing on R. Hence f(x) is one-one.

(ii) As range $\equiv$ codomain, therefore given function is ONTO



(iii) $f'(x) = 2(x - 1)$; $0 \leq x \leq 3$

$\therefore$ $f'(x) = \begin{cases} -\text{ve} & ; \quad 0 \leq x < 1 \\ +\text{ve} & ; \quad 1 < x < 3 \end{cases}$

$\therefore$ f(x) is non monotonic. Hence it is not injective.

For f(x) to be surjective, A should be equal to its range. By graph range is [2, 6]

$\therefore$ $A \equiv [2, 6]$

Self practice problems :

(10) For each of the following functions find whether it is one-one or many-one and also into or onto

(i) $f(x) = 2 \tan x$; $(\pi/2, 3\pi/2) \to R$ (ii) $f(x) = \dfrac{1}{1+x^2}$; $(-\infty, 0) \to R$

(iii) $f(x) = x^2 + \ell n\, x$

Answers : (i) one-one onto (ii) one-one into (iii) one-one onto

Odd and Even Functions :

(i) If $f(-x) = f(x)$ for all x in the domain of 'f', then f is said to be an even function. e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.

(ii) If $f(-x) = -f(x)$ for all x in the domain of 'f', then f is said to be an odd function. e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

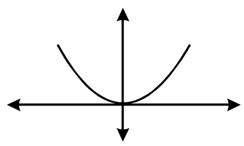
Note : (i) A function may neither be odd nor even. (e.g. $f(x) = e^x$, $\cos^{-1}x$)

(ii) If an odd function is defined at x = 0, then f(0) = 0

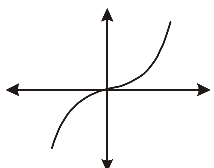
Properties of Even/Odd Function

(a) The graph of every even function is symmetric about the y–axis and that of every odd function is symmetric about the origin.

For example graph of $y = x^2$ is symmetric about y-axis, while graph of $y = x^3$ is symmetric about origin



$y = x^2$



$y = x^3$

(b) All functions (whose domain is symmetrical about origin) can be expressed as the sum of an even and an odd function, as follows

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

(c) The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

(d) If f and g both are even or both are odd, then the function f.g will be even but if any one of them is odd and the other even then f.g will be odd.

(e) If f(x) is even then $f'(x)$ is odd while derivative of odd function is even. Note that same cannot be said for integral of functions.

Example#14 : Show that $\log\left(x+\sqrt{x^2+1}\right)$ is an odd function.

Solution : Let $f(x) = \log\left(x+\sqrt{x^2+1}\right)$.

Then $f(-x) = \log\left(-x+\sqrt{(-x)^2+1}\right)$

$$= \log\left(\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}\right) = \log\frac{1}{\sqrt{x^2+1}+x} = -\log\left(x+\sqrt{x^2+1}\right) = -f(x)$$

or $f(x) + f(-x) = 0$
Hence f(x) is an odd function.

Example#15 : Show that $a^x + a^{-x}$ is an even function.
Solution : Let $f(x) = a^x + a^{-x}$
Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$.
Hence f(x) is an even function

Example#16 : Show that $\cos^{-1} x$ is neither odd nor even.
Solution : Let $f(x) = \cos^{-1}x$. Then $f(-x) = \cos^{-1}(-x) = \pi - \cos^{-1} x$ which is neither equal to f(x) nor equal to $-f(x)$.
Hence $\cos^{-1} x$ is neither odd nor even

Self practice problems

(11) Determine whether the following functions are even or odd?

(i) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ (ii) $\log\left(\sqrt{x^2+1}-x\right)$

(iii) $x\log\left(x+\sqrt{x^2+1}\right)$ (iv) $\sin^{-1} 2x\sqrt{1-x^2}$

Answers (i) Odd (ii) Odd
(iii) Even (iv) Odd

Even extension / Odd extension :

Let f(x) be defined in [a, b] where $ab \ge 0$. Even extension of this function implies to define the function in [–b, –a] to make it even. In order to get even extension replace x by –x in the given definition. Similarly, odd extension implies to define the function in [–b, –a] to make it odd. In order to get odd extension, multiply the definition of even extension by –1

Example #17 : What is even and odd extensions of $f(x) = x^3 - 6x^2 + 5x - 11$, $x > 0$
Solution : Even extension of f(x) :
$f(-x) = -x^3 - 6x^2 - 5x - 11 \quad ; x < 0$
Odd extension of f(x) :
$-f(-x) = x^3 + 6x^2 + 5x + 11 \quad ; x < 0$

Periodic Functions :

A function f(x) is called periodic with a period T if there exists a real number T > 0 such that for each x in the domain of f the numbers x – T and x + T are also in the domain of f and f(x) = f(x + T) for all x in the domain of f(x). Graph of a periodic function with period T is repeated after every interval of 'T'.

e.g. The function sin x and cos x both are periodic over 2π and tan x is periodic over π.

The least positive period is called the principal or fundamental period of f(x) or simply the period of the function.

Note : Inverse of a periodic function does not exist.

Properties of Periodic Functions :

(a) If f(x) has a period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(b) If f(x) has a period T, then f (ax + b) has a period $\frac{T}{|a|}$.

(c) Every constant function defined for all real x, is always periodic, with no fundamental period.

(d) If f (x) has a period T_1 and g (x) also has a period T_2 then period of f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is L.C.M. of T_1 and T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is nonperiodic.

L.C.M. of $\left(\frac{a}{b}, \frac{p}{q}, \frac{\ell}{m}\right) = \frac{\text{L.C.M. } (a, p, \ell)}{\text{H.C.F. } (b, q, m)}$

e.g. |sinx| has the period π, | cosx | also has the period π

$\therefore$ |sinx| + |cosx| also has a period π. But the fundamental period of |sinx| + |cosx| is $\frac{\pi}{2}$.

(e) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods.

Example #18 : Find period of the following functions

(i) $f(x) = \sin\frac{x}{2} + \cos\frac{x}{3}$

(ii) f(x) = {x} + sin x, where {.}denotes fractional part function

(iii) f(x) = cos x . cos 3x

(iv) $f(x) = \sin\frac{3x}{2} - \cos\frac{x}{3} - \tan\frac{2x}{3}$

Solution : **(i)** Period of $\sin\frac{x}{2}$ is 4π while period of $\cos\frac{x}{3}$ is 6π . Hence period of $\sin\frac{x}{2} + \cos\frac{x}{3}$ is 12π

{L.C.M. of 4 and 6 is 12}

(ii) Period of sin x $= 2\pi$

Period of {x} = 1

but L.C.M. of 2π and 1 is not possible as their ratio is irrational number

$\therefore$ it is aperiodic

(iii) f(x) = cos x . cos 3x

period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be fundamental periodic, but fundamental period = $\frac{2\pi}{n}$, where $n \in N$. Hence cross-checking for n = 1, 2, 3, ...we find π to be fundamental period $f(\pi + x) = (-\cos x)(-\cos 3x) = f(x)$

(iv) Period of f(x) is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{2/3}$ = L.C.M. of $\frac{4\pi}{3}, 6\pi, \frac{3\pi}{2}$ = 12π

Self practice problems :

(12) Find the period of following function.

(i) $f(x) = \sin x + |\sin x|$ (ii) $f(x) = \sqrt{3}\cos x - \sin\frac{x}{3}$

(iii) $\sin\frac{2x}{5} - \cos\frac{3x}{7}$ (iv) $f(x) = \sin^2 x + \cos^4 x$

Answers : (i) 2π (ii) 6π (iii) 70π (iv) π

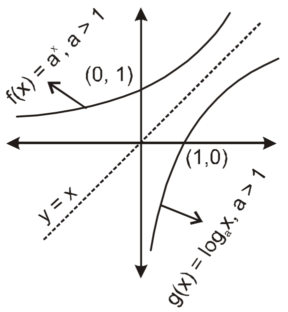
Inverse of a Function :

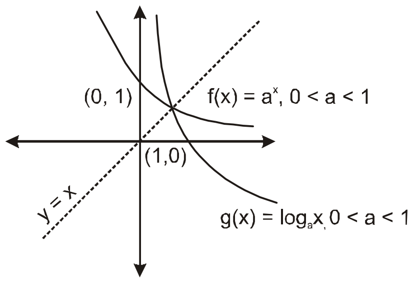
Let $y = f(x) : A \to B$ be a one-one and onto function. i.e. bijection, then there will always exist bijective function $x = g(y) : B \to A$ such that if (p, q) is an element of f, (q, p) will be an element of g and the functions f(x) and g(x) are said to be inverse of each other. g(x) is also denoted by $f^{-1}(x)$ and f(x) is denoted by $g^{-1}(x)$

Note : (i) The inverse of a bijection is unique.
(ii) Inverse of an even function is not defined.

