CLASSROOM STUDY PACKAGE

MATHEMATICS

APPLICATION OF DERIVATIVES

JEE EXPERT

APPLICATION OF DERIVATIVES

CONTENTS

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APPLICATION OF DERIVATIVES

Rate of change, Tangent and Normal, Error and Approximation, Rolle's Theorem, LMVT, Monotonicity, Maxima Minima.

Derivative as rate of change

In various fields of applied mathematics one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.

An economist may want to study how the investment changes with respect to variations in interest rates.

A physician may want to know, how small changes in dosage can affect the body's response to a drug.

A physicist may want to know the rate of charge of distance with respect to time.

All questions of the above type can be interpreted and represented using derivatives.

Definition : The average rate of change of a function f(x) with respect to x over an interval [a, a + h] is defined as $\dfrac{f(a+h)-f(a)}{h}$.

Definition : The instantaneous rate of change of f(x) with respect to x is defined as $f'(x) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists.

Note : To use the word 'instantaneous', x may not be representing time. We usually use the word 'rate of change' to mean 'instantaneous rate of change'.

Example # 1 How fast the area of a circle increases when its radius is 5cm;
(i) with respect to radius (ii) with respect to diameter

Solution : (i) $A = \pi r^2$, $\dfrac{dA}{dr} = 2\pi r$

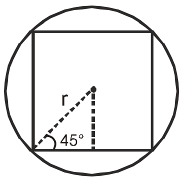
$$\therefore \quad \left.\frac{dA}{dr}\right]_{r=5} = 10\pi \text{ cm}^2/\text{cm}.$$

(ii) $A = \dfrac{\pi}{4}D^2$, $\dfrac{dA}{dD} = \dfrac{\pi}{2}D$

$$\therefore \quad \left.\frac{dA}{dD}\right]_{D=10} = \frac{\pi}{2} \,.\, 10 = 5\pi \text{ cm}^2/\text{cm}.$$

Example # 2 If area of circle increases at a rate of $2\text{cm}^2/\text{sec}$, then find the rate at which area of the inscribed square increases.

Solution : Area of circle, $A_1 = \pi r^2$. Area of square, $A_2 = 2r^2$ (see figure)



$$\frac{dA_1}{dt} = 2\pi r \frac{dr}{dt}, \qquad \frac{dA_2}{dt} = 4r \,.\, \frac{dr}{dt}$$

$$\because \quad 2 = 2\pi r \,.\, \frac{dr}{dt} \quad \Rightarrow \quad r\frac{dr}{dt} = \frac{1}{\pi}$$

$$\therefore \quad \frac{dA_2}{dt} = 4 \,.\, \frac{1}{\pi} = \frac{4}{\pi} \text{ cm}^2/\text{sec}$$

$\therefore$ Area of square increases at the rate $\dfrac{4}{\pi}$ cm^2/sec.

Example # 3 The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Solution. Let at some time t, the length of edge is x cm.

$v = x^3 \quad \Rightarrow \quad \frac{dv}{dt} = 3x^2 \frac{dx}{dt} \quad$ (but $\frac{dv}{dt} = 7$)

$\Rightarrow \quad \frac{dx}{dt} = \frac{7}{3x^2}$ cm/sec.

Now $\quad S = 6x^2$

$\frac{dS}{dt} = 12x \frac{dx}{dt} \quad \Rightarrow \quad \frac{dS}{dt} = 12x.\ \frac{7}{3x^2} = \frac{28}{x}$

when x = 4 cm, $\quad \frac{dS}{dt} = 7$ cm^2/sec.

Example # 4 Sand is pouring from pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one - sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?

Solution. $V = \frac{1}{3} \pi r^2 h$

but $\quad h = \frac{r}{6}$

$\Rightarrow \quad V = \frac{1}{3} \pi (6h)^2 h$

$\Rightarrow \quad V = 12\pi h^3$

$\frac{dV}{dt} = 36\pi h^2.\ \frac{dh}{dt}$

when, $\frac{dV}{dt} = 12$ cm^3/s $\quad$ and $\quad h = 4$ cm

$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi}$ cm/sec.

Self practice problems :

(1) Radius of a circle is increasing at rate of 3 cm/sec. Find the rate at which the area of circle is increasing at the instant when radius is 10 cm.

(2) A ladder of length 5 m is leaning against a wall. The bottom of ladder is being pulled along the ground away from wall at rate of 2cm/sec. How fast is the top part of ladder sliding on the wall when foot of ladder is 4 m away from wall.

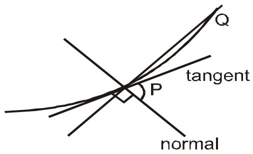
(3) Water is dripping out of a conical funnel of semi-vertical angle 45° at rate of 2cm^3/s. Find the rate at which slant height of water is decreasing when the height of water is $\sqrt{2}$ cm.

(4) A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment.

Answers : (1) 60π cm²/sec (2) $\frac{8}{3}$ cm/sec (3) $\frac{1}{\sqrt{2\pi}}$ cm/sec. (4) 140 ft/min.

Tangent and Normal

Let y = f(x) be function with graph as shown in figure. Consider secant PQ. If Q tends to P along the curve passing through the points Q_1, Q_2, ...
I.e. $Q \to P$, secant PQ will become tangent at P. A line through P perpendicular to tangent is called normal at P.



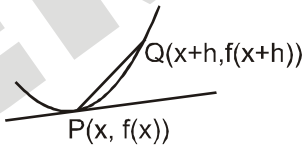
Figure

Geometrical Meaning of $\frac{dy}{dx}$

As $Q \to P$, $h \to 0$ and slope of chord PQ tends to slope of tangent at P (see figure).

Slope of chord PQ = $\frac{f(x+h)-f(x)}{h}$

$\lim\limits_{Q \to P}$ slope of chord PQ = $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$



$\Rightarrow$ slope of tangent at P = $f'(x) = \frac{dy}{dx}$

Equation of tangent and normal

$\left.\frac{dy}{dx}\right]_{(x_1, y_1)} = f'(x_1)$ denotes the slope of tangent at point (x_1, y_1) on the curve y = f(x). Hence the equation of tangent at (x_1, y_1) is given by

$(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

Also, since normal is a line perpendicular to tangent at (x_1, y_1) so its equation is given by

$(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

If $f'(x_1) = 0$, then tangent is the line $y = y_1$ and normal is the line $x = x_1$.

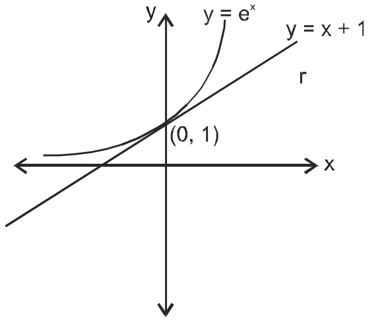
If $\lim\limits_{h \to 0} \frac{f(x_1+h)-f(x_1)}{h} = \infty$ or $-\infty$, then $x = x_1$ is tangent **(VERTICAL TANGENT)** and $y = y_1$ is normal.

Example # 5 Find equation of tangent to $y = e^x$ at x = 0. Hence draw graph

Solution At x = 0 $\Rightarrow$ $y = e^0 = 1$

Hence point of tangent is (0, 1)

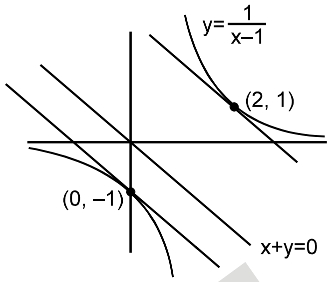
$\frac{dy}{dx} = e^x$ $\Rightarrow$ $\left.\frac{dy}{dx}\right|_{x=0} = 1$



Hence equation of tangent is
1 (x – 0) = (y – 1)
$\Rightarrow$ y = x + 1

Example # 6 Find the equation of all straight lines which are tangent to curve $y = \frac{1}{x-1}$ and which are parallel to the line $x + y = 0$.

Solution : Suppose the tangent is at (x_1, y_1) and it has slope – 1.



$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -1.$

$\Rightarrow \quad -\frac{1}{(x_1-1)^2} = -1.$

$\Rightarrow \quad x_1 = 0$ or 2

$\Rightarrow \quad y_1 = -1$ or 1

Hence tangent at (0, – 1) and (2, 1) are the required lines (see figure) with equations

$-1(x-0) = (y+1)$ and $-1(x-2) = (y-1)$

$\Rightarrow \quad x + y + 1 = 0$ and $y + x = 3$

Example # 7 Find equation of normal to the curve $y = |x^2 - |x||$ at $x = -2$.

Solution : In the neighborhood of $x = -2$, $y = x^2 + x$.

Hence the point of contact is (– 2, 2)

$\frac{dy}{dx} = 2x + 1 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=-2} = -3.$

So the slope of normal at (– 2, 2) is $\frac{1}{3}$.

Hence equation of normal is

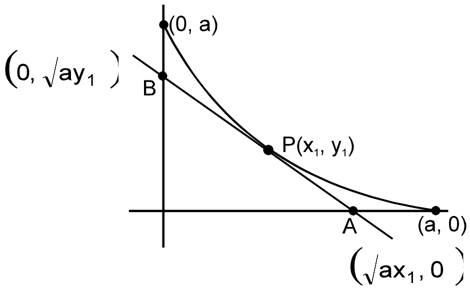
$\frac{1}{3}(x+2) = y - 2.$

$\Rightarrow \quad 3y = x + 8.$

Example # 8 Prove that sum of intercepts of the tangent at any point to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ on the coordinate axis is constant.

Solution : Let $P(x_1, y_1)$ be a variable point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, as shown in figure.

$\Rightarrow$ equation of tangent at point P is $-\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1) = (y - y_1)$



$$\Rightarrow \quad -\frac{x}{\sqrt{x_1}} + \sqrt{x_1} = \frac{y}{\sqrt{y_1}} - \sqrt{y_1}$$

$$\Rightarrow \quad \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1}$$

$$\Rightarrow \quad \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{a} \qquad (\because \quad \sqrt{x_1} + \sqrt{y_1} = \sqrt{a})$$

Hence point A is $\left(\sqrt{ax_1}, 0\right)$ and coordinates of point B is $(0, \sqrt{ay_1})$. Sum of intercepts

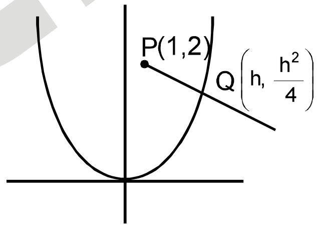
$$= \sqrt{a}\,(\sqrt{x_1} + \sqrt{y_1}) = \sqrt{a}\,.\,\sqrt{a} = a \qquad \text{(which is constant)}$$

Example # 9 Find the equation of all possible normal/s to the parabola $x^2 = 4y$ drawn from point (1, 2).

Solution : Let point Q be $\left(h, \frac{h^2}{4}\right)$ on parabola $x^2 = 4y$ as shown in figure

Now, m_{PQ} = slope of normal at Q.

Slope of normal $= -\left.\frac{dx}{dy}\right|_{x=h} = -\frac{2}{h}$



$$\Rightarrow \quad \frac{\frac{h^2}{4} - 2}{h-1} = -\frac{2}{h}$$

$$\Rightarrow \quad \frac{h^3}{4} - 2h = -2h + 2$$

$\Rightarrow \quad h^3 = 8 \qquad \Rightarrow \quad h = 2$

Hence coordinates of point Q is (2, 1) and so equation of required normal becomes $x + y = 3$.

Note : The equation gives only one real value of h, hence there is only one point of contact implying that only one real normal is possible from point (1, 2).

Self practice problems :

(5) Find the slope of the normal to the curve $x = 1 - a \sin \theta$, $y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$.

(6) Find the equation of the tangent and normal to the given curves at the given points.

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(ii) $y^2 = \frac{x^3}{4-x}$ at (2, – 2).

(7) Prove that area of the triangle formed by any tangent to the curve $xy = c^2$ and coordinate axes is constant.

(8) A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q . Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.

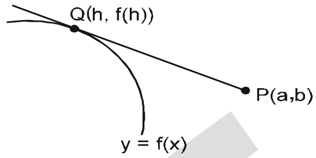
Answers : (5) $-\dfrac{a}{2b}$

(6) (i) Tangent : y = 2x + 1, Normal :x + 2y = 7

(ii) Tangent : 2x + y = 2, Normal :x – 2y = 6

Tangent from an external point

Given a point P(a, b) which does not lie on the curve y = f(x), then the equation of possible tangents to the curve y = f(x), passing through (a, b) can be found by solving for the point of contact Q.



Figure

$$f'(h) = \frac{f(h) - b}{h - a}$$

And equation of tangent is $y - b = \dfrac{f(h) - b}{h - a}(x - a)$

Example # 10 Find value of c such that line joining points (0, 3) and (5, – 2) becomes tangent to curve $y = \dfrac{c}{x+1}$.

Solution : Equation of line joining A & B is x + y = 3

Solving this line and curve we get

$3 - x = \dfrac{c}{x+1} \quad \Rightarrow \quad x^2 - 2x + (c - 3) = 0$ ......(i)

For tangency, roots of this equation must be equal.

Hence discriminant of quadratic equation = 0

$\Rightarrow \quad 4 = 4(c - 3) \quad \Rightarrow \quad c = 4$

Putting c = 4, equation (i) becomes

$x^2 - 2x + 1 = 0 \Rightarrow \quad x = 1$

Hence point of contact becomes (1, 2).

Note : If a line touches a curve then on solving the equation of line and curve we get at least two repeated roots corresponding to point of contact.

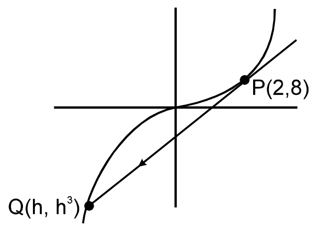
Example # 11 Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Solution : Equation of tangent at (2, 8) is

y = 12x – 16

Solving this with $y = x^3$

$x^3 - 12x + 16 = 0$



This cubic will give all points of intersection of line and curve $y = x^3$ i.e., point P and Q. (see figure)

But, since line is tangent at P so x = 2 will be a repeated root of equation $x^3 - 12x + 16 = 0$ and another root will be x = h. Using theory of equations :

sum of roots $\Rightarrow \quad 2 + 2 + h = 0 \quad \Rightarrow \quad h = -4$

Hence coordinates of Q are (– 4, – 64)

Self practice problems :

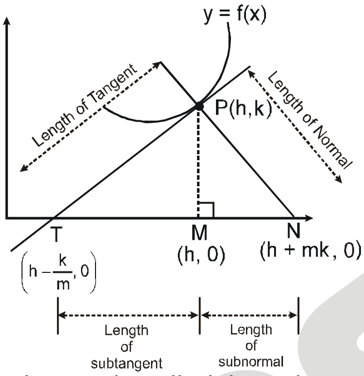
(9) How many tangents are possible from origin to the curve $y = (x + 1)^3$. Also find the equation of these tangents.

(10) Find the equation of tangent to the hyperbola $y = \frac{x+9}{x+5}$ which passes through (0, 0) origin

Answers : (9) $y = 0,\ 4y = 27x$ (10) $x + y = 0;\ 25y + x = 0$

Lengths of tangent, normal, subtangent and subnormal :

Let P (h, k) be any point on curve y = f(x). Let tangent drawn at point P meets x-axis at T & normal at point P meets x-axis at N. Then the length PT is called the length of tangent and PN is called length of normal. (as shown in figure)



Projection of segment PT on x-axis, TM, is called the subtangent and similarly projection of line segment PN on x axis, MN is called subnormal.

Let $m = \left.\frac{dy}{dx}\right]_{(h,k)}$ = slope of tangent.

Hence equation of tangent is $m(x - h) = (y - k)$.

Putting y = 0, we get x - intercept of tangent is $x = h - \frac{k}{m}$

Similarly, the x-intercept of normal is $x = h + km$

Now, length PT, PN, TM, MN can be easily evaluated using distance formula

(i) $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent (ii) $PN = |k|\sqrt{1+m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent (iv) $MN = |km|$ = Length of subnormal.

Example # 12 Find the length of tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point x = 0.

Solution : Here, $m = \left.\frac{dy}{dx}\right|_{x=0}$

$\frac{dy}{dx} = 3x^2 + 6x + 4 \quad \Rightarrow \quad m = 4$

and, $k = y(0) \quad \Rightarrow \quad k = -1$

$\ell = |k|\sqrt{1 + \frac{1}{m^2}} \quad \Rightarrow \quad \ell = |(-1)|\sqrt{1+\frac{1}{16}} = \frac{\sqrt{17}}{4}$

Example # 13 Prove that for the curve $y = be^{x/a}$, the length of subtangent at any point is always constant.

Solution : $y = be^{x/a}$

Let the point be (x_1, y_1)

$$\Rightarrow \quad m = \left.\frac{dy}{dx}\right|_{x=x_1} = \frac{b.e^{x_1/a}}{a} = \frac{y_1}{a}$$

Now, length of subtangent = $\left|\frac{y_1}{m}\right| = \left|\frac{y_1}{y_1/a}\right| = |\,a\,|$; which is always constant.

Example # 14 For the curve $y = a\,\ell n\,(x^2 - a^2)$ show that sum of lengths of tangent & subtangent at any point is proportional to coordinates of point of tangency.

Solution : Let point of tangency be (x_1, y_1)

$$m = \left.\frac{dy}{dx}\right|_{x=x_1} = \frac{2ax_1}{x_1^2 - a^2}$$

Length of tangent + subtangent = $|y_1|\sqrt{1 + \frac{1}{m^2}} + \left|\frac{y_1}{m}\right|$

$$= |y_1|\sqrt{1 + \frac{(x_1^2 - a^2)^2}{4a^2x_1^2}} + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right|$$

$$= |y_1|\frac{\sqrt{x_1^4 + a^4 + 2a^2x_1^2}}{2|\,ax_1\,|} + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right|$$

$$= \left|\frac{y_1(x_1^2 + a^2)}{2ax_1}\right| + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right| = \frac{|\,y_1\,|\,(2x_1^2)}{2\,|\,ax_1\,|} = \left|\frac{x_1\,y_1}{a}\right|$$

Self practice problems :

(11) For the curve $x^{m+n} = a^{m-n}\,y^{2n}$, where a is a positive constant and m, n are positive integers, prove that the m^{th} power of subtangent varies as n^{th} power of subnormal.

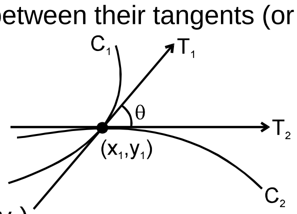
(12) Prove that the segment of the tangent to the curve $y = \frac{a}{2}\,\ell n\,\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}} - \sqrt{a^2-x^2}$ contained between the y-axis & the point of tangency has a constant length .

(13) Find the length of the subnormal to the curve $y^2 = x^3$ at the point (4, 8).

Answer : (13) 24

Angle between the curves

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves (as shown in figure).



$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1).

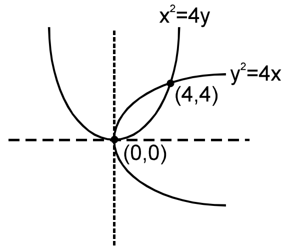
Notes : (i) The angle is defined between two curves if the curves are intersecting. This can be ensured by finding their point of intersection or graphically.

(ii) If the curves intersect at more than one point then angle between curves is found out with respect to the point of intersection.

(iii) Two curves are said to be orthogonal if angle between them at **each** point of intersection is right angle. i.e. $m_1\,m_2 = -1$.

Example # 15 Find angle between $y^2 = 4x$ and $x^2 = 4y$. Are these two curves orthogonal?

Solution : $y^2 = 4x$ and $x^2 = 4y$ intersect at point (0, 0) and (4, 4) (see figure).



$C_1 : y^2 = 4x$ $\qquad$ $C_2 : x^2 = 4y$

$\frac{dy}{dx} = \frac{2}{y}$ $\qquad$ $\frac{dy}{dx} = \frac{x}{2}$

$\left.\frac{dy}{dx}\right|_{(0,\,0)} \to \infty$ $\qquad$ $\left.\frac{dy}{dx}\right|_{(0,\,0)} = 0$

Hence tan θ = 90° at point (0, 0)

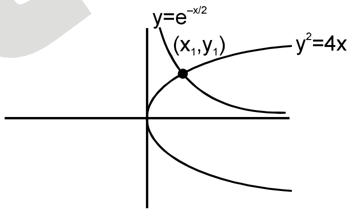
$\left.\frac{dy}{dx}\right|_{(4,\,4)} = \frac{1}{2}$ $\qquad$ $\left.\frac{dy}{dx}\right|_{(4,\,4)} = 2$

$$\tan\theta = \left|\frac{2-\frac{1}{2}}{1+2.\frac{1}{2}}\right| = \frac{3}{4}$$

Two curves are not orthogonal because angle between them at (4, 4) is not 90°.

Example # 16 Find the angle between curves $y^2 = 4x$ and $y = e^{-x/2}$

Solution : Let the curves intersect at point (x_1, y_1) (see figure).



for $y^2 = 4x$, $\qquad \left.\frac{dy}{dx}\right|_{(x_1,\,y_1)} = \frac{2}{y_1}$

and for $y = e^{-x/2}$ $\qquad \left.\frac{dy}{dx}\right|_{(x_1,\,y_1)} = -\frac{1}{2}\ e^{-x_1/2} = -\frac{y_1}{2}$

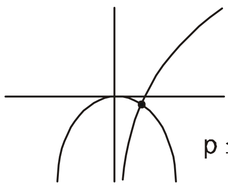
$\Rightarrow$ $\quad m_1 m_2 = -1$ $\qquad$ Hence θ = 90°

Note that we have not actually found the intersection point but geometrically we can see that the curves intersect.

Example # 17 Find possible values of p such that the equation $px^2 = \ell nx$ has exactly one solution.

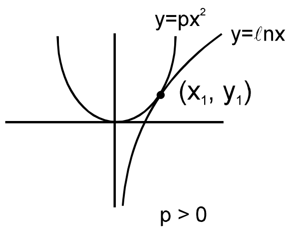
Solution : Two curves must intersect at only one point.

(i)



$p \le 0$

(ii)



I. If $p \le 0$ then there exists only one solution (see graph - (i))

II. If $p > 0$

then the two curves must only touch each other

i.e. $\quad$ tangent at $y = px^2$ and $y = \ell nx$ must have same slope at point (x_1, y_1)

$\Rightarrow$ $\quad 2px_1 = \frac{1}{x_1}$

$\Rightarrow$ $\quad x_1^{\,2} = \frac{1}{2p}$ $\qquad$ ....(i) also $y_1 = px_1^2$ $\Rightarrow$ $y_1 = p\left(\frac{1}{2p}\right)$

$\Rightarrow$ $y_1 = \frac{1}{2}$ ....(ii)

and $y_1 = \ell n x_1$ $\Rightarrow$ $\frac{1}{2} = \ell n x_1$

$\Rightarrow$ $x_1 = e^{1/2}$ ....(iii)

$\because$ $x_1^2 = \frac{1}{2p}$ $\Rightarrow$ $e = \frac{1}{2p}$ $\Rightarrow$ $p = \frac{1}{2e}$

Hence possible values of p are $(-\infty, 0] \cup \left\{\frac{1}{2e}\right\}$

Self practice problem :

(14) Find the angle of intersection of the following curves:

(i) $y = x^2$ & $6y = 7 - x^3$ at (1, 1)

(ii) $x^2 - y^2 = 5$ & $\frac{x^2}{18} + \frac{y^2}{8} = 1$.

