CLASSROOM STUDY PACKAGE

MATHEMATICS

Polynomial

JEE EXPERT

POLYNOMIAL

CONTENTS

KEY-CONCEPTS

An algebraic expression of the form

$p(x) = a_nx^n + a_{n-1}\,x^{n-1} + a_{n-2}\,x^{n-2} + ... + a_1x^1 + a_0x^0$, where

(i) $a_n \neq 0$

(ii) $a_0, a_1, a_2, ... a_n$ are real numbers

(iii) n is whole number, is called a polynomial.

a_n, a_{n-1}, a_{n-2}, are coefficients of x^n, x^{n-1} ...... x^0 respectively and a_nx^n, $a_{n-1}x^{n-1}$, $a_{n-2}\,x^{n-2}$, ... are terms of the polynomial. Here the term a_nx^n is called the **leading term** and its coefficient a_n, the **leading coefficient**.

Example -

(i) $p(x) = \frac{1}{2}x^3 - 3x^2 + 2x - 4$ is a polynomial in variable x.

(ii) $\frac{1}{2}x^3, -3x^2, 2x, -4$ are known as terms of polynomial and $\frac{1}{2}, -3, 2, -4$ are their coefficients.

Types of Polynomials : Generally we divide the polynomials in two categories.

Types of polynomials

Based on number of terms | Based on degree

(A) BASED ON NUMBER OF TERMS :- These are follows :

(a) Monomial : Polynomials having only one term are called monomials. ('Mono' means 'one')
Eg. $2, 2x, 5x^2, -5x^2, y, u^4$ etc.

(b) Binomial : A polynomial of two terms is called binomial.
Eg. $p(x) = x + 1$, $q(y) = 2y^7 + 5y^6$ etc.

(c) Trinomial : A polynomial of three terms is called a trinomial.
Eg. $p(x) = 2x^2 + x + 6$
$q(y) = 9y^6 + 4y^2 + 1$ etc.

(B) BASED ON DEGREE :-

Degree of Polynomials : The highest power of variable in a polynomial is known as it degree.

For example :

(a) $p(y) = 2y^2 - 3y + 7$ is a polynomial in the variable y of degree 2.

(b) $q(x) = \sqrt{2}x + 13x^4 + 5x^6$ is a polynomial in variable x of degree 6.

Polynomials classified by their degree:-

(i) Linear Polynomial : A polynomial of degree one is called a linear polynomial.
Ex. $p(x) = 4x + 5$

(ii) Quadratic Polynomial : A polynomial of degree two is called a quadratic polynomial.
Ex. $p(x) = 2x^2 + 5$

(iii) Cubic Polynomial : A polynomial of degree three is called a cubic polynomial.
Ex. $4x^2 + 2x^3 + 1$, $5x^3 + x^2$

(vi) Biquadrate polynomial : A polynomial of fourth degree is called a biquadrate polynomial.
Ex. $4x^4 + 2x^3 + 5x^2 + x + 1$

VALUE OF POLYNOMIALS :

If p(x) is a polynomial in variable x and α is any real number, then the value obtained by replacing x by α in p(x) is called value of p(x) at = α and is denoted by $p(\alpha)$.

For example : Find the value of $p(x) = x^3 - 6\,x^2 + 11x - 6$ at $x = -2$

$\Rightarrow$ $p(-2) = (-2)^3 - 6\,(-2)^2 + 11(-2) - 6 = -8 - 24 - 22 - 6$

$\Rightarrow$ $p(-2) = -60$

Zeros/Roots of a polynomial/equation

Consider a polynomial $f(x) = 3x^2 - 4x + 2$. If we replace x by 3 everywhere in the above expression, we get

$f(3) = 3 \times (3)^2 - 4 \times (3) + 2 = 27 - 12 + 2 = 17$

We can say that the value of the polynomial f(x) at x = 3 is 17.

Similarly the value of polynomial $f(x) = 3x^2 - 4x + 2$

at $x = -2$ is $f(-2) = 3(-2)^2 - 4 \times (-2) + 2 = 12 + 8 + 2 = 22$

at $x = 0$ is $f(0) = 3(0)^2 - 4(0) + 2 = 0 - 0 + 2 = 2$

at $x = \frac{1}{2}$ is $f\left(\frac{1}{2}\right) = 3 \times \left(\frac{1}{2}\right)^2 - 4 \times \left(\frac{1}{2}\right) + 2 = \frac{3}{4} - 2 + 2 = \frac{3}{4}$

In general, we can say $f(\alpha)$ if the value of the polynomial f(x) at $x = \alpha$, where α is a real number.

A real number α is zero of a polynomial f(x) if the value of the polynomial f(x) is zero at $x = \alpha$ i.e. $f(\alpha) = 0$.

OR

The value of the variable x, for which the polynomial f(x) becomes zero is called zero of the polynomial.

E.g. : consider, a polynomial $p(x) = x^2 - 5x + 6$; replace x by 2 and 3.

$p(2) = (2)^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$,

$p(3) = (3)^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

$\therefore$ 2 and 3 are the zeros of the polynomial p(x).

Roots of a polynomial equation

An expression f(x) = 0 is called a polynomial equation if f(x) is a polynomial of degree $n \geq 1$.

A real number α is a root of a polynomial f(x) = 0 if $f(\alpha) = 0$ i.e. α is a zero of the polynomial f(x).

E.g. consider the polynomial $f(x) = 3x-2$, then $3x - 2 = 0$ is the corresponding polynomial equation.

Here, $f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right) - 2 = 0$

i.e. $\frac{2}{3}$ is a zero of the polynomial $f(x) = 3x - 2$

or $\frac{2}{3}$ is a root of the polynomial equation $3x - 2 = 0$

Ex. Find q(0), q(1) and q(2) for each of the following polynomials:

(i) $q(x) = x^2 + 3x$ (ii) $q(y) = 2 + y + 2y^2 - 5y^3$

Sol. (i) $q(x) = x^2 + 3x$

$\therefore$ $q(0) = (0)^2 + 3 \times 0 = 0$

$q(1) = (1)^2 + 3 \times 1 = 4$

$q(2) = (2)^2 + 3 \times 2 = 4 + 6 = 10$

(ii) $q(y) = 2 + y + 2y^2 - 5y^3$

$\therefore$ $q(0) = 2 + 0 + 2(0)^2 - 5(0)^3 = 2$

$q(1) = 2 + 1 + 2(1)^2 - 5(1)^3 = 2 + 1 + 2 - 5 = 0$

and $q(2) = 2 + 2 + 2(2)^2 - 5(2)^3 = 2 + 2 + 8 - 40 = -28$

REMAINDER THEOREM :

Statement : Let p(x) be a polynomial of degree $\geq$ 1and 'a' is any real number. If p(x) is dividided by (x – a), then the remainder is p(a).

Dividend = Divisor × quotient + Remainder

E.g. Let p(x) be $x^3 - 7x^2 + 6x + 4$.

Divide p(x) with (x – 6) and to find the remainder, put x = 6 in p(x) i.e. p(6) will be the remainder.

$\therefore$ required remainder be

$p(6) = (6)^3 - 7.6^2 + 6.6 + 4 = 216 - 252 + 36 + 4 = 256 - 252 = 4$

$$
\begin{array}{r|l}
x-6 & x^3-7x^2+6x+4\ (x^2-x \\
 & \underline{\underset{-\quad +}{x^3-6x^2}} \\
 & \quad -x^2+6x+4 \\
 & \quad \underline{\underset{+\quad -}{-x^2+6x}} \\
 & \text{Remainder} = 4
\end{array}
$$

Thus, p(a) is remainder on dividing p(x) by (x – a).

Ex. Find remainder $3x^4 - 4x^3 - 3x - 1$by (x–1)

Sol. By long division

$$
\begin{array}{r|l}
 & \quad 3x^3 - x^2 - x - 4 \\ \hline
x-1 & 3x^4-4x^3-3x-1 \\
 & \underline{\underset{-\quad\quad +}{3x^4-3x^3}} \\
 & \quad -x^3-3x-1 \\
 & \quad \underline{\underset{+\quad\ -}{-x^3+x^2}} \\
 & \quad -x^2-3x-1 \\
 & \quad \underline{\underset{+\quad\ -}{-x^2+x}} \\
 & \quad\quad -4x-1 \\
 & \quad\quad \underline{\underset{+\quad -}{-4x+4}} \\
 & \quad\quad\quad -5
\end{array}
$$

Here, the remainder is -5. Now, the zero of (x - 1) is 1. So, putting x = 1 is p(x), we see that $p(1) = 3(1)^4 - 4(1)^3 - 3(1) - 1$

$= 3 - 4 - 3 - 1$

$= -5$, which is the remainder.