Properties of Inverse Function :

(a) The graphs of f and g are the mirror images of each other in the line y = x. For example $f(x) = a^x$ and $g(x) = \log_a x$ are inverse of each other, and their graphs are mirror images of each other on the line y = x as shown below.





(b) Normally points of intersection of f and f^{-1} lie on the straight line y = x. However it must be noted that f(x) and $f^{-1}(x)$ may intersect otherwise also. e.g f(x) = 1/x

(c) In general fog(x) and gof(x) are not equal. But if f and g are inverse of each other, then gof = fog. fog(x) and gof(x) can be equal even if f and g are not inverse of each other. e.g. f(x) = x + 1, g(x) = x + 2. However if fog(x) = gof(x) = x, then $g(x) = f^{-1}(x)$

(d) If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1} \circ g^{-1}$.

(e) If f(x) and g(x) are inverse function of each other, then $f'(g(x)) = \frac{1}{g'(x)}$

Example # 19 : (i) Determine whether $f(x) = \frac{2x+3}{4}$ for $f : R \to R$, is invertible or not? If so find it.

(ii) Is the function $f(x) = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ invertible?

(iii) Let $f(x) = x^2 + 2x$; $x \geq -1$. Draw graph of $f^{-1}(x)$ also find the number of solutions of the equation, $f(x) = f^{-1}(x)$

(iv) If $y = f(x) = x^2 - 3x + 1$, $x \geq 2$. Find the value of $g'(1)$ where g is inverse of f

Solution : (i) Given function is one-one and onto, therefore it is invertible.

$$y = \frac{2x+3}{4}$$

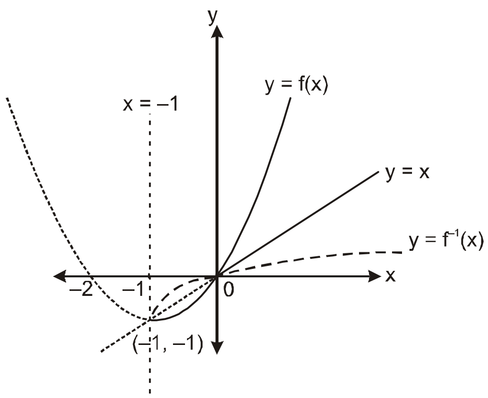
$$\Rightarrow \quad x = \frac{4y-3}{2} \qquad \therefore \qquad f^{-1}(x) = \frac{4x-3}{2}$$

(ii) Domain of f is [–1, 1]

$f(0) = 0 = f(1)$

$\Rightarrow$ f is not one – one $\Rightarrow$ f is not invertible

(iii)



$f(x) = f^{-1}(x)$ is equivalent to $f(x) = x$

$\Rightarrow \quad x^2 + 2x = x \quad \Rightarrow \quad x(x + 1) = 0 \quad \Rightarrow \quad x = 0, -1$

Hence two solution for $f(x) = f^{-1}(x)$

(iv) $y = 1 \quad \Rightarrow \quad x^2 - 3x + 1 = 1$

$\Rightarrow \quad x(x - 3) = 0 \quad \Rightarrow \quad x = 0, 3$

But $x \geq 2 \qquad \therefore \qquad x = 3$

Now $g(f(x)) = x$

Differentiating both sides w.r.t. x

$$\Rightarrow \quad g'(f(x)).\ f'(x) = 1 \qquad \Rightarrow \qquad g'(f(x)) = \frac{1}{f'(x)}$$

$$\Rightarrow \quad g'(f(3)) = \frac{1}{f'(3)} \qquad \Rightarrow \qquad g'(1) = = \frac{1}{6-3} = \frac{1}{3} \quad (\text{As } f'(x) = 2x - 3)$$

Alternate Method

$y = x^2 - 3x + 1$

$x^2 - 3x + 1 - y = 0$

$$x = \frac{3 \pm \sqrt{9 - 4(1-y)}}{2}$$

$$= \frac{3 \pm \sqrt{5+4y}}{2}$$

$x \geq 2$

$$x = \frac{3 + \sqrt{5+4y}}{2}$$

$$g(x) = \frac{3 + \sqrt{5+4x}}{2}$$

$$g'(x) = 0 + \frac{1}{4\sqrt{5+4x}}\ 4$$

$$g'(1) = \frac{1}{\sqrt{5+4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

Self practice problems :

(13) Determine $f^{-1}(x)$, if given function is invertible

(i) $f : (-\infty, -1) \rightarrow (-\infty, -2)$ defined by $f(x) = -(x + 1)^2 - 2$

(ii) $f : \left[\frac{\pi}{6}, \frac{7\pi}{6}\right] \rightarrow [-1, 1]$ defined by $f(x) = \sin\left(x + \frac{\pi}{3}\right)$

Answers : (i) $-1 - \sqrt{-x-2}$ (ii) $\frac{2\pi}{3} + \sin^{-1}x$

General tips :

If x, y are independent variables, then:

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x).\ f(y) \Rightarrow f(x) = x^n,\ n \in R$

(iii) $f(x + y) = f(x).\ f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

Example # 20 : If f(x) is a polynomial function satisfying $f(x) . f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and $f(2) = 9$, then find f (3)

Solution : $f(x) = 1 \pm x^n$

As $f(2) = 9 \quad \therefore \quad f(x) = 1 + x^3$

Hence $f(3) = 1 + 3^3 = 28$

Self practice problems

(14) If f(x) is a polynomial function satisfying $f(x) . f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and $f(3) = -8$, then find f(4)

(15) If $f(x + y) = f(x) . f(y)$ for all real x, y and $f(0) \neq 0$, then prove that the function, $g(x) = \frac{f(x)}{1+f^2(x)}$ is an even function.

Answer : (14) -15

EXERCISE-I

Practice on Domain:

Discuss common domains of all trigonometric; inverse trigonometric; exponential; logarithmic; polynomials; rational functions along with special functions like [x]; {x} ; sgn x & |x| together with their graphs.

(i) $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$ (ii) $f(x) = \sqrt{\log_2\left(\dfrac{5x - x^2}{4}\right)}$ or $\sqrt{\log_{\frac{1}{2}} \dfrac{5x - x^2}{4}}$

(iii) $f(x) = \cos^{-1}\left(\dfrac{2-|x|}{4}\right) + (\ln(3 - x))^{-1}$ [Ans. $[-6, 2] \cup (2, 3)$;

(iv) $f(x) = \cos^{-1}[x]$ (domain & range) [Ans. $-1 \le x < 2$, $\{\pi, \pi/2, 0\}$]

(v) $f(x) = \dfrac{1}{\sqrt{|x|-x}}$ $(x < 0)$; $h(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$

$g(x) = \dfrac{1}{\sqrt{x-|x|}}$ [Ans: ϕ] ; $l(x) = \log_{2\{x\}-3}(x^2 - 5x + 13)$ [Ans: ϕ]

(vi) $f(x) = \sqrt{\log_x(\cos 2\pi x)}$ [Ans. $(0, \dfrac{1}{4}) \cup (\dfrac{3}{4}, 1) \cup \{2, 3, 4, ... \infty\}$] **[T/S]**

(vii) $f(x) = \sqrt{(x^2 - 3x - 10)\ln^2(x-3)}$; $\{4\} \cup [5, \infty)$ **[T/S]**

(viii) $f(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{x^2 + t^2}} = \ln\left[\frac{x + \sqrt{2}|x|}{|x|}\right] = \left[\begin{array}{c} \ln(\sqrt{2}+1) \text{ if } x > 0 \\ \text{not defined at } x = 0 \\ \ln(\sqrt{2}-1) \text{ if } x<0 \end{array}\right.$

[Ans. Domain is R– {0} ; Range is consists of two $\{\ln(\sqrt{2}-1), \ln(\sqrt{2}+1)\}$; $\ln\left[t + \sqrt{x^2+t^2}\right]_0^x$]

Note : $f(0) = \displaystyle\int_0^0 \frac{dt}{|t|}$, as $\dfrac{1}{|t|}$ is not defined at x = 0 hence f (0) is not defined.