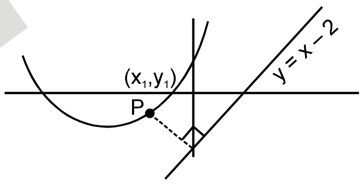
Answers : (14) (i) $\pi/2$ (ii) $\pi/2$

Shortest distance between two curves

Shortest distance between two non-intersecting differentiable curves is always along their common normal.
(Wherever defined)

Example # 18 Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Solution : Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$



then $\left.\frac{dy}{dx}\right|_{(x_1, y_1)}$ = slope of line

$\Rightarrow$ $2x_1 + 3 = 1$ $\Rightarrow$ $x_1 = -1$ $\Rightarrow$ $y_1 = 0$

Hence point (– 1, 0) is the closest and its perpendicular distance from the line $y = x - 2$ will give the shortest distance

$\Rightarrow$ $p = \frac{3}{\sqrt{2}}$.

Self practice problem :

(15) Find the minimum & maximum values of $(x + 2)^2 + (y - 1)^2$, if $(x - 2)^2 + (y + 1)^2 \le 4$.

Answer : (15) $2\sqrt{5} - 2,\ 2\sqrt{5} + 2$.

Error and Approximation :

Let y = f(x) be a function. If these is an error δx in x then corresponding error in y is $\delta y = f(x + \delta x) - f(x)$.

We have $\lim\limits_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{dy}{dx} = f'(x)$

We define the differential of y, at point x, corresponding to the increment δx as $f'(x)\,\delta x$ and denote it by dy.

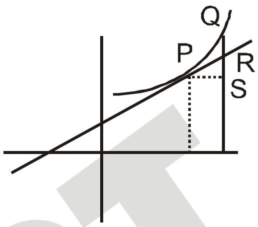
i.e. $dy = f'(x)\,\delta x$.

Let $P(x, f(x))$, $Q((x + \delta x), f(x + \delta x))$ (as shown in figure)

$\delta y = QS$,

$\delta x = PS$,

$dy = RS$



In many practical situations, it is easier to evaluate dy but not δy.

Rolle's Theorem :

If a function f defined on [a, b] is

(i) continuous on [a, b]

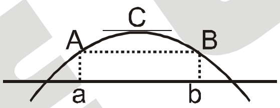
(ii) derivable on (a, b) and

(iii) f(a) = f(b),

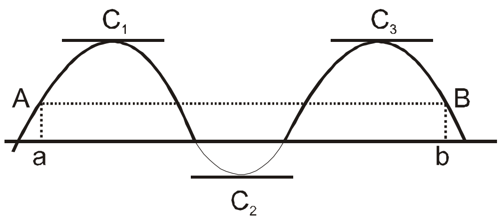
then there exists at least one real number c between a and b (a < c < b) such that $f'(c) = 0$

Geometrical Explanation of Rolle's Theorem :

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn (as shown in figure).



A(a, f(a)), B(b, f(b)), f(a) = f(b), C(c, f(c)), $f'(c) = 0$.



C_1 $(c_1, f(c_1))$, $f'(c_1) = 0$

C_2 $(c_2, f(c_2))$, $f'(c_2) = 0$

C_3 $(c_3, f(c_3))$, $f'(c_3) = 0$

The theorem simply states that between two points with equal ordinates on the graph of f(x), there exists at least one point where the tangent is parallel to x-axis.

Algebraic Interpretation of Rolle's Theorem :

Between two zeros a and b of f(x) (i.e. between two roots a and b of f(x) = 0) there exists at least one zero of $f'(x)$

Example # 19 : Verify Rolle's theorem for $f(x) = (x - a)^n (x - b)^m$, where m, n are positive real numbers, for $x \in [a, b]$.

Solution : Being a polynomial function f(x) is continuous as well as differentiable. Also f(a) = f(b)

$\Rightarrow$ $f'(x) = 0$ for some $x \in (a, b)$

$n(x - a)^{n-1} (x - b)^m + m(x - a)^n (x - b)^{m-1} = 0$

$\Rightarrow$ $(x - a)^{n-1} (x - b)^{m-1} \; [(m + n) x - (nb + ma)] = 0$

$\Rightarrow$ $x = \dfrac{nb + ma}{m + n}$, which lies in the interval (a, b), as $m, n \in R^+$.

Example # 20 : If $2a + 3b + 6c = 0$ then prove that the equation $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

Solution : Let $f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx$

$f(0) = 0$ and $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c = 2a + 3b + 6c = 0$

If $f(0) = f(1)$ then $f'(x) = 0$ for some value of $x \in (0, 1)$

$\Rightarrow$ $ax^2 + bx + c = 0$ for at least one $x \in (0, 1)$

Self Practice Problems :

(16) If f(x) satisfies condition in Rolle's theorem then show that between two consecutive zeros of $f'(x)$ there lies at most one zero of f(x).

(17) Show that for any real numbers λ, the polynomial $P(x) = x^7 + x^3 + \lambda$, has exactly one real root.

Lagrange's Mean Value Theorem (LMVT) :

If a function f defined on [a, b] is

(i) continuous on [a, b] and

(ii) derivable on (a, b)

then there exists at least one real numbers between a and b (a < c < b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof : Let us consider a function $g(x) = f(x) + \lambda x$, $x \in [a, b]$

where λ is a constant to b determined such that g(a) = g(b).

$\therefore$ $\lambda = -\dfrac{f(b) - f(a)}{b - a}$

Now the function g(x), being the sum of two continuous and derivable functions it self

(i) continuous on [a, b]

(ii) derivable on (a, b) and

(iii) g(a) = g(b).

Therefore, by Rolle's theorem there exists a real number $c \in (a, b)$ such that $g'(c) = 0$

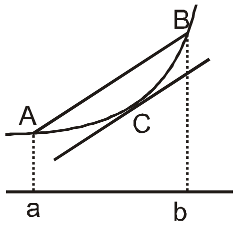
But $g'(x) = f'(x) + \lambda$

$\therefore$ $0 = g'(c) = f'(c) + \lambda$

$$f'(c) = -\lambda = \frac{f(b) - f(a)}{b - a}$$

Geometrical Interpretation of LMVT :

The theorem simply states that between two points A and B of the graph of f(x) there exists at least one point where tangent is parallel to chord AB.



C(c, f(c)), $f'(c)$ = slope of AB.

Alternative Statement : If in the statement of LMVT, b is replaced by a + h, then number c between a and b may be written as a + θh, where 0 < θ < 1. Thus

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h) \qquad \text{or} \qquad f(a+h) = f(a) + hf'(a+\theta h),\ 0<\theta<1$$

Example # 21 : Verify LMVT for $f(x) = -x^2 + 4x - 5$ and $x \in [-1, 1]$

Solution : $f(1) = -2$; $f(-1) = -10$

$\Rightarrow \quad f'(c) = \dfrac{f(1)-f(-1)}{1-(-1)}$

$\Rightarrow \quad -2c + 4 = 4 \quad \Rightarrow \quad c = 0$

Example # 22 : Using Lagrange's mean value theorem, prove that if b > a > 0,

then $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$

Solution : Let $f(x) = \tan^{-1} x$; $x \in [a, b]$ applying LMVT

$f'(c) = \dfrac{\tan^{-1}b - \tan^{-1}a}{b-a}$ for a < c < b and $f'(x) = \dfrac{1}{1+x^2}$,

Now $f'(x)$ is a monotonically decreasing function

Hence if a < c < b $\Rightarrow$ $f'(b) < f'(c) < f'(a)$

$\Rightarrow \quad \dfrac{1}{1+b^2} < \dfrac{\tan^{-1}b - \tan^{-1}a}{b-a} < \dfrac{1}{1+a^2}$ Hence proved

Example # 23 : Let $f : R \to R$ be a twice differentiable function such that f(2) = 8, f(4) > 64, f(7) = 343 then show that there exists a $c \in (2, 7)$ such that $f''(c) < 6c$.

Solution: Consider $g(x) = f(x) - x^3$

By LMVT

$\dfrac{g(4)-g(2)}{4-2} = g'(c_1)$, $2 < c_1 < 4$

and

$\dfrac{g(7)-g(4)}{7-4} = g'(c_2)$, $4 < c_2 < 7$

$g'(c_1) > 0$, $g'(c_2) < 0$

By LMVT

$\dfrac{g'(c_2)-g'(c_1)}{c_2-c_1} = g''(c)$, $c_1 < c < c_2$

$\Rightarrow g''(c) < 0$

$\Rightarrow f''(c) - 6c < 0$ for same $c \in (c_1, c_2)$ c (2, 7)

Self Practice Problems

(18) If a function f(x) satisfies the conditions of LMVT and $f'(x) = 0$ for all $x \in (a, b)$, then f(x) is constant on [a, b].

(19) Using LMVT, prove that if two functions have equal derivatives at all points of (a, b), then they differ by a constant

(20) If a function f is
(i) continuous on [a, b],
(ii) derivable on (a, b) and (iii) $f'(x) > 0$, $x \in (a, b)$, then show that f(x) is strictly increasing on [a, b].

Monotonicity of a function :

Let f be a real valued function having domain $D(D \subset R)$ and S be a subset of D. f is said to be monotonically increasing (non decreasing) (increasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$. f is said to be monotonically decreasing (non increasing) (decreasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

f is said to be strictly increasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Similarly, f is said to be strictly decreasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Notes : (i) f is strictly increasing $\Rightarrow$ f is monotonically increasing (non decreasing). But converse need not be true.

(ii) f is strictly decreasing $\Rightarrow$ f is monotonically decreasing (non increasing). Again, converse need not be true.

(iii) If f(x) = constant in S, then f is increasing as well as decreasing in S

(iv) A function f is said to be an increasing function if it is increasing in the domain. Similarly, if f is decreasing in the domain, we say that f is monotonically decreasing

(v) f is said to be a monotonic function if either it is monotonically increasing or monotonically decreasing

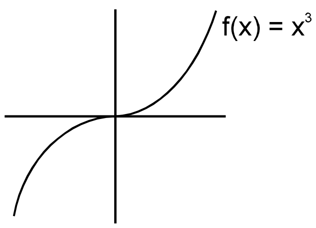
(vi) If f is increasing in a subset of S and decreasing in another subset of S, then f is non monotonic in S.

Application of differentiation for detecting monotonicity :

Let I be an interval (open or closed or semi open and semi closed)
(i) If $f'(x) > 0 \ \forall \ x \in I$, then f is strictly increasing in I
(ii) If $f'(x) < 0 \ \forall \ x \in I$, then f is strictly decreasing in I

Note : Let I be an interval (or ray) which is a subset of domain of f. If $f'(x) > 0$, $\forall \ x \in I$, except for countably many points where $f'(x) = 0$, then f(x) is strictly increasing in I.
{$f'(x) = 0$ at countably many points $\Rightarrow f'(x) = 0$ does not occur on an interval which is a subset of I}

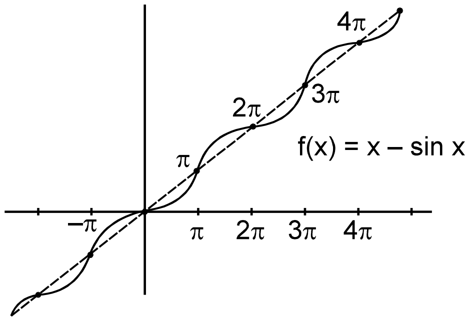
Example # 24 : Let $f(x) = x^3$. Find the intervals of monotonicity.
Solution : $f'(x) = 3x^2$
$f'(x) > 0$ everywhere except at x = 0. Hence f(x) will be strictly increasing function for $x \in R$ {see figure}



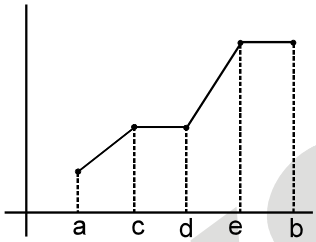
Example # 25 : Let $f(x) = x - \sin x$. Find the intervals of monotonicity.

Solution : $f'(x) = 1 - \cos x$

Now, $f'(x) > 0$ every where, except at $x = 0, \pm 2\pi, \pm 4\pi$ etc. But all these points are discrete (countable) and do not form an interval. Hence we can conclude that f(x) is strictly increasing in R. In fact we can also see it graphically.



Example # 26 : Let us consider another function whose graph is shown below for $x \in (a, b)$.



Solution : Here also $f'(x) \geq 0$ for all $x \in (a, b)$. But, note that in this case, $f'(x) = 0$ holds for all $x \in (c, d)$ and (e, b). Thus the given function is increasing (monotonically increasing) in (a, b), but not strictly increasing.

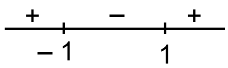
Example # 27 : Find the intervals in which $f(x) = x^3 - 3x + 2$ is increasing.

Solution : $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

$f'(x) = 3(x - 1)(x + 1)$

for M.I. $f'(x) \geq 0 \quad \Rightarrow \quad 3(x - 1)(x + 1) \geq 0$



$\Rightarrow \quad x \in (-\infty, -1] \cup [1, \infty)$, thus f is increasing in $(-\infty, -1]$ and also in $[1, \infty)$

Example # 28 : Find the intervals of monotonicity of the following functions.

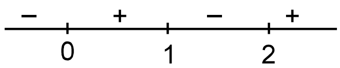
(i) $f(x) = x^2 (x - 2)^2$ (ii) $f(x) = x\,\ell n\, x$

(iii) $f(x) = \sin x + \cos x$; $x \in [0, 2\pi]$

Solution : (i) $f(x) = x^2 (x - 2)^2$

$f'(x) = 4x (x - 1)(x - 2)$

observing the sign change of $f'(x)$



Hence increasing in $[0, 1]$ and in $[2, \infty)$

and decreasing for $x \in (-\infty, 0]$ and $[1, 2]$

(ii) $f(x) = x\,\ell n\, x$

$f'(x) = 1 + \ell n\, x$

$f'(x) \geq 0 \quad \Rightarrow \quad \ell n\, x \geq -1 \quad \Rightarrow \quad x \geq \dfrac{1}{e}$

$\Rightarrow$ increasing for $x \in \left[\dfrac{1}{e}, \infty\right)$ and decreasing for $x \in \left(0, \dfrac{1}{e}\right]$.

(iii) $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

for increasing $f'(x) \geq 0 \Rightarrow \cos x \geq \sin x$

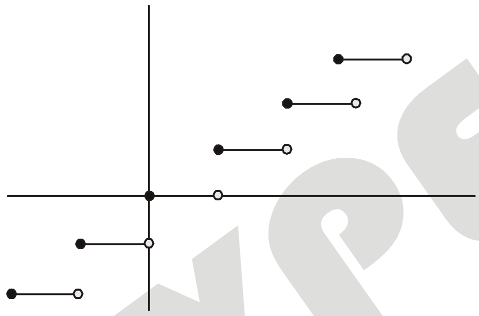
$\Rightarrow$ f is increasing in $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$

f is decreasing in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Note : If a function f(x) is increasing in (a, b) and f(x) is continuous in [a, b], then f(x) is increasing on [a, b]

Example # 29: f(x) = [x] is a step up function. Is it a strictly increasing function for $x \in R$.

Solution : No, f(x) = [x] is increasing (monotonically increasing) (non-decreasing), but not strictly increasing function as illustrated by its graph.



Example # 30 : If $f(x) = \sin^4x + \cos^4x + bx + c$, then find possible values of b and c such that f(x) is monotonic for all $x \in R$

Solution : $f(x) = \sin^4x + \cos^4x + bx + c$

$f'(x) = 4\sin^3x \cos x - 4\cos^3x \sin x + b = -\sin 4x + b.$

<u>Case - (i)</u> : for M.I. $f'(x) \geq 0$ for all $x \in R$

$\Leftrightarrow b \geq \sin 4x$ for all $x \in R \Leftrightarrow b \geq 1$

<u>Case - (ii)</u> : for M.D. $f'(x) \leq 0$ for all $x \in R$

$\Leftrightarrow b \leq \sin 4x$ for all $x \in R \Leftrightarrow b \leq -1$

Hence for f(x) to be monotonic $b \in (-\infty, -1] \cup [1, \infty)$ and $c \in R$.

Example # 31: Find possible values of 'a' such that $f(x) = e^{2x} - (a + 1)e^x + 2x$ is monotonically increasing for $x \in R$

Solution : $f(x) = e^{2x} - (a + 1)e^x + 2x$

$f'(x) = 2e^{2x} - (a + 1)e^x + 2$

Now, $2e^{2x} - (a + 1)e^x + 2 \geq 0$ for all $x \in R$

$\Rightarrow 2\left(e^x + \frac{1}{e^x}\right) - (a + 1) \geq 0$ for all $x \in R$

$(a + 1) \leq 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow a + 1 \leq 4 \quad \left(\because \; e^x + \frac{1}{e^x} \text{ has minimum value } 2\right)$

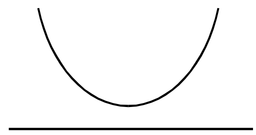
$\Rightarrow a \leq 3$

<u>Aliter</u> (Using graph)

$2e^{2x} - (a + 1)e^x + 2 \geq 0$ for all $x \in R$

putting $e^x = t$; $t \in (0, \infty)$

$2t^2 - (a + 1)t + 2 \geq 0$ for all $t \in (0, \infty)$



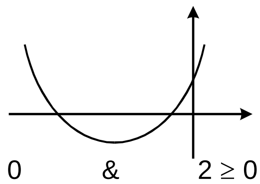
Case - (i) : $D \le 0$

$\Rightarrow$ $(a + 1)^2 - 4 \le 0$

$\Rightarrow$ $(a + 5)(a - 3) \le 0$

$\Rightarrow$ $a \in [-5, 3]$

or



Case - (ii) : both roots are non positive

$D \ge 0$ & $-\dfrac{b}{2a} < 0$ & $f(0) \ge 0$

$\Rightarrow$ $a \in (-\infty, -5] \cup [3, \infty)$ & $\dfrac{a+1}{4} < 0$ & $2 \ge 0$

$\Rightarrow$ $a \in (-\infty, -5] \cup [3, \infty)$ & $a < -1$ & $a \in R$

$\Rightarrow$ $a \in (-\infty, -5]$

Taking union of (i) and (ii), we get $a \in (-\infty, 3]$.

Self practice problems :

(21) Find the intervals of monotonicity of the following functions.

(i) $f(x) = -x^3 + 6x^2 - 9x - 2$ (ii) $f(x) = x + \dfrac{1}{x+1}$

(iii) $f(x) = x \,.\, e^{x-x^2}$ (iv) $f(x) = x - \cos x$

(22) Let $f(x) = x - \tan^{-1}x$. Prove that f(x) is monotonically increasing for $x \in R$.

(23) If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ monotonically increases for $\forall x \in R$, then find range of values of a

(24) Let $f(x) = e^{2x} - ae^x + 1$. Prove that f(x) cannot be monotonically decreasing for $\forall x \in R$ for any value of 'a'.

(25) The values of 'a' for which function $f(x) = (a + 2)x^3 - ax^2 + 9ax - 1$ monotonically decreasing for $\forall x \in R$.

Answers : (21) (i) I in [1, 3] ; D in $(-\infty, 1] \cup (3, \infty)$

(ii) I in $(-\infty, -2] \cup [0, \infty)$; D in $[-2, -1) \cup (-1, 0]$

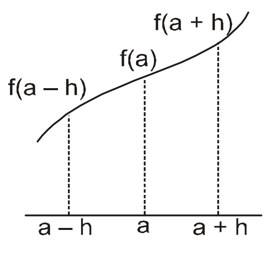
(iii) I in $\left[-\dfrac{1}{2}, 1\right]$; D in $\left(-\infty, -\dfrac{1}{2}\right] \cup [1, \infty)$

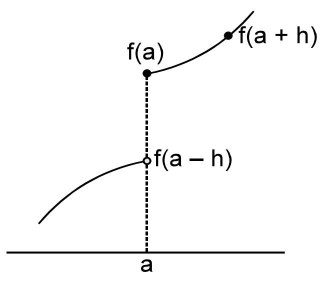
(iv) I for $x \in R$

(23) $a \ge 0$ (25) $-\infty < a \le -3$

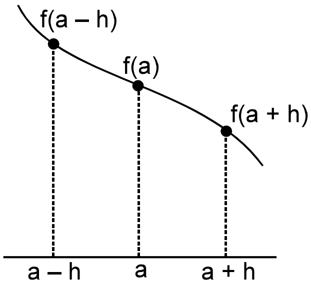
Monotonicity of function about a point :

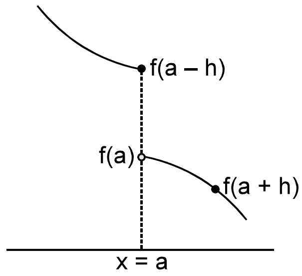
1. A function f(x) is called as a strictly increasing function about a point (or at a point) $a \in D_f$ if it is strictly increasing in an open interval containing a (as shown in figure).



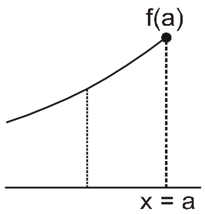


2. A function f(x) is called a strictly decreasing function about a point x = a, if it is strictly decreasing in an open interval containing a (as shown in figure).

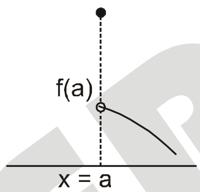




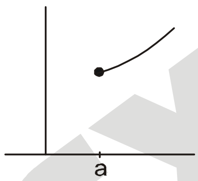
Note : If x = a is a boundary point then use the appropriate one sided inequality to test monotonicity of f(x).



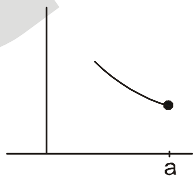
f is increasing at x = a



f is decreasing at x = a



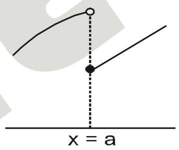
f is increasing at x = a



f is decreasing at x = a

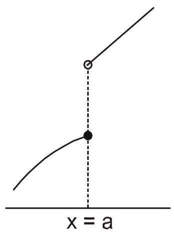
e.g. : Which of the following functions (as shown in figure) is increasing, decreasing or neither increasing nor decreasing at
x = a.

(i)



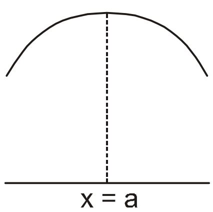
neither increasing
nor decreasing at x = a

(ii)



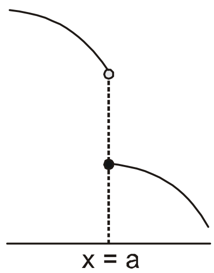
Increasing at x = a

(iii)



neither increasing
nor decreasing at x = a

(iv)



decreasing at x = a

Test for increasing and decreasing functions about a point

Let f(x) be differentiable.

(1) If $f'(a) > 0$ then f(x) is increasing at x = a.

(2) If $f'(a) < 0$ then f(x) is decreasing at x = a.

(3) If $f'(a) = 0$ then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a.

   (i) If $f'(x)$ is positive on both the neighbourhoods, then f is increasing at $x = a$.
   (ii) If $f'(x)$ is negative on both the neighbourhoods, then f is decreasing at $x = a$.
   (iii) If $f'(x)$ have opposite signs on these neighbourhoods, then f is non-monotonic at $x = a$.

Example # 32: Let $f(x) = x^3 - 3x + 2$. Examine the monotonicity of function at points $x = 0, 1, 2$.