Ex. Find the remainder when

(i) $x^3 - ax^2 + 6x - a$ is divided by $x - a$

(ii) $2x^4 + x^3 - 2x^2 + x + 1$ by $2x - 1$

Solution

(i) Let $p(x) = x^3 - ax^2 + 6x - a$

zero of $x - a$ is a

$p(a) = a^3 - a(a)^2 + 6(a) - a$

$= a^3 - a^3 + 6a - a = 5a$

So, by the remainder theorem, remainder = 5a

(ii) Let $p(x) = 2x^4 + x^3 - 2x^2 + x + 1$

zero of $2x - 1$ is 1/2

$$\text{So, } p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) + 1$$

$$= \frac{1}{8} + \frac{1}{8} - \frac{1}{2} + \frac{1}{2} + 1 = \frac{1}{4} + 1 = \frac{5}{4}$$

So, by the remainder theorem remainder = $\frac{5}{4}$

Factor Theorem :

Statement: Let f(x) be a polynomial of degree ≥ 1 and a be any real constant such that f(a) = 0, then (x–a) is a factor of f(x). conversely, if (x–a) is a factor of f(x), then f(a) = 0.

Proof : By remainder theorem, if f(x) is divided by (x–a), the remainder will be f(a). let q(x) be the quotient. Then, we can write,

f(x) = (x–a) × q (x) + f(a) ($\because$ Dividend = Divisor × Quotient + Remainder)

If f(a) = 0, then f(x) = (x – a) × q(x)

Thus, (x – a) is a factor of q(x).

Converse Let (x–a) is a factor of f(x).

Then we have a polynomial q(x) such that f(x) = (x – a) × q(x)

Replacing x by a, we get f(a) = 0.

Hence, proved.

Ex. Determine the value of a for which the polynomial $2x^4 - ax^3 + 4x^2 + 2x + 1$ is divisible by 1 - 2x.

Sol. Let $p(x) = 2x^4 - ax^3 + 4x^2 + 2x + 1$. If the polynomial p(x) is divisible by (1 - 2x), then (1 - 2x) is a factor of p(x).

$$\therefore \quad p\left(\frac{1}{2}\right) = 0$$

$$\Rightarrow \quad 2\left(\frac{1}{2}\right)^4 - a\times\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^2 + 2\times\frac{1}{2} + 1 = 0$$

$$\Rightarrow \quad \frac{2}{16} - \frac{a}{8} + \frac{4}{4} + \frac{2}{2} + 1 = 0 \Rightarrow \frac{1}{8} - \frac{a}{8} + 1 + 1 + 1 = 0 \Rightarrow \quad \frac{25}{8} = \frac{a}{8} \Rightarrow a = 25$$

Hence, the given polynomial will be divisible by 1–2x, if a = 25.

Ex. Use the factor theorem to determine whether (x – 1) is a factor of

f(x) = $2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$

Sol. By using factor theorem, (x–1) is a factor of f(x) , only when f(1) = 0

f(1) = $2\sqrt{2}(1)^3 + 5\sqrt{2}(1)^2 - 7\sqrt{2}$ = $2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2} = 0$

Hence, (x–1) is a factor of f(x).

Ex. For what value of k, (x–1) is a factor of $p(x) = kx^2 – 3x + k$?

Sol. Here $p(x) = kx^2 – 3x + k$

$\because$ x –1 is a factor of p(x)

$\therefore$ x –1 = 0 $\quad\therefore$ x = 1

$\therefore$ p(1) = 0

or $k(1)^2 – 3(1) + k = 0$

$\Rightarrow$ K–3 + K = 0

or 2k – 3 = 0

$\therefore$ k = $\frac{3}{2}$

TYPE OF FACTORIZATION :-

(i) Factorization by taking out the common factors

Ex. $ab(a^2 + b^2 - c^2) + bc(a^2 + b^2 - c^2) - ca(a^2 + b^2 - c^2)$

Sol. We have

$ab(a^2 + b^2 - c^2) + bc(a^2 + b^2 - c^2) - ca(a^2 + b^2 - c^2)$

$= (a^2 + b^2 - c^2)(ab + bc - ca)$

(ii) Factorization by grouping the terms

Ex. $(x^2 + 3x)^2 - 5(x^2 + 3x) - y(x^2 + 3x) + 5y$

Sol. We have

$(x^2 + 3x)^2 - 5(x^2 + 3x) - y(x^2 + 3x) + 5y$

$= (x^2 + 3x)\{(x^2 + 3x) - 5\} - y\{(x^2 + 3x) - 5\} = (x^2 + 3x - 5)(x^2 + 3x - y)$

(iii) Factorization by making a perfect square

Ex. $a^2 + b^2 - 2(ab - ac + bc)$

Sol. We have

$a^2 + b^2 - 2(ab - ac + bc)$

$= a^2 + b^2 - 2ab + 2ac - 2bc$

$= (a - b)^2 + 2c\,(a - b)$

$= (a - b)\,\{(a - b) + 2c\} = (a - b)\,(a - b + 2c)$

(iv) Factorization the difference of two squares

Ex. $x^8 - y^8$

Sol. We have

$x^8 - y^8 = \{(x^4)^2 - (y^4)^2\} = (x^4 - y^4)\,(x^4 + y^4)$

$= \{(x^2)^2 - (y^2)^2\,(x^4 + y^4) = (x^2 - y^2)\,(x^2 + y^2)\,(x^4 + y^4)$

$= (x - y)\,(x + y)\,(x^2 + y^2)\,(x^4 + y^4)$

$= (x - y)\,(x + y)\,(x^2 + y^2)\,\{(x^2)^2 + (y^2)^2 + 2x^2y^2 - 2x^2y^2)$

$= (x - y)\,(x + y)\,(x^2 + y^2)\,\{(x^2 + y^2)^2 - (\sqrt{2}\,xy)^2\}$

$= (x - y)\,(x + y)\,(x^2 + y^2)\,(x^2 + y^2 - \sqrt{2}\,xy)\,(x^2 + y^2 + \sqrt{2}\,xy)$

(v) Factorization of quadratic polynomials by splitting the middle term

Ex. $x^2 + 3\sqrt{3}\,x - 30$

Sol. In order to factorize $x^2 + 3\sqrt{3}\,x - 30$, we have to find two numbers p and q such that $p + q = 3\sqrt{3}$ and $pq = -30$. Clearly, $5\sqrt{3} + (-2\sqrt{3}) = 3\sqrt{3}$ and $5\sqrt{3} \times -2\sqrt{3} = -30$

So. we write the middle term $3\sqrt{3}$ as $5\sqrt{3}\ -2\sqrt{3}\,x$

$\therefore\quad x^2 + 3\sqrt{3}\,x - 30$

$= x^2 + 5\sqrt{3}\,x - 2\sqrt{3}x - 30$

$= (x^2 + 5\sqrt{3}\,x) - (2\sqrt{3}\,x + 30)$

$= (x^2 + 5\sqrt{3}\,x)\,(2\sqrt{3}\,x + 10\sqrt{3} \times \sqrt{3})$

$= x(x + 5\sqrt{3}) - 2\sqrt{3}\,(x + 5\sqrt{3}) = (x + 5\sqrt{3})\,(x - 2\sqrt{3})$

Algebraic identities :

An algebraic identity is an algebraic equation that is true for all values of the variables present in the equation.

I. (i) $(x + y)^2 = x^2 + 2xy + y^2$; (ii) $(x-y)^2 = x^2 - 2xy + y^2$

II. $x^2 - y^2 = (x + y)\,(x - y)$

III. $(x + a)\,(x + b) = x^2 + (a+b)\,x + ab$

IV. $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

V. (i) $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$

(ii) $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$

VI. (i) $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$

(ii) $x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$

VII. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$

VIII. If $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$

SOLVED EXAMPLES

Ex.1 Factorise $x^2 - 5x - 24$ by using the factor theorem.

Sol. $p(x) = x^2 - 5x - 24$.

Here, coefficient of the leading term is 1 and the constant term is –24. A zero of the polynomial p(x) will be a factor of the number –24, by inspection, we find that the number 8 is a divisor of –24 and also we have,

$p(8) = (8)^2 - 5(8) - 24$

$\Rightarrow\ 64 - 40 - 24 = 0$

i.e. 8 is a zero of the polynomial p(x).

Then (x–8) is a factor of the polynomial. We can express

$x^2 - 5x - 24 = (x^2 - 8x) + (3x - 24)$

$= x(x - 8) + 3(x - 8)$

$= (x - 8)(x + 3)$

We can also find the second factor of the polynomial by dividing $x^2 - 5x - 24$ by (x–8).

Ex.2 Factorise $x^3 - 23x^2 + 142x - 120$.

Sol. $p(x) = x^3 - 23x^2 + 142x - 120$.

The coefficient of the leading term is 1 and the constant term is –120. Factors of –120 are many but we find a suitable factor of –120 which is a zero of the polynomial.

By inspection, we find that

$p(1) = (1)^3 - 23(1)^2 + 142(1) - 120$

$= 1 - 23 + 142 - 120 = 0$

$\Rightarrow$ (x –1) is a factor of the polynomial.