(x) $f(x) = \left(\log_{\frac{x-2}{x+3}} 2\right) + \sqrt{9 - x^2}$ {Ans. (2, 3] }

(xi) $f(x) = \dfrac{1}{[x]} + \log_{1-\{x\}}(x^2 - 3x + 10) + \dfrac{1}{\sqrt{2-|x|}} + \dfrac{1}{\sqrt{\sec(\sin x)}}$ **[T/S]**

[Ans: $(-2, -1) \cup (-1, 0) \cup (1, 2)$]

(xii) $f(x) = \dfrac{1}{\sqrt{\sin(\cos x)}} + \sin^{-1}\left(\dfrac{2x}{\pi}\right) + \dfrac{1}{\{-x\}} + \dfrac{1}{\ell n\left(1 - \left[\tan\dfrac{x}{2}\right] - \left[-\tan\dfrac{x}{2}\right]\right)}$

[Hint: Start with domain of $\sin^{-1}\left(\dfrac{2x}{\pi}\right)$

[Ans : $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) - \{-1, 0, 1\}$ Note that $[x] + [-x] = \left[\begin{array}{l} 0 \text{ if } x \in I \\ -1 \text{ if } x \notin I \end{array}\right.$]

SOME MORE PRACTICE PROBLEMS ON DOMAIN AND RANGE :

(1) $f(x) = {}^{x+1}C_{2x-8}$ and $g(x) = {}^{2x-8}C_{x+1}$ and $h(x) = f(x) \cdot g(x)$ then domain of $h(x) = \{9\}$ and range of $h(x) = \{1\}$.

(2) Domain of function: $f(x) = \sqrt[5]{x} + \sqrt[3]{x}$ $(x \in R)$; $\sqrt[x]{2}$; $(x \geq 2, x \in N)$

(3) $f(x) = \dfrac{3x + |x|}{x} = \begin{cases} 4 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ 2 & \text{if } x < 0 \end{cases}$ [Domain $R - \{0\}$; range $\{2, 4\}$]

(4) The domain of the function $f(x) = \max. \{\sin x, \cos x\}$ is $(-\infty, \infty)$. The range of $f(x)$ is

(A*) $\left[-\dfrac{1}{\sqrt{2}}, 1\right]$ (B) $\left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$ (C) $[0, 1]$ (D) $(-1, 1]$

[Hint: Plot the graph of $y = \sin x$ and $y = \cos x$ in $[0, 2\pi]$ and interpret the range of $f(x)$]

SOME PROBLEMS ON RANGE

(i) $f(x) = 4 \tan x \cos x$; Ans. $(-4, 4)$ (ii) $f(x) = \cos^4 \dfrac{x}{5} - \sin^4 \dfrac{x}{5}$; Ans. $[-1, 1]$

(iii) $f(x) = \sin\sqrt{x}$; Ans. $[-1, 1]$ (iv) $f(x) = \cos(2\sin x)$; Ans. $[\cos 2, 1]$

(v) $f(x) = \sin(\log_2 x)$; Ans. $[-1, 1]$ (vi) $f(x) = 3 - 2^x$; Ans. range $(-\infty, 3]$

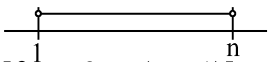
(vii) $f(x) = \dfrac{3x + |x|}{x}$; Ans. D : $R - \{0\}$; R : $\{2, 4\}$

(viii) Let n be a positive integer. If the number of integers in the domain of the function $f(x) = ln\left((1-x)(x-n)\right)$ is $2n - 11$, then the value of n is

(A) 8 (B*) 9 (C) 10 (D) 11

[Sol. $(1 - x)(x - n) > 0$

or $(x - 1)(x - n) < 0$



hence number of integers $= (n - 2)$ [from 2 to $(n - 1)$]

$\therefore$ $n - 2 = 2n - 11$

$\therefore$ $n = 9$ Ans.]

(ix) $f(x) = \cos 2x - \sin 2x$; Ans. $[-\sqrt{2}, \sqrt{2}]$

(x) $f(x) = \cot^2\left(x - \dfrac{\pi}{4}\right)$ Ans. $[0, \infty)$

(xi) $f(x) = ln(5x^2 - 8x + 4) = ln\left(5\left\{\left(x - \dfrac{4}{5}\right)^2 + \dfrac{4}{25}\right\}\right) = \left[ln\,\dfrac{4}{5}, \infty\right)$ [Domain: $x \in R$]

(xii) $f(x) = \log_2(2 - \log_{\sqrt{2}}(16\sin^2 x + 1))$; $(-\infty, 1]$

Note that f is even and decreasing. Also $-\dfrac{1}{4} < \sin x < \dfrac{1}{4}$.

(xiii) Find the range of the function $f(x) = \log_2\left(4^{x^2} + 4^{(x-1)^2}\right)$.

[Sol. Consider $4^{x^2} + 4^{(x-1)^2}$

AM $\geq$ GM for two positive numbers 4^{x^2} and $4^{(x-1)^2}$

$$\frac{4^{x^2}+4^{(x-1)^2}}{2} \ge \left[4^{x^2}\cdot 4^{(x-1)^2}\right]^{1/2} = 2^{x^2}\cdot 2^{(x-1)^2} = 2^{x^2+(x-1)^2};\ 4^{x^2}+4^{(x-1)^2} \ge 2^{x^2+(x-1)^2+1}$$

now $z = x^2 + (x-1)^2 + 1 \Rightarrow 2x^2 - 2x + 2 = 2[x^2 - x + 1] = 2\left[\left(x-\frac{1}{x}\right)^2 + \frac{3}{4}\right]$

$\therefore$ $z_{max} \to \infty$ hence $z_{min} = \frac{3}{2}$

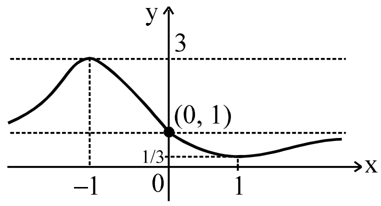
$\therefore$ $4^{x^2}+4^{(x-1)^2}$ has the minimum value $= 2^{3/2}$

hence $f(x) \ge \log_2 (2)^{3/2} = \frac{3}{2}$

$\therefore$ $y \ge \frac{3}{2} \Rightarrow$ range is $\left[\frac{3}{2}, \infty\right)$ **Ans.**]

(xiv) $f(x) = \frac{x^2-x+1}{x^2+x+1}$; $f'(x) = \frac{2(x^2-1)}{(x^2+x+1)^2} = f(1) = \frac{1}{3}$; $f(-1)=3$

D : $x \in R$, Range $\left[\frac{1}{3}, 3\right]$



fig(xiii)

[Note: Graph of $f(x) = \frac{ax+b}{cx+d}$ is always monotonic.]

(xv) $f(x) = \frac{x}{\ell n x}$ [Range : $(-\infty, 0) \cup [e, \infty)$]

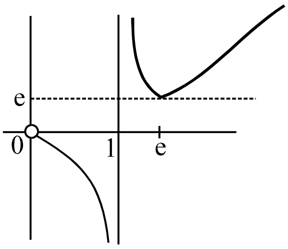


fig (xiv)

(xvi) $f(x) = \sin^{-1}x^2 + \left[\left\{\ell n \sqrt{x-[x]}\right\}\right] + \cot^{-1}\left(\frac{1}{1+\sqrt{2}x^2}\right)$

[Ans. range : $\left(\frac{\pi}{4}, \frac{7\pi}{8}\right)$; Domain is $(-1,1) - \{0\}$]

[Hint: Domain of function is $(-1, 1) - \{0\}$. because $x - [x] = 0$ for integral value of x, hence middle term will not be defined.

Also $[\{f\}] = 0$, whenever f is meaningful.

$\therefore$ value of $f(x) = \sin^{-1} x^2 + \tan^{-1} (1 + \sqrt{2}\, x^2)$.

$\left(\begin{array}{l}\cot^{-1} x = \tan^{-1}\frac{1}{x} \\ \text{when } x > 0\end{array}\right)$

Function is continuous and is even.

Least value of the function will occur when $x \to 0$ and is $\pi/4$.