Solution : $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

(i) $f'(0) = -3 \Rightarrow$ decreasing at $x = 0$

(ii) $f'(1) = 0$

also, $f'(x)$ is positive on left neighbourhood and $f'(x)$ is negative in right neighbourhood.

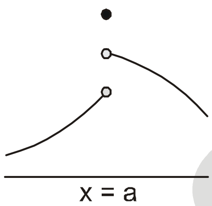
$\Rightarrow$ neither increasing nor decreasing at $x = 1$.

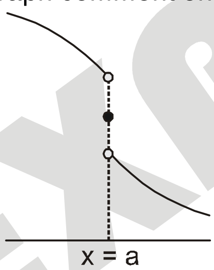
(iii) $f'(2) = 9 \Rightarrow$ increasing at $x = 2$

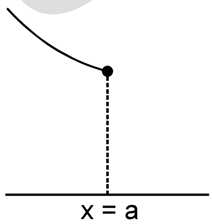
Note : Above method is applicable only for functions those are continuous at $x = a$.

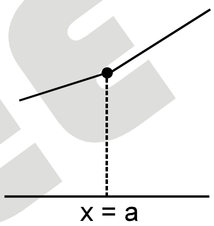
Self practice problems :

(26) For each of the following graph comment on monotonicity of f(x) at x = a.

(i)


(ii)


(iii)


(iv)


(27) Let $f(x) = x^3 - 3x^2 + 3x + 4$, comment on the monotonic behaviour of f(x) at (i) $x = 0$ (ii) $x = 1$.

(28) Draw the graph of function $f(x) = \begin{cases} x & 0 \le x \le 1 \\ [x] & 1 \le x \le 2 \end{cases}$. Graphically comment on the monotonic behaviour of f(x) at $x = 0, 1, 2$. Is f(x) M.I. for $x \in [0, 2]$?

Answers : (26) (i) neither M.I. nor M.D. (ii) M.D. (iii) M.D (iv) M.I.
(27) M.I. both at $x = 0$ and $x = 1$.
(28) M.I. at $x = 0, 2$; neither M.I. nor M.D. at $x = 1$. No, f(x) is not M.I. for $x \in [0, 2]$.

Use of monotonicity for proving inequalities

Comparison of two functions f(x) and g(x) can be done by analysing the monotonic behaviour of $h(x) = f(x) - g(x)$

Example # 33 : For $x \in \left(0, \frac{\pi}{2}\right)$ prove that $\sin x < x < \tan x$

Solution : Let $f(x) = x - \sin x$ $\Rightarrow$ $f'(x) = 1 - \cos x$

$f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ f(x) is M.I. $\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $x - \sin x > 0$ $\Rightarrow$ $x > \sin x$

Similarly consider another function $g(x) = x - \tan x$ $\Rightarrow$ $g'(x) = 1 - \sec^2 x$

$g'(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$ $\Rightarrow$ g(x) is M.D.

Hence $g(x) < g(0)$

$x - \tan x < 0$ $\Rightarrow$ $x < \tan x$

$\sin x < x < \tan x$ Hence proved

Example # 34 : For $x \in (0, 1)$ prove that $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$ hence or otherwise find $\lim\limits_{x\to 0} \left[\frac{\tan^{-1}x}{x}\right]$

Solution : Let $f(x) = x - \frac{x^3}{3} - \tan^{-1}x$

$f(x) = 1 - x^2 - \frac{1}{1+x^2}$

$f'(x) = -\frac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1)$ $\Rightarrow$ f(x) is M.D.

$\Rightarrow$ $f(x) < f(0)$

$\Rightarrow$ $x - \frac{x^3}{3} - \tan^{-1}x < 0$

$\Rightarrow$ $x - \frac{x^3}{3} < \tan^{-1}x$ ....(i)

Similarly $g(x) = x - \frac{x^3}{6} - \tan^{-1}x$

$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2}$

$g'(x) = \frac{x^2(1-x^2)}{2(1+x^2)}$

$g'(x) > 0$ for $x \in (0, 1)$ $\Rightarrow$ g(x) is M.I.

$\Rightarrow$ $g(x) > g(0)$

$x - \frac{x^3}{6} - \tan^{-1}x > 0$

$x - \frac{x^3}{6} > \tan^{-1}x$ ....(ii)

from (i) and (ii), we get

$$x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6} \qquad \text{Hence Proved}$$

Also, $1 - \frac{x^2}{3} < \frac{\tan^{-1}x}{x} < 1 - \frac{x^2}{6}$, for $x > 0$

Hence by sandwich theorem we can prove that $\lim\limits_{x\to 0} \frac{\tan^{-1}x}{x} = 1$ but it must also be noted that as $x \to 0$, value of $\frac{\tan^{-1}x}{x} \longrightarrow 1$ from left hand side i.e. $\frac{\tan^{-1}x}{x} < 1$

$$\Rightarrow \quad \lim_{x\to 0} \left[\frac{\tan^{-1}x}{x}\right] = 0$$

NOTE : In proving inequalities, we must always check when does the equality takes place because the point of equality is very important in this method. Normally point of equality occur at end point of the interval or will be easily predicted by hit and trial.

Example # 35 : For $x \in \left(0, \frac{\pi}{2}\right)$, prove that $\sin x > x - \frac{x^3}{6}$

Solution : Let $f(x) = \sin x - x + \frac{x^3}{6}$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

we cannot decide at this point whether $f'(x)$ is positive or negative, hence let us check formonotonic nature of $f'(x)$

$f''(x) = x - \sin x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is M.I. for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x) > f'(0)$

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is M.I.

$\Rightarrow f(x) > f(0)$

$\Rightarrow \sin x - x + \frac{x^3}{6} > 0$

$\Rightarrow \sin x > x - \frac{x^3}{6}$ Hence proved

Example # 36 : Examine which is greater : $\sin x \tan x$ or x^2. Hence evaluate $\lim\limits_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \frac{\pi}{2}\right)$

Solution : Let $f(x) = \sin x \tan x - x^2$

$f'(x) = \cos x \,.\, \tan x + \sin x \,.\, \sec^2 x - 2x$

$\Rightarrow f'(x) = \sin x + \sin x \sec^2 x - 2x$

$\Rightarrow f''(x) = \cos x + \cos x \sec^2 x + 2\sec^2 x \sin x \tan x - 2$

$\Rightarrow f''(x) = (\cos x + \sec x - 2) + 2 \sec^2 x \sin x \tan x$

Now $\cos x + \sec x - 2 = \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2$ and $2\sec^2 x \tan x \,.\, \sin x > 0$ because $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ $f''(x) > 0$ $\Rightarrow$ $f'(x)$ is M.I.

Hence $f'(x) > f'(0)$

$\Rightarrow$ $f'(x) > 0$ $\Rightarrow$ $f(x)$ is M.I.

$\Rightarrow$ $f(x) > 0$ $\Rightarrow$ $\sin x \tan x - x^2 > 0$

Hence $\sin x \tan x > x^2$

$\Rightarrow$ $\frac{\sin x \tan x}{x^2} > 1$ $\Rightarrow$ $\lim\limits_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right] = 1$

Example # 37 : Prove that $f(x) = \left(1+\frac{1}{x}\right)^x$ is monotonically increasing in its domain. Hence or otherwise draw graph of f(x) and find its range

Solution : $f(x) = \left(1+\frac{1}{x}\right)^x$, for Domain of f(x), $1 + \frac{1}{x} > 0$

$\Rightarrow$ $\frac{x+1}{x} > 0$ $\Rightarrow$ $(-\infty, -1) \cup (0, \infty)$

Consider $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}}\frac{-1}{x^2}\right]$

$\Rightarrow$ $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$

Now $\left(1+\frac{1}{x}\right)^x$ is always positive, hence the sign of $f'(x)$ depends on sign of $\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

i.e. we have to compare $\ell n\left(1+\frac{1}{x}\right)$ and $\frac{1}{1+x}$

So lets assume $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$

$g'(x) = \frac{1}{1+\frac{1}{x}}\,\frac{-1}{x^2} + \frac{1}{(x+1)^2}$ $\Rightarrow$ $g'(x) = \frac{-1}{x(x+1)^2}$

(i) for $x \in (0, \infty)$, $g'(x) < 0$ $\Rightarrow$ g(x) is M.D. for $x \in (0, \infty)$

$g(x) > \lim\limits_{x\to\infty} g(x)$

$g(x) > 0$.

and since $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$

(ii) for $x \in (-\infty, -1)$, $g'(x) > 0$ $\Rightarrow$ g(x) is M.I. for $x \in (-\infty, -1)$

$\Rightarrow$ $g(x) > \lim\limits_{x\to-\infty} g(x)$

$\Rightarrow$ $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$

Hence from (i) and (ii) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$

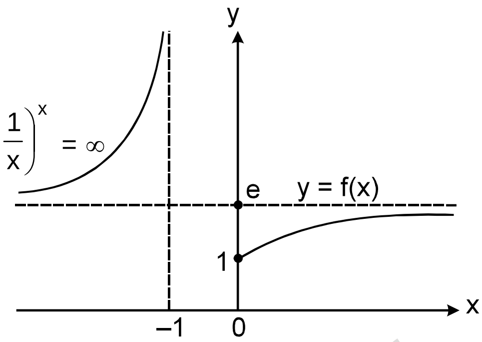
$\Rightarrow$ f(x) is M.I. in its Domain

For drawing the graph of f(x), its important to find the value of f(x) at boundary points i.e. $\pm\infty, 0, -1$

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x = 1 \quad \text{and} \quad \lim_{x\to -1}\left(1+\frac{1}{x}\right)^x = \infty$$

so the graph of f(x) is



Range is $y \in (1, \infty) - \{e\}$

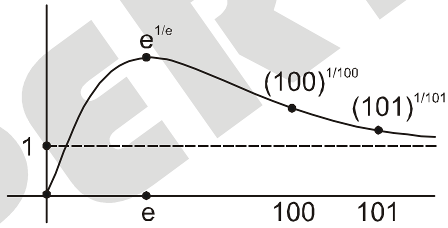
Example # 38 : Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

Solution : Assume $f(x) = x^{1/x}$ and let us examine monotonic nature of f(x)

$$f'(x) = x^{1/x} \cdot \left(\frac{1-\ell nx}{x^2}\right)$$

$f'(x) > 0 \quad \Rightarrow \quad x \in (0,e)$
and $f'(x) < 0 \quad \Rightarrow \quad x \in (e,\infty)$
Hence f(x) is M.D. for $x \ge e$

and since $100 < 101$
$\Rightarrow \quad f(100) > f(101)$
$\Rightarrow \quad (100)^{1/100} > (101)^{1/101}$



Self practice problems :

(29) Prove the following inequalities

(i) $x < -\ell n(1-x)$ for $x \in (0, 1)$
(ii) $x > \tan^{-1}(x)$ for $x \in (0, \infty)$
(iii) $e^x > x + 1$ for $x \in (0, \infty)$
(iv) $\dfrac{x}{1+x} \le \ell n(1+x) \le x$ for $x \in (0, \infty)$
(v) $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$ for $x \in \left(0, \dfrac{\pi}{2}\right)$

(30) Using $f(x) = x^{1/x}$, identify which is larger e^π or π^e.

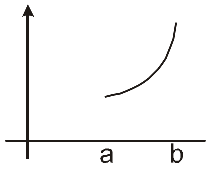
Answer : (30) e^π

Concavity, convexity, point of inflection

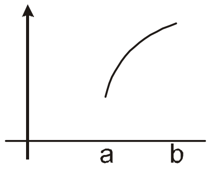
A function f(x) is concave in (a, b) if tangent drawn at every point $(x_0, (f(x_0))$, for $x_0 \in (a, b)$ lie below the curve. f(x) is convex in (a,b) if tangent drawn at each point $(x_0, f(x_0))$, $x_0 \in (a, b)$ lie above the curve.

A point (c, f(c)) of the graph y = f(x) is said to be a point of inflection of the graph, if f(x) is concave in $(c - \delta, c)$ and convex in $(c, c + \delta)$ (or vice verse), for some $\delta \in R^+$.

Results : **1.** If $f''(x) > 0 \ \forall \ x \in (a, b)$, then the curve $y = f(x)$ is concave in (a, b)



2. If $f''(x) < 0 \ \forall \ x \in (a, b)$ then the curve $y = f(x)$ is convex in (a, b)



3. If f is continuous at $x = c$ and $f''(x)$ has opposite signs on either sides of c, then the point (c, f(c)) is a point of inflection of the curve

4. If $f''(c) = 0$ and $f'''(c) \neq 0$, then the point (c, f(c)) is a point of inflection

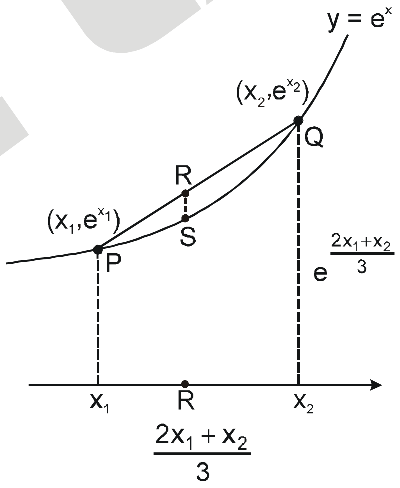
Proving Inequalities using curvature :

Generally these inequalities involve comparison between values of two functions at some particular points.

Example # 39 : Prove that for any two numbers x_1 & x_2, $\dfrac{2e^{x_1} + e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$

Solution : Assume $f(x) = e^x$ and let x_1 & x_2 be two points on the curve $y = e^x$.

Let R be another point which divides $\overline{PQ}$ in ratio 1 : 2.



y coordinate of point R is $\dfrac{2e^{x_1} + e^{x_2}}{3}$ and y coordinate of point S is $e^{\frac{2x_1+x_2}{3}}$. Since f(x) = e^x is concave up, the point R will always be above the point S.

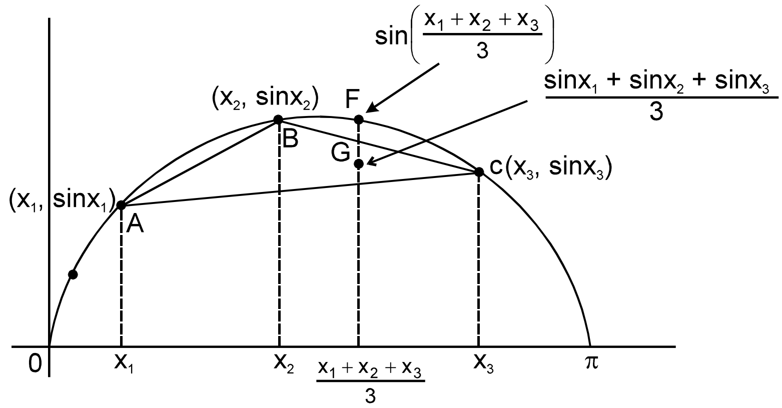
$$\Rightarrow \quad \frac{2e^{x_1} + e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$$

Alternate : Above inequality could also be easily proved using AM and GM.

Example # 40 : If $0 < x_1 < x_2 < x_3 < \pi$ then prove that $\sin\left(\dfrac{x_1 + x_2 + x_3}{3}\right) > \dfrac{\sin x_1 + \sin x_2 + \sin x_3}{3}$.

Hence prove that : if A, B, C are angles of a triangle then maximum value of sinA + sinB + sinC is $\frac{3\sqrt{3}}{2}$.

Solution :



Point A, B, C form a triangle.

y coordinate of centroid G is $\frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$ and y coordinate of point F is

sin $\left(\frac{x_1+x_2+x_3}{3}\right)$.

Hence sin $\left(\frac{x_1+x_2+x_3}{3}\right) \geq \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$.

If A + B + C = π, then

sin $\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A + \sin B + \sin C}{3}$ $\Rightarrow$ sin $\frac{\pi}{3} \geq \frac{\sin A + \sin B + \sin C}{3}$

$\Rightarrow$ $\frac{3\sqrt{3}}{2} \geq \sin A + \sin B + \sin C$

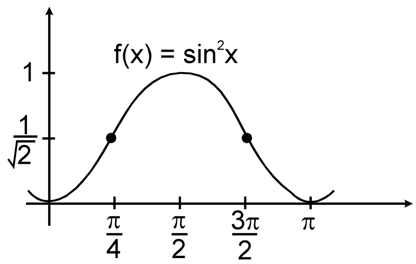
$\Rightarrow$ maximum value of (sinA + sinB + sinC) = $\frac{3\sqrt{3}}{2}$

Example # 41 : Find the points of inflection of the function f(x) = $\sin^2 x$ $x \in [0, 2\pi]$

Solution : f(x) = $\sin^2 x$
$f'(x)$ = sin2x
$f''(x)$ = 2 cos2x

$f''(0) = 0$ $\Rightarrow$ $x = \frac{\pi}{4}, \frac{3\pi}{4}$

both these points are inflection points as sign of $f''(x)$ change on either sides of these points.



Example # 42 : Find the inflection point of f(x) = $3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to concavity and point of inflection.

Solution : f(x) = $3x^4 - 4x^3$
$f'(x) = 12x^3 - 12x^2$
$f'(x) = 12x^2 (x - 1)$

$f''(x) = 12(3x^2 - 2x)$

$f''(x) = 12x(3x - 2)$

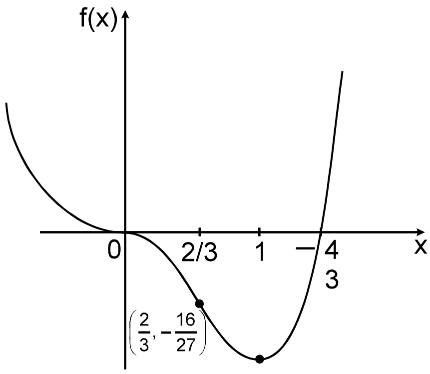
$f''(x) = 0 \quad \Rightarrow \quad x = 0, \frac{2}{3}.$

Again examining sign of $f''(x)$

$$\begin{array}{cccc} + & & - & & + \\ & 0 & & 2/3 & \end{array}$$

thus $x = 0, \frac{2}{3}$ are the inflection points

Hence the graph of f(x) is



Self practice problems :

(31) Identify which is greater $\frac{1+e^2}{e}$ or $\frac{1+\pi^2}{\pi}$

(32) If $0 < x_1 < x_2 < x_3 < \pi$, then prove that $\sin\left(\frac{2x_1 + x_2 + x_3}{4}\right) > \frac{2\sin x_1 + \sin x_2 + \sin x_3}{4}$

(33) If f(x) is monotonically decreasing function and $f''(x) > 0$. Assuming $f^{-1}(x)$ exists prove that

$$\frac{f^{-1}(x_1) + f^{-1}(x_2)}{2} > f^{-1}\left(\frac{x_1 + x_2}{2}\right).$$

Answer : (31) $\frac{1+e^2}{e}$

Global Maximum :

A function f(x) is said to have global maximum on a set E if there exists at least one $c \in E$ such that $f(x) \leq f(c)$ for all $x \in E$.

We say global maximum occurs at x = c and global maximum (or global maximum value) is f(c).

Local Maxima :

A function f(x) is said to have a local maximum at x = c if f(c) is the greatest value of the function in a small neighbourhood (c – h, c + h), h > 0 of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \leq f(c)$.

i.e. $f(c - \delta) < f(c) > f(c + \delta)$, $0 < \delta < h$

Note : If x = c is a boundary point then consider (c – h, c) or (c, c + h) (h > 0) appropriately.

Global Minimum :

A function f(x) is said to have a global minimum on a set E if there exists at least one $c \in E$ such that $f(x) \geq f(c)$ for all $x \in E$.

Local Minima :

A function f(x) is said to have a local minimum at x = c if f(c) is the least value of the function in a small neighbourhood (c – h, c + h), h > 0 of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \geq f(c)$.

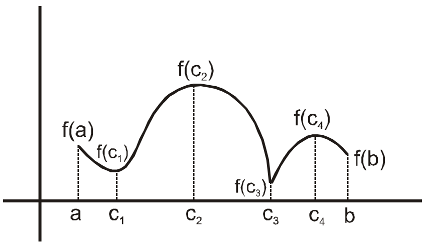
i.e. $f(c - \delta) > f(c) < f(c + \delta)$, $0 < \delta < h$

Extrema :

A maxima or a minima is called an extrema.

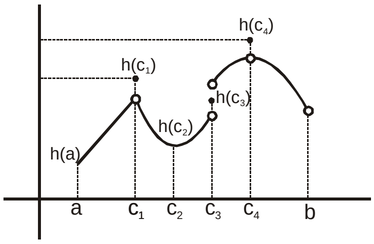
Explanation :

Consider graph of $y = f(x)$, $x \in [a, b]$



$x = a$, $x = c_2$, $x = c_4$ are points of local maxima, with maximum values $f(a)$, $f(c_2)$, $f(c_4)$ respectively.
$x = c_1$, $x = c_3$, $x = b$ are points of local minima, with minimum values $f(c_1)$, $f(c_3)$, $f(b)$ respectively
$x = c_2$ is a point of global maximum
$x = c_3$ is a point of global minimum

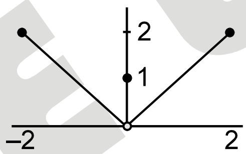
Consider the graph of $y = h(x)$, $x \in [a, b)$



$x = c_1$, $x = c_4$ are points of local maxima, with maximum values $h(c_1)$, $h(c_4)$ respectively.
$x = a$, $x = c_2$ are points of local minima, with minimum values $h(a)$, $h(c_2)$ respectively.
$x = c_3$ is neither a point of maxima nor a minima.
Global maximum is $h(c_4)$
Global minimum is $h(a)$

Example # 43: Let $f(x) = \begin{cases} |x| & 0 < |x| \le 2 \\ 1 & x = 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

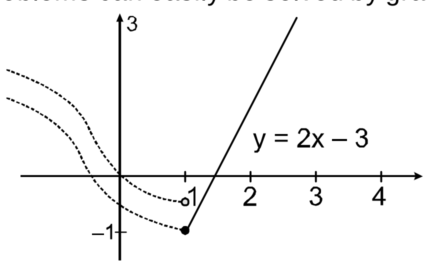
Solution : f(x) has local maxima at x = 0 (see figure).



Example # 44: Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$

Find all possible values of b such that f(x) has the smallest value at x = 1.

Solution. Such problems can easily be solved by graphical approach (as in figure).



Hence the limiting value of f(x) from left of x = 1 should be either greater or equal to the value of function at x = 1.

$$\lim_{x \to 1^-} f(x) \ge f(1)$$

$$\Rightarrow \qquad -1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} \geq -1$$

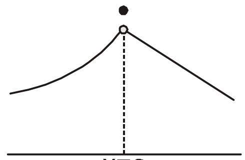
$$\Rightarrow \qquad \frac{(b^2+1)(b-1)}{(b+1)\,(b+2)} \geq 0$$

$$\Rightarrow \qquad b \in (-2, -1) \cup [1, +\infty)$$

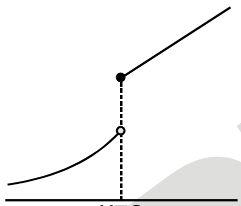
Self practice problems :

(34) In each of following graphs identify if x = a is point of local maxima, minima or neither

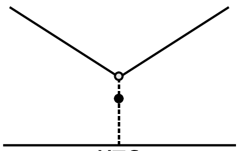
(i)



(ii)

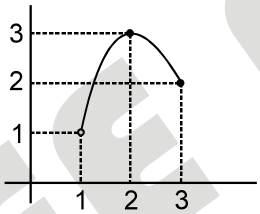


(iii)

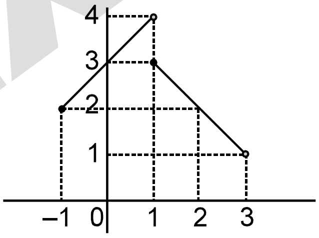


(35) Examine the graph of following functions in each case identify the points of global maximum/minimum and local maximum / minimum.