Now, we can express the given polynomial as below:

$x^3 - 23x^2 + 142x - 120 = (x^3 - x^2) + (-22x^2 + 22x) + (120x - 120)$

$= x^2(x - 1) - 22x(x - 1) + 120(x - 1)$

$= (x-1)(x^2 - 22x + 120)$

$= (x-1)\{x^2 + (-12 - 10)x + 120\}$

$= (x-1)\{(x^2 - 12x) + (-10x + 120)\}$

$= (x-1)\{x(x-12) - 10(x - 12)\}$

$= (x-1)\{(x - 12)(x-10)\}$

$= (x - 1)(x-10)(x-12)$

Ex.3 Find the product using appropriate identities:

(i) $(x + 8)(x + 8)$ (ii) $(3x - 2y)(3x - 2y)$ (iii) $(x + 0.1)(x - 0.1)$

Sol. (i) $(x + 8)(x + 8) = (x + 8)^2 = x^2 + 2(x)(8) + (8)^2 = x^2 + 16x + 64$.

(ii) $(3x - 2y)(3x - 2y) = (3x - 2y)^2$

$= (3x)^2 - 2(3x)(2y) + (2y)^2 = 9x^2 - 12xy + 4y^2$

(iii) $(x + 0.1)(x-0.1) = (x)^2 - (0.1)^2$

$= x^2 - 0.01$

Ex.4 Expand each of the following using suitable identities

(i) $(3x - 4y)^2$ (ii) $(3x - y)^3$ (iii) $(3x + 4y + 5z)^2$

Sol. (i) $(3x - 4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2$

$= 9x^2 - 24xy + 16y^2$

(ii) $(3x - y)^3 = (3x)^3 - (y)^3 - 3(3x)(y)(3x-y)$

$= 27x^3 - y^3 - 9xy(3x - y)$

$= 27x^3 - y^3 - (9xy)(3x) + (9xy)(y)$

$= 27x^3 - y^3 - 27x^2y + 9xy^2$

(iii) $(3x + 4y + 5z)^2 = (3x)^2 + (4y)^2 + (5z)^2 + 2(3x)(4y) + 2(4y)(5z) + 2(5z)(3x)$

$= 9x^2 + 16y^2 + 25z^2 + 24xy + 40yz + 30zx$

Ex.5 Factorize the following:

(i) $4x^2 + 20xy + 25y^2$ (ii) $25x^2y^2z^2 - 36u^2$

(iii) $125x^3y^3 + 27z^3$ (iv) $125x^3 + 225x^2y + 135xy^2 + 27y^3$

(v) $8x^3 + y^3 + 27z^3 - 18xyz$

(vii) $(a - b)^3 + (b - c)^3 + (c - a)^3$

Sol. (i) $4x^2 + 20xy + 25y^2 = (2x)^2 + 2(2x)(5y) + (5y)^2$

$= (2x + 5y)^2$

(ii) $25x^2y^2z^2 - 36u^2 = (5xyz)^2 - (6u)^2$

$= (5xyz + 6u)(5xyz - 6u)$

(iii) $125x^3y^3 + 27z^3 = (5xy)^3 + (3z)^3$

$= (5xy + 3z)\{(5xy)^2 + (5xy)(3z) + (3z)^2\}$

$= (5xy + 3z)(25x^2y^2 + 15xyz + 9z^2)$

(iv) $125x^3 + 225x^2y + 135xy^2 + 27y^3$

$= (5x)^3 + 45xy(5x + 3y) + (3y)^3$

$= (5x)^3 + 3(5x)(3y)(5x + 3y) + (3y)^3$

$= (5x + 3y)^3$

(v) $8x^3 + y^3 + 27z^3 - 18xyz$

$= (2x)^3 + (y)^3 + (3z)^3 - 3(2x)(y)(3z)$

$= (2x + y + 3z)\{(2x)^2 + (y)^2 + (3z)^2 - (2x)(y) - (y)(3z) - (3z)(2x)\}$

$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 3yz - 6zx)$

(vii) $(a - b)^3 + (b - c)^3 + (c - a)^3$

Here, $(a - b) + (b - c) + (c - a) = 0$

So, $(a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a)$

Ex.6 If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x-3)$, find the value of a.

Sol. Let $p(x) = ax^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + a$ be the given polynomials. The remainders when p(x) and q(x) are divided by $(x-3)$ are p(3) and q(3) respectively.

By the given condition, we have $p(3) = q(3)$

$\Rightarrow$ $a \times (3)^3 + 4 \times (3)^2 + 3 \times 3 - 4 = (3)^3 - 4 \times 3 + a$

$\Rightarrow$ $27a + 36 + 9 - 4 = 27 - 12 + a$

$\Rightarrow$ $26a + 26 = 0$

$\Rightarrow$ $26a = -26$

$\Rightarrow$ $a = -1$

Ex.7 Find $q(a + 1) - 2q(a)$ if $q(x) = x^2 + 3x + 4$.

Sol. To evaluate q (a + 1), replace x in q(x) with a + 1.

$q(x) = x^2 + 3x + 4$

$q(a + 1) = (a + 1)^2 + 3(a + 1) + 4$

$= a^2 + 2a + 1 + 3a + 3 + 4 = a^2 + 5a + 8$

To evaluate 2q(a). replace x with a in q(x), then multiply the expression by 2.

$q(x) = x^2 + 3x + 4$

$2q(a) = 2(a^2 + 3a + 4) = 2a^2 + 6a + 8$

Now evaluate q (a + 1) – 2q(a)

$q(a + 1) - 2q(a) = a^2 + 5a + 8 - (2a^2 + 6a + 8)$

$= a^2 + 5a + 8 - 2a^2 - 6a - 8 = -a^2 - a$

Ex.8 Verify whether the indicated values of variables are zeros of the polynomials corresponding to them:

(i) $p(y) = 4y - 4\pi$, $y = 4, \pi$

(ii) $q(u) = (u + 1)(u + 2)$, $u = -1, 2$

Sol. (i) $p(y) = 4y - 4\pi$

$p(4) = 4(4) - 4\pi = 16 - 4\pi \neq 0$

$p(\pi) = 4\pi - 4\pi = 0$

$\Rightarrow$ π is a zero and 4 is not a zero of the polynomial

(ii) $q(u) = (u + 1)(u + 2)$

$q(-1) = (-1 + 1)(-1 + 2) = (0)(1) = 0$

$q(2) = (2 + 1)(2 + 2) = (3)(4) = 12 \neq 0$

$\Rightarrow$ -1 is a zero and 2 is not a zero of the polynomial.

Ex.9 Which of the number 1, –1, and –3 are zeroes of the polynomial $2x^4 + 9x^3 + 11x^2 + 4x - 6$.

Sol. Let $f(x) = 2x^4 + 9x^3 + 11x^2 + 4x - 6$

$f(1) = 2(1)^4 + 9(1)^3 + 11(1)^2 + 4(1) - 6$

$= 2 + 9 + 11 + 4 - 6 = 20 \neq 0$

$\therefore$ 1 is not a zero of the polynomial f(x)

Again $f(-1) = 2(-1)^4 + 9(-1)^3 + 11(-1)^2 + 4(-1) - 6$

$= 2 - 9 + 11 - 4 - 6 = -6 \neq 0$

$\therefore$ -1 is not a zero the polynomial f(x)

Also $f(-3) = 2(-3)^4 + 9(-3)^3 + 11(-3)^2 + 4(-3) - 6$

$= 162 - 243 + 99 - 12 - 6 = 0$

$\therefore$ -3 is a zero of the polynomial f(x).

Thus 1 and –1 are not zeroes of f(x) whereas –3 is a zero of f(x).

Ex.10 Let R_1 and R_2 are the remainder when the polynomials $x^3 + 2x^2 - 5ax - 7$ and $x^3 + ax^2 - 12x + 6$ are divided by $x + 1$ and $x - 2$ respectively. If $2R_1 + R_2 = 6$, find the value of a.

Sol. Let $p(x) = x^3 + 2x^2 - 5ax - 7$

and $q(x) = x^3 + ax^2 - 12x + 6$ be the given polynomials,

Now, R_1 = Remainder when p(x) is divided by x + 1.

$\Rightarrow$ $R_1 = p(-1)$

$\Rightarrow$ $R_1 = (-1)^3 + 2(-1)^2 - 5a(-1) - 7$

$[\because p(x) = x^3 + 2x^2 - 5ax - 7]$

$\Rightarrow$ $R_1 = -1 + 2 + 5a - 7$

$\Rightarrow$ $R_1 = 5a - 6$

And, R_2 = Remainder when q(x) is divided by x–2

$\Rightarrow$ $R_1 = q(2)$

$\Rightarrow$ $R_2 = (2)^3 + a \times 2^2 - 12 \times 2 + 6$

$[\because q(x) = x^3 + ax^2 - 12x - 6]$

$\Rightarrow$ $R_2 = 8 + 4a - 24 + 6$

$\Rightarrow$ $R_2 = 4a - 10$

Substitution the values of R_1 and R_2 in $2R_1 + R_2 = 6$, we get

$\Rightarrow$ $2(5a - 6) + (4a - 10) = 6$

$\Rightarrow$ $10a - 12 + 4a - 10 = 6$

$\Rightarrow$ $14a - 22 = 6$

$\Rightarrow$ $14a = 28$

$\Rightarrow$ $a = 2$

Ex.11 If $f(x) = x^4 - 2x^3 + 3x^2 - ax + b$ is a polynomial such that when it is divided by $x - 1$ and $x + 1$, the remainder are respectively 5 and 19. Determine the remainder when f(x) is divided by $(x-2)$.

