LIMITS CONTINUITY & DIFFERENTIABILITY

Limits

Let y = f(x) be a given function defined in the neighbourhood of x = a, but not necessarly at the point x = a. The limiting behaviour of the function in the neighbourhood of x = a when |x - a| is small, is called the limit of the function when x approaches 'a' and we write this as $\lim\limits_{x\to a} f(x)$.

Let $\lim\limits_{x\to a} f(x) = \ell$. It would simply mean that when we approach the point x = a from the values which are just greater than or just smaller than x = a, f(x) would have a tendency to move closer to the value 'ℓ'. This is same as saying, "difference between f(x) and ℓ can be made as small as we like by suitably choosing x in the neighbourhood of x = a". Mathematically, we write this as, $\lim\limits_{x\to a} f(x)= \ell$, which is equivalent to saying that $|f(x)-\ell|<\varepsilon\ \forall$ x such that $0 < |x-a| < \delta$ and ε depends on δ where ε and δ are sufficiently small positive numbers.

It should be clear that the limit of f(x) at x = a would exist if any only if, f(x) is well defined in the neighbourhood of x = a (not necessarily at x = a) and has a unique behaviour in the neighbourhood of x = a.

Remarks :Normally students have the perception that limit should be a finite number. But it is not really so. It is quite possible that f(x) had infinite limit as $x \to a$. If $\lim\limits_{x\to a} f(x) = \infty$, it would simply mean that functions has tendency to assume very large positive values in the neighbourhood of x = a, as for example $\lim\limits_{x\to a} 1/(x) = \infty$.

Left and Right Limit

Let y = f(x) be a given function, and x =a is the point under consideration. Left tendency of f(x) at x = a is called it's left limit and right tencency is called it's right limit.

Left tendency (left limit) is denoted by f(a-0) or f(a-) and right tendency (right limit) is denoted by f(a+0) or f(a+) and are written as

$$\left.\begin{aligned} f(a-0) &= \lim_{h\to 0} f(a-h) \\ f(a+0) &= \lim_{h\to 0} f(a+h) \end{aligned}\right\} \text{where 'h' is a small positive number.}$$

Thus for the existence of the limit of f(x) at x = a, it is necessary and sufficient that f(a-0) = f(a+0), if these are finite or f(a-0) and f(a+0) both should be either $+\infty$ or $-\infty$.

Remark : For the existence of the limit at x = a, f(x) need not be defined at x = a. However if f(a) exists, limit need not exist or even if it exists then it need not be equal to f(a).

Illustration 1. For what values of m does the $\lim\limits_{x\to 2} f(x)$ exist when $f(x) = \begin{cases} mx-3, & \text{when } x \le 2 \\ \dfrac{x}{m}, & \text{when } x > 2 \end{cases}$.

Solution:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(mx-3) = 2m-3;\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\frac{x}{m} = \frac{2}{m}$$

$\lim\limits_{x\to 2} f(x)$ exists when $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

$$\Rightarrow 2m - 3 = \frac{2}{m} \Rightarrow 2m^2 - 3m - 2 = 0 \Rightarrow m = -\frac{1}{2}, 2.$$

Algebra of limits

The following are some of the Basic Theorems on limits which are widely used to calculate the limit of the given functions.

Let $\lim\limits_{x\to a} f(x) = \ell_1$ and $\lim\limits_{x\to a} g(x) = \ell_2$ where ℓ_1 and ℓ_2 are finite, then

- $\lim\limits_{x\to a}(c_1 f(x) \pm c_2 g(x)) = c_1\ell_1 \pm c_2\ell_2$. where c_1 and c_2 are given constants.
- $\lim\limits_{x\to a} f(x).g(x) = \lim\limits_{x\to a} f(x).\lim\limits_{x\to a} g(x) = \ell_1.\ell_2$
- $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{\ell_1}{\ell_2}, \ell_2 \neq 0.$
- $\lim\limits_{x\to a} f(g(x)) = f(\lim\limits_{x\to a} g(x)) = f(\ell_2)$, if and only if f(x) is continuous at x = ℓ_2.

In particular, $\lim\limits_{x\to a} \ln(g(x)) = \ln(\lim\limits_{x\to a} g(x)) = \ln \ell_2$ if $\ell_2 > 0$

Illustration 2. Find $\lim\limits_{x\to 3}(2x^3 - 3x^2 - x - 1)$.

Solution : $\lim\limits_{x\to 3}(2x^3 - 3x^2 - x - 1) = 2\lim\limits_{x\to 3} x^3 - 3\lim\limits_{x\to 3} x^2 - \lim\limits_{x\to 3} x - \lim\limits_{x\to 3}(1)$

$$= 2(\lim_{x\to 3} x)^3 - 3(\lim_{x\to 3} x)^2 - 3 - 1 = 2\times 3^3 - 3\times 3^2 - 3 - 1 = 23.$$

Some Important Results on Limits

- If p(x) is a polynomial, $\lim\limits_{x\to a} p(x) = p(a)$.
- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$
- $\lim\limits_{x\to 0}\dfrac{e^x - 1}{x} = 1$
- $\lim\limits_{x\to 0}\dfrac{(1+x)^n - 1}{x} = n$
- $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)}{x} = 1$
- $\lim\limits_{x\to 0}\dfrac{\sin^{-1} x}{x} = 1 = \lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x}$
- $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = \lim\limits_{x\to 0}\dfrac{\tan x}{x} = \lim\limits_{x\to 0}\cos x = 1$
  (where 'x' is in radians)
- $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$
- $\lim\limits_{x\to 0}\dfrac{a^x - 1}{x} = \ell n(a), a \in R^+$
- $\lim\limits_{x\to a}\dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n} a^{m-n}$
- $\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x} = \log_a e, a > 0, \neq 1$

If $\lim\limits_{x\to a} f(x) = 0$ then the following results will be holding true :

- $\lim\limits_{x\to a}\dfrac{\sin f(x)}{f(x)} = \lim\limits_{x\to a}\dfrac{\tan f(x)}{f(x)} = \lim\limits_{x\to a}\cos f(x) = 1$
- $\lim\limits_{x\to a}\dfrac{\sin^{-1} f(x)}{f(x)} = \lim\limits_{x\to a}\dfrac{\tan^{-1} f(x)}{f(x)} = 1$
- $\lim\limits_{x\to a}\dfrac{b^{f(x)} - 1}{f(x)} = \ln b (b > 0)$

$\Rightarrow \lim\limits_{x\to a}(1+f(x))^{\frac{1}{f(x)}} = e$

Frequently Used Series Expansions

Following are some of the frequently used series expansions :

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots$$

$$\tan x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots\ldots\ldots\ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$$

$$a^x = 1 + x.\ln a + (\ln a)^2 \frac{x^2}{2!} + \ldots\ldots\ldots, a \in R^+$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots., n \in R, |x| < 1, \text{ n is any real number}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots., -1 < x \le 1$$

Illustration 3. Examine $\lim\limits_{x\to 0}\frac{\sqrt{1-\cos x}}{x}$.

Solution : $\sqrt{1-\cos x} = \begin{cases} -\sqrt{2}\sin\frac{x}{2}, & x < 0 \\ \sqrt{2}\sin\frac{x}{2}, & x \ge 0 \end{cases}$, Therefore $\lim\limits_{x\to 0^-}\frac{\sqrt{1-\cos x}}{x} = \lim\limits_{x\to 0^-}\frac{-\sqrt{2}\sin\frac{x}{2}}{x} = -\frac{1}{\sqrt{2}}$,

$\lim\limits_{x\to 0+}\frac{\sqrt{1-\cos x}}{x} = \lim\limits_{x\to 0+}\frac{\sqrt{2}\sin\frac{x}{2}}{x} = \frac{1}{\sqrt{2}}$; Since, $\lim\limits_{x\to 0^-}\frac{\sqrt{1-\cos x}}{x} \ne \lim\limits_{x\to 0+}\frac{\sqrt{1-\cos x}}{x}$,

$\lim\limits_{x\to 0}\frac{\sqrt{1-\cos x}}{x}$ does not exist.

Illustration 4. Evaluate the following limits, if these exist.

(i) $\lim\limits_{x\to 0}|x|^{\sin x}$, (ii) $\lim\limits_{x\to 0}(\sin x)^x$

Solution : (i) $\lim\limits_{x\to 0}|x|^{\sin x} =$

$$\lim_{x\to 0} e^{\ln(|x|)^{\sin x}} = \lim_{x\to 0} e^{\sin x.\ln|x|} = e^{\lim\limits_{x\to 0}\frac{\ln|x|}{\text{cosec}\,x}} = e^{\lim\limits_{x\to 0}\frac{1/x}{-\cot x\,\text{cosec}\,x}} = e^{\lim\limits_{x\to 0} -\frac{\sin^2 x}{x\cos x}} = e^{\lim\limits_{x\to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \frac{x}{\cos x}} = e^0 = 1$$

(ii) $\lim\limits_{x\to 0}(\sin x)^x$

Clearly in this case $(\sin x)^x$ in not defined towards the left of x = 0.
Hence the given limit will not exist.

Illustration :5. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n,\ a_0 \ne 0$.

Solution : Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x^n\left(a^0 + \frac{a_1}{x}+\frac{a_2}{x^2}+\ldots+\frac{a_n}{x^n}\right)$

$= \left(\lim\limits_{x\to\infty} x\right)^n (a_0+0+0\ldots+0) = \infty$. And $\lim\limits_{x\to-\infty} f(x) = \left(\lim\limits_{x\to-\infty} x\right)^n\left[\lim\limits_{x\to-\infty}\left(a^0+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots+\frac{a_n}{x^n}\right)\right]$

$= (-\infty)^n(a_0+0+\ldots+0) = \begin{cases}\infty \text{ if n is even}\\ -\infty \text{ if n is odd.}\end{cases}$

Illustration6. Examine $\lim\limits_{x\to\infty}\frac{e^{x^2}-1}{e^{x^2}+1}$.

Solution : This is of the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\frac{e^{x^2}-1}{e^{x^2}+1} = \lim\limits_{x\to\infty}\frac{1-e^{-x^2}}{1+e^{-x^2}} = \lim\limits_{x\to\infty}\frac{1-\left(\frac{1}{e}\right)^{x^2}}{1+\left(\frac{1}{e}\right)^{x^2}} = \frac{1-0}{1+0} = 1$.

Illustration 7. Evaluate $\lim\limits_{n\to\infty}\frac{1^3+2^3+\ldots+n^3}{(n^2+1)^2}$

Solution : $\lim\limits_{n\to\infty}\frac{1^3+2^3+\ldots+n^3}{(n^2+1)^2} = \lim\limits_{n\to\infty}\frac{n^2(n+1)^2}{4(n^2+1)^2} = \lim\limits_{n\to\infty}\frac{n^4\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{4n^4\left(1+\frac{2}{n^2}+\frac{1}{n^4}\right)} = \frac{1}{4}$.