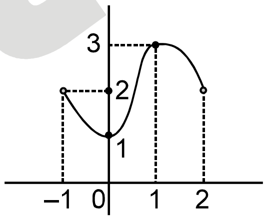
(i)



(ii)



(iii)



Answers : (34) (i) Maxima (ii) Neither maxima nor minima
(iii) Minima

(35) (i) Local maxima at x = 2, Local minima at x = 3, Global maximum at x = 2. No global minimum
(ii) Local minima at x = – 1, No point of Global minimum, no point of local or Global maxima
(iii) Local & Global maximum at x = 1, Local & Global minimum at x = 0.

Maxima, Minima for differentiable functions :

Mere definition of maxima, minima becomes tedious in solving problems. We use derivative as a tool to overcome this difficulty.

A necessary condition for an extrema :

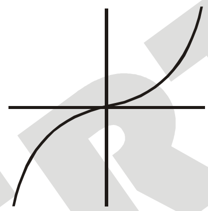
Let f(x) be differentiable at x = c.

Theorem : A necessary condition for f(c) to be an extremum of f(x) is that $f'(c) = 0$.

i.e. f(c) is extremum $\Rightarrow$ $f'(c) = 0$

Note : $f'(c) = 0$ is only a necessary condition but not sufficient

i.e. $f'(c) = 0 \not\Rightarrow$ f(c) is extremum.



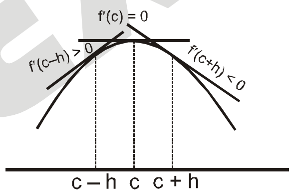
Consider $f(x) = x^3$
$f'(0) = 0$
but f(0) is not an extremum (see figure).

Sufficient condition for an extrema :

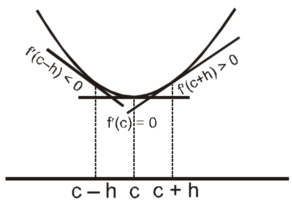
Let f(x) be a differentiable function.

Theorem : A sufficient condition for f(c) to be an extremum of f(x) is that $f'(x)$ changes sign as x passes through c.

i.e. f(c) is an extrema (see figure) $\Leftrightarrow$ $f'(x)$ changes sign as x passes through c.



x = c is a point of maxima. $f'(x)$ changes sign from positive to negative.



x = c is a point of local minima (see figure), $f'(x)$ changes sign from negative to positive.

Stationary points :

The points on graph of function f(x) where $f'(x) = 0$ are called stationary points.
Rate of change of f(x) is zero at a stationary point.

Example # 45: Find stationary points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$.

Solution : $f'(x) = 12x^2 - 12x - 24$
$f'(x) = 0 \quad \Rightarrow \quad x = -1, 2$
$f(-1) = 23$, $f(2) = -31$
$(-1, 23)$, $(2, -31)$ are stationary points

Example # 46: If $f(x) = x^3 + ax^2 + bx + c$ has extreme values at $x = -1$ and $x = 3$. Find a, b, c.

Solution. Extreme values basically mean maximum or minimum values, since f(x) is differentiable function so

$f'(-1) = 0 = f'(3)$

$f'(x) = 3x^2 + 2ax + b$

$f'(3) = 27 + 6a + b = 0$

$f'(-1) = 3 - 2a + b = 0$

$\Rightarrow \quad a = -3,\ b = -9,\ c \in R$

First Derivative Test :

Let f(x) be continuous and differentiable function.

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$, let $x = c$ be a solution. (i.e. Find stationary points)

Step - III. Observe change of sign

(i) If $f'(x)$ changes sign from negative to positive as x crosses c from left to right then $x = c$ is a point of local minima

(ii) If $f'(x)$ changes sign from positive to negative as x crosses c from left to right then $x = c$ is a point of local maxima.

(iii) If $f'(x)$ does not changes sign as x crosses c then $x = c$ is neither a point of maxima nor minima.

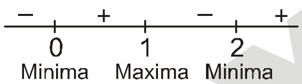
Example # 47: Find the points of maxima or minima of $f(x) = x^2 (x - 2)^2$.

Solution. $f(x) = x^2 (x - 2)^2$

$f'(x) = 4x (x - 1) (x - 2)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 2$

examining the sign change of $f'(x)$



Hence x = 1 is point of maxima, x = 0, 2 are points of minima.

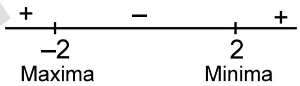
Note : In case of continuous functions points of maxima and minima are alternate.

Example # 48: Find the points of maxima, minima of $f(x) = x^3 - 12x$. Also draw the graph of this functions.

Solution. $f(x) = x^3 - 12x$

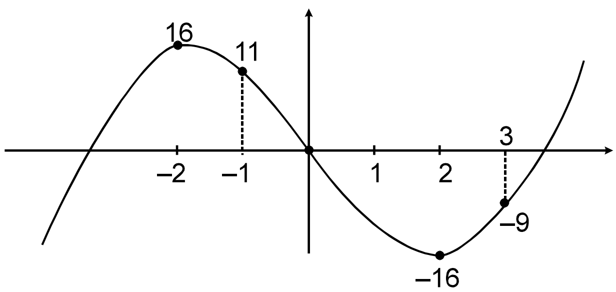
$f'(x) = 3(x^2 - 4) = 3(x - 2)(x + 2)$

$f'(x) = 0 \quad \Rightarrow \quad x = \pm 2$



For tracing the graph let us find maximum and minimum values of f(x).

x	f(x)
2	–16
–2	+16



Example # 49 : Show that $f(x) = (x^3 - 6x^2 + 12x - 8)$ does not have any point of local maxima or minima. Hence draw graph

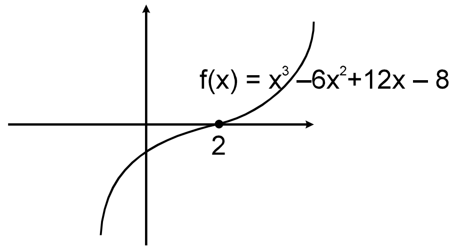
Solution. $f(x) = x^3 - 6x^2 + 12x - 8$

$f'(x) = 3(x^2 - 4x + 4)$

$f'(x) = 3(x - 2)^2$

$f'(x) = 0 \quad \Rightarrow \quad x = 2$

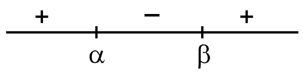
but clearly $f'(x)$ does not change sign about $x = 2$. $f'(2^+) > 0$ and $f'(2^-) > 0$. So $f(x)$ has no point of maxima or minima. In fact $f(x)$ is a monotonically increasing function for $x \in R$.



Example # 50 : Let $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If $f(x)$ has positive point of maxima, then find possible values of 'a'.

Solution. $f'(x) = 3[x^2 + 2(a - 7)x + (a^2 - 9)]$

Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.



Maxima occurs at smaller root α which has to be positive. This basically implies that both roots of $f'(x) = 0$ must be positive and distinct.

(i) $D > 0 \quad \Rightarrow \quad a < \dfrac{29}{7}$

(ii) $-\dfrac{b}{2a} > 0 \quad \Rightarrow \quad a < 7$

(iii) $f'(0) > 0 \quad \Rightarrow \quad a \in (-\infty, -3) \cup (3, \infty)$

from (i), (ii) and (iii) $\quad \Rightarrow \quad a \in (-\infty, -3) \cup \left(3, \dfrac{29}{7}\right)$

Self practice problems :

(36) Find the points of local maxima or minima of following functions

(i) $f(x) = (x - 1)^3 (x + 2)^2$

(ii) $f(x) = x^3 + x^2 + x + 1$.

Answer : 36. (i) Maxima at $x = -2$, Minima at $x = -\dfrac{4}{5}$

(ii) No point of local maxima or minima.

Maxima, Minima for continuous functions :

Let f(x) be a continuous function.

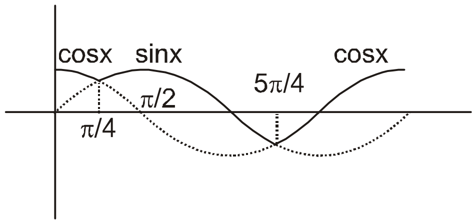
Critical points :

The points where $f'(x) = 0$ or $f(x)$ is not differentiable are called critical points.

Stationary points $\subseteq$ Critical points.

Example # 51 : Find critical points of $f(x) = \max(\sin x, \cos x)\ \forall,\ x \in (0, 2\pi)$.

Solution :



From the figure it is clear that f(x) has three critical points $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$.

Important Note :

For f(x) defined on a subset of R, points of extrema (if exists) occur at critical points

Example # 52 : Find the possible points of Maxima/Minima for $f(x) = |x^2 - 2x|$ $(x \in R)$

Solution. $f(x) = \begin{cases} x^2 - 2x & x \geq 2 \\ 2x - x^2 & 0 < x < 2 \\ x^2 - 2x & x \leq 0 \end{cases}$

$f'(x) = \begin{cases} 2(x-1) & x > 2 \\ 2(1-x) & 0 < x < 2 \\ 2(x-1) & x < 0 \end{cases}$

$f'(x) = 0$ at $x = 1$ and $f'(x)$ does not exist at $x = 0, 2$. Thus these are critical points.

Example # 53 : Let $f(x) = \begin{cases} x^3 + x^2 - 10x & x < 0 \\ 3\sin x & x \geq 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

Solution. f(x) is continuous at x = 0.

$f'(x) = \begin{cases} 3x^2 + 2x - 10 & x < 0 \\ 3\cos x & x > 0 \end{cases}$

$f'(0^+) = 3$ and $f'(0^-) = -10$ thus f(x) is non-differentiable at x = 0 $\Rightarrow$ x = 0 is a critical point.

Also derivative changes sign from negative to positive, so x = 0 is a point of local minima.

Example # 54 : Find the critical points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$ if (i) $x \in [0, 3]$ (ii) $x \in [-3, 3]$
(iii) $x \in [-1, 2]$.

Solution. $f'(x) = 12(x^2 - x - 2)$
$= 12(x - 2)(x + 1)$
$f'(x) = 0 \Rightarrow x = -1$ or 2

(i) if $x \in [0, 3]$, x = 2 is critical point.

(ii) if $x \in [-3, 3]$, then we have two critical points x = – 1, 2.

(iii) If $x \in [-1, 2]$, then no critical point as both x = 1 and x = 2 become boundary points.

Note : Critical points are always interior points of an interval.

Global extrema for continuous functions :

(i) Function defined on closed interval

Let f(x), $x \in [a, b]$ be a continuous function

Step - I : Find critical points. Let it be $c_1, c_2, \ldots, c_n$

Step - II : Find $f(a), f(c_1), \ldots, f(c_n), f(b)$

Let $M = \max \cdot \{ f(a), f(c_1), \ldots, f(c_n), f(b)\}$

$m = \min \cdot \{f(a), f(c_1), \ldots f(c_n), f(b)\}$

Step - III : M is global maximum.

m is global minimum.

(ii) Function defined on open interval.

Let f(x), $x \in (a, b)$ be continuous function.

Step - I Find critical points . Let it be $c_1, c_2, \ldots c_n$

Step - II Find $f(c_1), f(c_2), \ldots, f(c_n)$

Let $M = \max \cdot \{f(c_1), \ldots f(c_n)\}$

$m = \min \cdot \{f(c_1), \ldots, f(c_n)\}$

Step - III $\lim\limits_{x \to a^+} f(x) = \ell_1$ (say), $\lim\limits_{x \to b^-} f(x) = \ell_2$ (say).

Let $\ell = \min. \{\ell_1, \ell_2\}$, $L = \max. \{\ell_1, \ell_2\}$

Step - IV

(i) If $m \le \ell$ then m is global minimum

(ii) If $m > \ell$ then f(x) has no global minimum

(iii) If $M \ge L$ then M is global maximum

(iv) If $M < L$, then f(x) has no global maximum

Example # 55 : Find the greatest and least values of $f(x) = x^3 - 12x$ $x \in [-1, 3]$

Solution. The possible points of maxima/minima are critical points and the boundary points.

for $x \in [-1, 3]$ and $f(x) = x^3 - 12x$

$x = 2$ is the only critical point.

Examining the value of f(x) at points x = –1, 2, 3. We can find greatest and least values.

x	f(x)
–1	11
2	–16
3	–9

$\therefore$ Minimum f(x) = – 16 & Maximum f(x) = 11.

Self Practice Problems :

(37) Let $f(x) = 2x^3 - 9x^2 + 12x + 6$

(i) Find the possible points of Maxima/Minima of f(x) for $x \in R$.

(ii) Find the number of critical points of f(x) for $x \in [0, 2]$.

(iii) Discuss absolute (global) maxima/minima value of f(x) for $x \in [0, 2]$

(iv) Prove that for $x \in (1, 3)$, the function does not has a Global maximum.

Answers :

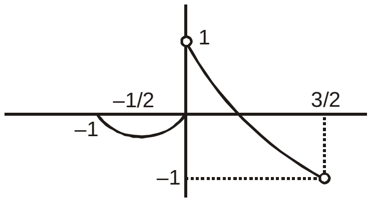
(37) (i) x = 1, 2 (ii) one

(iii) f(0) = 6 is the global minimum, f(1) = 11 is global maximum

Example # 56 : Let $f(x) = \begin{cases} x^2 + x & ; \ -1 \le x < 0 \\ \lambda & ; \ x = 0 \\ \log_{1/2}\left(x + \dfrac{1}{2}\right) & ; \ 0 < x < \dfrac{3}{2} \end{cases}$

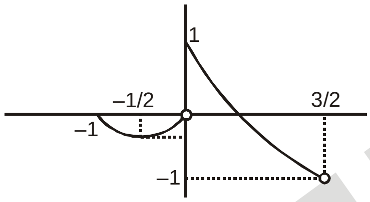
Discuss global maxima, minima for $\lambda = 0$ and $\lambda = 1$. For what values of λ does f(x) has global maxima

Solution : Graph of y = f(x) for $\lambda = 0$



No global maxima, minima

Graph of y = f(x) for $\lambda = 1$



Global maxima is 1, which occurs at x = 0

Global minima does not exists

$\lim\limits_{x\to 0^-} f(x) = 0$, $\lim\limits_{x\to 0^+} f(x) = 1$, $f(0) = \lambda$

For global maxima to exists

$f(0) \geq 1 \quad \Rightarrow \quad \lambda \geq 1$.

Example # 57 : Find extrema of $f(x) = 3x^4 + 8x^3 - 18x^2 + 60$. Draw graph of $g(x) = \dfrac{40}{f(x)}$ and comment on its local and global extrema.

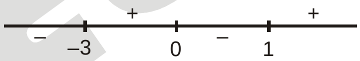
Solution : $f'(x) = 0$

$\Rightarrow \quad 12x\,(x^2 + 2x - 3) = 0$

$\Rightarrow \quad 12x\,(x - 1)\,(x + 3) = 0$

$\Rightarrow \quad x = -3, 0, 1$

$f'(x) = 12(x + 3)\,x(x - 1)$



local minima occurs at x = –3, 1

local maxima occurs at x = 0

f(–3) = – 75, f(1) = 53 are local minima

f(0) = 60 is local maxima

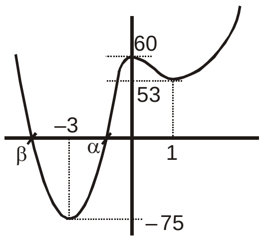
$\lim\limits_{x\to\infty} f(x) = \infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$

Hence global maxima does not exists : Global minima is – 75

$g'(x) = \dfrac{-40}{(f(x))^2}\, f'(x)$

$\Rightarrow$ g(x) has same critical points as that of f(x).

A rough sketch of y = f(x) is



Let zeros of f(x) be α, β

$g(\alpha)$, $g(\beta)$ are undefined,

$\lim\limits_{x\to\beta^-} g(x) = \infty$, $\lim\limits_{x\to\beta^+} g(x) = -\infty$, $\lim\limits_{x\to\alpha^-} g(x) = -\infty$, $\lim\limits_{x\to\alpha^+} g(x) = \infty$

$x = \alpha$, $x = \beta$ are asymptotes of y = g(x).

$\lim\limits_{x\to\infty} g(x) = 0, \quad \lim\limits_{x\to-\infty} g(x) = 0$

$\Rightarrow$ y = 0 is also an asymptote.

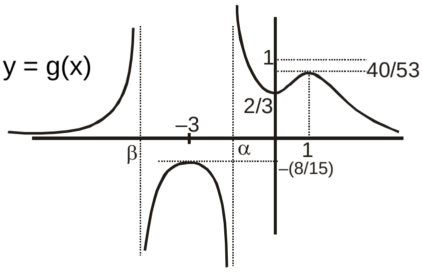
$\because$ x = –3, x = 1 are local minima of

y = f(x) $\Rightarrow$ x = – 3, x = 1 are local maxima of y = g(x)

similarly, x = 0 is local minima of y = g(x)

Global extrema of g(x) does not exists.

A rough sketch of y = g(x) is



Self Practice Problems :

(38) Let $f(x) = \frac{x}{2} + \frac{2}{x}$. Find local maximum and local minimum value of f(x). Can you explain this discrepancy of locally minimum value being greater than locally maximum value.

(39) If $f(x) = \begin{cases} (x+\lambda)^2 & x < 0 \\ \cos x & x \ge 0 \end{cases}$, find possible values of λ such that f(x) has local maxima at x = 0.

Answers : (38) Local maxima at x = – 2, f(–2) = – 2; Local minima at x = 2, f(2) = 2.

(39) $\lambda \in [-1, 1)$

Maxima, Minima by higher order derivatives :

Second derivative test :

Let f(x) have derivatives up to second order

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$. Let x = c be a solution

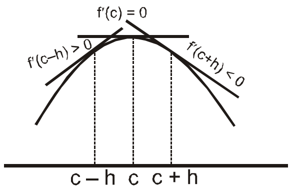
Step - III. Find $f''(c)$

Step - IV.

(i) If $f''(c) = 0$ then further investigation is required

(ii) If $f''(c) > 0$ then x = c is a point of minima.

(iii) If $f''(c) < 0$ then x = c is a point of maxima.



For maxima $f'(x)$ changes from positive to negative (as shown in figure).

$\Rightarrow$ $f'(x)$ is decreasing hence $f''(c) < 0$

Example # 58 : Find the points of local maxima or minima for f(x) = sin2x – x, $x \in (0, \pi)$.

Solution. f(x) = sin2x – x

$f'(x) = 2\cos 2x - 1$

$f'(x) = 0 \quad \Rightarrow \quad \cos 2x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$

$f''(x) = -4\sin 2x$

$f''\left(\frac{\pi}{6}\right) < 0 \quad \Rightarrow \quad$ Maxima at $x = \frac{\pi}{6}$

$f''\left(\frac{5\pi}{6}\right) > 0 \quad \Rightarrow \quad$ Minima at $x = \frac{5\pi}{6}$

Self practice problems :

(40) Find the points of local maxima or minima of f(x) = sin 2x – x

(41) Let f(x) = sinx (1 + cosx) ; $x \in (0, 2\pi)$. Find the number of critical points of f(x). Also identify which of these critical points are points of Maxima/Minima.

Answer : 40 Maxima at $x = n\pi + \frac{\pi}{6}$; Minima at $x = n\pi - \frac{\pi}{6}$

41. Three

$x = \frac{\pi}{3}$ is point of maxima.

$x = \pi$ is not a point of extrema.

$x = \frac{5\pi}{3}$ is point of minima.

n^{th} Derivative test :

Let $f(x)$ have derivatives up to nth order

If $f'(c) = f''(c) = \ldots\ldots\ldots = f^{n-1}(c) = 0$ and

$f^n(c) \neq 0$ then we have following possibilities

(i) n is even, $f^{(n)}(c) < 0 \Rightarrow x = c$ is point of maxima

(ii) n is even, $f^{(n)}(c) > 0 \Rightarrow x = c$ is point of minima.

(iii) n is odd, $f^{(n)}(c) < 0 \Rightarrow f(x)$ is decreasing about $x = c$

(iv) n is odd, $f^{(n)} > 0 \Rightarrow f(x)$ is increasing about $x = c$.

Example # 59 : Find points of local maxima or minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution. $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^2 (x - 1) (x - 3)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 3$

$f''(x) = 10x (2x^2 - 6x + 3)$

Now, $f''(1) < 0 \quad \Rightarrow \quad$ Maxima at $x = 1$

$f''(3) > 0 \quad \Rightarrow \quad$ Minima at $x = 3$

and, $f''(0) = 0 \quad \Rightarrow \quad$ IInd derivative test fails

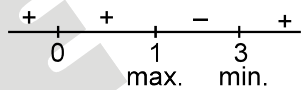
so, $f'''(x) = 30 (2x^2 - 4x + 1)$

$f'''(0) = 30$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

Note : It was very convenient to check maxima/minima at first step by examining the sign change of $f'(x)$ no sign change of $f'(x)$ at $x = 0$

$f'(x) = 5x^2 (x - 1) (x - 3)$



Application of Maxima, Minima :

For a given problem, an objective function can be constructed in terms of one parameter and then extremum value can be evaluated by equating the differential to zero. As discussed in n^{th} derivative test maxima/minima can be identified.

Useful Formulae of Mensuration to Remember :

1. Volume of a cuboid = ℓbh.

2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.

3. Volume of cube = a^3

4. Surface area of cube = $6a^2$

5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.

6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height)

7. Curved surface area of a cylinder = $2\pi rh$.

8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

9. Volume of a sphere = $\frac{4}{3}\pi r^3$.

10. Surface area of a sphere = $4\pi r^2$.

11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

12. Volume of a prism = (area of the base) × (height).

13. Lateral surface area of a prism = (perimeter of the base) × (height).

14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
(Note that lateral surfaces of a prism are all rectangle).

15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).

16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
(Note that slant surfaces of a pyramid are triangles).

Example # 60: If the equation $x^3 + px + q = 0$ has three real roots, then show that $4p^3 + 27q^2 < 0$.

Solution: $f(x) = x^3 + px + q$, $f'(x) = 3x^2 + p$

$\therefore$ f(x) must have one maximum > 0 and one minimum < 0. $f'(x) = 0$

$\Rightarrow$ $x = \pm\sqrt{\frac{-p}{3}}, p < 0$

f is maximum at $x = -\sqrt{\frac{-p}{3}}$ and minimum at $x = \sqrt{\frac{-p}{3}}$

$$f\left(-\sqrt{\frac{-p}{3}}\right) f\left(\sqrt{\frac{-p}{3}}\right) < 0$$

$$\left(q - \frac{2p}{3}\sqrt{\frac{-p}{3}}\right)\left(q + \frac{2p}{3}\sqrt{\frac{-p}{3}}\right) < 0$$

$q^2 + \frac{4p^3}{27} < 0$, $4p^3 + 27q^2 < 0$.

Example # 61 : Find two positive numbers x and y such that x + y = 60 and xy^3 is maximum.