Sol. When f(x) is divided by $x - 1$ and $x + 1$ the remainder are 5 and 19 respectively.

$\therefore$ $f(1) = 5$ and $f(-1) = 19$

$\Rightarrow$ $(1)^4 - 2 \times (1)^3 + 3 \times (1)^2 - a \times 1 + b = 5$

and $(-1)^4 - 2 \times (-1)^3 + 3 \times (-1)^2 - a \times (-1) + b = 19$

$\Rightarrow$ $1 - 2 + 3 - a + b = 5$

and $1 + 2 + 3 + a + b = 19$

$\Rightarrow$ $2 - a + b = 5$ and $6 + a + b = 19$

$\Rightarrow$ $-a + b = 3$ and $a + b = 13$

Adding these two equations, we get

$(-a + b) + (a + b) = 3 + 13$

$\Rightarrow$ $2b = 16$ $\Rightarrow$ $b = 8$

Putting $b = 8$ in $-a + b = 3$, we get

$-a + 8 = 3$ $\Rightarrow$ $a = -5$ $\Rightarrow$ $a = 5$

Putting the values of a and b in

$f(x) = x^4 - 2x^3 + 3x^2 - 5x + 8$

The remainder when f(x) is divided by $(x-2)$ is equal to f(2).

So, Remainder $= f(2) = (2)^4 - 2 \times (2)^3 + 3 \times (2)^2 - 5 \times 2 + 8 = 16 - 16 + 12 - 10 + 8 = 10$

Ex.12 Without actual division, prove that the polynomial $2x^3 + 13x^2 + x - 70$ is exactly divisible by $x - 2$.

Sol. The polynomial $p(x) = 2x^3 + 13x^2 + x - 70$ is exactly divisible by $x - 2$ means that $x - 2$ is a factor of $p(x) = 2x^3 + 13x^2 + x - 70$.

Now $p(2) = 2(2)^3 + 13(2)^2 + 2 - 70 = 16 + 52 + 2 - 70 = 0$

$\therefore$ By factor theorem, $x - 2$ is a factor of p(x) i.e. $p(x) = 2x^3 + 13x^2 + x - 70$ is exactly divisible by $x - 2$.

Ex.13 Show that $(x + 1)$ and $2x - 3$ are factors of $2x^3 - 9x^2 + x + 12$

Sol. Let $p(x) = 2x^3 - 9x^2 + x + 12$ be the given polynomial. In order to prove that $x + 1$ and $2x - 3$ are factors of p(x), it is sufficient to show that $p(-1)$ and $p(3/2)$ both are equal to zero.

Now, $p(x) = 2x^3 - 9x^2 + x + 12$

$\Rightarrow$ $p(-1) = 2 \times (-1)^3 - 9 \times (-1)^2 + (-1) + 12$

and $p\left(\frac{3}{2}\right) = 2 \times \left(\frac{3}{2}\right)^3 - 9 \times \left(\frac{3}{2}\right)^2 + \frac{3}{2} + 12$

$\Rightarrow$ $p(-1) = -12 + 12$

and $p\left(\frac{3}{2}\right) = \frac{54 - 162 + 12 + 96}{8} = 0$

$\Rightarrow$ $p(-1) = 0$ and $p\left(\frac{3}{2}\right) = 0$

Hence, $(x+1)$ and $(2x-3)$ are factors of the given polynomial.

Ex.14 The polynomial $ax^3 + bx^2 + x - 6$ has $(x + 2)$ as a factor and leaves a remainder 4 when divided by $(x-2)$. Find a and b.

Sol. Let $p(x) = ax^3 + bx^2 + x - 6$

By using factor theorem, $(x + 2)$ is a factor of p(x),

only when $p(-2) = 0$

$p(-2) = a(-2)^3 + b(-2)^2 + (-2) - 6 = 0$

$\Rightarrow$ $-8a + 4b - 8 = 0$

$\therefore$ $-2a + b = 2$ .....(i)

Also when p(x) is divided by (x–2) the remainder is 4.

$\therefore$ $p(2) = 4$

$\Rightarrow$ $a(2)^3 + b(2)^2 + 2 - 6 = 4$

$\Rightarrow$ $8a + 4b + 2 - 6 = 4$

$\Rightarrow$ $8a + 4b = 8$

$\Rightarrow$ $2a + b = 2$ .....(ii)

Adding equation (i) and (ii) we get $(-2a + b) + (2a + b) = 2 + 2$

$\Rightarrow$ $2b = 4$ $\Rightarrow$ $b = 2$

Putting b = 2 in (i) we get

$-2a + 2 = 2$

$-2a = 0$

$a = 0$

Hence, a = 0 and b = 2

Ex.15 Factorise the polynomial $x^2 + 3\sqrt{3}x + 6$ by spliting the middle term.

Sol. $p(x) = x^2 + 3\sqrt{3}x + 6$.

the coefficient of the middle term is $3\sqrt{3}x$. Now, we find two numbers l and m such that

$l + m = 3\sqrt{3}x$ and $l \times m = 1 \times 6 = 6$

By inspection, we find $l = \sqrt{3}$ and $m = 2\sqrt{3}$.

Then we have

$x^2 + 3\sqrt{3}x + 6 = x^2 + (\sqrt{3} + 2\sqrt{3})x + 6$

[By splitting the middle term]

$= x^2 + \sqrt{3}x + 2\sqrt{3}x + (\sqrt{3})(2\sqrt{3})$

$= \{x^2 + \sqrt{3}x\} + \{2\sqrt{3}x + (\sqrt{3})(2\sqrt{3})\}$

$= x\{x + \sqrt{3}\} + 2\sqrt{3}\{x + \sqrt{3}\}$

$= \{x + \sqrt{3}\}\{x + 2\sqrt{3}\}$

$\therefore$ $x^2 + 3\sqrt{3}x + 6 = (x + \sqrt{3})(x + 2\sqrt{3})$

Ex.16 Factorise $15x^2 - 8x + 1$ by using the factor theorem.

Sol. $p(x) = 15x^2 - 8x + 1 = 15 \times \left\{x^2 - \frac{8}{15}x + \frac{1}{15}\right\} = 15 \times q(x)$

where $q(x) = x^2 - \frac{8}{15}x + \frac{1}{15}$

we have made the coefficient of the leading term of the polynomial q(x) equal to 1. The constant term of the quadratic polynomial q(x) is $\frac{1}{15}$. Some of the factors of the number $\frac{1}{15}$ are ± 1. $\pm \frac{1}{3}$, $\pm$, $\frac{1}{5}$, $\pm \frac{1}{15}$. we find that

$$q\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 - \frac{8}{15}\left(\frac{1}{3}\right) + \frac{1}{15} = \frac{1}{9} - \frac{8}{45} + \frac{1}{15} = \frac{5-8+3}{45} = 0 \text{ and } q\left(\frac{1}{5}\right) = \left(\frac{1}{5}\right)^2 - \frac{8}{15}\left(\frac{1}{5}\right) + \frac{1}{15}$$

$= \frac{1}{25} - \frac{8}{75} + \frac{1}{15} = \frac{3-8+5}{75} = 0$

Thus, $\frac{1}{3}$ and $\frac{1}{5}$ are zeros of q(x). It implies that $\left(x - \frac{1}{3}\right)$ and $\left(x - \frac{1}{5}\right)$ are two factors of the polynomial

$q(x) = x^2 - \frac{8}{15}x + \frac{1}{15}$. {By factor theorem}.

i.e., $x^2 - \frac{8}{15}x + \frac{1}{15} = \left(x - \frac{1}{3}\right)\left(x - \frac{1}{5}\right)$

$\Rightarrow \; 15x^2 - 8x + 1 = 15 \times \left\{\left(x - \frac{1}{3}\right)\left(x - \frac{1}{5}\right)\right\} = 15 \times \left\{\frac{(3x-1)}{3} \times \frac{(5x-1)}{5}\right\} = (3x-1)\,(5x-1)$

Therefore, the polynomial $15x^2 - 8x + 1$ is factorised into two linear factors as $(3x-1)(5x-1)$

Ex.17 Factorise the polynomial $2x^3 + x^2 - 2x - 1$.