***Illustration* 8.** Evaluate $\lim\limits_{n\to\infty}\frac{[x]+[2x]+[3x]+\ldots[nx]}{n^2}$ where [.] denotes the greatest integer function.

Solution : We know that $x-1 \le [x] < x \;\Rightarrow\; x+2x+\ldots+nx-n \le \sum\limits_{r=1}^{n}[rx] < x+2x+\ldots+nx$

$\Rightarrow \frac{x.n(n+1)}{2} - n \le \sum\limits_{r=1}^{n}[rx] < \frac{x.n(n+1)}{2} \;\Rightarrow\; \frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{1}{n} \le \frac{1}{n^2}\sum\limits_{r=1}^{n}[rx] < \frac{x}{2}\left(1+\frac{1}{n}\right)$.

Now, $\lim\limits_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) = \frac{x}{2}$ and $\lim\limits_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{1}{n} = \frac{x}{2}$

Using Sandwich theorem we find that $\lim\limits_{n\to\infty}\frac{[x]+[2x]+[3x]+\ldots[nx]}{n^2} = \frac{x}{2}$

Alternative solution :

We know that $[rx] = rx - \{x_r\}$ for $r = 1, 2, 3, \ldots, n$

and $0 \le \{x_r\} < 1$ for each r. Also $\sum\limits_{r=1}^{n}[rx] = \sum\limits_{r=1}^{n}((rx)-\{x_r\}) = x\sum\limits_{r=1}^{n} r - \sum\limits_{r=1}^{n}\{x_r\} = x\frac{n(n+1)}{2} - k$

where $k < n$ (since each $\{x_r\} < 1$). Hence $\lim\limits_{n\to\infty}\frac{1}{n^2}\sum\limits_{r=1}^{n}[rx] = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\left[\frac{x}{2}\left(1+\frac{1}{n}\right)-\frac{k}{n^2}\right] = \frac{x}{2}$.

Illustration 9. Evaluate the following limits, if these exist. Here {x} denotes the fractional part and [.] the greatest integer part.

(i) $\lim\limits_{x\to 0}\frac{|x|^{\alpha}}{e^x}, \alpha\in R^+$, (ii) $\lim\limits_{x\to 0}|x|^{[\cos x]}$ (iii) $\lim\limits_{x\to 3}\frac{[x]-3}{(x-3)}$

Solution : $\lim\limits_{x\to 0}\dfrac{|x|^{\alpha}}{e^{x}} = 0$ as $\lim\limits_{x\to 0}|x|^{\alpha} = 0$ and $\lim\limits_{x\to 0}e^{x} = 1$.

(ii) Since [cosx] = 0 in the neighborhood of x = 0, except at x = 0,

we are dealing with a form (finite)^0. Thus $\lim\limits_{x\to 0}|x|^{[\cos x]} = 1$

(iii) $\lim\limits_{x\to 3}\dfrac{[x]-3}{x-3}$. Towards the right of x = 3, [x] = 3

$\Rightarrow$ [x] -3 = 0, in the right neighbourhood of x = 3

$\Rightarrow \lim\limits_{x\to 3+0}\dfrac{[x]-3}{x-3} = 0.$ Towards the left of x = 3, [x] = 2

$\Rightarrow$ [x] -3 = -1, in the left neighbourhood of x = 3

$\Rightarrow \lim\limits_{x\to 3-0}\dfrac{[x]-3}{x-3} = \lim\limits_{x\to 3-0}\dfrac{-1}{x-3} = \infty$. Thus $\lim\limits_{x\to 3}\dfrac{[x]-3}{x-3}$ does not exist.

Exercise 1

(i) **Evaluate** $\lim\limits_{x\to\infty}\dfrac{\cos x+\sin x}{x^{2}}$,

(ii) **Evaluate** $\lim\limits_{x\to 2}\dfrac{\sqrt{x-2}+\sqrt{x}-\sqrt{2}}{\sqrt{x^{2}-4}}$

(iii) **Evaluate** $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x+1}\right)^{x+2}$

(iv) **Evaluate** $\lim\limits_{x\to\infty}\dfrac{(2+x)^{40}(4+x)^{5}}{(2-x)^{45}}$

(v) **If** $\lim\limits_{x\to 0}\dfrac{\left((a-n)nx-\tan x\right)\sin nx}{x^{2}} = 0$**, where n is non zero real number, then find value of 'a'.**

(vi) **Evaluate** $\lim\limits_{x\to 1}\dfrac{\sin\{x\}}{\{x\}}$ **, where {x} is the fractional part of x.**

(vii) **Evaluate** $\lim\limits_{x\to[a]}\dfrac{e^{\{x\}}-\{x\}-1}{\{x\}^{2}}$ **where {x} denotes the fractional part of x and [a] denotes the integral part of a.**

Continuity

A function f(x) is said to be continuous at x = a if $\lim\limits_{x\to a^-}f(x) = \lim\limits_{x\to a^+}f(x) = f(a)$ i.e. L.H.L. = R.H.L.=

f(a) = value of the function at a i.e. $\lim\limits_{x\to a}f(x) = f(a)$.

If f(x) is not continuous at x = a, we say that f(x) is discontinuous at x = a.
f(x) will be discontinuous at x = a in any of the following cases:

- $\lim\limits_{x\to a^-}f(x)$ and $\lim\limits_{x\to a^+}f(x)$ exist but are not equal.
- $\lim\limits_{x\to a^-}f(x)$ and $\lim\limits_{x\to a^+}f(x)$ exist and are equal but not equal to f(a).
- f(a) is not defied.
- At least one of the limits does not exist.

Properties of Continuous Functions

Let f(x) and g(x) be functions, both continuous at x = a. Then

- $cf(x)$ is continuous at $x = a$ where c is any constant.
- $f(x) \pm g(x)$ is continuous at $x = a$.
- $f(x) \,.\, g(x)$ is continuous at $x = a$.
- $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

Continuity in an Interval

f(x) is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.
f(x) is said to be continuous in the closed interval [a,b] if

- f(x) is continuous in (a, b)
- $\lim\limits_{x \to a^+} f(x) = f(a)$
- $\lim\limits_{x \to b^-} f(x) = f(b)$

Function f(x)	**Interval in which f(x) is continuous.**
Constant C	R
x^n, n is an integer ≥ 0.	R
$\lvert x - a\rvert$	R
x^{-n}, n is a positive integer.	R - {0}
$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$	R
p(x)/q(x), p(x) and q(x) are polynomials in x	R - {x: q(x)=0}
sinx	R
cos x	R
tanx	$R-\left\{\frac{(2n+1)\pi}{2}:n=0,\pm1,\pm2,..\right\}$
cotx	$R-\{n\pi:n=0,\pm1,\pm2,...\}$
secx	$R-\left\{\frac{(2n+1)\pi}{2}:n=0,\pm1,\pm2,..\right\}$
cosecx	$R-\{n\pi:n=0,\pm1,\pm2,...\}$
e^x	R
ln x	$(0,\infty)$

Illustration 10. Let $f(x) = \begin{cases} \left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}} & , x \neq 0 \\ k & , x = 0 \end{cases}$

For what value of k is f(x) continuous at x = 0 ?

Solution : $\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}} = \lim\limits_{x\to0}\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}}$

$$\left[\lim_{x\to0}\left\{(1+\tan x)^{1/\tan x}\right\}^{\tan x/x}\right]\left[\lim_{x\to0}\left\{(1-\tan x)^{-1/\tan x}\right\}^{\tan x/x}\right] = e \times e = e^2$$

Since f(x) is continuous at x = 0, $\lim\limits_{x\to a} f(x) = f(0) \Rightarrow e^2 = k$.

Hence f(x) is continuous at x = 0 when $k = e^2$.

Illustration 11. Discuss the continuity of f(x) = $\begin{cases} |x+1|, & x \le -2 \\ 2x+3, & -2 < x \le 0 \\ x^2+3, & 0 < x \le 3 \\ x^3-15, & x > 3. \end{cases}$

Solution : We rewrite f(x) as f(x) = $\begin{cases} -x-1, & x \le -2 \\ 2x+3 & -2 < x \le 0 \\ x^2+3, & 0 < x \le 3 \\ x^3-15, & x > 3. \end{cases}$

As we can see, f(x) is defined as a polynomial function in each of the intervals $(-\infty,-2)$, (-2, 0), (0, 3) and $(3, \infty)$. Therefore it is continuous in each of these four intervals.
At the point x = -2,

$\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-}(-x-1) = 1$, and $\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+}(2x+3) = -1$,

Therefore, $\lim\limits_{x\to -2}$ f(x) does not exist. Thus f(x) is discontinuous at x= -2.

At the point x = 0. $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0+} f(x) = f(0) = 3$, Therefore f(x) is continuous at x = 0.

At the point x = 3. $\lim\limits_{x\to 3-} f(x) = \lim\limits_{x\to 3+} f(3) = 12$, Therefore, f(x) is continuous at x = 3.

Considering that $R = (-\infty,-2) \cup \{-2\} \cup (-2,0) \cup \{0\} \cup (0,3) \cup \{3\} \cup (3,\infty)$,
we conclude that f(x) is continuous at all points in R except at x = -2.

Illustration 12. Let f(x) be a continuous function and g(x) be a discontinuous function. Prove that f(x) + g(x) is a discontinuous function.

Solution : Suppose that h(x) = f(x) + g(x) is continuous. Then, in view of the fact that f(x) is continuous, g(x) = h(x) - f(x), a difference of continuous functions, is continuous. But this is acontradiction since g(x) is given as a discontinuous function.
Hence h(x) = f(x) + g(x) is discontinuous.

Continuity of Composite Functions

If the function u = f(x) is continuous at the point x = a, and the function y = g(u) is cntinuous at the point u = f(a), then the composite function y = (gof)(x) = f(f(x) is continuous at the point x = a.

Illustration 13: Find the points of discontinuity of $y = \frac{1}{u^2+u-2}$ where $u = \frac{1}{x-1}$.

Solution : The function u = f(x) = $\frac{1}{x-1}$ is discontinuous at the point x = 1.

The function y = g(u) $= \frac{1}{u^2+u-2} = \frac{1}{(u+2)(u-1)}$

is discontinuous at u = -2 and u = 1. When u = -2

$\Rightarrow \frac{1}{x-1} = -2 \Rightarrow x = \frac{1}{2}$; When u = 1 $\Rightarrow \frac{1}{x-1} = 1 \Rightarrow x = 2$;
Hence the composite function y = g(f(x)) is discontinuous at three points x = 1/2, x = 1 and x = 2.

Removable discontinuity

If $\lim_{x \to a} f(x)$ exists but is not equal to f(a), then f(x) has removable discontinuity at x = a and it can be removed by redefining f(x) for x = a.

Illustration14. Redefine the function f(x) = [x] + [-x] in such a way that it becomes continuous for $x \in (0, 2)$.

Solution : Here $\lim_{x \to 1} f(x) = -1$ but f(1) = 0. Hence, f(x) has a removable discontinuity at x = 1.