Solution. $x + y = 60$

$\Rightarrow$ $x = 60 - y$ $\Rightarrow$ $xy^3 = (60 - y)y^3$

Let $f(y) = (60 - y)\, y^3$; $y \in (0, 60)$

for maximizing f(y) let us find critical points

$f'(y) = 3y^2 (60 - y) - y^3 = 0$

$f'(y) = y^2 (180 - 4y) = 0$

$\Rightarrow$ $y = 45$

$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at y = 45.

So x = 15 and y = 45.

Example # 62 : Rectangles are inscribed inside a semicircle of radius r. Find the rectangle with maximum area.

Solution. Let sides of rectangle be x and y (as shown in figure).

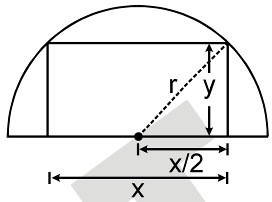
$\Rightarrow$ A = xy.

Here x and y are not independent variables and are related by Pythogorus theorem with r.

$$\frac{x^2}{4} + y^2 = r^2 \quad \Rightarrow \quad y = \sqrt{r^2 - \frac{x^2}{4}}$$

$$\Rightarrow \quad A(x) = x\sqrt{r^2 - \frac{x^2}{4}}$$

$$\Rightarrow \quad A(x) = \sqrt{x^2r^2 - \frac{x^4}{4}}$$



Let $f(x) = r^2x^2 - \frac{x^4}{4}$; $x \in (0, r)$

A(x) is maximum when f(x) is maximum

Hence $f'(x) = x(2r^2 - x^2) = 0$

$\Rightarrow$ $x = r\sqrt{2}$

also $f'(r\sqrt{2^+}) < 0$ and $f'(r\sqrt{2^-}) > 0$

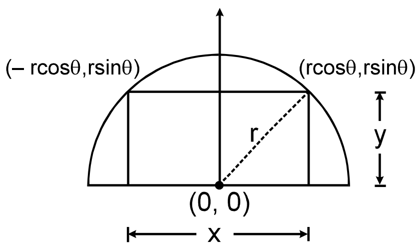
confirming at f(x) is maximum when $x = r\sqrt{2}$ & $y = \frac{r}{\sqrt{2}}$.

Aliter Let us choose coordinate system with origin as centre of circle (as shown in figure).

A = xy

$\Rightarrow$ $A = 2\,(r\cos\theta)\,(r\sin\theta)$

$\Rightarrow$ $A = r^2 \sin 2\theta, \quad \theta \in \left(0, \frac{\pi}{2}\right)$



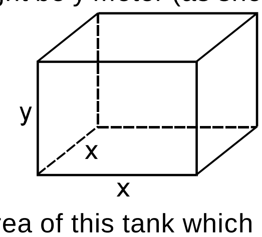
Clearly A is maximum when $\theta = \frac{\pi}{4}$

$\Rightarrow$ $x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$.

Example # 63 : A sheet of area 40 m^2 is used to make an open tank with square base. Find the dimensions of the base such that volume of this tank is maximum.

Solution. Let length of base be x meter and height be y meter (as shown in figure).

$V = x^2y$



again x and y are related to surface area of this tank which is equal to 40 m^2.

$\Rightarrow$ $x^2 + 4xy = 40$

$$y = \frac{40 - x^2}{4x} \qquad x \in (0, \sqrt{40})$$

$$\Rightarrow \quad V(x) = x^2\left(\frac{40 - x^2}{4x}\right)$$

$$V(x) = \frac{(40x - x^3)}{4}$$

maximizing volume,

$$V'(x) = \frac{(40 - 3x^2)}{4} = 0 \quad \Rightarrow \quad x = \sqrt{\frac{40}{3}}\ m$$

and $V''(x) = -\frac{3x}{2} \quad \Rightarrow \quad V''\left(\sqrt{\frac{40}{3}}\right) < 0.$

Confirming that volume is maximum at $x = \sqrt{\frac{40}{3}}$ m.

Example # 64 : If a right circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

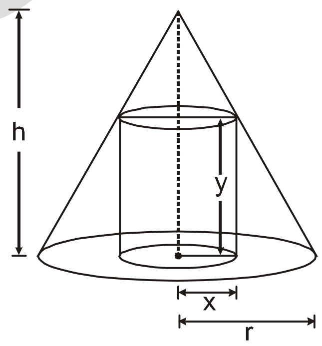
Solution. Let x be the radius of cylinder and y be its height

$v = \pi x^2 y$

x, y can be related by using similar triangles (as shown in figure).

$$\frac{y}{r - x} = \frac{h}{r}$$

$$\Rightarrow \quad y = \frac{h}{r}(r - x)$$



$$\Rightarrow \quad v(x) = \pi x^2 \frac{h}{r}(r - x) \qquad x \in (0, r)$$

$$\Rightarrow \quad v(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

$$v'(x) = \frac{\pi h}{r} x (2r - 3x)$$

$$v'\left(\frac{2r}{3}\right) = 0 \quad \text{and} \quad v''\left(\frac{2r}{3}\right) < 0$$

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

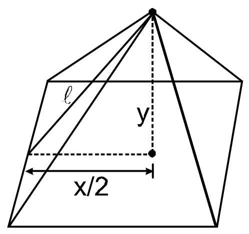
Note : Following formulae of volume, surface area of important solids are very useful in problems of maxima & minima.

Example # 65 : Among all regular square pyramids of volume $36\sqrt{2}$ cm^3. Find dimensions of the pyramid having least lateral surface area.

Solution. Let the length of a side of base be x cm and y be the perpendicular height of the pyramid (see figure).

$$V = \frac{1}{3} \times \text{ area of base x height}$$

$$\Rightarrow \quad V = \frac{1}{3} x^2 y = 36\sqrt{2}$$



$\Rightarrow \quad y = \dfrac{108\sqrt{2}}{x^2}$

and $\quad S = \dfrac{1}{2}$ × perimeter of base x slant height

$= \dfrac{1}{2}\,(4x).\,\ell$

but $\quad \ell = \sqrt{\dfrac{x^2}{4} + y^2}$

$\Rightarrow \quad S = 2x\sqrt{\dfrac{x^2}{4} + y^2} = \sqrt{x^4 + 4x^2y^2}$

$\Rightarrow \quad S = \sqrt{x^4 + 4x^2\left(\dfrac{108\sqrt{2}}{x^2}\right)^2}$

$S(x) = \sqrt{x^4 + \dfrac{8.(108)^2}{x^2}}$

Let $\quad f(x) = x^4 + \dfrac{8.(108)^2}{x^2}$ for minimizing f(x)

$f'(x) = 4x^3 - \dfrac{16(108)^2}{x^3} = 0$

$\Rightarrow \quad f'(x) = 4\,\dfrac{(x^6 - 6^6)}{x^3} = 0$

$\Rightarrow \quad x = 6$, which a point of minima

Hence x = 6 cm and $y = 3\sqrt{2}$.

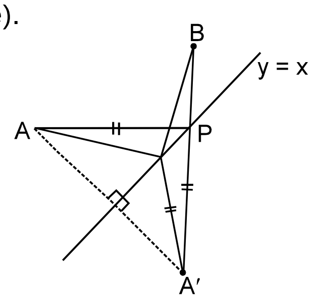
Example # 66 : Let A(1, 2) and B(– 2, – 4) be two fixed points. A variable point P is chosen on the straight line y = x such that perimeter of ΔPAB is minimum. Find coordinates of P.

Solution. Since distance AB is fixed so for minimizing the perimeter of ΔPAB, we basically have to minimize (PA + PB)

Let A′ be the mirror image of A in the line y = x (see figure).

F(P) = PA + PB

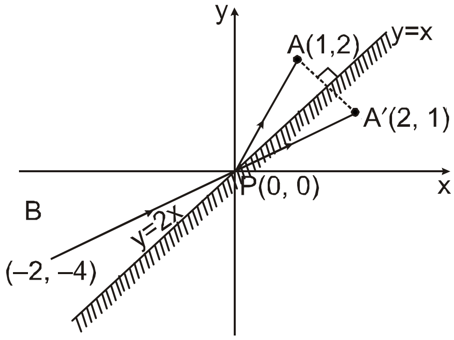
F(P) = PA′ + PB



But for ΔPA′B

PA′ + PB ≥ A′B and equality hold when P, A′ and B becomes collinear. Thus for minimum path length point P is that special point for which PA and PB become incident and reflected rays with respect to the mirror y = x.

Equation of line joining A′ and B is y = 2x intersection of this line with y = x is the point P. Hence P ≡ (0, 0).



Note : Above concept is very useful because such problems become very lengthy by making perimeter as a function of position of P and then minimizing it.

Self Practice Problems :

(42) Find the two positive numbers x and y whose sum is 35 and the product $x^2 y^5$ maximum.

(43) A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the slops to form a box. What should be the side of the square to be cut off such that volume of the box is maximum possible.

(44) Prove that a right circular cylinder of given surface area and maximum volume is such that the height is equal to the diameter of the base.

(45) A normal is drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Find the maximum distance of this normal from the centre.

(46) A line is drawn passing through point P(1, 2) to cut positive coordinate axes at A and B. Find minimum area of ΔPAB.

(47) Two towns A and B are situated on the same side of a straight road at distances a and b respectively perpendiculars drawn from A and B meet the road at point C and D respectively. The distance between C and D is c. A hospital is to be built at a point P on the road such that the distance APB is minimum. Find position of P.

Answers : (42) x = 25, y = 10. (43) 3 cm (45) 1 unit

(46) 4 units (47) P is at distance of $\frac{ac}{a+b}$ from C.

EXERCISE - 1

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Rate of change, Approximation

A-1. The length x of rectangle is decreasing at a rate of 3 cm/min and width y is increasing at a rate of 2 cm/min. When x = 10 cm and y = 6 cm, find the rate of change of (i) the perimeter, (ii) the area of rectangle.

A-2. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the first square.

A-3. A man 1.5 m tall walks away from a lamp post 4.5 m high at a rate of 4 km/hr.
(i) How fast is his shadow lengthening?
(ii) How fast is the farther end of shadow moving on the pavement?

A-4. If the radius of a sphere is measured as 8 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Section (B) : Tangent, Normal, Angle between curves, Orthogonality of Curves, Shortest distance, Length of Tangent & Normal

B-1. Find the equation of normal to the curve $x^3 + y^3 = 8xy$ at point where it is meet by the curve $y^2 = 4x$, other than origin.

B-2. If the tangent to the curve $xy + ax + by = 0$ at (1, 1) is inclined at an angle $\tan^{-1} 2$ with x-axis, then find a and b ?

B-3. The normal to the curve $5x^5 - 10x^3 + x + 2y + 6 = 0$ at the point P(0, –3) is tangent to the curve at the point(s). Find those point(s)?

B-4. Prove that the length of segment of all tangents to curve $x^{2/3} + y^{2/3} = a^{2/3}$ intercepted between coordinate axes is same.

B-5. Find equations of tangents drawn to the curve $y^2 - 2x^2 - 4y + 8 = 0$ from the point (1, 2).

B-6. If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then find coordinates of P

B-7. Find angle of intersection of the curves $y = 2\sin^2 x$ and $y = \cos 2x$.

B-8. Let f(x) and g(x) be two functions which cut each other orthogonally at their common point of intersection (x_1). Both f(x) and g(x) are equal to 0 at $x = x_1$. Also $|f'(x_1)| = |g'(x_1)|$, then find $\lim\limits_{x \to x_1} [f(x) \cdot g(x)]$, where [.] denotes greatest integer functions.

B-9. Find the point on hyperbola $3x^2 - 4y^2 = 72$ which is nearest to the straight line $3x + 2y + 1 = 0$

B-10. Find the shortest distance between the curves $f(x) = -6x^6 - 3x^4 - 4x^2 - 6$ and $g(x) = e^x + e^{-x} + 2$

B-11. For parabola $y^2 = 4ax$, prove that the ratio of subtangent to abscissa is constant. Also find the ratio.

Section (C) : Monotonicity on an interval, Monotonicity about a point

C-1. Show that $f(x) = \dfrac{x}{\sqrt{1+x}} - \ell n(1 + x)$ is an increasing function for $x > -1$.

C-2. Find the intervals of monotonicity for the following functions.

(i) $\dfrac{x^4}{4} + \dfrac{x^3}{3} - 3x^2 + 5$ (ii) $\log_3^2 x + \log_3 x$

C-3. Check monotonocity at following points for
(i) $f(x) = x^3 - 3x + 1$ at $x = -1, 2$
(ii) $f(x) = |x - 1| + 2|x - 3| - |x + 2|$ at $x = -2, 0, 3, 5$

C-4. Find the values of 'a' for which the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases for all real values of x.

C-5. Let $f(x) = \begin{cases} x^2 & x \geq 0 \\ ax & x < 0 \end{cases}$. Find real values of 'a' such that f(x) is strictly monotonically increasing at x = 0.

C-6. If g(x) is monotonically increasing and f(x) is monotonically decreasing for $x \in R$ and if (gof)(x) is defined for $x \in R$, then prove that (gof)(x) will be monotonically decreasing function. Hence prove that
$(g\circ f)(x + 1) \leq (g\circ f)(x - 1)$.

C-7. Prove the inequality, $\dfrac{\tan x_2}{\tan x_1} > \dfrac{x_2}{x_1}$ for $0 < x_1 < x_2 < \dfrac{\pi}{2}$.

C-8. For $x \in \left(0, \dfrac{\pi}{2}\right)$ identify which is greater (2sinx + tanx) or (3x). Hence find $\lim\limits_{x \to 0^+} \left[\dfrac{3x}{2\sin x + \tan x}\right]$ where [.] denote the GIF.

C-9. Let f and g be differentiable on R and suppose f(0) = g(0) and $f'(x) \leq g'(x)$ for all $x \geq 0$. Then show that $f(x) \leq g(x)$ for all $x \geq 0$.

Section (D) : Local maxima, Local minima, Global maxima, Global minima, Application of Maxima and Minima

D-1. Find the points of local maxima/minima of following functions
(i) $f(x) = 2x^3 - 21x^2 + 36x - 20$
(ii) $f(x) = -(x - 1)^3 (x + 1)^2$
(iii) $f(x) = x\,\ell nx$

D-2. Find the absolute maximum/minimum value of following functions
(i) $f(x) = x^3$; $x \in [-2, 2]$
(ii) $f(x) = \sin x + \cos x$; $x \in [0, \pi]$
(iii) $f(x) = 4x - \dfrac{x^2}{2}$; $x \in \left[-2, \dfrac{9}{2}\right]$
(iv) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $x \in [0, 3]$
(v) $f(x) = \sin x + \dfrac{1}{2}\cos 2x$; $x \in \left[0, \dfrac{\pi}{2}\right]$

D-3. Draw graph of $f(x) = x|x - 2|$ and, hence find points of local maxima/minima.

D-4. Let $f(x) = x^2$; $x \in [-1, 2)$. Then show that f(x) has exactly one point of local maxima but global maximum is not defined.

D-5. Find the minimum and maximum values of y in $4x^2 + 12xy + 10y^2 - 4y + 3 = 0$.

D-6. Let $f(x) = \begin{cases} 3 - x & 0 \leq x < 1 \\ x^2 + \ell nb & x \geq 1 \end{cases}$. Find the set of values of b such that f(x) has a local minima at x = 1.

D-7. John has 'x' children by his first wife and Anglina has 'x + 1' children by her first husband. They both marry and have their own children. The whole family has 24 children. It is given that the children of the same parents don't fight. Then find then maximum number of fights that can take place in the family.

D-8. If the sum of the lengths of the hypotenuse and another side of a right angled triangle is given, show that the area of the triangle is a maximum when the angle between these sides is $\pi/3$.

D-9. Find the volume of the largest cylinder that can be inscribed in a sphere of radius ' r ' cm.

D-10. Show that the semi vertical angle of a right circular cone of maximum volume, of a given slant height is $\tan^{-1}\sqrt{2}$.

D-11. A running track of 440 m. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end . If the area of the rectangular portion is to be maximum, find the length of its sides.

D-12. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the curve $y = 12 - x^2$.

D-13. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible when revolved around one of its side .

Section (E) : Curvature, Points of inflection,

E-1. Find number of point of inflection for the following functions
(i) $f(x) = (x-1)^3 (x-2)^2$ (ii) $f(x) = x + \sin x$ in $(0, 2\pi)$

E-2. Show that the locus of point of inflection of the curve $y = x \sin x$ is $y^2 (4 + x^2) = 4x^2$

Section (F) : Rolle's Theorem, LMVT

F-1. Verify Rolle's theorem for the function, $f(x) = \log_e \left(\frac{x^2+ab}{x(a+b)}\right) + p$, for [a, b] where $0 < a < b$.

F-2. Using Rolle's theorem prove that the equation $3x^2 + px - 1 = 0$ has at least one real root in the interval (– 1, 1).

F-3. If a, b are two real numbers with $a < b$ show that a real number 'c' can be found between a and b such that $3c^2 = b^2 + ab + a^2$.

F-4. Let f(x) be differentiable function and g(x) be twice differentiable function. Zeros of f(x), g′(x) be a, b respectively (a < b). Show that there exists at least one root of equation $f'(x)\, g'(x) + f(x)\, g''(x) = 0$ on (a, b).

F-5_. If $f(x) = \tan x$, $x \in \left[0, \frac{\pi}{5}\right]$ then show that $\frac{\pi}{5} < f\left(\frac{\pi}{5}\right) < \frac{2\pi}{5}$

F-6_. Show that the equation $x = a \sin x + b$ where $0 < a < 1$, $b > 0$, has at least one positive root

PART - II : OBJECTIVE QUESTIONS

*** Marked Questions may have more than one correct option.**

Section (A) : Rate of change, Approximation

A-1. Water is poured into an inverted conical vessel of which the radius of the base is 2 m and height 4 m, at the rate of 77 litre/minute. The rate at which the water level is rising at the instant when the depth is 70 cm is (use $\pi = 22/7$)
(A) 10 cm/min (B) 20 cm/min (C) 40 cm/min (D) none

A-2 On the curve $x^3 = 12y$. The interval in which abscissa changes at a faster rate then its ordinate
(A) (0, 2) (B) $(-\infty, -2) \cup (2, \alpha)$ (C) (–2, 2) (D) none of these

A-3. Using differentials, find the approximate value of $\sqrt{25.2}$.
(A) 5.02 (B) 5.01 (C) 5.03 (D) 5.04

A-4. The approximate change in the volume of a cube of side x meters caused by increasing the side by 4% is
(A) $0.06x^3m^3$ (B) $0.09x^3m^3$ (C) $0.12x^3m^3$ (D) $0.15x^3m^3$

A-5. A kite is 300 m high and there are 500 m of cord out. If the wind moves the kite horizontally at the rate of 5 km/hr. directly away from the person who is flying it, find the rate at which the cord is being paid?
(A) 4 (B) 8
(C) 3 (D) cannot be determined

Section (B) : Tangent, Normal, Angle between curves, Orthogonality of Curves, Shortest distance, Length of Tangent & Normal

B-1. Equation of the normal to the curve $y = -\sqrt{x} + 2$ at the point of its intersection with the curve $y = \tan(\tan^{-1} x)$ is
(A) $2x - y - 1 = 0$ (B) $2x - y + 1 = 0$ (C) $2x + y - 3 = 0$ (D) none

B-2. The curve $y - e^{xy} + x = 0$ has a vertical tangent at
(A) (1, 1) (B) (0, 1) (C) (1, 0) (D) no point

B-3. If the tangent to the curve $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ at $\theta = \frac{\pi}{3}$ makes an angle α $(0 \le \alpha < \pi)$ with x-axis, then α =
(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

B-4. Number of tangents drawn from the point (–1/2, 0) to the curve $y = e^{\{x\}}$. (Here { } denotes fractional part function).
(A) 2 (B) 1 (C) 3 (D) 4

B-5*. If tangent to curve $2y^3 = ax^2 + x^3$ at point (a, a) cuts off intercepts α, β on co-ordinate axes, where $\alpha^2 + \beta^2 = 61$, then the value of 'a' is equal to
(A) 20 (B) 25 (C) 30 (D) – 30

B-6*. The co-ordinates of point(s) on the graph of the function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cut off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign, is
(A) (2, 8/3) (B) (3, 7/2) (C) (1, 5/6) (D) none

B-7. If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is
(A) 1/2 (B) 1/3 (C) 2 (D) 3

B-8. The coordinates of the point of the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are
(A) (2 , – 4) (B) (18 , –12) (C) (2 , 4) (D) none of these

B-9*. The co-ordinates of a point on the parabola $2y = x^2$ which is nearest to the point (0, 3) is
(A) (2, 2) (B) $\left(-\sqrt{2}, 1\right)$ (C) $\left(\sqrt{2}, 1\right)$ (D) (– 2, 2)

Section (C) : Monotonicity an on interval, Monotonicity about a point

C-1. The function $\frac{|x-1|}{x^2}$ is monotonically decreasing at the point
(A) x = 3 (B) x = 1 (C) x = 2 (D) none of these

C-2. The values of p for which the function $f(x) = \left(\frac{\sqrt{p+4}}{1-p} - 1\right) x^5 - 3x + \ln 5$ decreases for all real x is

(A) $(-\infty, \infty)$
(B) $\left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$
(C) $\left[-3, \frac{5-\sqrt{27}}{2}\right] \cup (2, \infty)$
(D) $[1, \infty)$

C-3*. Which of the following statements is/are correct ?
(A) x + sinx is increasing function
(B) sec x is neither increasing nor decreasing function
(C) x + sinx is decreasing function
(D) sec x is an increasing function

C-4*. If $f(x) = 2x + \cot^{-1} x + \ell n \left(\sqrt{1+x^2} - x\right)$, then f(x) :
(A) increases in $[0, \infty)$
(B) decreases in $[0, \infty)$
(C) neither increases nor decreases in $[0, \infty)$
(D) increases in $(-\infty, \infty)$

C-5*. Let $g(x) = 2f(x/2) + f(1 - x)$ and $f''(x) < 0$ in $0 \le x \le 1$ then g(x)

(A) decreases in $\left[0, \frac{2}{3}\right]$
(B) decreases $\left[\frac{2}{3}, 1\right]$
(C) increases in $\left[0, \frac{2}{3}\right]$
(D) increases in $\left[\frac{2}{3}, 1\right]$

Section (D) : Local maxima, Local minima, Global maxima, Global minima, Application of Maxima and Minima

D-1. If $f(x) = 1 + 2x^2 + 4x^4 + 6x^6 + ... + 100x^{100}$ is a polynomial in a real variable x, then f(x) has:

(A) neither a maximum nor a minimum
(B) only one maximum
(C) only one minimum
(D) one maximum and one minimum

D-2. If $f(x) = \sin^3 x + \lambda \sin^2 x$; $-\pi/2 < x < \pi/2$, then the interval in which λ should lie in order that f(x) has exactly one minima and one maxima

(A) $(-3/2, 3/2) - \{0\}$
(B) $(-2/3, 2/3) - \{0\}$
(C) R
(D) $\left[-\frac{3}{2}, 0\right)$

D-3. The greatest, the least values of the function,

$f(x) = 2 - \sqrt{1+2x+x^2}$, $x \in [-2, 1]$ are respectively

(A) 2, 1
(B) 2, – 1
(C) 2, 0
(D) none

D-4. If $f(x) = a\, \ell n |x| + bx^2 + x$ has its extremum values at x = – 1 and x = 2, then
(A) a = 2, b = – 1
(B) a = 2, b = – 1/2
(C) a = – 2, b = 1/2
(D) none of these