Sol. $p(x) = 2x^3 + x^2 - 2x - 1$

$p(1) = 2(1)^3 + (1)^2 - 2(1) - 1 = 2 + 1 - 2 - 1 = 0$

$\Rightarrow$ $(x-1)$ is a factor of the polynomial p(x).

Now, $2x^3 + x^2 - 2x - 1$

$= (2x^3 - 2x^2) + (3x^2 - 3x) + (x - 1)$

$= 2x^2(x-1) + 3x(x-1) + 1(x-1)$

$= (x-1)\{2x^2 + 3x + 1\}$

$= (x-1)\{2x^2 + (2+1)x + 1\}$

{By splitting the middle term}

$= (x-1)(2x^2 + 2x + x + 1)$

$= (x-1)(2x(x+1) + 1(x+1))$

$= (x-1)\{(x+1)(2x+1)\}$

$= (x-1)(x+1)(2x+1)$

Ex.18 Expand each of the following :

(i) $(4x - 5y)^2$ (ii) $(x-3)(x-5)$ (iii) $(-2x + 5y - 3z)^2$ (iv) $(2a - 3b)^3$

Sol. (i) $(4x - 5y)^2 = (4x)^2 + 2.4x.5y + (5y)^2 = 16x^2 - 40xy + 25y^2$

(ii) $(x-3)(x-5) = \{x + (-3)\}\{x + (-5)\} = x^2 + \{(-3) + (-5)\}x + (-3).(-5)$

$= x^2 + (-3-5)x + 15 = x^2 - 8x + 15$

(iii) $(-2x + 5y - 3z)^2 = (-2x)^2 + (5y)^2 + (-3z)^2 + 2.(-2x).5y + 2.5y.(-3z) + 2.(-3z).(-2x)$

$= 4x^2 + 25y^2 + 9z^2 - 20xy - 30yz + 12zx$

(iv) $(2a - 3b)^3 = (2a)^3 - (3b)^3 - 3 \times 2a \times 3b\,(2a - 3b)$

$= 8a^3 - 27b^3 - 18ab\,(2a - 3b)$

$= 8a^3 - 27b^3 - 18ab \times 2a + 18ab \times 3b$

$= 8a^3 - 27b^3 - 36a^2b + 54ab^2$

Ex.19 Find the product :

$(x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 6yz - 3zx)$

Sol. $(x + 2y + 3z)(x^2 + 4y^2 + 9z^2 - 2xy - 6yz - 3zx)$

$= (x + 2y + 3z)(x^2 + (2y)^2 + (3z)^2 - x \times 2y - 2y \times 3z \times x)$

$x^3 + (2y)^3 + (3z)^3 - 3 \times x + 2y \times 3z$

$= x^3 + 8y^3 + 27z^3 - 18xyz$

Ex.20 If $a^2 + b^2 + c^2 = 20$ and $a + b + c = 0$, find $ab + bc + ca$.

Sol. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow$ $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow$ $(0)^2 = 20 + 2\,(ab + bc + ca)$

$\Rightarrow$ $-20 = 2\,(ab + bc + ca)$

$\Rightarrow$ $ab + bc + ca = -10$

Ex.21 If $a + b + c = 8$ and $ab + bc + ca = 20$, find the value of $a^3 + b^3 + c^3 - 3abc$.

Sol. Since $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$a + b + c = 8$ and $ab + bc + ca = 20$,

$(8)^2 = a^2 + b^2 + c^2 + 2 \times (20)$

$\Rightarrow$ $64 = a^2 + b^2 + c^2 + 40$

$\therefore$ $a^2 + b^2 + c^2 = 64 - 40 = 24$

We know that

$a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)\,\{a^2 + b^2 + c^2 - (ab + bc + ca)\}$

$\therefore$ $a^3 + b^3 + c^3 - 3abc$

$= 8 \times (24 - 20) = 4 \times 8 = 32$

$[\because\ a + b + c = 8,\ ab + bc + ca = 20$ and $a^2 + b^2 + c^2 = 24]$

Thus, $a^3 + b^3 + c^3 - 3abc = 32$

Ex.22 If $x^2 + \frac{1}{x^2} = 27$, find the value of $x - \frac{1}{x}$

Sol. $\left(x - \frac{1}{x}\right)^2 = x^2 - 2 \times x\ \frac{1}{x} + \frac{1}{x^2}$

$\Rightarrow$ $\left(x - \frac{1}{x}\right)^2 = x^2 - 2 + \frac{1}{x^2}$

$\Rightarrow$ $\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2$

$\Rightarrow$ $\left(x - \frac{1}{x}\right)^2 = 27 - 2 \left[\because x^2 + \frac{1}{x^2} = 27(\text{given})\right]$

$\Rightarrow$ $\left(x - \frac{1}{x}\right)^2 = 25$

$\Rightarrow$ $\left(x - \frac{1}{x}\right)^2 = (\pm 5)^2 \Rightarrow x - \frac{1}{x} = \pm 5$

Ex.23 If $x + \frac{1}{x} = 3$, find the value of

(i) $x^2 + \frac{1}{x^2}$ (ii) $x^3 + \frac{1}{x^3}$ (iii) $x^4 + \frac{1}{x^4}$

Sol. (i) $\left(x + \frac{1}{x}\right) = 3 \Rightarrow \left(x + \frac{1}{x}\right)^2 = (3)^2$ [On squaring both side]

$\Rightarrow x^2 + 2(x)\left(\frac{1}{x}\right) + \frac{1}{x^2} = 9$

$\Rightarrow x^2 + 2 + \frac{1}{x^2} = 9$

$\Rightarrow x^2 + \frac{1}{x^2} = 9 - 2 = 7$

Thus, $x^2 + \frac{1}{x^2} = 7$ .....(1)

(ii) $x + \frac{1}{x} = 3 \Rightarrow \left(x + \frac{1}{x}\right)^3 = (3)^3$ [On cubing both sides]

$x^3 + \left(\frac{1}{x}\right)^3 + 3.x.\frac{1}{x}\left(x + \frac{1}{x}\right) = 27$

$\Rightarrow x^3 + \frac{1}{x^3} + 3.3 = 27$ $\left[\therefore x + \frac{1}{x} = 3\right]$

$\Rightarrow x^3 + \frac{1}{x^3} = 27 - 9 = 18$

Thus, $x^3 + \frac{1}{x^3} = 18$

(iii) From (1), we have $x^2 + \frac{1}{x^2} = 7$

$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = (7)^2$ [On squaring both side]

$\Rightarrow x^4 + \frac{1}{x^4} + 2.\ x^2.\frac{1}{x^2} = 49$

$\Rightarrow x^4 + \frac{1}{x^4} + 2 = 49$

$\Rightarrow x^4 + \frac{1}{x^4} = 49 - 2 = 47$

Thux , $x^4 + \frac{1}{x^4} = 47$

Ex.24 If $x + y = 12$ and $xy = 32$, find the value of $x^2 + y^2$.

Sol. $(x + y)^2 = x^2 + y^2 + 2xy$

$\Rightarrow$ $144 = x^2 + y^2 + 2 \times 32$ [Putting $x + y = 12$ and $xy = 32$]

$\Rightarrow$ $144 - 64 = x^2 + y^2$

$\Rightarrow$ $x^2 + y^2 = 80$

Ex.25 Prove that : $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = [(a - b)^2 + (b - c)^2 + (c - a)^2]$

Sol. L.H.S. = $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)$

[Re-arranging the terms]

$= (a - b)^2 + (b - c)^2 + (c - a)^2$ = R.H.S.

Hence, $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = [(a-b)^2 + (b-c)^2 + (c-a)^2]$

Ex.26 If $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Sol. $a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow$ $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 2 \times 0$ [Multiplying both sides by 2]

$\Rightarrow$ $(a^2 - 2ab + b^2) + (b^2 - 2ab + c^2) + (c^2 - 2ac + a^2) = 0$

$\Rightarrow$ $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow$ $a - b = 0,\ b - c = 0\ ,\ c - a = 0$

[$\because$ Sum of positive quantities is zero if and only if each quantity is zero]

$\Rightarrow$ $a = b$, $b = c$ and $c = a$

$\Rightarrow$ $a = b = c$

ILLUSTRATION : If $x + y = 10$ and $x^2 + y^2 = 58$, find the value of $x^3 + y^3$.

SOLUTION : We know that $(x + y)^2 = x^2 + y^2 + 2xy$

Putting $x + y = 10$ and $x^2 + y^2 = 58$, we get $(10)^2 = 58 + 2xy$

$\Rightarrow$ $100 = 58 + 2xy$ $\Rightarrow$ $100 - 58 = 2xy$

$\Rightarrow$ $2xy = 42$ $\Rightarrow$ $xy = 42/2 = 21$ $\therefore$ $xy = 21$

Now, $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

$\Rightarrow$ $(10)^3 = x^3 + y^3 + 3 \times 21 \times 10$

$\Rightarrow$ $1000 = x^3 + y^3 + 630$

$\Rightarrow$ $1000 - 630 = x^3 + y^3$ Thus $x^3 + y^3 = 370$.