To remove this we define f(x) as follows

$$f(x) = \begin{cases} [x]+[-x], & x \in (0,1) \cup (1,2) \\ -1, & x = 1 \end{cases}$$

Now, f(x) is continuous for $x \in (0, 2)$.

Non-Removable Discontinuity

If $\lim_{x \to a} f(x)$ does not exist, then we can not remove this disconinuity so that this becomes a non-removable or essential discontinuity

e.g. f(x) = [x + 3] has essential discontinuity at any $x \in I$.

Exercise 2

(i). **If** $f(x) = \begin{cases} -x, & x < 0 \\ x, & x > 0 \\ 0, & x = 0 \end{cases}$ **then test the continuity of f(x) at x = 0 .**

(ii). **Test the continuity of f(x) where**

$$f(x) = \begin{cases} x^2+x+1, & 0 \le x \le 1 \\ x^2+2, & 1 < x \le 2 \end{cases}$$

(iii). **If** $f(x) = \begin{cases} \dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x \neq 2 \\ k & x = 2 \end{cases}$ **and if f(x) is continuous at x = 2, find the value of k.**

(iv). **A function f(x) is defined as follows**

$$f(x) = \frac{\sin x}{x}, \text{ when } x \neq 0$$
$$= 2, \text{ when } x = 0$$

is, f(x) continuous at x = 0? If not, redefine it so hat it become continuous at x = 0.

(v). **Determine the values of a, b, c for which the function f(x) is continuous at x = 0, where**

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & ; \ x < 0 \\ c & ; \ x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & ; \ x > 0 \end{cases}.$$

(vi). **If** $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x & 2 < x \le 3 \end{cases}$ **determine the points of discontinuity of the function g(x) = f(f(x)).**

Differentiability

Let y=f(x) be continuous in (a, b). Then the derivative or differential coefficient of f(x) w.r.t. x at $x \in$ (a, b), denoted by dy/dx or f'(x), is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{.... (1)}$$

Provided the limit exists and is finite and the function is said to differentiable.

To find the derivative of f(x) from the first principle

If we obtain the derivative of y = f(x) using the formula $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, we say that we are finding the derivative of f(x) with respect to x from the definition or from the first principle.

For example, y = cos 2x.

Here f(x) = cos 2x and $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos 2(x+h) - \cos 2(x)}{h}$

$$= \lim_{h \to 0} \frac{2\sin\frac{2(x+h)+2x}{2}\sin\frac{2x-2(x+h)}{2}}{h} \quad = \lim_{h \to 0} -2\sin(2x+h).\left(\frac{\sin h}{h}\right) = -2\sin 2x$$

Right Hand Derivative

Right hand derivative of f(x) at x = a is denoted by, Rf'(a) or f'(a^+)and is defined as

$$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, h > 0.$$

Left Hand Derivative

Left hand derivative of f(x) at x = a is denoted by Lf'(a) or f'(a-) and is defined as

$$Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}, h > 0.$$

Clearly, f(x) is differentiable at x = a if and only if R f'(a) = Lf'(a).

Illustration 15. Show that the function defined by f(x) = $\begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable for every value of x but the derivative is not continuous at x = 0.

Solution : For $x \neq 0, f'(x) = 2x\sin\frac{1}{x} + x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

For $x = 0, f'(x) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$

$= \lim_{h \to 0} h\sin\frac{1}{h} = 0;$ Thus, $f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$

f'(x) is continuous at x = 0 if

(i) $\lim_{x \to 0} f'(x)$ exists.

(ii) The value of the limit is f'(0).

Now $\lim_{x \to 0} f'(x) = \lim_{x \to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$; which does not exist since $\lim_{x \to 0} \cos\frac{1}{x}$ does not ex ist. Hence, f'(x) is not continuous at x = 0.

Illustration 16. Discuss the continuity and differentiability of the function

$$f(x)=\begin{cases}\dfrac{|x|(3e^{1/|x|}+4)}{2-e^{1/|x|}} & x\neq 0\\ 0 & x=0\end{cases}\quad \text{at } x = 0.$$

Solution: The given function may be written as $f(x)=\begin{cases}\dfrac{-x(3e^{-1/x}+4)}{2-e^{-1/x}}, & x<0\\ 0, & x=0\\ \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}, & x>0\end{cases}$

For continuity, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\dfrac{-x(3e^{-1/x}+4)}{2-e^{-1/x}}$; $\lim\limits_{x\to 0^-}\dfrac{-x(3+4e^{1/x})}{2e^{1/x}-1}=0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\dfrac{x(3e^{1/x}+4)}{2-e^{1/x}} = \lim\limits_{x\to 0^+}\dfrac{x(3+4e^{-1/x})}{2e^{-1/x}-1}=0$

Since $\lim\limits_{x\to 0^-} f(x)=f(0)=\lim\limits_{x\to 0^+} f(x)$, $f(x)$ is continuous at x = 0.

For differentiability, $f'(0)=\lim\limits_{h\to 0^-}\dfrac{f(h)-f(0)}{h}=\lim\limits_{h\to 0^-}\dfrac{-h(3e^{-1/h}+4)}{h(2-e^{-1/h})}$

$=\lim\limits_{h\to 0^-}\dfrac{-(3+4e^{1/h})}{2e^{1/h}-1}=3$ and $f_+'(0)=\lim\limits_{h\to 0^+}\dfrac{h(3e^{1/h}+4)}{h(2-e^{1/h})}=\lim\limits_{h\to 0^+}\dfrac{(3+4e^{-1/h})}{2e^{-1/h}-1}=-3$

Since $f_-'(0)\neq f_+'(0)$, f(x) is not differentiable at x = 0.

Exercise 3

(i) **Fuction f(x) is defined as** $f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5}, & x\neq 1\\ -\dfrac{1}{3} & x=1\end{cases}$

Is f(x) differentiable at x = 1 if yes find f'(1).

(ii) **Check the differentiability of the following functions at x = 0**

(a). cos(|x|) + |x|, **(b). sin(|x|) - |x|.**

(iii) **If** $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, then find $\dfrac{dy}{dx}$.

(iv) **If** $y=\dfrac{\ln x}{x}$, then prove that $\dfrac{d^2y}{dx^2}=\dfrac{2\ln x-3}{x^3}$.

(v) **If** $y=\sqrt{(a-x)(x-b)}-(a-b)\tan^{-1}\sqrt{\left(\dfrac{a-x}{x-b}\right)}$, then find $\dfrac{dy}{dx}$.

List of Derivatives of Important Functions

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
- $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}, x > 0$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

General Theorems on Differentiation

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x)$
- $\frac{d}{dx}[f(x).g(x)] = f'(x).g(x) + f(x)\,g'(x)$
- $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
- $\frac{d}{dx}\left[(f(x))^{g(x)}\right] = (f(x))^{g(x)}\left[\frac{g(x)}{f(x)}f'(x) + g'(x)\ln f(x)\right]$

Chain Rule

If y = f(u) and u = g(x), then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = f'(g(x))g'(x)$.

e.g. Let $y = [f(x)]^n$. We put u = f(x), so that $y = u^n$.

Therefore, using chain rule, we get $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = nu^{n-1} f'(x) = n[f(x)]^{n-1} f'(x)$.

Differentiation of parametrically defined functions

- If x and y are function of parameter t, first find dx/dt and dy/dt separately.
- Then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

e.g., $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ where θ is a parameter.

$$\frac{dx}{d\theta} = a(1+\cos\theta),\ \frac{dy}{d\theta} = a(0+\sin\theta) = a\sin\theta \text{, or } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\ \sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}.$$

Higher Order Derivatives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right),\ \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$\dfrac{d^ny}{dx^n} = \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)$; $\dfrac{d^ny}{dx^n}$ is called the n^{th} order derivative of y with respect to x.

Illustration 17. If $y = (\sin^{-1}x)^2 + k\sin^{-1}x$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$.

Solution : Hence $y = (\sin^{-1}x)^2 + k\sin^{-1}x$.

Differentiating both sides with respect to x, we have $\dfrac{dy}{dx} = 2\dfrac{\sin^{-1}x}{\sqrt{1-x^2}} + \dfrac{k}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\left(\dfrac{dy}{dx}\right)^2 = 4y + k^2$. Differentiating this with respect to x, we get

$$(1-x^2)\ 2\frac{dy}{dx}.\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx} \Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$$

Illustration 18. If $y = e^{\sin^2 x}$, find $\dfrac{d^2x}{dy^2}$, in terms of x.

Solution: Here $y = e^{\sin^2 x}$.

Differentiating with respect to x, we get $\dfrac{dy}{dx} = \sin 2x . e^{\sin^2 x}, \Rightarrow \dfrac{dx}{dy} = \operatorname{cosec}2x . e^{-\sin^2 x}$

Differentiating with respect to y, we get $\dfrac{d^2y}{dx^2} = \dfrac{d}{dy}(\operatorname{cosec}2x . e^{-\sin^2 x}) = \dfrac{d}{dx}(\operatorname{cosec}2x . e^{-\sin^2 x})\dfrac{dx}{dy}$

$$= (-2\operatorname{cosec}2x\cot 2x e^{-\sin^2 x} - e^{-\sin^2 x})\operatorname{cosec}2x . e^{-\sin^2 x}$$

$$= -(2\operatorname{cosec}^2 2x\cot 2x + \operatorname{cosec}2x).e^{-2\sin^2 x}$$

Exercise 4

(i). **If** $y = \sqrt{\sin x^2}$, *find* $\dfrac{dy}{dx}$.

(ii). **For** $y = \sin^3\sqrt{ax^2 + bx + c}$, *find* $\dfrac{dy}{dx}$.

(iii). **If** $x\cos y = \sin(x+y)$, **find** $\dfrac{dy}{dx}$.

(iv). **If** $x = a\cos^{2}-, y = a\sin^{2}-$, *find* $\dfrac{dy}{dx}$.

L' Hospital's Rule

We have dealt with problems which had indeterminate form either 0/0 or $\dfrac{\infty}{\infty}$

The other indeterminate forms are $\infty - \infty,\ 0.\infty,\ 0^0, \infty^0,\ 1^\infty$.

We state below a rule, called L' Hospital's Rule, meant for problems on limit of the form 0/0 or $\frac{\infty}{\infty}$.
Let f(x) and g(x) be functions differentiable in the neighbourhood of the point a, except may be at the point a itself. If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or, $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided that the limit on the right either exists as a finite number or is $\pm\infty$.

Illustration 19. Evaluate $\lim_{x\to 1} \frac{1 - x + \ell nx}{1 + \cos \pi x}$.

Solution: $\lim_{x\to 1} \frac{1 - x + \ell nx}{1 + \cos \pi x}$ (of the form 0 / 0) = $\text{Lim}_{x\to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin \pi x}$ (still of the form 0/0).

$= \lim_{x\to 1} \frac{x - 1}{\pi x \sin \pi x}$ (algebraic simplification).

$= \lim_{x\to 1} \frac{1}{\pi \sin \pi x + \pi^2 x \cos \pi x}$ (L'Hospital's rule again). $= -\frac{1}{\pi^2}$.

Illustration 20. Evaluate $\lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$.