$\therefore$ maximum value $= \lim_{x\to\pm1} f(x) = \sin^{-1} 1 + \tan^{-1} (1 + \sqrt{2}) = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}$

$\therefore$ Range of $f(x)$ is $\left(\frac{\pi}{4}, \frac{7\pi}{8}\right)$]

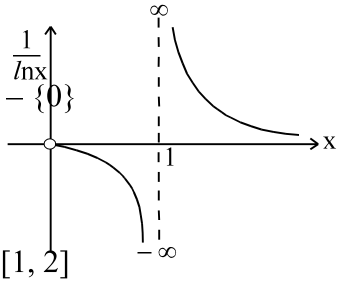
(xvii) $f(x) = \tan^{-1}\left(\frac{2x}{1+x^2}\right)$; range $= \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ **[T/S]**

(xviii) Find the range of the function $f(x) = |x-1| + |x-2| + |x-8| + |x-9|$.

[Ans. $[14, \infty)$]

[Hint: least value is 14 which occurs when $x \in [2, 8]$]

Examples on Domain & Range



(i) $f(x) = \dfrac{1}{l\text{n}\,x}$; Domain : $(0, 1) \cup (1, \infty)$; Range : $R - \{0\}$

(ii) $f(x) = \dfrac{1}{\sin^4 x + \cos^4 x}$ D : $x \in R$; R : [1, 2]

(iii) $f(x) = \log_2\left(\dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$ D: $x \in R$; R : [1, 2]

(iv) Suppose the domain of the function $y = f(x)$ is $-1 \le x \le 4$ and the range is $1 \le y \le 10$. Let $g(x) = 4 - 3f(x-2)$. If the domain of $g(x)$ is $a \le x \le b$ and the range of $g(x)$ is $c \le y \le d$ then which of the following relations hold good?

(A) $2a + 4b + c + d = 0$ (B*) $a + b + d = 8$

(C) $5b + c + d = 4$ (D*) $a + b + c + d + 18 = 0$

[Sol. for domain of g (x) is the set of x for which

$-1 \le x - 2 \le 4 \quad \Rightarrow \quad 1 \le x \le 6$

hence $a = 1$ and $b = 6$

for range, $1 \le y \le 10$

$\Rightarrow \quad 1 \le f(x) \le 10$

$\Rightarrow \quad 1 \le f(x-2) \le 10$

$\Rightarrow \quad 3 \le 3f(x-2) \le 30$

hence $-30 \le -3f(x-2) \le -3$

$\therefore \quad -26 \le 4 - 3f(x-2) \le 1$

hence $c = -26$ and $d = 1$

now verify]

(v) $f(x) = \dfrac{(x+2)(x-1)}{x(x+1)}$; $f'(x) = \dfrac{2(2x+1)}{[x(x+1)]^2}$

(vi) $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 2x - 3} = \dfrac{(x-4)(x-1)}{(x+3)(x-1)}$

[Ans: (vi) $D = R - \{1, -3\}$; Range $= R - \left\{-\dfrac{3}{4}, 1\right\}$; $f'(x) = \dfrac{7}{(x+3)^2}$]

fig (v)

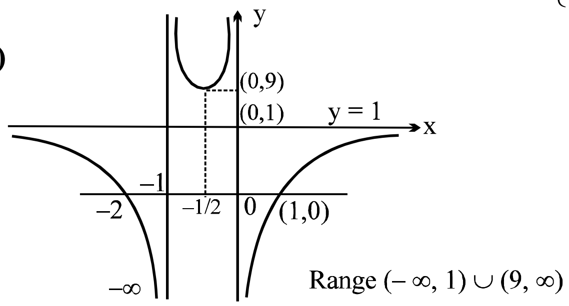
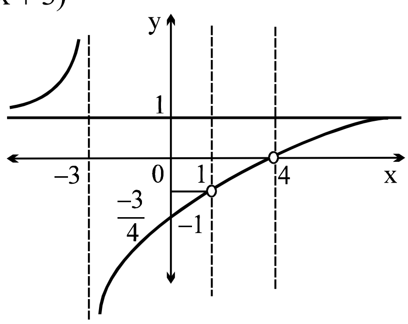


fig. (vi)



General tips for plotting the graph of a rational function :

(1) Examine whether denominator has a root or not. If no, then graph is continuous and f is non monotonic.

For eg. $f(x) = \dfrac{x}{x^2 - 5x + 9}$.

If denominator has roots then f (x) is discontinuous. Such functions can be monotonic / non monotonic.

For e.g. $f(x) = \dfrac{x^2 + 2x - 3}{x^2 + 2x - 8} = \dfrac{(x+3)(x-1)}{(x+4)(x-2)}$

(2) If numerator and denominator has a common factor (say x = a) it would mean removable discontinuity at x = a e.g. $f(x) = \frac{(x-2)(x-1)}{(x+3)\,(x-2)}$. Such a function will always be monotonic i.e. either increasing or decreasing and removable discontinuity at x = 2 .

(3) Compute points where the curve crosses the x-axis and also where it cuts the y-axis by putting y = 0 and x = 0 respectively and mark points accordingly.

(4) Compute $\frac{dy}{dx}$ and find the intervals where f (x) is increasing or decreasing and also where it has horizontal tangent.

(5) In regions where curve is monotonic compute y if $x \to \infty$ or $x \to -\infty$ to find whether y is asymptotic or not.

(6) If denominator vanishes say at x = a and (x – a) is not a common factor between numerator and denominator then examine $\underset{x\to a^-}{\text{Limit}}$ and $\underset{x\to a^+}{\text{Limit}}$ to find whether f approaches ∞ or $-\infty$. Now plot the graphs of the following functions

Home Work :

(1)	$f(x) = \frac{x+2}{2x^2+3x+6}$	(2)	$f(x) = \frac{x^2+2x-11}{2(x-3)}$	(3)	$f(x) = \frac{x^2+2x-3}{x^2+2x-8}$
(4)	$f(x) = \frac{x^2-3x+2}{x^2+x-6}$	(5)	$f(x) = \frac{x^2-x+1}{x^2+x+1}$	(6)	$f(x) = \frac{x^2-1}{x^2-3x}$

EXPLANATION:

1. $y = \frac{x+2}{2x^2+3x+6}$

[Hint: $y' = \frac{(2x^2+3x+6)-(x+2)(4x+3)}{(D^r)^2} \quad = \frac{(2x^2+3x+6)-(4x^2+11x++6)}{(D^r)^2}$

$$= \frac{-2x^2-8x}{(D^r)^2} = \frac{-2x(x+4)}{(D^r)^2}$$

$f'(x) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = -4 \quad \Rightarrow \quad f(0) = \frac{1}{3};\ f(-4) = \frac{-2}{32-12+6} = -\frac{1}{13}$]

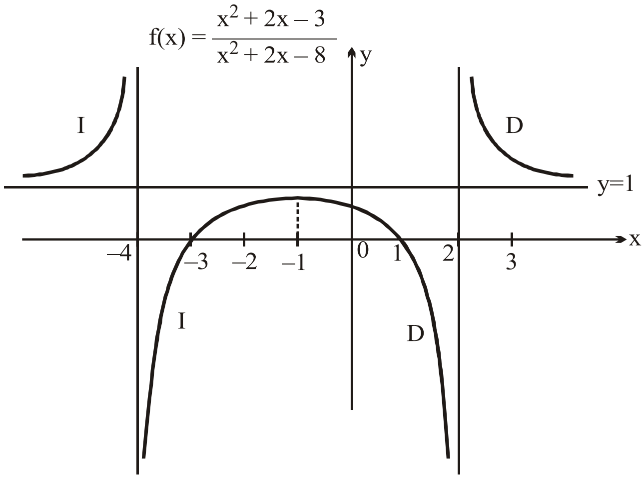
3. $y = \frac{x^2+2x-3}{x^2+2x-8} = \frac{(x+3)(x-1)}{(x+4)(x-2)}$

[Hint: $y = 1 + \frac{5}{(x^2+2x-8)}$

$y' = -\frac{5(2x+2)}{(x^2+2x-8)^2}$; $y'=0$ if $x = -1$

$x = -1 \quad f(-1) = 4/9$

$y' = \frac{-10(x+1)}{(x^2+2x-8)^2}$]