ILLUSTRATION : If $a + b + c = 0$ and $a^2 + b^2 + c^2 = 16$, find the value of $ab + bc + ca$

SOLUTION : We have $(a + b + c) = 0$ $\therefore$ $(a + b + c)^2 = (0)^2$

$\Rightarrow$ $(a + b + c)^2 = 0$ $\Rightarrow$ $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 0$

$\Rightarrow$ $16 + 2(ab + bc + ca) = 0$ $\Rightarrow$ $2(ab + bc + ca) = -16$

$\Rightarrow$ $ab + bc + ca = -16/2$ Thus, $ab + bc + ca = -8$

ILLUSTRATION : If $a^2 + b^2 + c^2 = 20$ and $a + b + c = 0$, find $ab + bc + ca$.

SOLUTION : We have $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow$ $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow$ $0^2 = 20 + 2(ab + bc + ca)$ $\Rightarrow$ $-20 = 2\ (ab + bc + ca)$

$\Rightarrow$ $-20/2 = (ab + bc + ca)$ $\Rightarrow$ $-10 = ab + bc + ca$

$\Rightarrow$ $ab + bc + ca = -10$.

ILLUSTRATION : If $4x^2 + y^2 = 40$ and $xy = 6$, find the value of $2x + y$.

SOLUTION : We have , $(2x + y)^2 = (2x)^2 + y^2 + 2 \times 2x \times y$

$\Rightarrow$ $(2x + y)^2 = (4x^2 + y^2) + 4xy$

$\Rightarrow$ $(2x + y)^2 = 40 + 4 \times 6$

$\Rightarrow$ $(2x + y)^2 = 64$

$\Rightarrow$ $(2x + y) = \pm\sqrt{64}$

$\Rightarrow$ (2x + y) = ± 8.

ILLUSTRATION : If $\left(x - \frac{1}{x}\right) = 3$, find the value of $\left(x^3 - \frac{1}{x^3}\right)$.

SOLUTION : We have, $\left[x - \frac{1}{x}\right]^3 = x^3 - \frac{1}{x^3} - 3x \times \frac{1}{x}\left[x - \frac{1}{x}\right]$

$\Rightarrow \left[x - \frac{1}{x}\right]^3 = x^3 - \frac{1}{x^3} - 3\left[x - \frac{1}{x}\right]$

$\Rightarrow$ Putting $x - \frac{1}{x} = 3$, We get

$\Rightarrow 3^3 = \left[x^3 - \frac{1}{x^3}\right] - 3 \times 3 \Rightarrow 27 = \left(x^3 - \frac{1}{x^3}\right) - 9$

$\Rightarrow \left(x^3 - \frac{1}{x^3}\right) = 27 + 9 \quad \Rightarrow \left(x^3 - \frac{1}{x^3}\right) = 36$

ILLUSTRATION : If $\left(x^2 + \frac{1}{x^2}\right) = 83$, find the value of $\left(x^3 - \frac{1}{x^3}\right)$

SOLUTION : We know that

$\Rightarrow \left[x - \frac{1}{x}\right]^2 = \left[x^2 + \frac{1}{x^2}\right] - 2 \quad \Rightarrow \left[x - \frac{1}{x}\right]^2 = 83 - 2 \quad \Rightarrow \left[x - \frac{1}{x}\right]^2 = 81$

$\Rightarrow \left[x - \frac{1}{x}\right]^2 = 9^2 \quad \Rightarrow \left[x - \frac{1}{x}\right] = 9 \Rightarrow \quad \left[x - \frac{1}{x}\right]^3 = 9^3$

$\Rightarrow \left[x^3 - \frac{1}{x^3}\right] - 3\left[x - \frac{1}{x}\right] = 729 \quad \Rightarrow \quad \left[x^3 - \frac{1}{x^3}\right] - 3 \times 9 = 729$

$\Rightarrow \left[x^3 - \frac{1}{x^3}\right] = 729 + 27 \quad \Rightarrow \quad \left[x^3 - \frac{1}{x^3}\right] = 756$

EXERCISE - I

Objective Type

1. A linear polynomial
(A) may have no zero
(B) may have one zero
(C) has one and only one zero always
(D) may have more than one zero

2. The coefficient of x^3 in the polynomial $5 + 2x + 3x^2 - 7x^3$ is
(A) 5 (B) 2 (C) 7 (D) – 7

3. The value of $P(x) = x^2 - 7x + 12$ at $x = 3$ is
(A) 42 (B) 0 (C) 8 (D) –6

4. A polynomial of degree 5 in x has at most
(A) 5 terms (B) 4 terms (C) 6 terms (D) 10 terms

5. Degree of the polynomial $4x^4 + 0x^3 + 0x^5 + 5x + 7$ is
(A) 4 (B) 5 (C) 3 (D) 7

6. Degree of the zero polynomial is
(A) 0 (B) 1 (C) any natural number (D) Not defined

7. Zero of the zero polynomial is
(A) 0 (B) 1 (C) Any real number (D) Not defined

8. One of the zeroes of the polynomial $2x^2 + 7x - 4$ is
(A) 2 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) –2

9. The zeroes of the polynomial $p(x) = x(x-1)(x-2)$ are
(A) 0 (B) 0 ,–1,–2 (C) 0, 1, –2 (D) 0, 1, 2

10. If $p(x) = x^2 - 2\sqrt{2}\,x + 1$, then $p(2\sqrt{2})$ is equal to
(A) 0 (B) 1 (C) $4\sqrt{2}$ (D) $8\sqrt{2}+1$

11. If $p(x) = x + 3$, then $p(x) + p(-x)$ is equal to
(A) 3 (B) 2x (C) 0 (D) 6

12. When the polynomial $x^3 + 3x^2 + 3x + 1$ is divided by $x + 1$, the remainder is
(A) 1 (B) 8 (C) 0 (D) –6

13. If $x^{51} + 51$ is divided by $x + 1$, the remainder is
(A) 0 (B) 1 (C) 49 (D) 50

14. The value of k for which $x - 1$ is a factor of the polynomial $4x^3 + 3x^2 - 4x + k$ is
(A) 3 (B) 0 (C) 1 (D) –3

15. The factors of $2x^2 - 3x^2 - 3x - 2$ are

(A) $(2x - 1)(x + 2)$ (B) $(2x + 1)(x - 2)$ (C) $(x + 1)(x - 2)$ (D) $(x - 1)(x + 2)$

16. The factors of $x^3 - 2x^2 - 13x - 10$ are

(A) $(x - 1)(x + 2)(x + 5)$ (B) $(x - 1)(x - 2)(x + 5)$ (C) $(x + 1)(x - 2)(x + 5)$ (D) $(x + 1)(x + 2)(x - 5)$

17. $(a - b)^3 + (b - c)^3 + (c - a)^3$ is equal to

(A) 3abc (B) $3a^3b^3c^3$

(C) $3(a - b)(b - c)(c - a)$ (D) $[a - (b + c)]^3$

18. $\dfrac{0.83 \times 0.83 \times 0.83 + 0.17 \times 0.17 \times 0.17}{0.83 \times 0.83 - 0.83 \times 0.17 + 0.17 \times 0.17}$ is equal to

(A) 1 (B) $(0.83)^3 + (0.17)^3$ (C) 0 (D) None of these

19. One of the factors of $(25x^2 - 1) + (1 + 5x)^2$ is

(A) $5 + x$ (B) $5 - x$ (C) $5x - 1$ (D) $10x$

20. Which of the following is a factor of $(x + y)^3 - (x^3 + y^3)$?

(A) $x^2 + y^2 + 2xy$ (B) $x^2 + y^2 - xy$ (C) xy^2 (D) $3xy$

21. If $\dfrac{x}{y} + \dfrac{y}{x} = -1$ $(x, y \neq 0)$, the value of $x^3 - y^3$ is

(A) 1 (B) -1 (C) 0 (D) 1/2

22. If $49x^2 - b = \left(7x + \dfrac{1}{2}\right)\left(7x - \dfrac{1}{2}\right)$, then the value of b is

(A) 0 (B) $\dfrac{1}{\sqrt{2}}$ (C) $\dfrac{1}{4}$ (D) $\dfrac{1}{2}$

23. If $x^2 + kx + 6 = (x + 2)(x + 3)$ for all x, then the value of k is

(A) 1 (B) -1 (C) 5 (D) 3

EXERCISE - II

1. The remainder when $P(x) = 4x^4 - 3x^3 - 2x^2 + x - 7$ is divided by $x-1$ is :
(A) – 7 (B) – 6 (C) 7 (D) 6

2. If the polynomials $2x^3 + ax^2 + 3x - 5$ and $x^3 + x^2 - 4x + a$ leave the same remainder when divided by $x-2$ then the value of a is :
(A) 3/13 (B) 3/14 (C) –13/3 (D) –3/13