Solution : $\lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$; [0/0] = $\lim_{x\to y} \frac{yx^{y-1} - y^x \log y}{x^x \log ex} = \frac{1 - \log y}{\log ey}$.

ANSWER TO EXERCISES

Exercise 1:

(i) 0 (ii) 1/2 (iii) e^{-2} (iv) -1 (v) $n+\frac{1}{n}$

Exercise 2:

(i) Continuous at x = 0

(ii) f(x) is continuous in [0, 2]

(iii) k = 7

(iv) 1

(v) $a=\frac{-3}{2}$, $b\in R$, $c=\frac{1}{2}$

(vi) 1, 2

Exercise 3:

(i) $f'(1)=-\frac{2}{9}$

(ii). (a) not differentiable (b). differentiable

(iii) $\frac{x}{\sqrt{1-x^4}}$

(v) $\sqrt{\frac{a-x}{x-b}}$

Exercise 4:

(i) $\frac{x\cos x^2}{\sqrt{\sin x^2}}$

(ii) $\frac{3\sin^2\sqrt{ax^2+bx+c}.\cos\sqrt{ax^2+bx+c}}{2\sqrt{ax^2+bx+c}}(2ax+b)$

(iii) $\frac{\cos y-\cos(x+y)}{x\sin y+\cos(x+y)}$

(iv) -1

SOLVED SUBJECTIVE PROBLEMS

Problem -1. Let $f(x) = \dfrac{|x^3 - 6x^2 + 11x - 6|}{x^3 - 6x^2 + 11x - 6}$. Find the set of points 'a' where $\lim\limits_{x\to a} f(x)$ does not exist.

Solution: We write, $f(x) = \left(\dfrac{|x-1|}{x-1}\right)\left(\dfrac{|x-2|}{x-2}\right)\left(\dfrac{|x-3|}{x-3}\right) = \begin{cases} -1 & , \quad x < 1 \\ 1 & , \quad 1 < x < 2 \\ -1 & , \quad 2 < x < 3 \\ 1 & , \quad x > 3 \end{cases}$

Therefore the limits exists at all points except at x = 1, 2, 3. For example, at x = 1.

$\lim\limits_{x\to 1^-} f(x) = -1$ and $\lim\limits_{x\to 1^+} f(x) = 1$, Since $\lim\limits_{x\to 1^-} f(x) \ne \lim\limits_{x\to 1^+} f(x)$

$\lim\limits_{x\to 1} f(x)$ does not exist. Similarly $\lim\limits_{x\to a} f(x)$ does not exist when a = 2, 3.

Problem -2. Find the values of a and b so that $\lim\limits_{x\to 0} \dfrac{x(1 + a\cos x) - b\sin x}{\sin^3 x}$ may be equal to 1.

Solution: We write, $\lim\limits_{x\to 0} \dfrac{x(1 + a\cos x) - b\sin x}{\sin^3 x} = \left[\lim\limits_{x\to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}\right]\left[\lim\limits_{x\to 0} \dfrac{x}{\sin x}\right]^3 =$

$\lim\limits_{x\to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}\left(\dfrac{0}{0}\text{form}\right)$, or

$\lim\limits_{x\to 0} \dfrac{1 + a\cos x - ax\sin x - b\cos x}{3x^2}$ (Using L' Hospital Rule)

The denominator being 0 for x = 0, the expression will lead to a finite limit if and only if the numbertor is also zero for x = 0. This happens when 1 + a - b = 0. ...(1)

Assuming that (1) is satisfied, we have $\lim\limits_{x\to 0} \dfrac{1 + (a-b)\cos x - ax\sin x}{3x^2}$ $\left(\dfrac{0}{0}\text{form}\right)$

$= \lim\limits_{x\to 0} \dfrac{(b-2a)\sin x - ax\cos x}{6x}$ $\left(\dfrac{0}{0}\text{form}\right) = \lim\limits_{x\to 0} \dfrac{(b-3a)\cos x + ax\sin x}{6} = \dfrac{b-3a}{6}$.

As Given $\dfrac{b-3a}{6} = 1 \Rightarrow b - 3a = 6$. ...(2)

from (1) and (2), we get, $a = -\dfrac{5}{2},\ b = -\dfrac{3}{2}$.

Alternative Solution :

We write $\lim\limits_{x\to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{x\left[1 + a\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \ldots\right)\right] - b\left(x - \dfrac{x^3}{6} + \ldots\right)}{x^3}$

$$= \lim_{x\to 0} \frac{x(1 + a - b) + x^3\left(\frac{b}{6} - \frac{a}{2}\right) + \text{ terms of order } x^4}{x^3} = 1 \text{ (given)}$$

$$\Rightarrow 1 + a - b = 0 \text{ and } \frac{b}{6} - \frac{a}{2} = 1 \Rightarrow a = -\frac{5}{2}, b = -\frac{3}{2}$$

Problem-3. Find the following limits

(i) $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x^2}}$ (ii) $\lim_{x\to 4}\frac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{(x-4)}, \alpha\in(0,\pi/2)$

Solution : (i) $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x^2}}$ (1^∞ form) $= e^{\lim_{x\to 0}\frac{\sin x}{x^2}} = e^{\lim_{x\to 0}\frac{1}{x}\left(\frac{\sin x}{x^2}\right)}$

$$=\begin{cases}0, & \text{when } x\to 0 \text{ from left,}\\ \infty, & \text{when } x\to 0 \text{ from right}\end{cases}$$

Thus thegiven limit does not exist.

(ii) $\lim_{x\to 4}\frac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{(x-4)},\left(\frac{0}{0}\text{from}\right)$

$$=\lim_{x\to 4}\frac{(\cos\alpha)^x-(\sin\alpha)^x-(\cos^2\alpha-\sin^2\alpha)(\cos^2\alpha+\sin^2\alpha)}{(x-4)}$$

$$=\lim_{x\to 4}\frac{(\cos\alpha)^x-(\sin\alpha)^x-\cos^4\alpha+\sin^4\alpha}{(x-4)} = \lim_{x\to 4}\frac{(\cos\alpha)^4-\left((\cos\alpha)^{x-4}-1\right)-\sin^4\alpha\left((\sin\alpha)^{x-4}-1\right)}{(x-4)}$$

$$=\cos^4\alpha.\lim_{x\to 4}\frac{(\cos\alpha)^{x-4}-1}{x-4}-\sin^4\alpha.\lim_{x\to 4}\frac{(\sin\alpha)^{x-4}-1}{x-4}$$

$$=\cos^4\alpha\ln(\cos\alpha)-\sin^4\alpha\ln(\sin\alpha).$$

Problem-4. Find $\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{\sin^3 x}$

Solution: $\lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{\sin^3 x} = \lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3\left(\frac{\sin^3 x}{x^3}\right)} = \lim_{x\to 0}\frac{\tan^{-1}x-\sin^{-1}x}{x^3} =$

$$\lim_{x\to 0}\frac{\tan^{-1}x-\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)}{x^3} = \lim_{x\to 0}\frac{\tan^{-1}\left(\frac{x-\frac{x}{\sqrt{1-x^2}}}{1+\frac{x^2}{\sqrt{1-x^2}}}\right)}{x^3} = \lim_{x\to 0}\frac{\tan^{-1}\left(\frac{x\sqrt{1-x^2}}{x^2+\sqrt{1-x^2}}\right)}{x^3} =$$

$$\lim_{x\to 0}\frac{\tan^{-1}\left(\frac{x\sqrt{1-x^2}-x}{x^2+\sqrt{1-x^2}}\right)\left(x\sqrt{1-x^2}-x\right)}{x^3\left(\frac{x\sqrt{1-x^2}-x}{x^2+\sqrt{1-x^2}}\right)\left(x^2+\sqrt{1-x^2}\right)} =$$

$$\lim_{x\to 0}\frac{\sqrt{1-x^2}-1}{x^2\left(x^2+\sqrt{1-x^2}\right)} = \lim_{x\to 0}\frac{1-x^2-1}{x^2\left(x^2+\sqrt{1-x^2}\right)\left(\sqrt{1-x^2}+1\right)} = -\frac{1}{2}.$$

Problem-5. If f(x) = $\lim_{n\to\infty}\frac{\left[x^2\right]+\left[(2x)^2\right]+....+\left[(nx)^2\right]}{n^3}$, then prove that f(x) is always continuous.

(Here [.] denotes the greatest integer function)

Solution : We have $\left[x^2\right]+\left[(2x)^2\right]+....+\left[(nx)^2\right]$ $=x^2+(2x)^2+....+(nx)^2-(\{x^2\}+\{(2x)^2\}+....+\{(nx)^2\})$

$$=\frac{x^2\left(n(n+1)(2n+1)\right)}{6}-(\{x^2\}+\{(2x)^2\}+.......+\{(nx)^2\}).$$

Now f(x) $=\lim\limits_{n\to\infty} x^2\frac{x^2\left(n(n+1)(2n+1)\right)}{6n^3}-\lim\limits_{n\to\infty}\frac{(\{x^2\}+\{(2x)^2\}+........+\{(nx)^2\})}{n^3}$

$=\frac{x^2}{3}-0=\frac{x^2}{3}$ as $0\le\{x^2\}+\{(2x)^2\}+........+\{(nx)^2\}<n,$ and $\frac{x^2}{3}$ is continuous every where.

Problem-6. Evaluate a,b,c and d, if $\lim\limits_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)=4$

Solution: Given that $4=\lim\limits_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)$

$$=\lim_{x\to\infty}\frac{(a-2)x^3+(3+c)x^2+(b-3)x+2+d}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}$$

Since the limit is finite, the degree of the numerator must be at the most 2 $\Rightarrow$ a-2 = 0 i.e., a = 2.

Hence 4 = $\lim\limits_{x\to\infty}\frac{(3+c)+\frac{b-3}{x}+\frac{2+d}{x^2}}{\sqrt{1+\frac{a}{x}+\frac{3}{x^2}+\frac{b}{x^3}+\frac{2}{x^4}}+\sqrt{1+\frac{2}{x}-\frac{c}{x^2}+\frac{3}{x^3}-\frac{d}{x^4}}}=\frac{3+c}{2}$

Therefore c= 5. Hence a = 2, c = 5 and b, d are any real numbers.

Problem-7. Let f(x +y) = f(x) + f(y) for all x and y. If the function f(x) is continuous at x = 0, show that f(x) is continuous for all x.

Solution: We are given that f(x+y) =f(x) + f(y) ; for all x and y.

Since f(x) is continuous at x = 0, we have $\lim\limits_{x\to0}$ f(x) = f(0).

To show that f(x) is continuous at any point a, we shall prove that $\lim\limits_{x\to a}f(x)=f(a)$

or, $\lim\limits_{h\to0}$ f(a+h) = f(a).

Indeed, $\lim\limits_{h\to0}$ f(a+h) = $\lim\limits_{h\to0}$ [f(a)+f(h)] = f(a) + $\lim\limits_{h\to0}$ f(h) = f(a) + f(0) = f(a+0) = f(a).