D-5*. Let $f(x) = (x^2 - 1)^n (x^2 + x + 1)$. f(x) has local extremum at x = 1 if
(A) n = 2
(B) n = 3
(C) n = 4
(D) n = 6

D-6*. If $f(x) = \frac{x}{1 + x\tan x}$, $x \in \left(0, \frac{\pi}{2}\right)$, then

(A) f(x) has exactly one point of minima
(B) f(x) has exactly one point of maxima
(C) f(x) is increasing in $\left(0, \frac{\pi}{2}\right)$
(D) maxima occurs at x_0 where $x_0 = \cos x_0$

D-7*. If $f(x) = \begin{cases} -\sqrt{1-x^2} & , \quad 0 \le x \le 1 \\ -x & , \quad x > 1 \end{cases}$, then

(A) Maximum of f(x) exist at x = 1
(B) Maximum of f(x) doesn't exists
(C) Minimum of $f^{-1}(x)$ exist at x = – 1
(D) Minimum of $f^{-1}(x)$ exist at x = 1

D-8*. If $f(x) = \tan^{-1}x - (1/2)\ \ell n\ x$. Then

(A) the greatest value of f(x) on $\left[1/\sqrt{3}, \sqrt{3}\right]$ is $\pi/6 + (1/4)\ \ell n\ 3$

(B) the least value of f(x) on $\left[1/\sqrt{3}, \sqrt{3}\right]$ is $\pi/3 - (1/4)\ \ell n\ 3$

(C) f(x) decreases on $(0, \infty)$
(D) f(x) increases on $(-\infty, 0)$

D-9. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let m(b) be the minimum value of f(x). As b varies, the range of m(b) is

(A) [0, 1]
(B) $\left(0, \frac{1}{2}\right]$
(C) $\left[\frac{1}{2}, 1\right]$
(D) (0, 1]

D-10. The radius of a right circular cylinder of greatest curved surface which can be inscribed in a given right circular cone is

(A) one third that of the cone
(B) $1/\sqrt{2}$ times that of the cone
(C) 2/3 that of the cone
(D) 1/2 that of the cone

D-11. The dimensions of the rectangle of maximum area that can be inscribed in the ellipse $(x/4)^2 + (y/3)^2 = 1$ are

(A) $\sqrt{8}, \sqrt{2}$
(B) 4, 3
(C) $2\sqrt{8}, 3\sqrt{2}$
(D) $\sqrt{2}$, $\sqrt{6}$

D-12. The largest area of a rectangle which has one side on the x–axis and the two vertices on the curve $y = e^{-x^2}$ is

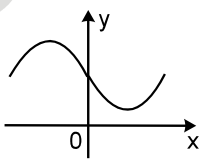
(A) $\sqrt{2}\ e^{-1/2}$
(B) $2\ e^{-1/2}$
(C) $e^{-1/2}$
(D) none

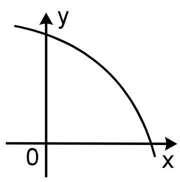
D-13. The maximum distance of the point (k, 0) from the curve $2x^2 + y^2 - 2x = 0$ is equal to

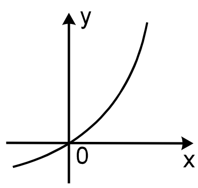
(A) $\sqrt{1+2k-k^2}$
(B) $\sqrt{1+2k+2k^2}$
(C) $\sqrt{1-2k+2k^2}$
(D) $\sqrt{1-2k+k^2}$

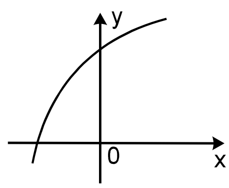
Section (E) : Curvature, Points of inflection, Inequalities

E-1. The curve y = f(x) which satisfies the condition $f'(x) > 0$ and $f''(x) < 0$ for all real x, is:

(A) 

(B) 

(C) 

(D) 

E-2. For which values of 'a' will the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ will be concave upward along the entire real line

(A) $a \in [0, \infty)$
(B) $a \in (-2, \infty)$
(C) $a \in [-2, 2]$
(D) $a \in (0, \infty)$

E-3. If the point (1, 3) serves as the point of inflection of the curve $y = ax^3 + bx^2$ then the value of 'a' and 'b' are:

(A) a = 3/2 & b = – 9/2
(B) a = 3/2 & b = 9/2
(C) a = – 3/2 & b = – 9/2
(D) a = – 3/2 & b = 9/2

E-4. If $f(x) = \ell n\,(x - 2) - \frac{1}{x}$, then

(A) f(x) is M.I. for $x \in (2, \infty)$
(B) f(x) is M.I. for $x \in [-1, 2]$
(C) f(x) is always concave downwards
(D) $f^{-1}(x)$ is M.I. wherever defined

Section (F) : Rolle's Theorem, LMVT

F-1. The function $f(x) = x^3 - 6x^2 + ax + b$ satisfy the conditions of Rolle's theorem on [1, 3]. Which of these are correct ?

(A) $a = 11, b \in R$ (B) $a = 11, b = -6$ (C) $a = -11, b = 6$ (D) $a = -11, b \in R$

F-2. The function $f(x) = x(x + 3)e^{-x/2}$ satisfies all the conditions of Rolle's theorem on [–3, 0]. The value of c which verifies Rolle's theorem, is

(A) 0 (B) – 1 (C) – 2 (D) 3

F-3. If f(x) satisfies the requirements of Lagrange's mean value theorem on [0, 2] and if f(0) = 0 and $f'(x) \le \frac{1}{2}\ \forall\ x \in [0, 2]$, then

(A) $|f(x)| \le 2$
(B) $f(x) \le 1$
(C) $f(x) = 2x$
(D) f(x) = 3 for at least one x in [0, 2]

PART - III : ASSERTION / REASONING

1. **Statement 1 :** The curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{1+a^2} + \frac{y^2}{1-b^2} = 1$ are orthogonal, for $b \in (-1,1)$.

Statement 2 : $ax^2 + by^2 = 1$ and $Ax^2 + By^2 = 1$ are orthogonal iff $ab(A - B) = AB(a - b)$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

2. **STATEMENT-1 :** If f(x) is increasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

STATEMENT-2 : If f(x) is decreasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

3. **STATEMENT-1 :** e^{π} is bigger than π^e.

STATEMENT-2 : $f(x) = x^{1/x}$ is a increasing function when $x \in [e, \infty)$

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

4. **Statement 1 :** ABC is given triangle having respective sides a,b,c. D,E,F are points of the sides BC,CA,AB respectively so that AFDE is a parallelogram. The maximum area of the parallelogram is $\frac{1}{4}$ bcsinA.

Statement 2 : Maximum value of $2kx - x^2$ is at $x = k$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

(E) Both STATEMENTS are false

5. Let $f(x) = x^{50} - x^{20}$

STATEMENT-1 : Global maximum of f(x) in [0, 1] is 0.

STATEMENT-2 : x = 0 is a stationary point of f(x).

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

(E) Both STATEMENTS are false

EXERCISE - 2

PART - I : SUBJECTIVE QUESTIONS

1. A light shines from the top of a pole 50 ft. high. A ball is dropped from the same height from a point 30 ft. away from the light. How fast is the shadow of the ball moving along the ground 1/2 sec. later? [Assume the ball falls a distance $s = 16\ t^2$ ft. in 't' sec.]

2. A variable ΔABC in the xy plane has its orthocentre at vertex 'B' , a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec . How fast is the area of the triangle increasing when $t = \frac{7}{2}$ sec .

3. Find equation of line which is tangent at a point on curve $4x^3 = 27\ y^2$ and normal at other point.

4. The tangent to curve $y = x - x^3$ at point P meets the curve again at Q. Prove that one point of trisection of PQ lies on y-axis. Find locus of other point of trisection

5. Find the equation of the common tangent to the parabolas $y = x^2 + 4x + 8$ and $y = x^2 + 8x + 4$, also find the coordinates of point of contact.

6. In the curve $x^a\ y^b = K^{a+b,}$ prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in a constant ratio. (All the constants being positive).

7. If f : $[0, \infty) \to R$ is the function defined by $f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$, then whether f(x) is injective or not.

8. If $f(x) = 2e^x - ae^{-x} + (2a + 1)\ x - 3$ monotonically increases for $\forall\ x \in R$, then find range of values of a

9. Find the set of values of p for which the equation $|\ell n\ x| - px = 0$ possess three distinct roots.

10. Find the set of all values of the parameter 'a' for which the function
$f(x) = \sin 2x - 8(a + 1) \sin x + (4a^2 + 8a - 14)\ x$ increases for all $x \in R$ and has no critical points for all $x \in R$

11. If $ax^2 + (b/x) \geq c$ for all positive x where a > 0 and b > 0 then show that $27ab^2 \geq 4c^3$.

12. Prove that $e^x + \sqrt{1+e^{2x}} \geq (1 + x) + \sqrt{2+2x+x^2}\ \ \forall\ x \in R$

13. Find which of the two is larger $\ell n\ (1 + x)$ or $\frac{\tan^{-1}x}{1+x}$.

14. Find the values of 'a' for which the function $f(x) = \frac{a}{3}\ x^3 + (a + 2)\ x^2 + (a - 1)\ x + 2$ possess a negative point of minimum.

15. Find the polynomial f (x) of degree 6, which satisfies $\lim_{x\to 0} \left(1 + \frac{f(x)}{x^3}\right)^{1/x} = e^2$ and has local maximum at x = 1 and local minimum at x = 0 and x = 2.

16. The three sides of a trapezium are equal each being 6 cms long, find the area of the trapezium when it is maximum.

17. A sheet of poster has its area 18 m². The margin at the top & bottom are 75 cms. and at the sides 50 cms. What are the dimensions of the poster if the area of the printed space is maximum?

18. Find the set of value(s) of 'a' for which the function f (x) = $\frac{ax^3}{3}$ + (a + 2) x^2 + (a – 1) x + 2 possess a negative point of inflection.

19. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall\ x \in \left(0, \frac{\pi}{2}\right)$ and g(x) = f(sin x) + f(cos x), then find the intervals of monotonicity of g(x).

20. Using Rolle's theorem show that the derivative of the function f(x) = $\begin{cases} x\sin\frac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$ vanishes at an infinite set of points of the interval (0, 1).

21. A function f is differentiable in the interval $0 \le x \le 5$ such that f(0) = 4 & f(5) = – 1. If g(x) = $\frac{f(x)}{x+1}$, then prove that there exists some $c \in (0, 5)$ such that $g'(c) = -\frac{5}{6}$.

22. Let f(x) and g(x) be differentiable functions having no common zeros so that $f(x)\,g'(x) \neq f'(x)\,g(x)$. Prove that between any two zeros of f(x), there exist atleast one zero of g(x).

23. f is continuous in [a, b] and differentiable in (a, b) (where a > 0) such that $\frac{f(a)}{a} = \frac{f(b)}{b}$. Prove that there exist $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(x_0)}{x_0}$.

24. If $\phi(x)$ is a differentiable function $\forall\ x \in R$ and $a \in R^+$ such that $\phi(0) = \phi(2a)$, $\phi(a) = \phi(3a)$ and $\phi(0) \neq \phi(a)$ then show that there is at least one root of equation $\phi'(x + a) = \phi'(x)$ in (0, 2a)

25. Let $-1 \le p \le 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it.

26_. The second derivative f''(x) of the function exists for all x is [0, 1] and satisfies $|f''(x)| \le 1$
If f(0) = f(1) then show that |f'(x)| < 1 for all x is (0, 1)

PART - II : OBJECTIVE QUESTIONS

Single choice type

1. If tangents are drawn from the origin to the curve y = sin x, then their points of contact lie on the curve

(A) x – y = xy
(B) x + y = xy
(C) $x^2 - y^2 = x^2y^2$
(D) $x^2 + y^2 = x^2y^2$

2. Let f(x) = $\begin{cases} -x^2 & , x < 0 \\ x^2 + 8 & , x \ge 0 \end{cases}$ Equation of tangent line touching both branches of y = f(x) is

(A) y = 4x + 1 (B) y = 4x + 4 (C) y = x + 4 (D) y = x + 1

3. If g(x) is a curve which is obtained by the reflection of f(x) = $\frac{e^x - e^{-x}}{2}$ by the line y = x then

(A) g(x) has more than one tangent parallel to x-axis
(B) g(x) has more than one tangent parallel to y-axis
(C) y = –x is a tangent to g(x) at (0, 0)
(D) g(x) has no extremum

4. Equation of normal drawn to the graph of the function defined as f(x) = $\frac{\sin x^2}{x}$, $x \neq 0$ and f(0) = 0 at the origin is

(A) x + y = 0 (B) x – y = 0 (C) y = 0 (D) x = 0

5. The line $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y = be^{-x/a}$ at the point

(A) $(-a, 2b)$ (B) $\left(\frac{a}{2},\frac{b}{2}\right)$ (C) $\left(a,\frac{b}{e}\right)$ (D) $(0, b)$

6. All points on curve $y^2 = 4a\left(x+a\sin\frac{x}{a}\right)$ at which tangents are parallel to the axis of x, lie on a

(A) circle (B) parabola (C) line (D) none of these

7. The ordinate of $y = (a/2)(e^{x/a} + e^{-x/a})$ is the geometric mean of the length of the normal and the quantity:

(A) a/2 (B) a (C) e (D) none of these.

8. Let $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an increasing function in the set of real numbers R. Then a & b satisfy the condition :

(A) $a^2 - 3b - 15 > 0$ (B) $a^2 - 3b + 15 \le 0$

(C) $a^2 + 3b - 15 < 0$ (D) $a > 0$ & $b > 0$

9. If $f(x) = a^{\{a^{|x|}\operatorname{sgn}x\}}$; $g(x) = a^{[a^{|x|}\operatorname{sgn}x]}$ for $a > 1$, $a \ne 1$and $x \in R$, where { } & [ ] denote the fractional part and integral part functions respectively, then which of the following statements holds good for the function h(x), where $(\ell n\, a)\, h(x) = (\ell n\, f(x) + \ell n\, g(x))$.

(A) 'h' is even and increasing (B) 'h' is odd and decreasing

(C) 'h' is even and decreasing (D) 'h' is odd and increasing

10. If $f(x) = (x-4)(x-5)(x-6)(x-7)$ then,

(A) $f'(x) = 0$ has four roots.

(B) three roots of $f'(x) = 0$ lie in $(4, 5) \cup (5, 6) \cup (6, 7)$.

(C) the equation $f'(x) = 0$ has only one real root.

(D) three roots of $f'(x) = 0$ lie in $(3, 4) \cup (4, 5) \cup (5, 6)$.

11. If $f : [1, 10] \to [1, 10]$ is a non-decreasing function and $g : [1, 10] \to [1, 10]$ is a non-increasing function. Let $h(x) = f(g(x))$ with $h(1) = 1$, then $h(2)$

(A) lies in (1, 2) (B) is more than 2 (C) is equal to 1 (D) is not defined

12. If $f(x) = \frac{x^2}{2-2\cos x}$; $g(x) = \frac{x^2}{6x-6\sin x}$ where $0 < x < 1$, then

(A) both 'f' and 'g' are increasing functions

(B) 'f' is decreasing & 'g' is increasing function

(C) 'f' is increasing & 'g' is decreasing function

(D) both 'f' & 'g' are decreasing function

13. Let $f(x) = \begin{cases} x^3-x^2+10x-5 & ,x\le1 \\ -2x+\log_2(b^2-2) & ,x>1\end{cases}$ the set of values of b for which f(x) has greatest value at x = 1 is given by :

(A) $1 \le b \le 2$ (B) $b = \{1, 2\}$

(C) $b \in (-\infty, -1)$ (D) $\left[-\sqrt{130}, -\sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{130}\right]$

14. The set of values of p for which the extremum of the function $f(x) = x^3 - 3px^2 + 3(p^2 - 1)x + 1$ lie in the interval $(-2, 4)$, is:

(A) $(-3, 5)$ (B) $(-3, 3)$ (C) $(-1, 3)$ (D) $(-1, 4)$

15. Four points A, B, C, D lie in that order on the parabola $y = ax^2 + bx + c$. The coordinates of A, B & D are known as $A(-2, 3)$; $B(-1, 1)$ and $D(2, 7)$. The coordinates of C for which the area of the quadrilateral ABCD is greatest, is

(A) (1/2, 7/4) (B) (1/2, – 7/4) (C) (–1/2, 7/4) (D) none

16. In a regular triangular prism the distance from the centre of one base to one of the vertices of the other base is ℓ. The altitude of the prism for which the volume is greatest, is :

(A) $\frac{\ell}{2}$ (B) $\frac{\ell}{\sqrt{3}}$ (C) $\frac{\ell}{3}$ (D) $\frac{\ell}{4}$

17. The lower corner of a leaf in a book is folded over so as to just reach the inner edge of the page. The fraction of width folded over if the area of the folded part is minimum is:

(A) 5/8 (B) 2/3 (C) 3/4 (D) 4/5

18. If x_1 and x_2 are abscissa of two points on the curve $f(x) = x - x^2$ in the interval [0, 1], then maximum value of the expression $(x_1 + x_2) - (x_1^2 + x_2^2)$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) 1 (D) 2

19. The maximum area of the rectangle whose sides pass through the angular points of a given rectangle of sides a and b is

(A) 2 (ab) (B) $\frac{1}{2}(a + b)^2$ (C) $\frac{1}{2}(a^2 + b^2)$ (D) none of these

20. Least value of the function, $f(x) = 2^{x^2} - 1 + \frac{2}{2^{x^2}+1}$ is:

(A) 0 (B) 3/2 (C) 2/3 (D) 1

21_. Square roots of 2 consecutive natural number greater than N^2 is differ by

(A) $> \frac{1}{2N}$ (B) $\geq \frac{1}{2N}$ (C) $< \frac{1}{2N}$ (D) None of these

22. Consider the following statements :

S_1 : The function $y = \frac{2x^2 - 1}{x^4}$ is neither increasing nor decreasing.

S_2 : If f(x) is strictly increasing real function defined on R and c is a real constant, then number of solutions of f(x) = c is always equal to one.

S_3 : Let f(x) = x ; $x \in (0, 1)$. f(x) does not has any point of local maxima/minima

S_4 : f(x) = {x} has maximum at x = 6 (here {.} denotes fractional part function).

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) TTFT (B) FTFT (C) TFTF (D) TFFT

More than one choice type

23. If P is a point on the curve $5x^2 + 3xy + y^2 = 2$ and O is the origin, then OP has

(A) minimum value $\frac{1}{2}$ (B) minimum value $\frac{2}{\sqrt{11}}$ (C) maximum value $\sqrt{11}$ (D) maximum value 2

24. For the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$, at point (2, – 1)

(A) length of subtangent is 7/6. (B) slope of tangent = 6/7

(C) length of tangent = $\sqrt{(85)}/6$ (D) none of these

25. Let $f(x) = x^{m/n}$ for $x \in R$ where m and n are integers, m even and n odd and 0 < m < n. Then

(A) f(x) decreases on $(-\infty, 0]$ (B) f(x) increases on $[0, \infty)$

(C) f(x) increases on $(-\infty, 0]$ (D) f(x) decreases on $[0, \infty)$

26. Let f and g be two differentiable functions defined on an interval I such that $f(x) \geq 0$ and $g(x) \leq 0$ for all $x \in I$ and f is strictly decreasing on I while g is strictly increasing on I then

(A) the product function fg is strictly increasing on I

(B) the product function fg is strictly decreasing on I

(C) fog(x) is monotonically increasing on I

(D) fog (x) is monotonically decreasing on I

27. Let $\phi(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3 \sin x + 4 \cos x \ \forall \ x \in R$, then
(A) ϕ is increasing whenever f is increasing
(B) ϕ is increasing whenever f is decreasing
(C) ϕ is decreasing whenever f is decreasing
(D) ϕ is decreasing if $f'(x) = -11$

28. For the function $f(x) = x^4 (12 \ln x - 7)$
(A) the point $(1, -7)$ is the point of inflection
(B) $x = e^{1/3}$ is the point of minima
(C) the graph is concave downwards in $(0, 1)$
(D) the graph is concave upwards in $(1, \infty)$

29. The curve $y = \dfrac{x+1}{x^2+1}$ has
(A) $x = 1$, as point of inflection
(B) $x = -2 + \sqrt{3}$, as point of inflection
(C) $x = -1$, as point of minimum
(D) $x = -2 - \sqrt{3}$, as point of inflection

30. Let $f(x) = 40/(3x^4 + 8x^3 - 18x^2 + 60)$. Which of the following statement(s) about f(x) is (are) correct ?
(A) f(x) has local minima at $x = 0$.
(B) f(x) has local maxima at $x = 0$.
(C) Absolute maximum value of f(x) is not defined.
(D) f(x) is local maxima at $x = -3$, $x = 1$.

31_. For the function $f(x) = x \cot^{-1}x$, $x \geq 0$
(A) there is atleast one $x \in (0, 1)$ for which $\cot^{-1}x = \dfrac{x}{1+x^2}$
(B) for atleast one x in the internal $(0, \infty)$, $f\left(x + \dfrac{2}{\pi}\right) - f(x) < 1$
(C) number of solution of the equation $f(x) = \sec x$ is 1
(D) $f'(x)$ is strictly decreasing is the internal $(0, \infty)$

32_. Which of the following statements are true :
(A) $|\tan^{-1} x - \tan^{-1} y| \leq |x - y|$, where x, y are real numbers.
(B) The function $x^{100} + \sin x - 1$ is strictly increasing in $[0, 1]$
(C) If a, b, c are is A.P, then at least one root of the equation $3ax^2 - 4bx + c = 0$ is positive
(D) Curve $y^2 = 4ax$ and $y = e^{-\frac{x}{2a}}$ are orthogonal curves.

PART - III : MATCH THE COLUMN

1.

	Column-I		Column-II
(A)	If θ is angle between the curves $y = [\,|\sin x| + |\cos x|\,]$, ([·] denote GIF) and $x^2 + y^2 = 5$ then $\text{cosec}^2\theta$ is	(p)	$\dfrac{5}{4}$
(B)	Length of subnormal to $x = \sqrt{2}\cos t$, $y = -3\sin t$ at $t = \dfrac{-\pi}{4}$ is	(q)	2
(C)	If [a, b], (b < 1) is largest interval in which $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 19$ is strictly increasing then $\dfrac{a}{b}$ is	(r)	$\dfrac{8}{3}$
(D)	If $a + b = 8$, $a, b > 0$ then minimum value of $\dfrac{a^3+b^3}{48}$ is	(s)	$\dfrac{9}{2}$

2. **Column – I** **Column – II**

(A) $f(x) = \frac{\sin x}{e^x}$, $x \in [0,\pi]$

(B) $f(x) = \text{sgn}((e^x - 1)\ \ell nx)$, $x \in \left[\frac{1}{2},\frac{3}{2}\right]$

(C) $f(x) = (x-1)^{2/5}$, $x \in [0,3]$

(D) $f(x) = \begin{cases} x\left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right), & x \in [-1,1]-\{0\} \\ 0, & x = 0 \end{cases}$

(p) Conditions in Rolle's theorem are satisfied.

(q) Conditions in LMVT are satisfied.

(r) At least one condition in Rolle's theorem is not satisfied.

(s) At least one condition in LMVT is not satisfied.