3. a + b is a factor of :
(A) $a^4(b^2 - c^2) + b^4 (c^2 - b^2) + c^4 (a^2 - b^2)$ (B) $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$
(C) $(a+b+c)^3 - (b+c+a)^3 - (c+a-b)^3 - (a+b-c)^3$ (D) $a(b^4 - c^4) + b (c^4 - a^4 + c (a^4 + b^4)$

4. If x + 2 is factor of $\{(x+1)^5 + 2x + k\}^3$, then the value of 'k' is :
(A) 1 (B) 3 (C) 4 (D) 5

5. If $2x + y = 3$, $xy = 1$, then the value of $8x^3 + y^3$ is :
(A) 16 (B) 9 (C) 4 (D) 1

6. The factors of $(2x^2 - 3x - 2)(2x^2 - 3x) - 63$ are :
(A) $(x-3)(2x+3)(x-1)(x-7)$ (B) $(x+3)(2x-3)(x-1)(x-7)$
(C) $(x+3)(2x+3)(2x^2 - 3x + 7)$ (D) $(x-3)(2x+3)(2x^2 - 3x + 7)$

7. The value of k for which x + k is a factor of $x^3 + kx^2 - 2x + k + 4$ is :
(A) –4/3 (B) –5 (C) 2 (D) 6/7

8. The remainder when $x^6 - 3x^5 + 2x^2 + 8$ is divided by x + 1 is :
(A) 24 (B) 14 (C) 8 (D) 18

9. The value of m, in order that $x^2 - mx - 2$ is the quotient when $x^3 + 3x^2 - 4$ is divided by x + 2, is :
(A) –1 (B) 1 (C) 0 (D) –2

10. If $x - \frac{1}{x} = 5$ then $x^3 - \frac{1}{x^3}$ equals :
(A) 125 (B) 130 (C) 135 (D) 140

11. $(x+y)^3 - (x-y)^3$ can be factorized as :
(A) $2y(3y^2 + x^2)$ (B) $2y(3x^2 + y^2)$ (C) $2x(3x^2 + y^2)$ (D) $2x(x^2+3y)^2$

12. If$x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$ then the condition is :
(A) a+c+e = b+d (B) a–c–e = b–d (C) a+c+–e = b–d (D) a+c+e = d–b

13. When $x^{2010} + 1$ is divided by x+1 then remainder is :
(A) 0 (B) 2 (C) 1 (D) –1

14. $f(x) = ax^7 + bx^3 + cx - 5$ where a, b, c are constants. If f(–7) = 7 then f(7) equals to :
(A) –17 (B) –7 (C) 14 (D) 21

15. If $11^7 + 4^7$ is divided by 15 then the remainder is :
(A) 0 (B) 1 (C) 2 (D) –2

16. Which one of the following algebraic expression is a polynomial in variable x ?
(A) $x^2 + \frac{2}{x^2}$ (B) $\frac{\sqrt{x}+1}{\sqrt{x}}$ (C) $x^2 + \frac{3x^{3/2}}{\sqrt{x}}$ (D) None of these

17. p(x) = $\sqrt{3}$ is a polynomial of degree

(A) 3 (B) 0 (C) 1 (D) None of these

18. Degree of the polynomial $(x+2)(x^2-2x+4)$

(A) 2 (B) 3 (C) 4 (D) None of these

19. If $p(x) = x^3 + 2x + 1$ is divided by $x - 2$ then the remainder is

(A) 13 (B) 10 (C) 12 (D) None of these

20. If $8x^4 - 8x^2 + 7$ is is divided by $2x + 1$, the remainder is

(A) 11/2 (B) 13/2 (C) 15/2 (D) 17/2

21. If $x^3 + y^3 + z^3 - 3xyz = K(x+y+z)\{(x-y)^2 + (y-z)^2 + (z-x)^2\}$ then K

(A) 3/2 (B) 1/2 (C) 5/2 (D) None of these

22. The remainder obtained when $t^6 + 3t^2$ 10 is divided by $t^3 + 1$ is

(A) $t^2 - 11$ (B) $t^3 - 1$ (C) $3t^2 + 11$ (D) None of these

23. The difference of the degrees of the polynomials is $3x^2y^3 + 5xy^2 - x^3$ and $3x^6 - 4x^3 + 2$ is

(A) 2 (B) 3 (C) 1 (D) None of these

24. If $\left(a + \frac{1}{a}\right)^2 = b$ then $a^3 + 1/a^3$ is equal to

(A) b^3 (B) $b^{3/2}$ (C) $b^{3/2} - 3b^{1/2}$ (D) $b^{3/2} + 3b^{1/2}$

25. If $x^{1/3} + y^{1/3} + z^{1/3} = 0$ then

(A) $x^3 + y^3 + z^3 = 0$ (B) $x+y+z = 27xyz$ (C) $(x+y+z)^3 = 27xyz$ (D) $x^3+y^3+z^3 = 27\,xyz$

True of false

1. A binomial can have atmost two terms.

2. Every polynomial is a binomial.

3. A binomial may have degree 5.

4. Zero of a polynomial is always 0.

5. A polynomial can not have more than one zero.

6. The degree of the sum of two polynomials each of degree 5 in always 5.

7. $x^2 - 5x + 6$ can not be written as a product of two linear factors.

8. $(2a + b)^2 - (2b + a)^2 = 3(a^2 - b)^2$ is an identity.

Match the column

1. **Column-I** **Column-II**

	Column-I		Column-II
1.	If (x–1) are factors of $f(x)= px^3 + x^2 – 2x + q$, then p + q =	(a)	3
2.	If (x + 2) is a factor of $p(x) = ax^3 + bx^2 + x – 6$ and p(x) when divided by x –2 leaves remainder 4, then a + b =	(b)	–3
3.	If $a^3 + b^3 + c^3 = 3abc$ and a + b + c = 0, then $\frac{(b+c)^2}{bc}+\frac{(c+a)^2}{ca}+\frac{(a+b)^2}{ab} =$	(c)	2
4.	If x –1 is a factor of $4x^3 + 3x^2 – 4x + k$, then k =	(d)	1

2. **Column-I** **Column-II**

	Column-I		Column-II
1.	If $p(x) = x^3 + x^2 + x + 1$ is divided by x + 1, then remainder =	(a)	$-2-\sqrt{2}$
2.	If $x^3 – x^2 + x –1$ is divided by x + 1, then remainder =	(b)	–2
3.	If x –1 is a factor of $2x^2 + kx + \sqrt{2}$, then k =	(c)	0
4.	If x –1 is a factor of $x^2 + x + k$, then k =	(d)	– 4

3. **Column-I** **Column-II**

	Column-I		Column-II
1.	Degree of $x^5 – x^4 + 3$	(a)	0
2.	Degree of $2 – y^2 – y^3 + 2y^8$	(b)	1
3.	Degree of 2	(c)	8
4.	Degree of x + 2	(d)	5

4. **Column-I** **Column-II**

	Column-I		Column-II
1.	If x –2 is a factor of $x^3 – 2kx^2 + kx –1$, then k =	(a)	– 4
2.	If x –2 is a factor of $x^5 – 3x^4 – kx^3 + 3kx^2 + 2kx – 4$, then k=	(b)	–7
3.	If x + 2 is a factor $x^3 – kx^2 + 6x –k$, then k =	(c)	5/2
4.	if 2x –1 is a factor of $2x^3 + kx^2 + 11x + k + 3$, then k =	(d)	7/6

EXERCISE - III

Subjective Type

1. Which of the following expressions are polynomial ?

(i) $11x + 1$ (ii) $7x^2 5x + \sqrt{5}$ (iii) $t^3 - 2t + 1$ (iv) $x^2 - \frac{1}{x^2}$