Problem-8. Given the function f(x) = $\frac{1}{x-1}$, find the points of discontinuity of the composite function y = f[f{f(x)}].

Solution : We know that f(x) = $\frac{1}{x-1}$ is discontinuous at x = 1.

For, $x\ne1, f\left\{f(x)\right\}=\frac{1}{\frac{1}{x-1}-1}=\frac{x-1}{2-x}$ is discontinuous at x = 2.

For $x\ne1,$ and $2, f\left[f\left\{f(x)\right\}\right]=\frac{1}{\frac{x-1}{2-x}-1}=\frac{2-x}{2x-3}$ which is discontinuous at x = $\frac{3}{2}$

Hence the points of discontinuity are x = 1, $\frac{3}{2}$ and x = 2.

Problem-9. Let f(x) = $\begin{cases} (1+|\tan x|)^{a/|\tan x|} & , \quad -\frac{\pi}{4}<x<0 \\ b & , \quad x=0 \\ e^{\cot 2x/\cot 3x} & , \quad 0<x<\frac{\pi}{4} \end{cases}$.

Detemine a and b such that f is continuous at x = 0.

Solution : f(x) is continuous at x = 0 when

$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x).$ ... (1)

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+|\tan x|)^{\frac{a}{|\tan x|}} \lim_{x\to 0^-} (1-\tan x)^{\frac{-a}{|\tan x|}} = e^a$

and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\cot 2x/\cot 3x} = \lim_{x\to 0^+} e^{(3/2)\left(\frac{2x}{\tan 2x}\right)\left(\frac{\tan 3x}{3x}\right)} = e^{3/2}$

Thus (1) becomes $e^a = b = e^{3/2} \Rightarrow a = 3/2,\ b = e^{3/2}$

Problem -10 Find the points of discontinuity (if any) of the function f(x) = $\lim_{n\to\infty} \frac{\log(2+x^2) - x^{2n}\sin x}{1+x^{2n}}$.

Solution : Using the result $\lim_{n\to\infty} x^{2n} = \begin{cases} 1 & x=\pm 1 \\ 0 & -1<x<1 \\ \infty & |x|>1 \end{cases}$. We can rewrite f(x) as follows :

$$f(x) = \begin{cases} \lim_{n\to\infty} \dfrac{\dfrac{\log(2+x^2)}{x^{2n}} - \sin x}{\dfrac{1}{x^{2n}}+1}, & |x|>1 \\ \log(2+x^2), & -1<x<1 \\ \dfrac{\log(2+x^2)-\sin x}{2}, & x=\pm 1 \end{cases} \Rightarrow f(x) = \begin{cases} -\sin x, & x<-1 \\ \dfrac{\log 3+\sin 1}{2} & x=-1 \\ \log(2+x^2), & -1<x<1 \\ \dfrac{\log 3-\sin 1}{2}, & x=1 \\ -\sin x, & x>1 \end{cases}$$

At $x=\pm 1$, LHL $\neq$ RHL $\Rightarrow f(x)$ is discontinuous at $x=\pm 1$

Problem-11. A function f: R $\to$ R satisfies the equation f(x+y) = f(x)f(y) for all x, y $\in R, f(x)\neq 0$.
Suppose that the function is differentiable at x = 0 and f'(0) = 2.
Prove that f'(x) = 2f(x).

Solution : We are given that

f(x+y) = f(x) f(y) ... (1)

and $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = 2.$ ... (2)

Now $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x+0)}{h}$

$= \lim_{h\to 0} \frac{f(x)f(h)-f(x)f(0)}{h},$ by (1)

$= f(x)\lim_{h\to 0} \frac{f(h)-f(0)}{h} = 2f(x)$ by (2)

Problem-12 Find the set of points where $f(x) = x^2|x|$ is thrice differentiable.

Solution : $f(x)=\begin{cases}-x^3 & , \quad x<0\\ 0 & , \quad x=0\\ x^3 & , \quad x>0\end{cases}$. This gives $f'(x)=\begin{cases}-3x^2 & , \quad x<0\\ 0 & , \quad x=0\\ 3x^2 & , \quad x>0\end{cases}$

So that f'(x) exists for all real x. $f''(x)=\begin{cases}-6x & , \quad x<0\\ 0 & , \quad x=0\\ 6x & , \quad x>0\end{cases}$

$\Rightarrow$ f''(x) exists for all real x. $f'''(x)=\begin{cases}-6, & x<0\\ 6, & x>0\end{cases}$. However, f'''(0) does not exist,since

$f_-'''(0) = -6$ and $f_+'''(0) = 6$, which are not equal.

Thus f'''(x) exists for all real x except for x = 0.

Hence, the set of points where f(x) is thrice differentiable, is R - {0}.

Problem -13. Differentiate $\tan^{-1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$ with respect to $\cot^{-1}\dfrac{2x}{1-x^2}$

Solution: Let y = $\tan^{-1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$ ....(1)

and , u = $\cot^{-1}\dfrac{2x}{1-x^2}$ ....(2)

We have to find $\dfrac{dy}{dx}$. In(1) put x = $\cos\theta$. We have, y = $\tan^{-1}\dfrac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}$

$$=\tan^{-1}\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}=\frac{\pi}{4}-\frac{\theta}{2}=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x.$$

This gives, $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{1-x^2}}$. In (2), put x = $\tan\phi$, then $u=\cot^{-1}\dfrac{2\tan\phi}{1-\tan^2\phi}=\cot^{-1}\{\tan 2\phi\}$

$$= \cot^{-1}\left\{\cot\left(\frac{\pi}{2}-2\phi\right)\right\}=\frac{\pi}{2}-2\phi=\frac{\pi}{2}-2\tan^{-1}x..$$

This given $\dfrac{du}{dx}=-\dfrac{2}{1+x^2}$. Hence, $\dfrac{dy}{du}=\dfrac{dy}{dx}\dfrac{dx}{du}=-\dfrac{1+x^2}{4\sqrt{1-x^2}}$.

Problem-14. Let f(x) = cosx and g(x) = $\begin{cases}\min.f(t); & 0\le t\le x \quad 0\le x\le\pi\\ \sin x-1, & x>\pi\end{cases}$.

Discuss the continuity of g(x).

Solution : Since cosx is a decreasing function in [0, π], the minimum of f(x) in [0, x] will be at x, for any $x\in[0,\pi]$. So that min. f(t) $(0\le t\le x)=f(x)\ \forall x\in[0,\pi]$.

Hence g(x) can be rewrittenas g(x) = $\begin{cases} \cos x, & 0 \le x \le \pi \\ \sin x - 1, & x > \pi \end{cases}$

$\text{LHL} = \lim_{h\to 0} g(\pi - h) = \lim_{h\to 0} \cos(\pi - h) = -\lim_{h\to 0} \cosh = -1.$

$\text{RHL} = \lim_{h\to 0} g(\pi + h) = \lim_{h\to 0} (\sin(\pi + h) - 1) = \lim_{h\to 0} (-\sinh - 1) = -1.$ $\Rightarrow$ LHL = RHL

so that g(x) is continuous for all x in $[0, \infty)$.

Problem-15. Let f(x) = $\begin{cases} -4, & -4 \le x < 0 \\ x^2 - 4, & 0 \le x \le 4 \end{cases}$

Discuss the continuity and differentiability of g(x) = f(|x|) +|f(x)|.

Solution : $-4 \le x \le 4 \Rightarrow 0 \le |x| \le 4 \Rightarrow |f(x)| = \begin{cases} |-4| & -4 \le x < 0 \\ |x^2 - 4| & 0 \le x \le 4 \end{cases}$

$\text{i.e.} |f(x)| = \begin{cases} 4 & -4 \le x < 0 \\ 4 - x^2 & 0 \le x < 2 \\ x^2 - 4 & 2 \le x \le 4 \end{cases}$ and $f(|x|) = x^2 - 4, \quad -4 \le x \le 4$

$$\Rightarrow g(x) = \begin{cases} x^2 & -4 \le x < 0 \\ 0 & 0 \le x < 2 \\ 2x^2 - 8 & 2 \le x \le 4 \end{cases}$$

At x = 0, g(x) is continuous as well as differentiable.
At x = 2, g(x) is continuous but not differentiable.

SOLVED OBJECTIVE PROBLEMS

Problem1. $\lim\limits_{x\to 5}\dfrac{x-5}{|x-5|}$ equals to

(a) 2 (b) 0 (c) -2 (d) none of these

Solution : $\lim\limits_{x\to 5^+}\dfrac{x-5}{|x-5|}=\lim\limits_{x\to 5^+}1=1$,

because for values to the right of 5, x-5 > 0, so |x - 5| = (x - 5).

$\lim\limits_{x\to 5^-}\dfrac{x-5}{|x-5|}=\lim\limits_{x\to 5^-}-1=-1$,

because for values to the left of 5,x-5<0, so |x - 5| = -(x - 5).

$\Rightarrow \lim\limits_{x\to 5}\dfrac{x-5}{|x-5|}$ doesn't exist. Hence (D) is the correct answer.

Problem2. If f(x) = $\begin{cases}\dfrac{\sin[x]}{[x]} & \text{for } [x]\neq 0\\ 0 & \text{for } [x]=0\end{cases}$, where[x] denotes the greatest integer less than or equal to x, then $\lim\limits_{x\to 0}$ f(x) equals to

(a) 1 (b) 0 (c) -1 (d) none of these

Solution : $\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^-}\dfrac{\sin[x]}{[x]}=\dfrac{\sin(-1)}{(-1)}=\sin 1$

and $\lim\limits_{x\to 0^+}f(x)$ = 0 as it is given that f(x) = 0 for [x] =0, or $\lim\limits_{x\to 0}$ f(x) doesn't exist.
Hence (d) is the correct answer.

Problem3. $\lim\limits_{x\to \frac{\pi}{3}}\dfrac{\sin\left(\frac{\pi}{3}-x\right)}{2\cos x-1}$ is equal to

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\dfrac{2}{\sqrt{3}}$

Solution : $\lim\limits_{x\to \frac{\pi}{3}}\dfrac{\sin\left(\frac{\pi}{3}-x\right)}{2\cos x-1}$ (form 0/0) By L, Hospital's rule

$$\frac{-\cos\left(\frac{\pi}{3}-x\right)}{-2\sin x}=\frac{-\cos 0}{-2\sin\frac{\pi}{3}}=\frac{1}{2.\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}.$$

Alternative solution : $\lim\limits_{x\to \frac{\pi}{3}}\dfrac{\sin\left(\frac{\pi}{3}-x\right)}{2\cos x-1}=\lim\limits_{x\to \frac{\pi}{3}}\dfrac{2\sin\dfrac{\left(\frac{\pi}{3}-x\right)}{2}\cos\dfrac{\left(\frac{\pi}{3}-x\right)}{2}}{2\left(\cos x-\cos\frac{\pi}{3}\right)}$

$$\lim_{x\to\frac{\pi}{3}} \frac{\sin\frac{\left(\frac{\pi}{3}-x\right)}{2}\cos\frac{\left(\frac{\pi}{3}-x\right)}{2}}{2\sin\frac{\left(\frac{\pi}{3}-x\right)}{2}\sin\frac{\left(\frac{\pi}{3}+x\right)}{2}} = \lim_{x\to\frac{\pi}{3}} \frac{\cos\frac{\left(\frac{\pi}{3}-x\right)}{2}}{2\sin\frac{\left(\frac{\pi}{3}+x\right)}{2}} = \frac{1}{2.\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Hence (B) is the correct answer.