3. **Column – I** **Column – II**

(A) A rectangle is inscribed in an equilateral triangle of side 4cm. Square of maximum area of such a rectangle is

(B) The volume of a rectangular closed box is 72 and the base sides are in the ratio 1 : 2. The least total surface area is

(C) Maximum value of $\left(\sqrt{-3+4x-x^2}+4\right)^2+(x-5)^2$ (where $1 \le x \le 3$) is

(D) The sides of a rectangle of greatest perimeter which is inscribed in a semicircle of radius $\sqrt{5}$ are a and b. Then $a^3 + b^3$ =

(p) 65

(q) 36

(r) 12

(s) 108

4. **Match the column**

Column - I

(A) $\sin^{-1} x - \cos^{-1}x$ is maximum at

(B) $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$ is minimum at

(C) $(\tan^{-1}x)^2 + (\cot^{-1}x)^2$ is minimum at

(D) $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$ is maximum at

Column - II

(p) $x = -1$

(q) $x = -\frac{1}{\sqrt{2}}$

(r) $x = 0$

(s) $x = \frac{1}{\sqrt{2}}$

(t) $x = 1$

PART - IV : COMPREHENSION

Comprehension # 1

Let a(t) be a function of t such that $\frac{da}{dt} = 2$ for all values of t and a = 0 when t = 0. Further $y = m(t)\,x + c(t)$ is tangent to the curve $y = x^2 - 2ax + a^2 + a$ at the point whose abscissa is 0. Then

1. If the rate of change of distance of vertex of $y = x^2 - 2ax + a^2 + a$ from the origin with respect to t is k, then k =

(A) 2 (B) $2\sqrt{2}$ (C) $\sqrt{2}$ (D) $4\sqrt{2}$

2. If the rate of change of c(t) with respect to t, when t = k, is ℓ, then ℓ =

(A) $16\sqrt{2} - 2$ (B) $8\sqrt{2} + 2$ (C) $10\sqrt{2} + 2$ (D) $16\sqrt{2} + 2$

3. The rate of change of m(t), with respect to t, at t = ℓ is

(A) –2 (B) 2 (C) – 4 (D) 4

Comprehension # 2

Consider a function f defined by $f(x) = \sin^{-1} \sin\left(\frac{x + \sin x}{2}\right)$, $\forall\ x \in [0, \pi]$, which satisfies

$f(x) + f(2\pi - x) = \pi$, $\forall\ x \in [\pi, 2\pi]$ and $f(x) = f(4\pi - x)$ for all $x \in [2\pi, 4\pi]$, then

4. If α is the length of the largest interval on which f(x) is increasing, then α =

(A) $\frac{\pi}{2}$ (B) π (C) 2π (D) 4π

5. If f(x) is symmetric about x = β, then β =

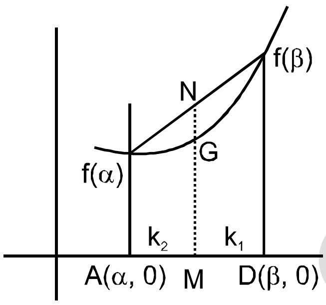
(A) $\frac{\alpha}{2}$ (B) α (C) $\frac{\alpha}{4}$ (D) 2α

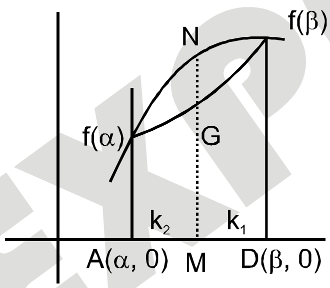
6. Maximum value of f(x) on $[0, 4\pi]$ is :

(A) $\frac{\beta}{2}$ (B) β (C) $\frac{\beta}{4}$ (D) 2β

Comprehension # 3

For a double differentiable function f(x) if $f''(x) \geq 0$ then f(x) is concave upward and if $f''(x) \leq 0$ then f(x) is concave downward





Here $M\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2}, 0\right)$

If f(x) is a concave upward in [a, b] and $\alpha, \beta \in [a, b]$ then $\frac{k_1 f(\alpha) + k_2 f(\beta)}{k_1 + k_2} \geq f\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2}\right)$, where k_1, $k_2 \in R^+$

If f(x) is a concave downward in [a, b] and $\alpha, \beta \in [a, b]$ then $\frac{k_1 f(\alpha) + k_2 f(\beta)}{k_1 + k_2} \leq f\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2}\right)$, where $k_1, k_2 \in R^+$

then answer the following

7. Which of the following is true

(A) $\frac{\sin\alpha + \sin\beta}{2} > \sin\left(\frac{\alpha + \beta}{2}\right)$; $\alpha, \beta\ (0, \pi)$

(B) $\frac{\sin\alpha + \sin\beta}{2} < \sin\left(\frac{\alpha + \beta}{2}\right)$; $\alpha, \beta \in (\pi, 2\pi)$

(C) $\frac{\sin\alpha + \sin\beta}{2} < \sin\left(\frac{\alpha + \beta}{2}\right)$; $\alpha, \beta \in (0, \pi)$

(D) none of these

8. Which of the following is true

(A) $\frac{2^{\alpha}+2^{\beta+1}}{3} \le 2^{\frac{\alpha+2\beta}{3}}$

(B) $\frac{2\ell n\,\alpha+\ell n\beta}{3} \ge \ell n\left(\frac{2\alpha+\beta}{3}\right)$

(C) $\frac{\tan^{-1}\alpha+\tan^{-1}\beta}{2} \le \tan^{-1}\left(\frac{\alpha+\beta}{2}\right)$ a, b $\in R^{-}$

(D) $\frac{e^{\alpha}+2e^{\beta}}{3} \ge e^{\frac{\alpha+2\beta}{3}}$

9. Let α , β and γ are three distinct real numbers and $f''(x) < 0$. Also f(x) is increasing function and let $A = \frac{f^{-1}(\alpha)+f^{-1}(\beta)+f^{-1}(\gamma)}{3}$ and $B = f^{-1}\left(\frac{\alpha+\beta+\gamma}{3}\right)$, then order relation between A and B is - (given $f^{-1}(x)$)

(A) A > B (B) A < B (C) A = B (D) none of these

EXERCISE - 3

PART - I : IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. In [0, 1] Lagranges Mean Value theorem is NOT applicable to **[IIT-JEE-2003, Scr.(3, –1) /84]**

(A) $f(x) = \begin{cases} \frac{1}{2} - x & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & x \ge \frac{1}{2} \end{cases}$ (B) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

(C) f(x) = x|x| (D) f(x) = |x|

2. Using the relation $2(1 - \cos x) < x^2$, $x \ne 0$ or otherwise, prove that $\sin(\tan x) \ge x$, $\forall x \in \left[0, \frac{\pi}{4}\right]$

[IIT-JEE-2003, Main (4, 0) /60]

3. Let f : [0,4] $\to$ R is a differentiable function **[IIT-JEE-2003, Main (4, 0) /60]**
For some a,b $\in$ (0,4), show that $f^2(4) - f^2(0) = 8f(a) f'(b)$

4. For the circle $x^2 + y^2 = r^2$, find the value of 'r' for which the area enclosed by the tangents drawn from the point P(6,8) to the circle and the chord of contact is maximum.
[IIT-JEE-2003, Main (2, 0) /60]

5. If f is differentiable and strictly increasing in a neighborhood of '0', then

$\lim\limits_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)} =$ **[IIT-JEE-2004, Scr.(3, –1) /84]**

(A) 0 (B) 1 (C) – 1 (D) 2

6. If $f(x) = x^\alpha \ell nx$ and f(0) = 0, If Rolle's theorem can be applied to f in [0, 1] , then value of α can be **[IIT-JEE-2004, Scr.(3, –1) /84]**
(A) –2 (B) –1 (C) 0 (D) 1/2

7. $P(x) = 51x^{101} - 2323x^{100} - 45x + 1035$
Using Rolle's theorem, prove that P(x) = 0 has at least one root in $(45^{1/100}, 46)$.
[IIT-JEE-2004, Main (2, 0) /60]

8. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$ **[IIT-JEE-2004, Scr.(3, –1) /84]**
(A) f(x) is a strictly increasing function (B) f(x) has a local maxima
(C) f(x) is a strictly decreasing function (D) f(x) is bounded

9. Prove that $\sin x + 2x \ge \frac{3x(x+1)}{\pi}$, $x \in \left[0, \frac{\pi}{2}\right]$. Justify the inequalities used in the relation.

[IIT-JEE-2004, Main (4, 0) /60]

10. If $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve y = f(x) at the point (1, 2). **[IIT-JEE-2005, Main (2, 0) /60]**

11. If f(x) be a twice differentiable function such that $f(x) = x^2$ for x = 1, 2, 3, then
[IIT-JEE-2005, Scr.(3, –1) /84]
(A) $f''(x) = 2 \quad \forall\ x \in [1, 3]$ (B) $f''(x) = 2$ for some $x \in (1, 3)$
(C) $f''(x) = 3\ \forall\ x \in (2, 3)$ (D) $f''(x) = f'(x)$ for $x \in (2, 3)$

12. If P(x) be a polynomial of degree 3 satisfying $P(-1) = 10$, $P(1) = -6$ and P(x) has maxima at $x = -1$ and $P'(x)$ has minima at $x = 1$. Find the distance between the local maxima and local minima of the curve. **[IIT-JEE-2005, Main (4, 0) /60]**

13. If f(x) is a twice differentiable function such that $f(a) = 0$, $f(b) = 2$, $f(c) = -1$, $f(d) = 2$, $f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zeroes of $g(x) = (f'(x))^2 + f''(x) f(x)$ in the interval [a, e] is **[IIT-JEE-2006, 6/184]**

14*. f(x) is cubic polynomial which has local maximum at $x = -1$, If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local minima at $x = 0$, then **[IIT-JEE-2006, (5, –1)/184]**

(A) the distance between (–1, 2) and (a, f(a)), where x = a is the point of local minima is $2\sqrt{5}$.

(B) f(x) is increasing for $x \in [1, 2\sqrt{5})$

(C) f(x) has local minima at x = 1

(D) the value of f(0) = 5

15. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ **[IIT-JEE 2007, Paper-1, (3, –1)/ 81]**

(A) on the left of x = c
(B) on the right of x = c
(C) at no point
(D) at all points

Comprehension # 1

If a continuous function f defined on the real line **R**, assumes positive and negative values in **R** then the equation f(x) = 0 has a root in **R**. For example, if it is known that a continuous function f on **R** is positive at some point and its minimum value is negative then the equation f(x) = 0 has a root in **R**.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

16. The line y = x meets $y = ke^x$ for $k \le 0$ at **[IIT-JEE 2007, Paper-2, (4, –1)/ 81]**

(A) no point
(B) one point
(C) two points
(D) more than two points

17. The positive value of k for which $ke^x - x = 0$ has only one root is **[IIT-JEE 2007, Paper-2, (4, –1)/ 81]**

(A) $\frac{1}{e}$
(B) 1
(C) e
(D) $\log_e 2$

18. For k > 0, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is **[IIT-JEE 2007, Paper-2, (4, –1)/ 81]**

(A) $\left(0, \frac{1}{e}\right)$
(B) $\left(\frac{1}{e}, 1\right)$
(C) $\left(\frac{1}{e}, \infty\right)$
(D) (0, 1)

19. Let f(x) = 2 + cos x for all real x. **[IIT-JEE 2007, Paper-2, (3, –1)/ 81]**

STATEMENT - 1 : For each real t, there exists a point c in $[t, t + \pi]$ such that $f'(c) = 0$.

because

STATEMENT - 2 : $f(t) = f(t + 2\pi)$ for each real t.

(A) Statement - 1 is True, Statement - 2 is True; Statement - 2 **is** a correct explanation for Statement - 1

(B) Statement - 1 is True, Statement - 2 is True; Statement - 2 is **NOT** a correct explanation for Statement - 1

(C) Statement - 1 is True, Statement - 2 is False

(D) Statement - 1 is False, Statement - 2 is True

20. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is
[IIT-JEE 2008, Paper-2, (3, –1)/ 163]
(A) even and is strictly increasing in $(0, \infty)$
(B) odd and is strictly decreasing in $(-\infty, \infty)$
(C) odd and is strictly increasing in $(-\infty, \infty)$
(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

21. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3 & , \ -3 < x \le -1 \\ x^{2/3} & , \ -1 < x < 2 \end{cases}$ is **[IIT-JEE 2008, Paper-1, (3, –1)/ 82]**
(A) 0 (B) 1 (C) 2 (D) 3

22*. For the function $f(x) = x\cos\frac{1}{x}$, $x \ge 1$, **[IIT-JEE 2009, Paper-2, (4, –1)/ 80]**
(A) for at least one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$
(B) $\lim_{x\to\infty} f'(x) = 1$
(C) for all x in the interval $[1, \infty)$, $f(x + 2) - f(x) > 2$
(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

23. Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2$. Then the value of p(2) is **[IIT-JEE 2009, Paper-2, (4, –1)/ 80]**

24. Let f be a function defined on **R** (the set of all real numbers) such that
$f'(x) = 2010\,(x - 2009)\,(x - 2010)^2\,(x - 2011)^3\,(x - 2012)^4$, for all $x \in$ **R**.
If g is a function defined on **R** with values in the interval $(0, \infty)$ such that $f(x) = \ell n\,(g(x))$, for all $x \in$ **R**, then the number of points in **R** at which g has a local maximum is
[IIT-JEE 2010, Paper-2, (3, 0)/ 79]

25. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then **[IIT-JEE 2010, Paper-1, (3, –1)/ 84]**
(A) a = b and c ≠ b (B) a = c and a ≠ b (C) a ≠ b and c ≠ b (D) a = b = c

26. Match the statements given in **Column-I** with the intervals/union of intervals given in **Column-II**
[IIT-JEE 2011, Paper-2, (8, 0), 80]

Column-I	**Column-II**
(A) The set $\left\{\mathrm{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number}, \lvert z\rvert = 1, z \ne \pm 1\right\}$ is	(p) $(-\infty, -1) \cup (1, \infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty, 0) \cup (0, \infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is	(r) $[2, \infty)$
(D) If $f(x) = x^{3/2}(3x - 10)$, $x \ge 0$, then f(x) is increasing in	(s) $(-\infty, -1] \cup [1, \infty)$
	(t) $(-\infty, 0] \cup [2, \infty)$

27. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is
[IIT-JEE 2011, Paper-2, (4, 0), 80]

28. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 p(3) = 2 , then p′(0) is **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

29. Let f : IR $\rightarrow$ IR be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

30. The number of points in $(-\infty, \infty)$, for which $x^2 - x \sin x - \cos x = 0$, is **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

(A) 6 (B) 4 (C) 2 (D) 0

31.* A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

(A) 24 (B) 32 (C) 45 (D) 60

32. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2 \le h \le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$

[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

33.* The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x = **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) – 2 (B) $\frac{-2}{3}$ (C) 2 (D) $\frac{2}{3}$

Paragraph for Question Nos. 34 to 35

Let f : [0, 1] $\rightarrow$ R (the set of all real numbers) be a function. Suppose the function f is twice differentiable, f(0) = f(1) = 0 and satisfies $f''(x) - 2f'(x) + f(x) \ge e^x$, $x \in [0, 1]$.

34. Which of the following is true for 0 < x < 1 ? **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $0 < f(x) < \infty$ (B) $-\frac{1}{2} < f(x) < \frac{1}{2}$ (C) $-\frac{1}{4} < f(x) < 1$ (D) $-\infty < f(x) < 0$

35. If the function $e^{-x} f(x)$ assumes its minimum in the interval [0, 1] at $x = \frac{1}{4}$, which of the following is true ? **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $f'(x) < f(x)$, $\frac{1}{4} < x < \frac{3}{4}$ (B) $f'(x) > f(x)$, $0 < x < \frac{1}{4}$

(C) $f'(x) < f(x)$, $0 < x < \frac{1}{4}$ (D) $f'(x) < f(x)$, $\frac{3}{4} < x < 1$

36. A line L : y = mx + 3 meets y - axis at E(0, 3) and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum

Match List I with List II and select the correct answer using the code given below the lists :

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

	List - I		**List - II**
P.	m =	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	y_0 =	3.	2
S.	y_1 =	4.	1

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

PART - II : AIEEE PROBLEMS (PREVIOUS YEARS)

1. The positive real number x when added to its reciprocal gives the minimum sum at x equals **[AIEEE 2003]**

(1) 2 (2) 1 (3) – 1 (4) – 2

2. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where a > 0, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals **[AIEEE 2003]**

(1) 3 (2) 1 (3) 2 (4) 1/2

3. If 2a + 3b + 6c = 0, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval

(1) (0, 1) (2) (1, 2) (3) (2, 3) (4) (1, 3)

[AIEEE 2004]

4. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa, is **[AIEEE 2004]**

(1) (2, 4) (2) (2, – 4) (3) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (4) $\left(\frac{9}{8}, \frac{9}{2}\right)$

5. The normal to the curve x = a (1 + cos θ), y = a sin θ at 'θ' always passes through the fixed point **[AIEEE 2004]**

(1) (a, 0) (2) (0, a) (3) (0, 0) (4) (a, a)

6. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is **[AIEEE 2004]**

(1) π/2 (2) π/6 (3) π/4 (4) π/3

7. A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched ? **[AIEEE 2005]**

Interval	**Function**
(1) $(-\infty, -4]$	$x^3 + 6x^2 + 6$
(2) $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(3) $[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(4) $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$

8. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm³/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases , is- **[AIEEE 2005]**

(1) $\frac{5}{6\pi}$ cm/min (2) $\frac{1}{54\pi}$ cm/min (3) $\frac{1}{18\pi}$ cm/min (4) $\frac{1}{36\pi}$ cm/min

9. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is **[AIEEE 2005]**

(1) 2ab (2) ab (3) $\sqrt{ab}$ (4) $\frac{a}{b}$

10. Let f(x) be differentiable for all x. If f(1) = – 2 and $f'(x) \geq 2$ for $x \in (1, 6)$ then **[AIEEE 2005]**

(1) f(6) = 5 (2) f(6) < 5 (3) $f(6) \geq 8$ (4) f(6) < 8

11*. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point 'θ' is such that

(1) it is at a constant distance from the origin (2) it passes through $\left(\frac{a\pi}{2}, -a\right)$ **[AIEEE 2005]**

(3) it makes angle $\pi/2 + \theta$ with the x-axis (4) it passes through the origin

12. The function $f(x) = \frac{x}{2}+\frac{2}{x}$ has a local minimum at **[AIEEE 2006]**

(1) x = – 2 (2) x = 0 (3) x = 1 (4) x = 2

13. A value of c for which the conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is **[AIEEE 2007]**

(1) $2\log_3 e$ (2) $\frac{1}{2}\log_e 3$ (3) $\log_3 e$ (4) $\log_e 3$

14. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in **[AIEEE 2007]**

(1) $(\pi/4, \pi/2)$ (2) $(-\pi/2, \pi/4)$ (3) $(0, \pi/2)$ (4) $(-\pi/2, \pi/2)$

15. Suppose the cubic $x^3 - px + q = 0$ has three distinct real roots where p > 0 and q > 0. Then, which one of the following holds ? **[AIEEE 2008]**

16. The shortest distance between the line y – x = 1 and the curve $x = y^2$ is **[AIEEE 2009]**

(1) $\frac{3\sqrt{2}}{8}$ (2) $\frac{2\sqrt{3}}{8}$ (3) $\frac{3\sqrt{2}}{5}$ (4) $\frac{\sqrt{3}}{4}$

17. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that x = 0 is the only real root of $P'(x) = 0$. If P(–1) < P(1), then in the interval [–1, 1] **[AIEEE 2009]**

(1) P (–1) is the minimum and P(1) is the maximum of P

(2) P (–1) is not minimum but P(1) is the maximum of P

(3) P (–1) is the minimum and P(1) is not the maximum of P

(4) neither P (–1) is the minimum nor P(1) is the maximum of P

18. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is **[AIEEE 2010]**

(1) y = 1 (2) y = 2 (3) y = 3 (4) y = 0

19. Let f : **R** ì **R** be defined by **[AIEEE 2010]**

$$f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at x = – 1, then a possible value of k is

(1) 0 (2) $-\frac{1}{2}$ (3) –1 (4) 1

20. Let f : **R** ì **R** be a continuous function defined by f(x) = $\frac{1}{e^x + 2e^{-x}}$ **[AIEEE 2010]**

Statement -1 : f(c) = $\frac{1}{3}$, for some c $\in$ R.

Statement -2 : $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all x $\in$ R.

(1) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement -1.
(2) Statement-1 is true, Statement-2 is false.
(3) Statement -1 is false, Statement -2 is true.
(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

21. The shortest distance between line y – x = 1 and curve x = y^2 is : **[AIEEE-2011]**

(1) $\frac{\sqrt{3}}{4}$ (2) $\frac{3\sqrt{2}}{8}$ (3) $\frac{8}{3\sqrt{2}}$ (4) $\frac{4}{\sqrt{3}}$

22. Let f be a function defined by - **[AIEEE-2011, II]**

$$f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Statement - 1 : x = 0 is point of minima of f
Statement - 2 : f '(0) = 0.

(1) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.
(2) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for statement-1
(3) Statement-1 is true, statement-2 is false.
(4) Statement-1 is false, statement-2 is true.

23. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is :**[AIEEE- 2012]**

(1) $\frac{9}{7}$ (2) $\frac{7}{9}$ (3) $\frac{2}{9}$ (4) $\frac{9}{2}$

24. Let a, b $\in$ R be such that the function f given by f(x) = $\ell n |x| + bx^2 + ax, x \neq 0$ has extreme values at x = – 1 and x = 2.
Statement-1 : f has local maximum at x = – 1 and at x = 2. **[AIEEE- 2012]**

Statement-2 : a = $\frac{1}{2}$ and b = $\frac{-1}{4}$.

(1) Statement-1 is false, Statement-2 is true.
(2) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(3) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(4) Statement-1 is true, statement-2 is false.

25. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]
(1) lies between 1 and 2 (2) lies between 2 and 3
(3) lies between –1 and 0 (4) does not exist.**[AIEEE - 2013, (4, –¼),120]**

ANSWERS

EXERCISE - 1

PART - I

A-1. (i) -2 cm/min (ii) 2 cm^2/min

A-2. $2x^2 - 3x + 1$

A-3. (i) 2 km/hr (ii) 6 km/h

A-4. $7.68\ \pi$cm^3

Section (B) :

B-1. $y = x$ **B-2.** $a = 1, b = -2$

B-3. $(1, -1)$, $(-1, -5)$

B-5. $2x + y = 4,\ y = 2x$ **B-6.** $(9/4, 3/8)$

B-7. $\frac{\pi}{3}$ **B-8.** -1 **B-9.** $(-6, 3)$

B-10. 10 **B-11.** 2 : 1

Section (C) :

C-2. (i) M.D. in $(-\infty, -3]$
M.I. in $[-3, 0]$
M.D. in $[0, 2]$
M.I. in $[2, \infty)$

(ii) M.D. in $\left(0, \frac{1}{\sqrt{3}}\right]$

M.I. in $\left[\frac{1}{\sqrt{3}}, \infty\right)$

C-3. (i) Neither increasing nor decreasing, increasing
(ii) at $x = -2$ decreasing
at $x = 0$ decreasing
at $x = 3$ neither increasing nor decreasing
at $x = 5$ increasing

C-4. $(-\infty, -3]$ **C-5.** $a \in R^+$

C-8. $2\sin x + \tan x$, 0

Section (D) :

D-1. (i) local max at $x = 1$, local min at $x = 6$

(ii) local max. at $x = -\frac{1}{5}$, local min. at $x = -1$

(iii) local mini at $x = \frac{1}{e}$, No local maxima

D-2. (i) max = 8, min. = -8

(ii) max = $\sqrt{2}$, min = -1

(iii) max. = 8, min. = -10

(iv) max. = 25, min = -39

(v) max. at $x = \pi/6$, max. value = 3/4; min. at $x = 0$ and $\pi/2$, min. value = 1/2

D-3. local max at $x = 1$, local min at $x = 2$.