(v) $\sqrt{y} + 5y - 1$ (vi) $z^{11} - 5z^7 + \frac{1}{4}$

2. Classify the following as linear, quadratic and cubic polynomials :

(i) $x^3 - 4$ (ii) $x^2 + 1$ (iii) $5x^2 - 3x + \sqrt{7}$ (iv) $1 + 5x$

(v) $4r^3$

3. Find the value of the following polynomial at the indicate value of variables:

(i) $p(x) = 5x^2 - 3x + 7$ at $x = 1$

(ii) $q(y) = 3y^2 - 4y + \sqrt{11}$ at $y = 2$

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$ at $t = a$

4. Find the zeroes of each of the following polynomials:

(i) $p(x) = x - 4$ (ii) $g(x) = 2x + 1$

(iii) $p(x) = (x + 1)(x + 2)$ (iv) $p(x) = (x - 1)(x - 2)(x - 3)$

(v) $p(x) = 7x^2$

5. Verify whether the following are zeroes of the polynomial indicated against them :

(i) $p(x) = 5x - 1$, $x = \frac{1}{5}$

(ii) $p(x) = (x - 2)(x - 5)$, $x = 2, 5$

(iii) $s(x) = x^2$, $x = 0 , 1$

(iv) $p(x) = 3x^2 - 1$, $x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$

(v) $g(x) = 5x^2 + 7x$ $x = 0, -\frac{7}{5}$

6. Show that -1 is a zero of the polynomial $2x^3 - x^2 + x + 4$

7. Show that 1 is not a zero of the polynomial $4x^4 - 3x^3 + 2x^2 - 5x + 1$

8. Use remainder theorem to find remainder when p(x) is divided by q(x) in the following questions:

(i) $p(x) = 2x^2 - 5x + 7$, $q(x) = x - 1$

(ii) $p(x) = x^9 - 5x^4 + 1$, $q(x) = x + 1$

(iii) $p(x) = 4x^3 - 12x^2 + 11x - 5$, $q(x) = x - \frac{1}{2}$

9. Find the value of k if $(x - 2)$ is a factor of $2x^3 - 6x^2 + 5x + k$

10. For what value of m is $2x^3 + mx^2 + 11x + m + 3$ exactly divisible by $(2x - 1)$?

11. Using factor theorem, show that $(a - b)$ is a factor of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$

12. Factorize each of the following expressions :

(i) $p^4 - 81q^4$

(ii) $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$

(iii) $24\sqrt{3}x^3 - 125y^3$

(iv) $125(x - y)^3 + (5y \;\; 3z)^3 + (3z - 5x)^3$

(v) $(x - y)^3 + (y - z)^3 + (z - x)^3$

13. If one of the factors of $x^2 + x - 20$ is $(x + 5)$, find other factor

14. Simplify :

(i) $\sqrt{2a^2 + 2\sqrt{6}ab + 3b^2}$ (iii) $\sqrt{36x^2 + 60x + 25}$

15. Write the expansion of the following

(i) $(9x + 2y + z)^2$ (ii) $(3x - 2y - z)^2$

16. Find the product of following :

$(5a - 3b)(25a^2 + 15ab + 9b^2)$

17. Let A and B are the remainders when the polynomial $x^3 + 2x^2 - 5ax - 7$ and $x^3 + ax^2 - 12x + 6$ are divided by $x + 1$ and $x - 2$ respectively. If $2A + B = 6$, find the value of a

18. With out actual division, prove that $a^4 + 2a^3 - 2a^2 + 2a - 3$ is exactly divisible $(a^2 + 2a - 3)$

19. If $(x + 1)$ and $(x - 1)$ are the factors of $mx^3 + x^2 - 2x + n$, find the value of m and n

20. If $x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, show that $a + c + e = b + d = 0$.

21. Find the value of $a^3 - 27b^3$ if $a - 3b = -6$ and $ab = -10$

22. If $x + y + z = 8$ and $xy + yz + zx = 20$, find the value of $x^3 + y^3 + z^3 - 3xyz$

23. Find the value of :

(i) $x^3 + y^3 - 12xy + 64$ when $x + y = -4$

(ii) $(x - a)^3 + (x - b)^3 + (x - c)^3 - 3(x - a)$

$(x - b)(x - c)$ when $a + b + c = 3x$

(iii) $(25)^3 - (29)^3 + (4)^3$

EXERCISE - IV

Factorize the following

1. $ap^2 + bp^2 + aq^2 + bq^2$

2. $1 + a + b + c + ab + bc + ca + abc$

3. $ab(x^2 + y^2) - xy(a^2 + b^2)$

4. $16(x + y)^2 - 40(x + y)(x - y) + 25(x - y)^2$

7. $a^3x^3 - 3a^2bx^2 + 3ab^2x - b^3$

11. $2\sqrt{2}a^3 + 16\sqrt{2}b^3 + c^3 - 12abc$

12. $3xy(1 + a^2 - b^2) + 6yz(-1 - a^2 + b^2) - 9zx(1 + a^2 - b^2)$

14. $a^2x^2 - b^2z^2 - b^2y^2 + c^2z^2 - a^2z^2 + c^2y^2 - a^2y^2 + b^2x^2 - c^2x^2$

18. $x^4 + x^2y^2 + y^4$

22. $(x^2 - ax - 5)(x^2 - ax - 11) - 16$

23. $x^2 - (a+1/a)x + 1$

24. Prove that : $(y + z - x)^3 + (z + x - y)^3 + (x + y - z)^3$
$= 3(y + z - x)(z + x - y)(x + y - z) = -24xyz$, if $x + y + z = 0$

25. Simplify : $\dfrac{\{(a^2-b^2)^3 + (b^2-c^2) + (c^2-a^2)^3\}}{\{(a-b)^3 + (b-c)^3 + (c-a)^3\}}$

26. If $x + y + z = 0$, prove that : $\dfrac{x^2}{yz} + \dfrac{y^2}{zx} + \dfrac{z^2}{xy} = 3$

28. Prove that : $a^3 + b^3 + c^3 - 3abc = \dfrac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

35. Use the factor theorem to prove that : $x + a$ is a factor of $x^{2n} - a^{2n}$ and $x^{2n+1} + a^{2n+1}$, where n is any integer.

36. Factorise $y^3 - 2y^2 - 29y - 42$ by using factor theorem.

37. What must be added to $(x^4 + 2x^3 - 2x^2 - 2x - 1)$ to obtain a polynomial which is exactly divisible be (x^2+2x-3) ?

38. If $2x^3 + ax^2 + bx - 6$ has $(x-1)$ as a factor and leaves a remainder 2 when divided by $(x-2)$, find the values of 'a' & 'b'.

39. A quadratic polynomial when divided by $x-1$ leaves a remainder of 1 & when divided by $x+1$, leaves a remainder of -3. Find the remainder that it will leave when divided by x^2-1.

40. A polynomial of degree greater than 2, when divided by $x-1$ leaves a remainder of 2 & when divided by $x-2$, leaves a remainder of 1. Find the remainder that it will leave when divided by $(x-1)(x-2)$.

Answer key

EXERCISE- I

Objective Type

1.	C	**2.**	D	**3.**	B	**4.**	C	**5.**	A	**6.**	D	**7.**	C
8.	B	**9.**	D	**10.**	B	**11.**	D	**12.**	C	**13.**	D	**14.**	D
15.	B	**16.**	D	**17.**	C	**18.**	A	**19.**	D	**20.**	D	**21.**	C
22.	C	**23.**	C										

EXERCISE- II

1.	A	**2.**	C	**3.**	A	**4.**	D	**5.**	B				
6.	D	**7.**	A	**8.**	B	**9.**	A	**10.**	D	**11.**	B	**12.**	A
13.	B	**14.**	A	**15.**	A	**16.**	C	**17.**	B	**18.**	B	**19.**	A
20.	A	**21.**	B	**22.**	C	**23.**	C	**24.**	C	**25.**	C		

True of False

1. False **2.** False **3.** True **4.** False **5.** False **6.** True **7.** False **8.** True

Match the column

1. $(1\to d), (2\to c), (3\to a), (4\to b)$ **2.** $(1\to c), (2\to d), (3\to a), (4\to b)$

3. $(1\to d), (2\to c), (3\to a), (4\to b)$ **4.** $(1\to d), (2\to c), (3\to a), (4\to b)$

EXERCISE-III

1. (i, (ii), (iii), (vi) **2.** (i) Cubic, (ii) Quadratic, (iii) Quadratic, (iv) Linear, (v) Cubic

3. (i) 9, (ii) $4 + \sqrt{11}$, (iii) $4a^4 + 5a^3 - a^2 + 6$ **4.** (i) 4, (ii) –1/2 , (iii) –1, –2, (iv) 1, 2, 3, (v) 0

5. (i) yes, (ii) Yes, both (iii) x = 0, x = 1 is not zero, (iv) $x = -\frac{1}{\sqrt{3}}$, $x = \frac{2}{\sqrt{3}}$ is not zero, (v) Yes, both

8. (i) 4, (ii) –5, (iii) –2 **9.** – 2 **10. – 7**

12. (i) $(p + 3q)(p - 3q)(p^2 + 9q^2)$ (ii) $\left(x-\sqrt{2}\right)\left(7\sqrt{2}x+4\right)$ (iii) $\left(2\sqrt{3}x-5y\right)\left(12x^2+10\sqrt{3}xy+25y^2\right)$

(iv) $15(x - y)(5y - 3z)(3z - 5x)$ (v) $3(x - y)(y - z)(z - x)$

13. $(x-4)$ **14.** (i) $\sqrt{2}a+\sqrt{3}b$ (iii) $(6x + 5)$

15. (i) $81x^2 + 4y^2 + z^2 + 36xy + 4yz + 18zx$ (ii) $9x^2 + 4y^2 + z^2 - 12xy + 4yz - 6xz$

16. $125a^3 - 27b^3$ **19.** m = 2, n = – 1 **21.** 324 **22.** 32 **23.** (i) 0 (iii) 0 (iv) – 8700

21. 324 **22.** 32 **23.** (i) 0 (ii) 0 (iii) – 8700

JEE EXPERT