Problem 4. The number of points at which the function $f(x) = \frac{1}{\log|x|}$ is discontinuous is

(a) 1 (b) 2 (c) 3 (d) 4

Solution : The function log|x| is not defined at x = 0, so, x = 0 is a point of discontinuity
Also, for f(x) to be defined, log|x| $\neq 0$ that is $x \neq \pm 1$
Hence, 1 and -1 are also points of discontinuity.
Clearly f(x) is continuous for $x \in R$ -{0, 1, -1}.
Thus, there are three points of discontinuity.
Hence (C) is the correct answer.

Problem5. The set of all points where the function f(x) = x|x| is differentiable is

(a) $(-\infty,\infty)$ (b) $(-\infty, 0)\cup(0, \infty)$ (c) $(0, \infty)$ (d) $[0, \infty)$

Solution : $f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$, $f'(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x & \text{if } x < 0 \end{cases}$

f(x) is differentiable for all $x \in R$ except possibly at x = 0.
But $f'(0^+)=f'(0^-)=0$.
Hence f is differentiable every where.
Hence (A) is the correct answer.

Problem6. Let f(x) = [$\tan^2 x$], where[.] denotes the greatest integer function, Then

(A) $\lim_{h\to 0}$ f(x) doesn't exist (b) f(x) is continuous at x = 0
(c) f(x) is not differentiable at x = 0 (d) f'(0) = 1

Solution : $\lim_{h\to 0}[\tan^2(0+h)] = \lim_{h\to 0}[\tan^2(0-h)] = [\tan^2 0] = 0$

$\Rightarrow$ f(x) is continuous at x = 0.
Since f(x) = 0 in the neighbourhood of 0, f'(0) = 0
Hence (b) is the correct answer.

Problem 7: If f(x) = |x-25| and g(x) = f(f(x)) then for x > 50, g'(x) is equal to
(a) 0 (b) 1 (c) 25 (d) none of these

Solution : g(x) = f(f(x)) = f(|x-25|) = ||x-25|-25|
and for x > 50, g(x) = |x - 25 - 25| = |x - 50| = x - 50 $\Rightarrow$ g'(x) = 1.
Hence (b) is the correct answer.

Problem 8. In order that the function f(x) = $(x+1)^{\cot x}$ is continuous at x = 0, f(0) must be defined as
(a) 0 (b) e (c) 1/e (d) none of these

Solution : $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x+1)^{\cot sx}$ (1^∞ form) or $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left[(1+x)^{1/x}\right]^{x/\tan x} = e^1 \Rightarrow f(0) = e.$ Hence (b) is the correct answer.

Problem 9 If $f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$, then

(a) R f'(0) exists but L f'(0) does not exist
(b) L f'(0) exists bu R f'(0) does not exist
(c) f(x) is differentiable at x = 0.
(d) none of these

Solution : Since domain of f(x) is $[0, \infty)$, Lf'(0^-) does not exist.
Hence (a) is the correct answer.

Problem10. If f(x) = $\dfrac{\sin(2\pi[\pi^2 x])}{5+[x]^2}$ ([.] denotes the greatest integer function), then f(x) is

(a) discontinuous at some x.
(b) continuous at all x, but the derivative f'(x) doesn't exist for some x.
(c) f'(x) exists for all x, but f"(x) doesn't exist some x.
(d) f"(x) exist for all x.

Solution : Since $\left[\pi^2 x\right]$ is an integer whatever be the value of x and so $2\pi\left[\pi^2 x\right]$ is an integral mul tiple of π.

Thus, sin $\left(2\pi\left[\pi^2 x\right]\right) = 0$ and $5 + [x]^2 \neq 0$ for all x.

Hence f(x) = 0 $\forall x \in R$.
Thus, f(x) is a constant function and so it is continuous and differentiable any number of times for all $x \in R$. Hence (d) is the correct answer.

Problem :11 The function f(x) = $\dfrac{\log(1+ax) - \log(1-bx)}{x}$ is not defined at x = 0.The value which should be assigned to f at x = 0, so that it is continuous at x = 0 is

(a) a - b (b) a + b (c) log a + log b (d) none of these

Solution : $f(x) = a\left[\dfrac{\log(1+ax)}{ax}\right] + b\left[\dfrac{\log(1-bx)}{-bx}\right]$

so that $\lim\limits_{x\to 0} f(x) = a.1 + b.1 = a + b = f(0).$ $\left[\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x} = 1\right]$

Hence (b) is the correct answer.

Alternative Solution :

$\lim\limits_{x\to 0}\dfrac{a - abx + b + abx}{(1+ax)(1-bx)}$ = a + b (by L'Hospital's Rule) $\Rightarrow$ f(0) = a+b, if f is continuous.

Problem 12. The number of points at which the function f(x) = $\dfrac{1}{x-[x]}$ ([.] denotes, the greatest integer function) is not continuous is

(a) 1 (b) 2 (c) 3 (d) none of these

Solution : x-[x] = 0 when x is an intger,
so that, f(x) is discontinuous for all $x \in I$. i.e. f(x) is discontinuous at infinite number of points. Hence (d) is the correct answer.

Problem 13. If $f(x) = [x\sin\pi x]$, (where[.] denotes the greatest integer function) then f(x) is

(a) continuous in (-1, 1) (b) differentiable at x = -1

(c) differentiable at x = 1 (d) none of these

Solution : By the definition of [x], it is obvious that f(x) = $[x\sin\pi x]$ = 0 when $-1 \le x \le 1$ and f(x) $= [x\sin\pi x]$ = -1 when 1 < x < 1 + h, (h small).

Thus f(x) is constant and equal to 0 in [-1, 1] and hence f(x) is continuous and differ entiable in (-1, 1).

At x = 1, clearly f(x) is discontinuous since $\lim\limits_{x\to1^+} f(x) = -1$ and $\lim\limits_{x\to1^-} f(x) = 0$.

Hence (a) is the correct answer.

Problem 14. For $m,n \in I, \lim\limits_{x\to0}\dfrac{\sin x^n}{(\sin x)^m}$ is equal to

(a) 1, if n < m (b) 0, if n > m (c) n/m (d) none of these

Solution : Writing the given expression in the form $\left(\dfrac{\sin x^n}{x^n}\right)\left(\dfrac{x^n}{x^m}\right)\left(\dfrac{x}{\sin x}\right)^m$ and noting that the

$\lim\limits_{\theta\to0}\dfrac{\sin\theta}{\theta}$ =1, we see that the required limit equals to 1 if n = m, and 0 if n > m.

Hence (b) is correct answer.

Problem15. The function f defined by f(x)=$\dfrac{\sin x^2}{x}$ for $x \ne 0$

= 0 for x = 0 is

(a) continuous and derivable at x = 0

(b) neither continuous nor derivable at x =0

(c) continuous but not derivable at x = 0

(d) none of these

Solution : The function is continuous at x = 0, because

$$\lim_{x\to0} f(x) = \lim_{x\to0}\frac{\sin x^2}{x} = \lim_{x\to0}\left(\frac{\sin x^2}{x^2}\right).x = 0 = f(0).$$

Also, $Rf'(0) = \lim\limits_{h\to0}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to0}\dfrac{\frac{\sin h^2}{h}-0}{h} = \lim\limits_{h\to0}\dfrac{\sin h^2}{h^2} = 1$

and $Lf'(0) = \lim\limits_{h\to0}\dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to0}\dfrac{\frac{\sin h^2}{-h}-0}{-h} = \lim\limits_{h\to0}\dfrac{\sin h^2}{h^2} = 1$

so that, f(x) is derivable at x = 0. Hence (a) is the correct answer.

Problem 16. If f(x) = $\begin{cases} x\sin\dfrac{1}{x} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$. Then

. . . . .. $^+$) and f'(0^-) do not exist (b) f'(0^+) exists but f'(0^-) does not exist

(c) f'(0^+)=f'(0^-) (d) none of these

Solution : As for the derivative f'(0) we have,

$f(0^+) = \lim_{h \to 0} \dfrac{h \sin\frac{1}{h} - 0}{h} = \lim_{h \to 0} \sin\dfrac{1}{h}$ which doesn't exist, Similarly, the limit f'(0$^-$) doesn't ex ist. Hence (a) is the correct answer.

Problem 17 The function f(x) = $|x^3|$ is

(a) differentiable everywhere (b) continuous but not differentiable at x = 0
(c) not a continuous function (d) none of these

Solution: The range of the function x^3 is $(-\infty, \infty)$, and the range of f(x) is $[0, \infty)$, f is clearly differentiable except possibly at the point x = 0.
Now, clearly by definition Rf'(0) = Lf'(0) = 0
so that, f is differentiable at x = 0 and hence every where.
Hence (a) is the correct answer.

Alternative Solution : Here f(x) = $\begin{cases} -x^3, & x < 0 \\ x^3, & x > 0, \end{cases}$ so that f'(x) = $\begin{cases} -3x^2, & x < 0 \\ 3x^2, & x > 0, \end{cases}$

$\Rightarrow$ the function is differentiable everywhere including x = 0.

Problem18. $\sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is

(a) Continuous but not differentiable at x = 1 (b) Differentiable at x = 1
(c) Neither continuous nor differentiable at x = 1 (d) continuous every where

Solution : $\sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is difined only for x = -1 and x = 1. Hence (c) is the correct answer**.**

Problem 19. The number of points where f(x) = [sinx + cosx] (where [.] denotes the greatest inte ger function), $x \in (0, 2\pi)$ is not continuous is

(a) 3 (b) 4 (c) 5 (d) 6

Solution : f(x) will be discontinuous at those points, where sinx + cosx is an integer, which is the case for $x \in \left\{\dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}\right\}$. Thus f(x) is discontinuous exactly for five values of x. Hence (c) is correct answer.

Problem 20 If $\lim_{x \to 0}(1 + ax + bx^2)^{2/x} = e^3$, then

(a) a = 3, b = 0 (b) $a = \dfrac{3}{2}, b \neq 1$ (c) a = 3/2, b = 4 (d) a = 2, b = 3.

Solution : $\lim_{x \to 0}(1 + ax + bx^2)^{2/x} = \lim_{x \to 0}(1 + ax + bx^2)^{\frac{1}{ax+bx^2} \cdot \frac{2(ax+bx^2)}{x}}$

$= e^{\lim_{x \to 0} \frac{2(ax+bx^2)}{x}} = e^3 \Rightarrow a = \dfrac{3}{2}$, b any real number

Hence (c) is the correct answer.