D-5. 3, 1 (respective) **D-6.** $b \in (0, e]$

D-7. F = 191 **D-9.** $\frac{4\pi r^3}{3\sqrt{3}}$

D-11. 110 m , $\frac{220}{\pi}$ m **D-12.** 32 sq. units

D-13. 12cm, 6 cm

Section (E) :

E-1. (i) 3 (ii) 1

PART - II

Section (A) :

A-1. (B) **A-2** (C) **A-3.** (A)
A-4. (C) **A-5.** (A)

Section (B) :

B-1. (A) **B-2.** (C) **B-3.** (D) **B-4.** (B)
B-5*. (CD) **B-6*.** (AB) **B-7.** (B)
B-8. (A) **B-9*.** (A) (D)

Section (C) :

C-1. (A) **C-2.** (B) **C-3*.** (AB)
C-4*. (AD) **C-5*.** (BCD)

Section (D) :

D-1. (C) **D-2.** (A) **D-3.** (C)
D-4. (B) **D-5*.** (ACD) **D-6*.** (BD)
D-7*. (AC) **D-8*.** (ABC) **D-9.** (D)
D-10. (D) **D-11.** (C) **D-12.** (A)
D-13. (C)

Section (E) :

E-1. (D) **E-2.** (C) **E-3.** (D)
E-4. (ACD)

Section (F) :

F-1. (A) **F-2.** (C) **F-3.** (B)

PART - III

1. (A) **2.** (D) **3.** (C) **4.** (A) **5.** (B)

EXERCISE - 2

PART - I

1. -1500 ft/sec **2.** $\frac{66}{7}$ cm^2/sec

3. $y = \sqrt{2}\,x - 2\sqrt{2}$, $y = -\sqrt{2}\,x + 2\sqrt{2}$

4. $y = x - 5x^3$

5. y = 8x + 4; point of contact (2, 20) and (0, 4)

7. Injective **8.** $[0, \infty)$

9. $p \in (0, 1/e)$ **10.** $a < -\left(2+\sqrt{5}\right)$ or $a > \sqrt{5}$

13. $\ell n\,(1 + x)$ **14.** $(1, \infty)$

15. $f(x) = 2x^4 - \frac{12}{5}x^5 + \frac{2}{3}x^6$

16. $27\sqrt{3}$ sq. cms **17.** Width $2\sqrt{3}$ m, length $3\sqrt{3}$ m

18. $(-\infty, -2) \cup (0, \infty)$

19. Increasing when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$,

decreasing when $x \in \left(0, \frac{\pi}{4}\right)$.

25. $\cos\left(\frac{1}{3}\cos^{-1}p\right)$

PART - II

1. (C) **2.** (B) **3.** (D) **4.** (A) **5.** (D) **6.** (B)

7. (B) **8.** (B) **9.** (D) **10.** (B) **11.** (C) **12.** (C)

13. (D) **14.** (C) **15.** (A) **16.** (B) **17.** (B)

18. (A) **19.** (B) **20.** (D) **21.** (C) **22.** (C)

23. (BD)

24. (ABC) **25.** (AB) **26.** (AD) **27.** (AD)

28. (ABCD) **29.** (ABD) **30.** (ACD) **31.** (BD)

32. (ABCD)

PART - III

1. (A) $\to$ (p), (B) $\to$ (s), (C) $\to$ (q), (D) $\to$ (r)

2. (A) $\to$ (p,q), (B) $\to$ (r,s),
(C) $\to$ (r,s), (D) $\to$ (r,s)

3. (A) $\to$ (r), (B) $\to$ (s), (C) $\to$ (q), (D) $\to$ (p)

4. (A) $\to$ (t), (B) $\to$ (s), (C) $\to$ (t), (D) $\to$ (p)

PART - IV

1. (B) **2.** (D) **3.** (C) **4.** (C) **5.** (B) **6.** (A)

7. (C) **8.** (D) **9.** (A)

EXERCISE - 3

PART - I

1. (A) **4.** 5 **5.** (C) **6.** (D) **8.** (A) **10.** y = 2

11. (B) **12.** $4\sqrt{65}$ **13.** 6 **14*.** (BC)

15. (A) **16.** (B) **17.** (A) **18.** (A) **19.** (B) **20.** (C)

21. (C) **22*.** (BCD) **23.** 0 **24.** 1

25. (D)

26. (A) $\to$ (s), (B) $\to$ (t), (C) $\to$ (r), (D) $\to$ (r)

27. 2 **28.** (9) **29.** (5) **30.** (C) **31.*** (A, C)

32. 9 **33.*** (AB) **34.** (D) **35.** (C) **36.** (A)

PART - II

1. (2) **2.** (3) **3.** (1) **4.** (4) **5.** (1) **6.** (1)

7. (2) **8.** (3) **9.** (1) **10.** (3) **11*.** (13) **12.** (4)

13. (1) **14.** (2) **15.** (1) **16.** (1) **17.** (2) **18.** (3)

19. (3) **20.** (4) **21.** (2) **22.** (2) **23.** (3) **24.** (2)

25. (4)

Advanced Level Problems

PART - I : OBJECTIVE QUESTIONS

Single choice type

1. The interval in which the function x^3 increases less rapidly than $6x^2 + 15x + 5$ is
(A) $(-\infty, -1)$ (B) $(-5, 1)$ (C) $(-1, 5)$ (D) $(5, \infty)$

2. On the interval [0, 1) the function $f(x) = x^{25}(1 - x)^{75}$ takes its maximum value at
(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{1}{4}$

3. The difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $[-\pi/2, \pi/2]$ is
(A) $\frac{\sqrt{3}+\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}+\sqrt{2}}{2} + \pi/6$ (C) $\frac{\pi}{2}$ (D) π

4. The abscissa of point on curve $ay^2 = x^3$, normal at which cuts off equal intercepts from the coordinate axes is
(A) $\frac{2a}{9}$ (B) $\frac{4a}{9}$ (C) $-\frac{4a}{9}$ (D) $-\frac{2a}{9}$

5. The x-intercept of tangent at arbitrary point of curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to
(A) square of the abscissa of the point of tangency
(B) square root of the abscissa of the point of tangency
(C) cube of the abscissa of the point of tangency
(D) cube root of the abscissa of the point of tangency.

6. The slope of the normal at the point with abscissa x = – 2 of the graph of the function $f(x) = \left|x^2 - |x|\right|$ is
(A) $-\frac{1}{6}$ (B) $-\frac{1}{3}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

7. A particle moving on a curve has the position at time t given by $x = f'(t) \sin t + f''(t) \cos t$, $y = f'(t) \cos t - f''(t) \sin t$, where f is a thrice differentiable function. Then the velocity of the particle at time t is :
(A) $f'(t) + f''(t)$ (B) $f'(t) - f'''(t)$ (C) $f'(t) + f'''(t)$ (D) $f'(t) - f''(t)$

8. If at any point on a curve the subtangent and subnormal are equal, then the tangent is equal to
(A) $\sqrt{2}$ ordinate (B) $\sqrt{2}$ |ordinate| (C) $\sqrt{2\,(\text{ordinate})}$ (D) none of these

9. The number of values of c such that the straight line $3x + 4y = c$ touches the curve $\frac{x^4}{2} = x + y$ is:
(A) 0 (B) 1 (C) 2 (D) 4

10. The beds of two rivers (within a certain region) are a parabola $y = x^2$ and a straight line $y = x - 2$. These rivers are to be connected by a straight canal. The coordinates of the ends of the shortest canal can be:
(A) $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $\left(-\frac{11}{8}, \frac{5}{8}\right)$ (B) $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $\left(\frac{11}{8}, -\frac{5}{8}\right)$
(C) (0, 0) & (1, –1) (D) none of these

11. The lines $y = -\frac{3}{2}x$ and $y = -\frac{2}{5}x$ intersect the curve $3x^2 + 4xy + 5y^2 - 4 = 0$ at the points P and Q respectively. The tangents drawn to the curve at P and Q:
(A) intersect each other at angle of 45º
(B) are parallel to each other
(C) are perpendicular to each other
(D) none of these

12. At (0, 0), the curve $y^2 = x^3 + x^2$
(A) touches x-axis
(B) bisects the angle between the axes
(C) makes an angle of 60° with Ox
(D) none of these

13. The longest interval in which $f(x) = x\sqrt{4ax - x^2}$ $(a < 0)$ is decreasing is –
(A) [4a, 0] (B) [3a, 0] (C) $(-\infty, 3]$ (D) none of these

14. Number of roots of the equation $4\cos(e^x) = 2^x + 2^{-x}$, is
(A) 0 (B) 1 (C) 2 (D) infinite

15. The set of values of the parameter 'a' for which the function ;
$f(x) = 8ax - a \sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in R$, is
(A) [–1, 1] (B) $(-\infty, -6)$ (C) $(6, +\infty)$ (D) $[6, +\infty)$

16. $f : [0, 4] \to R$ is a differentiable function. Then for some $a, b \in (0, 4)$, $f^2(4) - f^2(0) =$
(A) $8f'(a) . f(b)$ (B) $4f'(b) f(a)$ (C) $2f'(b) f(a)$ (D) $f'(b) f(a)$

17. The values of 'a' and 'b' for which all the extrema of the function $f(x) = a^2x^3 - \frac{a}{2}x^2 - 2x - b$ are positive and the minimum is at the point $x_0 = \frac{1}{3}$
(A) when $a = -2 \Rightarrow b < \frac{-11}{27}$ and when $a = 3 \Rightarrow b < -\frac{1}{2}$
(B) when $a = 3 \Rightarrow b < \frac{-11}{27}$ and when $a = 2 \Rightarrow b < -\frac{1}{2}$
(C) when $a = -2 \Rightarrow b < \frac{-11}{27}$ and when $a = 2 \Rightarrow b < -\frac{1}{2}$
(D) None of these

18. If $f(x) = \begin{cases} 3 + |x - k|, & x \le k \\ a^2 - 2 + \frac{\sin(x-k)}{x-k}, & x > k \end{cases}$ has minimum at x = k , then
(A) $a \in R$ (B) $|a| < 2$ (C) $|a| > 2$ (D) $1 < |a| < 2$

19. Let $f : R \to R$
$f(x) = \sqrt{4x^2 + 28x + 85} + \sqrt{4x^2 - 28x + 113}$
The minimum value of f is -
(A) 96 (B) 14 (C) $96\sqrt{2}$ (D) $14\sqrt{2}$

20. The equation $x^3 - 3x + [a] = 0$, where [.] denotes the greatest integer function, will have three real and distinct roots if
(A) $a \in (-\infty, 2)$
(B) $a \in (0, 2)$
(C) $a \in (\infty, -2) \cup (0, \infty)$
(D) $a \in [-1, 2)$

21. Let $f(x) = \sin\frac{\{x\}}{a} + \cos\frac{\{x\}}{a}$ where $a > 0$ and $\{\,.\,\}$ denotes the fractional part function. Then the set of values of a for which f can attain its maximum values is

(A) $\left(0, \frac{4}{\pi}\right)$ (B) $\left(\frac{4}{\pi}, \infty\right)$ (C) $(0, \infty)$ (D) none of these

22. The values of the parameter 'k' for which the equation $x^4 + 4x^3 - 8x^2 + k = 0$ has all roots real is given by

(A) $k \in (0,3)$ (B) $k \in (0, 128)$ (C) $k \in (3, 128)$ (D) $k \in (128, \infty)$

More than one choice type

23. The equation of normal to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ $(n \in N)$ at the point with abscissa equal to 'a' can be:

(A) $ax + by = a^2 - b^2$ (B) $ax + by = a^2 + b^2$
(C) $ax - by = a^2 - b^2$ (D) $bx - ay = a^2 - b^2$

24. For the curve represented parametrically by the equations,
$x = 2\ \ell n \cot t + 1$ & $y = \tan t + \cot t$
(A) tangent at $t = \pi/4$ is parallel to x – axis
(B) normal at $t = \pi/4$ is parallel to y – axis
(C) tangent at $t = \pi/4$ is parallel to the line $y = x$
(D) tangent and normal intersect at the point (2, 1)

25. Let f(x) be a differentiable function and $f(\alpha) = f(\beta) = 0$ $(\alpha < \beta)$, then in the interval (α, β)
(A) $f(x) + f'(x) = 0$ has at least one root
(B) $f(x) - f'(x) = 0$ has at least one real root
(C) $f(x) \,.\, f'(x) = 0$ has at least one real root
(D) none of these

26. If p, q, r be real, then the intervals in which, $f(x) = \begin{vmatrix} x+p^2 & pq & pr \\ pq & x+q^2 & qr \\ pr & qr & x+r^2 \end{vmatrix}$,

(A) increase is $x < -\frac{2}{3}(p^2 + q^2 + r^2)$, $x > 0$ (B) decrease is $\left(-\frac{2}{3}(p^2 + q^2 + r^2), 0\right)$

(C) decrease is $x < -\frac{2}{3}(p^2 + q^2 + r^2)$, $x > 0$ (D) increase is $\left(-\frac{2}{3}(p^2 + q^2 + r^2), 0\right)$

27. Which of the following inequalities are valid –
(A) $|\tan^{-1} x - \tan^{-1} y| \le |x - y|\ \forall\ x, y \in R$ (B) $|\tan^{-1} x - \tan^{-1} y| \ge |x - y|$
(C) $|\sin x - \sin y| \le |x - y|$ (D) $|\sin x - \sin y| \ge |x - y|$

28. The values of the parameter 'a' for which the point of minimum of the function $f(x) = 1 + a^2x - x^3$ satisfies the inequality $\frac{x^2 + x + 2}{x^2 + 5x + 6} < 0$ are -

(A) $(2\sqrt{3}, 3\sqrt{3})$ (B) $(-3\sqrt{3}, -2\sqrt{3})$ (C) $(-2\sqrt{3}, 3\sqrt{3})$ (D) $(-3\sqrt{2}, 2\sqrt{3})$

29. A function $f(x) = \frac{x^2 - 3x + 2}{x^2 + 2x - 3}$ is -

(A) maximum at $x = -3$ (B) minimum at $x = -3$ and maximum at $x = 1$
(C) no point of maxima and minima (D) increasing in its domain

Comprehension

A function f(x) having the following properties;

(i) f(x) is continuous except at x = 3

(ii) f(x) is differentiable except at x = – 2 and x = 3

(iii) $f(0) = 0, \lim\limits_{x\to 3} f(x) \to -\infty, \lim\limits_{x\to -\infty} f(x) = 3, \lim\limits_{x\to \infty} f(x) = 0$

(iv) $f'(x) > 0\ \forall\ x \in (-\infty, -2) \cup (3, \infty)$ and $f'(x) \le 0\ \forall\ x \in (-2, 3)$

(v) $f''(x) > 0\ \forall\ x \in (-\infty, -2) \cup (-2, 0)$ and $f''(x) < 0\ \forall\ x \in (0, 3) \cup (3, \infty)$

then answer the following questions

30. Maximum possible number of solutions of $f(x) = |x|$ is

(A) 2 (B) 1 (C) 3 (D) 4

31. Graph of function $y = f(-|x|)$ is

(A) differentiable for all x, if $f'(0) = 0$

(B) continuous but not differentiable at two points, if $f'(0) = 0$

(C) continuous but not differentiable at one points, if $f'(0) = 0$

(D) discontinuous at two points, if $f'(0) = 0$

32. $f(x) + 3x = 0$ has five solutions if

(A) $f(-2) > 6$ (B) $f'(0) < -3$ and $f(-2) > 6$

(C) $f'(0) > -3$ (D) $f'(0) > -3$ and $f(-2) > 6$

True/False

33. The combined resistance R of two resistors R_1 & R_2 ($R_1, R_2 > 0$) is given by, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If $R_1 + R_2$ = constant. The maximum resistance R is obtained by choosing $R_1 = R_2$.

PART - II : SUBJECTIVE QUESTIONS

1. A figure is bounded by the curves, $y = x^2 + 1$, y = 0, x = 0 and x = 1. At what point (a, b), a tangent should be drawn to the curve $y = x^2 + 1$ for it to cut off a trapezium of the greatest area from the figure.

2. The fuel charges for running a train are proportional to the square of the speed generated in m.p.h. and costs Rs. 48/- per hour at 16 mph. What is the most economical speed if the fixed charges i.e. salaries etc. amount to Rs. 300/- per hour.

3. The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point x = 2 and is bisected by that point . Find 'a' .

4. If $y = \frac{ax + b}{(x-1)(x-4)}$ has a turning value at (2, –1) find a and b, show that the turning value is a maximum.

5. If x and y are sides of two squares such that $y = x - x^2$. Find the rate of change of area of second square (side y) with respect to the first square (side x) when x = 1cm

6. With the usual meaning for a, b, c and s, if Δ be the area of a triangle, prove that the error in Δ resulting from a small error in the measurement of c, is given by

$$d\Delta = \frac{\Delta}{4}\left\{\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right\}dc$$

7. If p be the length of perpendicular drawn from the origin upon the tangent to the curve $x = a\cos t + a\,t\sin t$ and $y = a\sin t - at\cos t$ at the point t, then prove that :

(i) $p\,\alpha\,|t|$ (ii) $p\,\alpha\left|\frac{dx}{dt} + i\frac{dy}{dt}\right|$; where $i = \sqrt{-1}$

8. Show that the equation of the tangent to the curve represented parametrically by the equations.

$$x = a\left\{\frac{\phi(t)}{f(t)}\right\} \text{ and } y = a\left\{\frac{\psi(t)}{f(t)}\right\}$$

can be expressed in the form

$$\begin{vmatrix} x & y & a \\ \phi(t) & \psi(t) & f(t) \\ \phi'(t) & \psi'(t) & f'(t) \end{vmatrix} = 0$$

where f, g and h are the differentiable functions.

9. Show that the condition, for curves $x^{2/3} + y^{2/3} = c^{2/3}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to touch, is c = a + b.

10. Find the possible values of a such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution

11. If the relation between subnormal SN and subtangent ST at any point S on the curve $by^2 = (x + a)^3$ is $p(SN) = q\,(ST)^2$, then find value of $\frac{p}{q}$ in terms of b and a.

12. If $(m - 1)\,a_1^{\,2} - 2m\,a_2 < 0$, then prove that $x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots\ldots + a_{m-x} + a_0 = 0$ has at least one non real root $(a_1, a_2, \ldots\ldots, a_m \in R)$

13. If $f'(x) > 0, f''(x) > 0\ \forall\ x \in (0, 1)$ and $f(0) = 0, f(1) = 1$, then prove that $f(x)\,f^{-1}(x) < x^2\ \forall\ x \in (0, 1)$

14. Find the interval of increasing and decreasing for the function $g(x) = 2f\left(\frac{x^2}{2}\right) + f\left(\frac{27}{2} - x^2\right)$, where $f''(x) < 0$ for all $x \in R$.

15. Using calculus prove that $H.M \le G.M. \le A.M$ for positive real numbers.

16. Prove the following inequalities

(i) $1 + x^2 > (x\sin x + \cos x)$ for $x \in [0, \infty)$.

(ii) $\sin x - \sin 2x \le 2x$ for all $x \in \left[0, \frac{\pi}{3}\right]$

(iii) $\frac{x^2}{2} + 2x + 3 \ge (3 - x)e^x$ for all $x \ge 0$

(iv) $0 < x\sin x - \frac{\sin^2 x}{2} < \frac{1}{2}(\pi - 1)$ for $0 < x < \frac{\pi}{2}$

17. Find the interval to which b may belong so that the function

$$f(x) = \left(1 - \frac{\sqrt{21 - 4b - b^2}}{b + 1}\right)x^3 + 5x + \sqrt{6}$$ is increasing at every point of its domain.

18. If $0 < x < 1$ prove that $y = x\,\ell n\,x - \frac{x^2}{2} + \frac{1}{2}$ is a function such that $\frac{d^2y}{dx^2} > 0$. Deduce that $x\,\ell n\,x > \frac{x^2}{2} - \frac{1}{2}$.

19. Find positive real numbers 'a' and 'b' such that $f(x) = ax - bx^3$ has four extrema on [–1 , 1] at each of which $|f(x)| = 1$

20. For any acute angled ΔABC, find the maximum value of $\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}$

21. Suppose p,q,r,s are fixed real numbers such that a quadrilateral can be formed with p,q,r,s in clockwise order. Prove that the vertices of the quadrilateral of maximum area lie on a circle .

22. For what real values of 'a' and 'b' all the extrema of the function $f(x) = \frac{5a^2}{3}x^3 + 2ax^2 - 9x + b$ are positive and the maximum is at the point $x_0 = \frac{-5}{9}$

23. Find the minimum value of $f(x) = 8^x + 8^{-x} - 4(4^x + 4^{-x})\ \forall\ x \in R$

24. Using calculus , prove that $\log_2 3 < \log_3 5 < \log_4 7$.

25. Show that the volume of the greatest cylinder which can be inscribed in a cone of height ' h ' and semi – vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$.

26. Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length ℓ of the median drawn to its lateral side .

27. A tangent to the curve $y = 1 - x^2$ is drawn so that the abscissa x_0 of the point of tangency belongs to the interval (0, 1]. The tangent at x_0 meets the x–axis and y–axis at A & B respectively. Then find the minimum area of the triangle OAB, where O is the origin

28. A cone is made from a circular sheet of radius $\sqrt{3}$ by cutting out a sector and keeping the cut edges of the remaining piece together. Then find the maximum volume attainable for the cone

29. Suppose velocity of waves of wave length λ in the Atlantic ocean is $k\sqrt{\left\{\left(\frac{\lambda}{a}\right)+\left(\frac{a}{\lambda}\right)\right\}}$, where k and a are constants. Show that minimum velocity attained by the waves is independent of the constant a.

ANSWERS

PART - I

1.	(C)	**2.**	(D)	**3.**	(D)	**4.**	(B)	**5.**	(C)	**6.**	(D)
7.	(C)	**8.**	(B)	**9.**	(B)	**10.**	(B)	**11.**	(C)	**12.**	(B)
13.	(D)	**14.**	(C)	**15.**	(D)	**16.**	(A)	**17.**	(A)	**18.**	(C)
19.	(D)	**20.**	(D)	**21.**	(A)	**22.**	(A)	**23.**	(AC)	**24.**	(AB)
25.	(ABC)	**26.**	(AB)	**27.**	(AC)	**28.**	(AB)	**29.**	(CD)	**30.**	(C)
31.	(B)	**32.**	(D)	**33.**	True						

PART - II

1. $\left(\frac{1}{2}, \frac{5}{4}\right)$ **2.** 40 mph **3.** a = 1 **4.** a = 1, b = 0

5. 0 **10.** $a \in \left(-\frac{13}{4}, 3\right)$ **11.** $\frac{8}{27}\ |b|$

14. g(x) is increasing if $x \in (-\infty, -3] \cup [0, 3]$
g(x) is decreasing if $x \in [-3, 0] \cup [3, \infty)$

17. $[-7, -1) \cup [2, 3]$ **19.** a = 3 , b = 4 **20.** $\frac{9\sqrt{3}}{2\pi}$

22. If $a = \frac{-9}{5}$, then $b > \frac{36}{5}$; If $a = \frac{81}{25}$ then $b > \frac{400}{243}$ **23.** -10

26. cos A = 0.8 **27.** $\frac{4\sqrt{3}}{9}$ **28.** $2\pi/3$