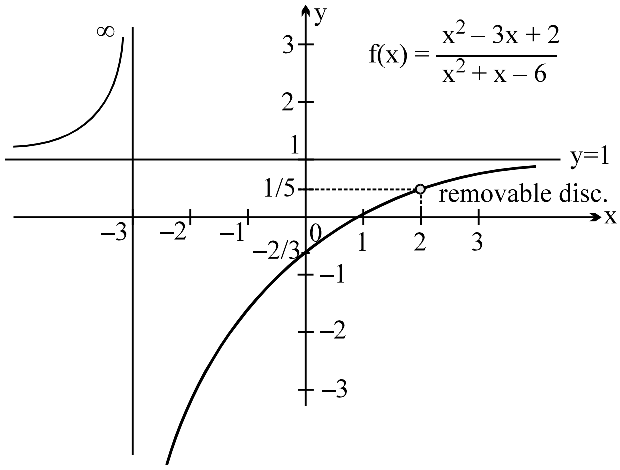


4. $y = \dfrac{(x-2)(x-1)}{(x+3)(x-2)}$

[Hint: $y = \dfrac{x+1}{x+3}$

$y' = \dfrac{(x+3)-(x-1)}{(x+3)^2} = \dfrac{4}{(x+3)^2}$

as $x \to 2$, $y \to 1/5$]



PRACTICE PROBLEMS ON IDENTICAL AND NON IDENTICAL FUNCTION:

(i) $f(x) = ln\, x^2$; $g(x) = 2ln\, x$ (N.I.) $\quad$ $f(x) = \text{cosec}x$; $g(x) = \dfrac{1}{\sin x}$ (I)

(ii) $f(x) = \cot(\cot^{-1}x)$; $g(x) = x$ (I) $\quad$ $f(x) = \tan x$; $g(x) = \dfrac{1}{\cot x}$ (NI)

$f(x) = ln\, e^x$; $g(x) = e^{ln\, x}$ (N.I.) $\quad$ $f(x) = \sec x$; $g(x) = \dfrac{1}{\cos x}$ (I)

(iii) $f(x) = \sin^{-1}(3x - 4x^3)$; $g(x) = 3\sin^{-1}x$ (N.I.)

(iv) $f(x) = \sec^{-1}x + \text{cosec}^{-1}x$; $g(x) = \dfrac{\pi}{2}$ (N.I.)

(v) $f(x) = \cot^2x.\cos^2x$; $g(x) = \cot^2x - \cos^2x$ (I)

(vi) $f(x) = \text{Sgn}\,(x^2 + 1)$; $g(x) = \sin^2x + \cos^2x$ (I)

(vii) $f(x) = \tan^2x.\sin^2x$; $g(x) = \tan^2x - \sin^2x$ (I)

(viii) $f(x) = \sec^2x - \tan^2x$; $g(x) = 1$ (N.I.)

(ix) $f(x) = \log_x e$; $g(x) = \dfrac{1}{\log_e x}$ (I) $\quad$ $f(x)=\text{sgn}(\cot^{-1}x)$; $g(x)=\text{sgn}(x^2 - 4x + 5)$ (I)

$f(x) = \log_e x$; $g(x) = \dfrac{1}{\log_x e}$ (N.I.)

(x) $f(x) = \tan(\cot^{-1}x)$; $g(x) = \cot(\tan^{-1}x)$ (I)

(xi) $f(x) = \sqrt{x^2 - 1}$; $g(x) = \sqrt{x-1} \cdot \sqrt{x+1}$ (N.I.)

$f(x) = \sqrt{1-x^2}$; $g(x) = \sqrt{1-x} \cdot \sqrt{1+x}$ (I)

(xii) $f(x) = \tan x \cdot \cot x$; $g(x) = \sin x \cdot \text{cosec}x$ (N.I.)

(xiii) $f(x) = e^{ln\, e^x}$; $g(x) = e^x$ (I)

(xiv) $f(x) = \sqrt{\dfrac{1-\cos 2x}{2}}$; $g(x) = \sin x$ (N.I.)

(xv) $f(x) = \sqrt{x^2}$; $g(x) = (\sqrt{x}\,)^2$ (N.I.)

(xvi) $f(x) = \log(x + 2) + \log(x - 3)$; $g(x) = \log(x^2 - x - 6)$ (N.I.)

(xvii) $f(x) = \frac{1}{|x|}$; $g(x) = \sqrt{x^{-2}}$ (I)

(xviii) $f(x) = x|x|$; $g(x) = x^2 \text{sgn } x$ (I)

(xix) $f(x) = \lim\limits_{n\to\infty} \text{it} \frac{x^{2n}-1}{x^{2n}+1}$; $g(x) = \text{sgn}(|x| - 1)$ (I)

(xx) $f(x) = \sin(\sin^{-1} x)$; $g(x) = \cos(\cos^{-1}x)$ (I)

(xxi) $f(x) = \frac{1}{1+\frac{1}{x}}$; $g(x) = \frac{x}{1+x}$ (N.I.)

(xxii) $f(x) = [\{x\}]$; $g(x) = \{[x]\}$ (I) (note that f(x) and g(x) are constant functions)

(xxiii) $f(x) = e^{\ln \cot^{-1} x}$; $g(x) = \cot^{-1}x$ (I)

(xxiv) $f(x) = e^{\ln \sec^{-1} x}$; $g(x) = \sec^{-1}x$ (N.I.) Identical if $x \in (-\infty, -1] \cup (1, \infty)$

(xxv) $F(x) = (fog)(x)$; $G(x) = (gof)(x)$ where $f(x) = e^x$; $g(x) = \ln x$ (N.I.)

Examples on classification

(1)

(a) $f(x) = e^x + e^{-x}$ (b) $f(x) = \sqrt{1+x^2}$ (c) $f(x) = x^3$

(d) $f(x) = |x| \text{ Sgn } x$

(e) f: $[-1, 1] \to [-1, 1]$ $f(x) = \sin 2x$ is many one onto (draw graphs)

(f)$_{86/func}$ The function $f : [2, \infty) \to Y$ defined by $f(x) = x^2 - 4x + 5$ is both one-one and onto if :

(A) Y = R (B*) $Y = [1, \infty)$ (C) $Y = [4, \infty)$ (D) $[5, \infty)$

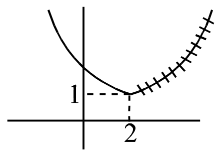
[Hint : $f : [2, \infty) \to y$

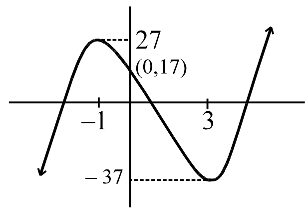
$f(x) = x^2 - 4x + 5$

$= (x - 2)^2 + 1$

If the function is onto

$Y \to [1, \infty) \to$ B]





(2)(a) $f(x) = x^3 - 2x^2 + 5x + 3$ is one-one-onto (as $f'(x) > 0$)

(b) $f : R \to R$ $f(x) = 2x^3 - 6x^2 - 18x + 17$

many one onto function

(3) If the function $f(x) = x^2 + bx + 3$ is not injective for values of x in the interval $0 \le x \le 1$ then b lies in

(A) $(-\infty, \infty)$ (B) $(-2, \infty)$ (C*) $(-2, 0)$ (D) $(-\infty, 2)$

[Sol. For many one in [0, 1]

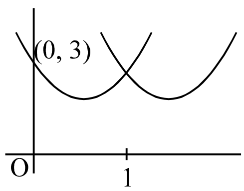
0 < vertex < 1

$0 < -\frac{b}{2} < 1$

$0 < -b < 2$

$-2 < b < 0$

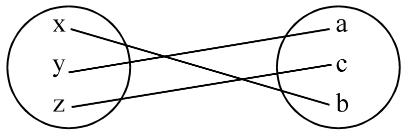
Note: If f is injective then $b \in R - (-2, 0)$]



(4) Let f: $\{x, y, z\} \to \{a, b, c\}$ be a one–one function.

It is known that only one of these statements is true and the remaining two are false.

(i) $f(x) \ne b$ (ii) $f(y) = b$; (iii) $f(z) \ne a$ find $f(x)$ / $f^{-1}(x)$



(5) $f: R \to R$ $f(x) = \dfrac{2x^2 - x + 5}{7x^2 + 2x + 10}$ is many one into

(Note denominator & numerator both +ve $\Rightarrow$ f(x) is always +ve)

(6) $f: R \to R$ is defined as $f(x) = \begin{cases} x^2 + 2mx - 1 & \text{for } x \le 0 \\ mx - 1 & \text{for } x > 0 \end{cases}$.