SUBJECTIVE ASSIGNMENTS

LEVEL - I

1. Determine the following limits :

(i) $\lim\limits_{x\to 0}\dfrac{\sin(\alpha+x)-\sin(\alpha-x)}{\cos(\alpha+x)-\cos(\alpha-x)}$ (ii) $\lim\limits_{\alpha\to\pi/4}\dfrac{\sin\alpha-\cos\alpha}{\alpha-\frac{\pi}{4}}$ (iii) $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$

(iv) $\lim\limits_{x\to\infty} x(e^{1/x}-1)$ (v) $\lim\limits_{n\to\infty}\dfrac{1+\frac{1}{2}+\frac{1}{4}+.....\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+.....+\frac{1}{3^n}}$ (vi) $\lim\limits_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\frac{1}{\sin x}}$

(vii) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{(1-\sin x)^2}{\left(\frac{\pi}{2}-x\right)^2}$ (viii) $\lim\limits_{x\to 0}\dfrac{\cos(m+2)x-\cos mx}{\cos(m+4)x-\cos(m+2)x}$

(ix) $\lim\limits_{x\to 3^-}\dfrac{x-[x]}{x-3}$ (x) $\lim\limits_{x\to 1}\dfrac{|x^3-x|}{x-x^3}$ (xi) $\lim\limits_{x\to 1}\left[\tan\left(\frac{\pi}{4}+\log x\right)\right]^{\frac{1}{\log x}}$

(xii) $\lim\limits_{n\to\infty}\dfrac{1.n^2+2(n-1)^2+3(n-2)^2+...n.1^2}{1^3+2^3+3^3+.....+n^3}$

(xiii) $\lim\limits_{n\to\infty}\left(\dfrac{1}{1.4}+\dfrac{1}{4.7}+...+\dfrac{1}{(3n-2)(3n+1)}\right)$

(xiv) $\lim\limits_{x\to 0}\left(\dfrac{3^x+4^x}{2}\right)^{2/x}$ (xv) $\lim\limits_{x\to 0}\left(\dfrac{1^x+2^x+3^x+.....+n^x}{n}\right)^{a/x}$

2. Find the set of points where the following functions are discontinuous

(i) f(x) = tan2x (ii) f(x) = {3x) (iii) $f(x)=\dfrac{x}{\sin x}$ (iv) $f(x)=\tan[\pi^2]x-\tan[-\pi^2]x$

(v) $f(x)=\dfrac{1}{1-e^{-\frac{x-1}{x-2}}}$ (vi) $f(x)=\dfrac{3\sin^2 x+\cos^2 x+1}{2\cos^2 x-1}$

where [.] denotes the greatest integer function and {.} denotes the fractional part.

3. Discuss the continuity of the following functions.

(i) $f(x)=\dfrac{|x^2-1|}{x+1}$ (ii) $f(x)=\dfrac{|x|}{x}(x^2-1)$

(iii) $f(x)=\left[[x]\right]-[x-1]$ (iv) $f(x)=\dfrac{\sqrt{x}}{\sqrt{1+\sqrt{x}}-1}$

4. Let $f(x)=\begin{cases}\dfrac{\sin ax^2}{x^2}, & x\neq 0\\ \dfrac{3}{4}+\dfrac{1}{4a}, & x=0\end{cases}$. For what value of a is f(x) continuous at x = 0 ?

5. (i) Let $f(x)=\begin{cases}\dfrac{x^2-4}{x+2} & , \quad x<-3 \\ \ln a & , \quad x=-3 \\ a+bx & , \quad x>-3\end{cases}$. For what value of a and b is f(x) continuous on the real line ?

(ii). Let $f(x)=\begin{cases}ax+1 & , \quad x<1 \\ 3 & , \quad x=1 \\ bx^2+1 & , \quad x>1\end{cases}$. For what values of a and b is f(x) continuous at x = 1.

6. Let $f(x)=\begin{cases}\dfrac{|x^2+5x+6|}{x^2+5x+6}, & x\neq -2,-3 \\ 1 & , \quad x=-2,-3\end{cases}$. Find f'(-2) if it exists.

7. Find the constants a and b such that (i) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=0$

(ii) $\lim\limits_{x\to\infty}\left(\sqrt{x^2-x+1}-ax-b\right)=0$

8. Let $f(x)=\begin{cases}e^{x^2+x}, & x<0 \\ ax+b, & x\geq 0\end{cases}$ Find a and b so that f(x) is continuous and has a derivative at x = 0.

9. If x = e^t sin t, y = e^t cos t, show that $\dfrac{d^2y}{dx^2}=\dfrac{2\left(x\dfrac{dy}{dx}-y\right)}{(x+y)^2}=\dfrac{-2(x^2+y^2)}{(x+y)^3}$.

10. Let $f(x)=\begin{cases}xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & x\neq 0 \\ 0 & x=0\end{cases}$. Discuss the continuity and differentiability at x = 0

Answers

1. (i) $-\cot\alpha$ (ii) $\sqrt{2}$ (iii) $\frac{2}{\pi}$ (iv) 1 (v) $\frac{4}{3}$

(vi) 1 (vii) 0 (viii) $\frac{m+1}{m+3}$ (ix) $-\infty$

(x) limit does not exist (xi) e^2 (xii) $\frac{1}{3}$ (xiii) $\frac{1}{3}$ (xiv) 12

(xv) $(n!)^{a/n}$

2. (i) $\left\{(2n+1)\frac{\pi}{4}: n \text{ any integer}\right\}$ (ii) $x=\frac{1}{3}$, is an integer (iii) $x = n\pi,\ n \in I$ (iv) $\frac{(2n+1)\pi}{18}, \frac{(2n+1)\pi}{20}, n \in I$

(v) Discontinuous at x = 1,2 (vi) Discontinuous at x = $n\pi \pm \frac{\pi}{4}, n \in I$

3. (i) Continuous in R-{-1} (ii) Continuous in R-{0}

(iii) Continuous in R (iv) Continuous in $(0, \infty)$

4. $a = -\frac{1}{4},\ 1$

5. (i) $a = e^{-5}, b = \frac{1}{3}\left(5+e^{-5}\right)$ (ii) a = b = 2

6. Does not exists

7. (i) a = 1, b= -1 (ii) a = 1, b= -1/2

8. a= 1, b =1

10. Continuous but not differentiable at x = 0

LEVEL - II

1. A function f(x) is defined as $f(x) = \begin{cases} \cos^2 x, & 0 < x \le \frac{\pi}{3} \\ ax + b, & \frac{\pi}{3} < x < \frac{\pi}{2} \end{cases}$.

   Determine the values of a and b so that f(x) is continuous and has a continuous derivative at $x = \frac{\pi}{3}$.

2. Let $f(x) = \begin{cases} x^4 + x^2 - x + 1, & x \le 1 \\ 2x^3 - x^2 + x, & x > 1 \end{cases}$. Show that f(x) is continuous and possesses a continuous first derivative at x = 1 but that second derivative does not exist at this point.

3. If $f(x) = \begin{cases} \min(x, x^2), & x \ge 0 \\ \min(2x, x^2 - 1), & x < 0 \end{cases}$, then find the number of non-differentiable points of f(x).

4. Evaluate $\lim\limits_{x \to 0} \frac{\cos(\sin x) - \cos x}{x^4}$.

5. If $f(x) = \begin{cases} \frac{\sin[x^2]\pi}{x^2 - 3x - 18} + ax^3 + b, & 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x, & 1 < x \le 2 \end{cases}$, where [.] denotes the greatest integer function, is differentiable in [0, 2], then find the value of 'a' and 'b'.

6. If $y = \log_2[\log_3(\log_5(\sin x + c))]$, find the range of c for which $\frac{dy}{dx}$ exists and hence find it.

7. If $f(x) = (1+x)^n$, show that $f(0) + \frac{f^1(0)}{1!} + \frac{f^2(0)}{2!} + ... + \frac{f^n(0)}{n!} = 2^n$.

8. (i) Let $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ 3x^2 - 4, & -1 < x \le 1 \\ x - 2, & 1 < x < 3 \\ 4x^2 + 5, & 3 \le x < 4 \\ x^3 + 5, & x \ge 4 \end{cases}$, Discuss the continuity of f(x) on the real line.

   (ii). Find the values of a and b so that the function $f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b, & \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x, & \frac{\pi}{2} < x \le \pi \end{cases}$ is continuous for $0 \le x \le \pi$.

9. If $y = 1+\frac{C_1}{x-C_1}+\frac{C_2x}{(x-C_1)(x-C_2)}+\frac{C_3x^2}{(x-C_1)(x-C_2)(x-C_3)}$, show that

$$\frac{dy}{dx}=\frac{y}{x}\left[\frac{C_1}{C_1-x}+\frac{C_2}{C_2-x}+\frac{C_3}{C_3-x}\right].$$

10 If(x) = $\begin{cases} x, & x<1 \\ 2-x, & x\ge 1 \end{cases}$ discuss the differentiability of fof(x).

ANSWERS

1. $A=-\frac{\sqrt{3}}{2},\ B=\frac{1}{4}+\frac{\pi}{2\sqrt{3}}$ 3. 3 4. 1/6 5. $a=\frac{1}{6}, b=\frac{\pi}{4}-\frac{13}{6}$

6. $c\in(6,\infty)$, $dy/dx = (\log_3\log_5(\sin x+c))^{-1}(\log_5(\sin x+c))^{-1}(\log_2 e)(\log_3 e)(\log_5 e)\frac{\cos x}{\sin x+c}$

8. (i) Continuous in R ~ {3} (ii). $a=\frac{\pi}{6}, b=-\frac{\pi}{12}$

10. fof(x) is differentiable $\forall$ x, expect at x = 1.

LEVEL - III

1. (i) Let f : R $\to$ R such that for all x, y $\in$ R. $|f(x)-f(y)| \le |x-y|^n$ where n $\in$ N and n > 1. Prove that f(x) is a constant function.

   (ii) Let f be an even function and let f'(0) exist. Then find f'(0).

2. Discuss the continuity and differentiability of f(x) = min (|x|, |x-1|, 2-|x-1|).

3. Let f(x) = $\begin{cases} [x], & -2 \le x \le -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \le 2 \end{cases}$ and g(x) = f(|x|) + |f(x)|. Check the continuity and differentiability of g (x) in (-2, 2).

4. Let f(x) = x (1 - x²), x $\in$ R and g(x) = $\begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$. Discuss the continuity and differentiability of f (g(x)) and g (f (x)).

5. Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all real x and y. If f'(0) exists and equals -1 and f(0) = 1, find f(2).

6. Draw the graph of the following function and discuss its continuity and differentiability at x = 1. y = $\begin{cases} 3^x, & -1 \le x \le 1 \\ 4-x, & 1 \le x \le 4 \end{cases}$.

7. Discuss the continuity and differentiability of the function y = f(x) defined parametrically as x = 2t -|t - 1|, y= 2t² + t | t | $\forall$ t $\in$ R.

8. Let f : R $\to$ R be a function satisfying $\frac{f(x)+2f(y)}{3} = f\left(\frac{x+2y}{3}\right) \forall$ x, y $\in$R and f'(0) = 1. Find f(x).

9. Suppose p(x) = $a_0 + a_1x + a_2x^2 + \ldots a_nx^n$. If |p(x)| $\le$ $|e^{x-1} - 1|$ for all x $\ge$0, prove that $|a_1+2a_2 + \ldots + na_n| \le 1$.

10. Let f(x) = $\begin{cases} x+a & \text{if} \quad x<0 \\ |x-1| & \text{if} \quad x \ge 0, \end{cases}$ and $g(x) = \begin{cases} x+1 & \text{if} \quad x<0 \\ (x-1)^2 + b & \text{if} \quad x \ge 0, \end{cases}$ where a and b are nonnegative real numbers. Determine the composite function gof. If (gof)(x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof(x) differentiable at x = 0? Justify your answer.

ANSWERS

1. (ii) 0
2. continuous for every x $\in$ R, not differentiable at x = $-\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.
3. g(x) is not continuous at x = -1, $-\frac{1}{2}$ and not differentiable at x = -1, $-\frac{1}{2}, \frac{1}{\sqrt{2}}$
4. f(g(x)) is continuous and differentiable everywhere, g(f(x)) is discontinuous and non-differentiable at x = 0, ± 1.
5. -1
6. Cont. but not diff. at x = 1.
7. f(x) is continuous for all x and differentiable for all x except x = 2
8. f(x) = x + c

10. $g(f(x)) = \begin{cases} x+a+1, & x<-a \\ (x+a-1)^2+b, & -a \le x < 0 \\ x^2+b, & 0 \le x < 1 \\ (x-2)^2+b, & x \ge 1 \end{cases}$ a= 1, b = 0, g(f(x)) is differentiable.

OBJECTIVE ASSIGNMENTS

LEVEL - I

1. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ is equal to

(A) 3/4 (B) 1/4 (C) 1/2 (D) 1

2. If $f(x)=\sqrt{\frac{x-\sin x}{x+\cos^2 x}}$, then $\lim_{x\to\infty} f(x)$ is

(A)0 (B) ∞ (C) 1 (D) none of these

3. $\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{2x+1}$ is equal to

(A) 1 (B) 0 (C) -1 (D) 1/2

4. $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\cos x-1}{\cot x-1}$ is equal to

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{2\sqrt{2}}$ (D) 1

5. $\lim_{x\to 0}\frac{x}{\tan^{-1}2x}$ is equal to

(A) 0 (B) 1/2 (C) 1 (D) ∞

6. $\lim_{x\to 0}\frac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}$ is equal to

(A) 10/3 (B) 3/10 (C) 6/5 (D) 5/6

7. $\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+....+\frac{n}{1-n^2}\right]$ is equal to

(A) 0 (B) -1/2 (C) 1/2 (D) none of these

8. $\lim_{x\to 0}\frac{\sqrt{(1-\cos 2x)}}{2x}$ is equal to

(A) 1 (B) -1 (C) 0 (D) none of these

9. $\lim_{x\to\infty} x^{\frac{1}{x}}$ equals

(A) 0 (B) 1 (C) e (D) ∞

10. $\lim_{x\to 2}(-1)^{[x]}$, where [x] is the greatest integer function, is equal to

(A) 1 (B) -1 (C) ± 1 (D) none of these

11. $\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3}$ is given by

(A) 1 (B) e^3 (C) e (D) e^4

12. The value of f(0) so that the function f(x) = $\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$, is continuous at each point in its domain, is equal to

(A) 2 (B) 1/3 (C) 2/3 (D) -1/3

13. Let f(x) = $\begin{cases} x+1 & \text{if} \quad x\le 1 \\ 3-ax^2 & \text{if} \quad x>1 \end{cases}$. The value of a for which f(x) is continuous is

(A) 1 (B) 2 (C) -1 (D) -2

14. Which of the following functions have finite number of points of discontinuity ?

(A) tan x (B) x[x] (C) $\frac{|x|}{x}$ (D) sin [n π x]

15. The function f(x) = $[x]^2$ - $[x^2]$ (where [.] denotes the greatest integer function) is discontinuous at

(A) all integers (B) all integers except 0 and 1

(C) all integers except 1 (D) all integers except 0

ANSWER

1.	C	2.	C	3.	D	4.	B	5.	B
6.	A	7.	B	8.	D	9.	B	10	D
11.	D	12.	B	13.	A	14.	C	15.	C

LEVEL - II

1. Let f(x + y) = f(x). f(y), for all x and y. If f(5) = 2 and f'(0) = 3, then f'(5) is equal to
(A) 5 (B) 6 (C) 0 (D) none of these

2. $\lim\limits_{x\to 1}([x] + |x|)$, (where [.] deontes the greatest integer function)
(A) is 0 (B) is 1 (C) does not exists (D) none of these

3. Number of points at which $f(x) = |x^2 + x| + |x - 1|$ is non-differentiable is
(A) 0 (B) 1 (C) 2 (D) 3

4. Let $f(x) = \lim\limits_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1}$, then
(A) f(x) = 1 for |x| >1 (B) f(x) = -1 for |x| <1
(c) f(x) is not defined for any value of x (D) f(x) = 1 for |x| = 1.

5. If $\lim\limits_{n\to\infty}\left(an - \frac{1+n^2}{1+n}\right)$ = b, a finite number, then
(A) a = 1, b = 1 (B) a = 1, b = 0 (C) a = -1, b = 1 (D) none of these

6. $\lim\limits_{x\to -1+} \frac{\sqrt{\pi} - \sqrt{\cos^{-1} x}}{\sqrt{x+1}}$ is equal to
(A) $\frac{1}{\sqrt{\pi}}$ (B) $\frac{1}{\sqrt{2\pi}}$ (C) 1 (D) 0

7. If [x] denotes the greatest integer less than or equal to x, then the value of $\lim\limits_{x\to 1}$ (1-x+[x-1] + [1-x] is
(A) 0 (B) 1 (C) -1 (D) none of these

8. $\lim\limits_{x\to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^2}$ is equal to
(A) 1/2 (B) - 1/2 (C) 0 (D) none of these

9. Let h (x) = min {x, x^2}, x $\in$R, then h (x) is
(A) differentiable everywhere (B) non- differentiable at three values of x
(C) non-differentiable at two values of x (D) none of these

10. If f(x) = $\lim\limits_{n\to\infty}(\sin x)^{2n}$, then f is
(A) continuous at x = $\frac{\pi}{2}$ (B) discontinuous at x = $\frac{\pi}{2}$
(C) discontinuous at x = π (D) none of these

11. If f(x) = x, x $\le$1 and f(x) = x^2 + bx + c (x>1) and f' (x) exists finitely for all x $\in$ R, then
(A) b = -1, c $\in$ R (B) c = 1, b $\in$ R (C) b = 1, c= -1 (D) b = -1, c = 1

12. If the function f(x) $= \begin{cases} Ax - B & , \quad x \le 1 \\ 3x & , \quad 1 < x < 2 \\ Bx^2 & , \quad x \ge \end{cases}$ be continuous at x = 1 and discountinuous at x = 2, then

(A) A = 3 + B, B $\neq$ 3 (B) a = 3 + B, B = 3 (C) A = 3 + B (D) none of these

13. If f(x) = $\begin{cases} ax^2 + b & , \quad x \le 1 \\ bx^2 + ax + c, & x > 1 \end{cases}$ $b \neq 0$, then f(x) is continuous and differentiable at x = 1 if

(A) c = 0, a = 2b (B) a = b, c $\in$R (C) a = b, c = 0 (D) a = b, c $\neq$ 0.

14. If f(x) = $\begin{cases} x^3 & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -x^3 & , \quad x < 0 \end{cases}$, then

(A) f is derivable at x= 0
(B) f is continuous, but not derivable at x = 0
(C) LHD at x = 0 is 1
(D) none of these

15. Let f'' (x) be continuous at x = 0 and f'' (0) = 4. Then $\lim_{x \to 0} \frac{2f(x) - 3f(2x) + f(4x)}{x^2}$ is equal to

(A) 11 (B) 2 (C) 12 (D) none of these

ANSWER

1.	B	2.	C	3.	D	4.	A, B	5.	A
6.	B	7.	C	8.	C	9.	C	10.	B
11.	D	12.	A	13.	A	14.	A	15.	C

DO YOURSELF

1. If α is a repeated root of $ax^2 + bx + c = 0$ then $\lim\limits_{x\to\alpha} \dfrac{\sin(ax^2+bx+c)}{(x-\alpha)^2}$ is

(a) 0 (b) a (c) b (d) c

2. Let $a = \min\{x^2 + 2x + 3,\ x \in R\}$ and $b = \lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta^2}$. Then the value of $\sum\limits_{r=0}^{n} a^r . b^{n-r}$ is.

(a) $\dfrac{2^{n+1}-1}{3.2^n}$ (b) $\dfrac{2^{n+1}+1}{3.2^n}$ (c) $\dfrac{4^{n+1}-1}{3.2^n}$ (d) $\dfrac{4^{n+1}+1}{3.2^n}$

3. At the point x = 1, the function $f(x) = \begin{cases} x^3 - 1; & 1 < x < \infty \\ x - 1; & -\infty < x \le 1 \end{cases}$

(a) continuous and differentiable (b) continuous and not differentiable
(c) discontinuous and differentiable (d) discontinuous and not differentiable

4. Let $f(x) = \begin{cases} \dfrac{1}{|x|} & ; \ |x| \ge 1 \\ ax^2 + b & ; \ |x| < 1 \end{cases}$ be continuous and differentiable everywhere. Then a and b are

(a) $-\dfrac{1}{2},\ \dfrac{3}{2}$ (b) $\dfrac{1}{2},\ -\dfrac{3}{2}$ (c) $\dfrac{1}{2},\ \dfrac{3}{2}$ (d) none of these

5. Let f(x) = maximum $\{4,\ 1+x^2,\ x^2-1\}\ \forall x \in R$. Total number of points, where f(x) is non-differentiable, is equal to

(a) 2 (b) 4 (c) 6 (d) none of these

6. The function $f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x < 1 \\ [2x-3]x, & x \ge 1 \end{cases}$, where [.] denotes the greatest integer function, is

(a) continous and differentiable at x = 1 (b) continuous but not differentiable at x = 1
(c) discontinuous at x = 1 (d) none of these

7. If $f(x) = \begin{cases} \dfrac{(1-\sin^3 x)}{3\cos^3 x} & ; \ x < \dfrac{\pi}{2} \\ a & ; \ x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi - 2x)^2} & ; \ x > \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \dfrac{\pi}{2}$. Then (a, b) is

(a) $\left(\dfrac{1}{2},\ 4\right)$ (b) $\left(1,\ \dfrac{1}{4}\right)$ (c) $\left(2,\ \dfrac{1}{4}\right)$ (d) none of these

1.	**B**	**2.**	**C**	**3.**	**B**	**4.**	**A**	**5.**	**A**	**6.**	**C**	**7.**	**A**

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