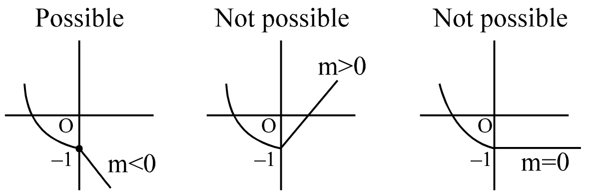
If $f(x)$ is one-one then m must lies in the interval

(A*) $(-\infty, 0)$ (B) $(-\infty, 0]$ (C) $(0, \infty)$ (D) $[0, \infty)$

[Sol. for f to be one-one $f'(x) > 0$ or $f'(x) < 0$ for all x

clearly f is continuous at x = 0 and f (0) = – 1

possible cases can be as shown



Note that for m < 0 the vertex of $f(x) = x^2 + 2mx - 1$ lie on + sides of x-axis]

ADVANCE PROBLEMS :

Let a function f defined from $R \to R$ as: $f(x) = \begin{cases} x + m & \text{for } x \le 1 \\ 2mx - 1 & \text{for } x > 1 \end{cases}$

If the function is surjective on R then *m* must lie in the interval

(A*) (0, 2] (B) $(-\infty, 0]$ (C) $(-\infty, 0)$ (D) $(0, \infty)$

[Sol. for f to be surjective range must be set of all real numbers

as shown only figure-2 is possible

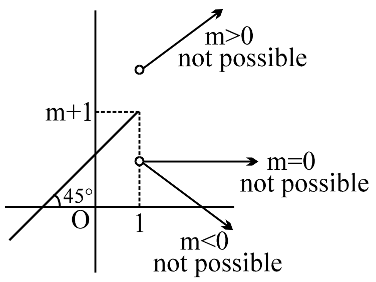


Figure-1

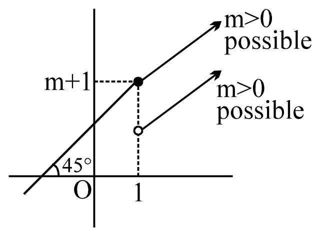


Figure-2

Hence $m + 1 \ge 2m - 1$

$2 \ge m \Rightarrow 0 < m \le 2$]

FUNCTIONAL EQUATIONS:

For $x \in R$, the function $f(x)$ satisfies $2f(x) + f(1 - x) = x^2$ then the value of $f(4)$ is equal to

(A) $\frac{13}{3}$ (B) $\frac{43}{3}$ (C*) $\frac{23}{3}$ (D) none

[Sol. $2f(x) + f(1-x) = x^2$ ....(1)

$x \to 1-x$ $f(x) + 2f(1-x) = (1-x)^2$ ....(2)

multiply (1) by (2) $4f(x) + 2f(1-x) = 2x^2$ ....(3)

(3) – (2)

$3f(x) = 2x^2 - (1-x)^2$

$3f(4) = 32 - 9 = 23$

$f(4) = 23/3$ Ans.]

(1) Given a function f (x) satisfying $f(x) + 2f\left(\frac{1}{1-x}\right) = x$. Find f (2). [Ans. 2/3]

[Sol. Given $f(x) + 2f\left(\frac{1}{1-x}\right) = x$ ....(1)

$x \to \frac{1}{1-x}$ $f\left(\frac{1}{1-x}\right) + 2f\left(\frac{x-1}{x}\right) = \frac{1}{1-x}$ ....(2)

again $x \to \frac{1}{1-x}$ $f\left(\frac{x-1}{x}\right) + 2f(x) = \frac{x-1}{x}$ ....(3)

from (1) and (2)

$f(x) - 4f\left(\frac{x-1}{x}\right) = x - \frac{2}{1-x}$ ....(4)

multiplying (3) by 4

$8f(x) + 4f\left(\frac{x-1}{x}\right) = \frac{4(x-1)}{x}$ ....(5)

add (4) and (5)

$9f(x) = x - \frac{2}{1-x} + \frac{4(x-1)}{x}$

$9f(2) = 2 + 2 + \frac{4(1)}{2} = 6 \quad \Rightarrow \quad f(2) = \frac{2}{3}$ **Ans.**]

(3) Let f (x) and g (x) be functions which take integers as arguments. Let $f(x+y) = f(x) + g(y) + 8$ for all integer x and y. Let $f(x) = x$ for all negative integers x, and let $g(8) = 17$. The value of $f(0)$ is

(A*) 17 (B) 9 (C) 25 (D) – 17

[Sol. put x = – 8 and y = 8 in

$f(x+y) = f(x) + g(y) + 8$

$f(0) = f(-8) + g(8) + 8 = -8 + 17 + 8 = 17$ (using f (x) = x)]

(4) Let $f(x) = ax^7 + bx^3 + cx - 5$, where a, b and c are constants. If $f(-7) = 7$, then $f(7)$ equals

(A*) –17 (B) –7 (C) 14 (D) 21

[Hint: $f(-x) = -ax^7 - bx^3 - cx - 5$

$\therefore$ $f(x) + f(-x) = -10$; Put x = 7

$f(7) = -10 - f(-x) = -17$]

(5) The function $f : R \to R$ satisfies the condition $m f(x-1) + n f(-x) = 2|x| + 1$. If $f(-2) = 5$ and $f(1) = 1$, then (m + n) equals

(A*) 4/3 (B) 3 (C) 4 (D) 6

[Hint: put x = 2 and x = – 1 and make two simultaneous equations]

PRACTICE PROBLEMS ON COMPOSITE FUNCTION:

(1) If $f(x) = x^2$ and $g(x) = x - 7$ find gof and fog $(x^2 - 7; (x - 7)^2)$

(2) $f : R \to R$ be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $(fof)(\sqrt{2}) = -\sqrt{2}$ then the value of 'a' is

(A) $\sqrt{2}$ (B) 2 (C) $\frac{1}{2}$ (D*) $\frac{1}{\sqrt{2}}$

[Sol. $f(\sqrt{2}) = 2a - \sqrt{2}$

$\therefore f\left(f\left(\sqrt{2}\right)\right) = f\left(2a - \sqrt{2}\right) = -\sqrt{2}$

$= a\left(2a - \sqrt{2}\right)^2 - \sqrt{2} = -\sqrt{2}$

$= a\left(4a^2 + 2 - 4a\sqrt{2}\right) - \sqrt{2} = -\sqrt{2}$

$= a\left(4a^2 - 4\sqrt{2}\,a + 2\right) = 0$

a = 0 (rejected)

or $4a^2 - 4\sqrt{2}\,a + 2 = 0 \Rightarrow \left(2a - \sqrt{2}\right)^2 = 0 \Rightarrow a = \frac{1}{\sqrt{2}}$ Ans.]

(3)(a) Let $f(x) = \sqrt{x}$; $g(x) = \sqrt{2 - x}$, find the domain of

(a) fog $(-\infty, 2]$ (b) gof $[0, 4]$ (c) fof $[0, \infty)$

(d) gog $[-2, 2]$

(b) Suppose that $f(x) = x^x$ and $g(x) = x^{2x}$. Which one of the following represents the composite function f [g (x)], is

(A) $x^{x^{2x+1}}$ (B) $x^{2x^{2x}}$ (C) $x^{2x^{x+1}}$ (D*) $x^{2x^{2x+1}}$

[Sol. $f(x) = x^x$; $g(x) = x^{2x}$

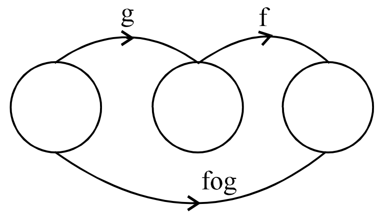
$f[g(x)] = (g(x))^{g(x)} = \left(x^{2x}\right)^{x^{2x}} = x^{2x \cdot x^{2x}} = x^{2x^{2x+1}}$ Ans.]

(4) If $f(x) = \log_{100x}\left(\frac{2\log_{10} x + 2}{-x}\right)$ and $g(x) = \{x\}$. If the function (fog)(x) exists then find the range of g(x). [Ans. $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{10}\right)$]

Hint: (i) $100x > 0$ & $100x \neq 1 \Rightarrow x \neq \frac{1}{100}$

(ii) $x > 0$ and $\log_{10}x + 1 < 0 \Rightarrow 0 < x < \frac{1}{10}$ & $x \neq \frac{1}{100}$

(fog)(x) exists $\Rightarrow$ range of g(x) $\subset$ domain of f (x)



(5) Let $f_1(x) = x$, $f_2(x) = 1 - x$; $f_3(x) = \frac{1}{x}$, $f_4(x) = \frac{1}{1-x}$; $f_5(x) = \frac{x}{x-1}$; $f_6(x) = \frac{x-1}{x}$

Suppose that $f_6(f_m(x)) = f_4(x)$ and $f_n(f_4(x)) = f_3(x)$ then find the value of m & n.

[Ans. m = 6; n = 5]

[Sol. Given $f_6(x) = \frac{x-1}{x}$ ....(1) $\Rightarrow$ $f_6(f_m(x)) = \frac{f_m(x)-1}{f_m(x)}$ ....(2)

but $f_6(f_m(x)) = f_4(x) = \frac{1}{1-x}$ (given)

$\therefore$ $f_6(f_m(x)) = \frac{f_m(x)-1}{f_m(x)} = \frac{1}{1-x}$

put $f_m(x) = k$, $\frac{k-1}{k} = \frac{1}{1-x}$

$k - kx - 1 + x = k$ $\Rightarrow$ $k = \frac{x-1}{x}$ $\Rightarrow$ $f_m(x) = \frac{x-1}{x} = f_6(x)$ $\Rightarrow$ m = 6

again $f_n(f_4(x)) = f_3(x) = \frac{1}{x}$

$f_n\left(\frac{1}{1-x}\right) = \frac{1}{x}$; let $\frac{1}{1-x} = t$ $\Rightarrow$ $t - tx = 1$ $\Rightarrow$ $x = \frac{t-1}{t}$

$\therefore$ $f_n(t) = \frac{t}{t-1}$ $\Rightarrow$ $f_n(x) = \frac{x}{x-1} = f_5(x)$

hence n = 5 Ans.]

(6) $f(x) = ax + b$; $g(x) = bx + a$, $a, b \in N$ if $f(g(50)) - g(f(50)) = 28$. Find ab.

[Ans. 12, 213]

(7) $f : R \to R$, $f(x^2 + x + 3) + 2f(x^2 - 3x + 5) = 6x^2 - 10x + 17$ $\forall\, x \in R$ then find f (x).

[Ans. f (x) = 0 has a root in (0, 2); f (x) is invertible]

[Sol. obviously f is a linear polynomial

let $f(x) = ax + b$ hence $f(x^2 + x + 3) + 2f(x^2 - 3x + 5) \equiv 6x^2 - 10x + 17$

or $[a(x^2 + x + 3) + b] + 2[a(x^2 - 3x + 5) + b] \equiv 6x^2 - 10x + 17$

$\left.\begin{array}{ll} a + 2a = 6 & ....(1) \\ a - 6a = -10 & ....(2) \end{array}\right] \Rightarrow a = 2$ (*comparing coeff. of* x^2 *and coeff. of* x *both sides*)

again, $3a + b + 10a + 2b = 17$ $\Rightarrow$ $6 + b + 20 + 2b = 17$

$26 + 3b = 17$ $\Rightarrow$ $b = -3$

$\therefore$ $f(x) = 2x - 3$ $\Rightarrow$ **(B)** and **(D)**]

PRACTICE PROBLEMS :

(i) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that fog = h.

[Sol. $f[2x + 1] = (2x + 1)^2 + 6$, put $2x + 1 = t$] [**Ans.** $f(x) = x^2 + 6$]

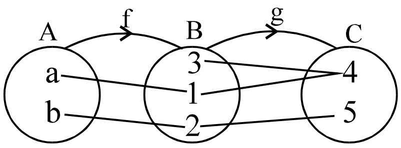
(ii) Let P and Q be polynomials such that P(x) and $Q(P(Q(x)))$ have the same roots. If the degree of P is 7, then the degree of Q, is

(A) 0 (B*) 1 (C) 2 (D) 7

[Hint: Let the degree of Q(x) is n i.e. $Q(x) = x^n$ and $P(x) = Qx^7$, $Q(P(Q(x)))$ has degree
$n \times 7n = 7n^2$
$\therefore \quad 7n^2 = 7 \quad \Rightarrow \quad n = 1$ Ans.]

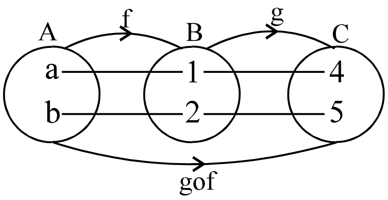
(iii) Let $f : A \to B$ and $g : B \to C$ be functions and $gof : A \to C$. Which of the following statements is **true**?
(A) If gof is one-one then f and g both are one-one.
(B*) If gof is one-one then f is one-one.
(C*) If gof is a bijection then f is one-one and g is onto.
(D*) If f and g are both one-one then gof is one-one.

[Sol. (A) As Shown gof is one-one but g is many-one



$\Rightarrow$ (A) is not correct

(B) If gof is one-one then f is also one-one,
if f is many-one then gof can not be one-one



(C) and (D) are obviously true.]

PRACTICE PROBLEMS ON COMPOSITE FUNCTION:

(1)(a) $f(x) = \begin{cases} 1+x & \text{if } 0 \le x \le 2 \\ 3-x & \text{if } 2 < x \le 3 \end{cases}$ find fof [Ans. $(fof)(x) = \begin{cases} x+2 & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \\ 4-x & 2 < x \le 3 \end{cases}$]

(b) $f(x) = \begin{cases} 1-x & \text{if } 0 \le x \le 1 \\ x+2 & \text{if } 1 < x < 2 \\ 4-x & \text{if } 2 \le x \le 4 \end{cases}$ find (fof) (x) [Ans: (fof) (x) = $\begin{cases} x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } 1 < x < 2 \\ 6-x & \text{if } 2 < x < 3 \\ x-3 & \text{if } 3 \le x \le 4 \\ 2 & \text{if } x = 2 \end{cases}$]

(c) $\left.\begin{aligned} f(x) &= -1+|x-2|, & 0 \le x \le 4 \\ g(x) &= 2-|x|, & -1 \le x \le 3 \end{aligned}\right\}$ find gof and fog

[Ans: $gof(x) = \begin{cases} 1+x & 0 \le x \le 1 \\ 3-x & 1 < x < 2 \\ x-1 & 2 \le x < 3 \\ 5-x & 3 \le x \le 4 \end{cases}$; $(fog)(x) = \begin{cases} -1 & x = 0 \\ -(1+x) & -1 \le x < 0 \\ x-1 & 0 \le x \le 2 \end{cases}$]

(d) $f(x) = \begin{cases} 1+x^3 & x < 0 \\ x^2-1 & x \ge 0 \end{cases}$ and $g(x) = \begin{cases} (x-1)^{1/3} & x < 0 \\ (x+1)^{1/2} & x \ge 0 \end{cases}$ find g (f(x))

ADVANCE PROBLEMS ON COMPOSITE FUNCTION:

(e)(i) If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$ $(x \neq 0)$ find f (2). [Ans. $-\frac{7}{4}$]

(ii) Let f be a real valued function of real and positive argument such that

$$f(x) + 3x\, f\left(\frac{1}{x}\right) = 2(x + 1) \text{ for all real } x > 0. \text{ The value of } f(10099) \text{ is}$$

(A) 550 (B) 505 (C*) 5050 (D) 10010

[Sol replace $x \to 1/x$ and solve to get

$f(x) = \frac{x+1}{2}$

$f(10099) = \frac{10099+1}{2} = \frac{10100}{2} = 5050$]

(f) Let $f : R \to R$ be a function such that $\left(f(x^3+1)\right)^{\sqrt{x}} = 5$, $\forall\ x \in (0, \infty)$ then the value of $\left(f\left(\frac{27+y^3}{y^3}\right)\right)^{\sqrt{\frac{27}{y}}}$

for $y \in (0, \infty)$ is equal to

(A) 5 (B) 5^2 (C*) 5^3 (D) 5^6

[Sol. $\left(f\left(\frac{27}{y^3}+1\right)\right)^{\sqrt{\frac{27}{y}}} = \left(f\left(\left(\frac{3}{y}\right)^3+1\right)\right)^{\frac{3\sqrt{3}}{\sqrt{y}}} = \left[f\left(\left(\frac{3}{y}\right)^3+1\right)^{\sqrt{\frac{3}{y}}}\right]^3$;

let $\frac{3}{y} = x$, then $\left(f\left(x^3+1\right)^{\sqrt{x}}\right)^3 = 5^3$ Ans.]

Example on Homogeneous function

EXAMPLES:

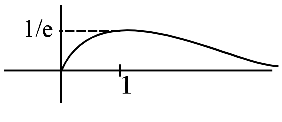
(1) Which of the following function(s) is(are) bounded on the intervals as indicated

(A*) $f(x) = 2^{\frac{1}{x-1}}$ on (0, 1) (B) $g(x) = x\cos\frac{1}{x}$ on $(-\infty, \infty)$

(C*) $h(x) = xe^{-x}$ on $(0, \infty)$ (D*) $l(x) =$ arc $\tan 2^x$ on $(-\infty, \infty)$

[Sol. (A) $\underset{x\to 0^+}{\text{Limit}}\ f(x) = \underset{h\to 0}{\text{Limit}}\ 2^{\frac{1}{h-1}} = \frac{1}{2}$; $\underset{x\to 1^-}{\text{Limit}}\ f(x) = \underset{h\to 0}{\text{Limit}}\ 2^{\frac{1}{-h}} = 0$

$\Rightarrow$ $f(x) \in (0, 1/2)$ $\Rightarrow$ bounded

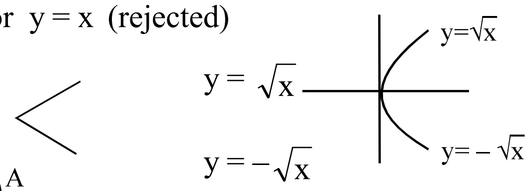


(C) $\underset{h\to 0}{\text{Limit}}\ x\,e^{-x} = \underset{h\to 0}{\text{Limit}}\ h\,e^{-h} = 0$; $\underset{x\to\infty}{\text{Limit}}\ x\,e^{-x} = \underset{x\to\infty}{\text{Limit}}\ \frac{x}{e^x} = 0$

$\Rightarrow$ Also $y = \frac{x}{e^x}$ $\Rightarrow$ $y' = \frac{e^x - xe^x}{e^{2x}}$ $e^{x(1-x)}$ $\Rightarrow h(x) = \left(0, \frac{1}{e}\right]$]

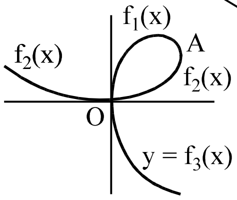
Examples on Implicit and explicit function $f(x, y) = 0$

(1) $x\sqrt{1+y} + y\sqrt{1+x} = 0$; explicit $y = -\dfrac{x}{1+x}$ or $y = x$ (rejected)



(2) $y^2 = x$ represents two separate branches.

(3) $x^3 + y^3 - 3xy = 0$
folium of descartes



(4) $x = 2y - y^2$

(5) Find the domain of the explicit form of the function is represented implictly by the equation $(1 + x)\cos y = x^2$ [Ans. $\left[\dfrac{1-\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right]$]

Examples on Odd & Even function

Odd	Even	Neither odd nor even
1. $ln\,(x + \sqrt{1+x^2}\,)$	1. $x\,\dfrac{2^x+1}{2^x-1}$	1. $2x^3 - x + 1$
2. $ln\,\dfrac{1-x}{1+x}$	2. $\sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$	2. sinx + cosx
3. $x\sin^2 x - x^3$	3. constant	
4. $\sqrt{1+x+x^2} - \sqrt{1-x+x^2}$	4. $x^2 - \lvert x \rvert$	
5. $\dfrac{1+2^{Kx}}{1-2^{Kx}}$	5. $\dfrac{(1+2^x)^2}{2^x}$	

SOME PRACTICE PROBLEMS :

(1) If $f(x) = (a - 2)x + 3a - 4$ is even/odd. Find 'a'. [Ans. Even (a=2), Odd (a=4/3)]

(2) Prove that $f(x) = \dfrac{2x(\sin x + \tan x)}{2[2+(x/\pi)]-3}$ is always odd.

[Hint: $f(x) = \dfrac{x(\sin x + \tan x)}{[x/\pi]+1/2}$; $x = n\pi,\ x \ne n\pi$; $f(x) = \begin{cases} 0 \text{ if } x = n\pi \\ \text{if } x \ne n\pi, \text{ then } f(-x) \end{cases}$]

EXAMPLE ON INVERSE

(1) $f : R \to R$ $f(x) = e^x$ and $g : R \to R$ $g(x) = 3x - 2$ find fog and gof. Also find domains of $(fog)^{-1}$ and $(gof)^{-1}$ [$(fog)(x) = e^{3x-2}$; $(gof)(x) = 3e^x - 2$]
domain of $(fog)^{-1} = R^+$; domain of $(gof)^{-1} = (-2, \infty)$

(2) If $f : R \to R$ $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$ is an invertible mapping find 'a'
$f'(x) \ge 0$ (note that $f'(x) \ge 0$ as leading coefficient f is > 0) [Ans: $a \in [1, 4]$]

(3) (a) $f : [0, \infty) \to [1, \infty)$ $f(x) = \dfrac{e^x + e^{-x}}{2}$ find $f^{-1}(x)$

(b) $f : R \to R$, $f(x) = \dfrac{e^{x/2} - e^{-x/2}}{2}$; $f^{-1}(x) = 2ln\left(x + \sqrt{1+x^2}\right)$

(4) For the function $f : R - \{4\} \rightarrow R - \{-2\}$; $f(x) = \dfrac{2x-5}{4-x}$. Find

(a) zero's of f (x), (b) intervals of monotonocity
(c) $f^{-1}(x)$, (d) maxima or minima
(e) intervals where concave upward and concave downward
(f) equation to the asymptotes,
(g) nature of function whether one-one or onto
(h) $\int_1^2 f(x)\,dx$, (i) garph

[Ans. **(a)** 5/2,
(b) $\uparrow$ in its domain i.e. $(-\infty, 4) \cup (4, \infty)$,
(c) $f^{-1}(x) = \dfrac{4x+5}{x+2}$, **(d)** no,
(e) $(-\infty, 4)$ upwards and $(4, \infty)$ downwards,
(f) $y = -2$, **(g)** one one, **(h)**

EXAMPLES ON PERIODIC FUNCTION:

(1) $f(x) = \cos\dfrac{2x}{3} - \sin\dfrac{4x}{5}$ (15π) (2) $f(x) = \cos(\sin x)$ (π)

(3) $f(x) = \sin(\cos x)$ (2π) (4) $f(x) = \sin^4x + \cos^4x$ $\left(\dfrac{\pi}{2}\right)$

(5) $f(x) = x - [x] = \{x\}$ (One)

(6) Period of the function, $f(x) = [x] + [2x] + [3x] + ... + [nx] - \dfrac{n(n+1)}{2}x$
where $n \in N$ and [ ] denotes the greatest integer function, is
(A*) 1 (B) n (C) 1/n (D) non periodic

[Sol. $([x]-x) + ([2x]-2x) + ... + ([nx]-nx) = (\{x\} + \{2x\} + ... \{nx\})$
$\downarrow \quad \downarrow \quad \downarrow$
LCM of 1 1/2 1/n $\rightarrow$ 1]

(7) f(x) = sinx + cos ax is a periodic function then prove that ‘a’ must be rational

(8)(a) If $f(x) = \dfrac{\sin nx}{\sin\frac{x}{n}}$ has its period $= 4\pi$ then find the integral values of n.

(b) For $a > 0$, if $f(x+a) = \dfrac{1}{2} + \sqrt{f(x) - f^2(x)}$, prove that f is periodic.

(9) TPT : $f(x) = \cos\sqrt{x}$; x sinx and sin x + {x} are aperiodic.

(10) $f(x) = 2\cos\left(\dfrac{x-\pi}{5}\right)$ (Ans: $p = 10\pi$)

(11) If $f(x) = (a+3)x + 5a$, $x \in R$ is periodic. [Ans. $a = -3$]
[**Hint**: $f(x) = mx + c$ is periodic only if $m = 0$]