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UNIT-1

1. Matrices

⇒ An arguement of mn numbers in m rows and n columns enclosed in square bracket is called a matrix of order $m \times n$. It is written as $A = [a_{ij}], 1 \le i \le m$
$1 \le j \le n$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

e.g. $m=3, n=4$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4}$$

Types of matrices.

1. Row matrix :
A matrix which has only one row and any number of column.
e.g $[1, 2]_{1 \times 2}$ 1row, 2column
$[1, 3]_{1 \times 3}$ 1row, 3column

2. Column matrix:
A matrix which has only one column and any number of rows.
e.g. $C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2 \times 1}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{4 \times 1}$

3. Rectangular Matrix:

A matrix in which number of rows is not equal to number of columns is called rectangular matrix.

e.g. $E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

$F = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & -1 & 5 \end{bmatrix}_{2\times 4}$

4. Square Matrix:

A matrix in which number of rows is equal to number of columns.

e.g.

$A = [2]_{1\times 1}$

$B = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}_{2\times 2}$

$C = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 0 & 1 & 0 \\ 2 & 4 & -3 \end{bmatrix}_{3\times 3}$

5. Diagonal Matrix:

A square matrix is said to be diagonal matrix if all non-diagonal elements are 0.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

6. Unit or Identity Matrix:

A square matrix is said to be unit matrix if all diagonal elements are unity and all non-diagonal elements are zero. It is denoted by $[I_n]$

e.g. $I_1 = [1]$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7 Scalar Matrix:

A diagonal matrix is said to be scalar, if all diagonal elements are equal.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

8. Transpose of Matrix:

A matrix obtained by interchanging rows and columns is called transpoes of a matrix. It is denoted by A^T or A'.

e.g. $= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

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9) Upper Triangular Matrix:

A square matrix is said to be upper triangular if all elements below the diagonal are zero is called upper triangular Matrix.

e.g. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}_{2\times2}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

10. Lower Triangular Matrix:

A square matrix is said to be lower triangular if all elements above the diagonal are zero.

e.g. $C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

⇒ Determinant of Matrix:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A) = |A| = ad - bc$

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$= a(ei - hf) - b(di - gf) + c(dh - ge)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 2 & 3 & 0 \end{bmatrix}$$

$$|A| = 1(0-6) - 2(0-4) + 3(0-8)$$
$$= -6 + 8 - 24$$
$$= 2 - 24$$
$$= -22$$

11. Singular Matrix:

A square Matrix 'A' is said to be singular if $|A| = 0$. A square Matrix 'A' is said to be non-singular if $|A| \neq 0$

Problem.

Prove that the matrix $\begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$ is a non-singular matrix

$\Rightarrow$ Let $A = \begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$

$$|A| = 1(9) - 4(6)$$
$$= 9 - 24$$
$$= -15$$
$$\neq 0$$

$\therefore$ A is non-singular

12. Conjugate of a matrix:

If a matrix 'A' is obtained by replacing each element by its conjugate we get complex conjugate of 'A'. denote by $\bar{A}$.

e.g $A = \begin{bmatrix} 2+3i & 0 \\ -4 & 7i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2-3i & 0 \\ -4 & -7i \end{bmatrix}$$

13. Symmetric Matrix:

A square matrix A is said to be symmetric if $A = A^T$

e.g. $A = \begin{bmatrix} 2 & 9 \\ 9 & 7 \end{bmatrix}$ ∴ A' or $A^T = \begin{bmatrix} 2 & 9 \\ 9 & 7 \end{bmatrix}$.

14. Skew Symmetric Matrix:

A square matrix A is said to be skew symmetric if $A = A^T$

e.g. $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$\therefore -A^T = \begin{bmatrix} -0 & 2 \\ -2 & 0 \end{bmatrix}$

$A = -A^T$

H.W

If $\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix}$

find a, b, c, d.

$a+b=4$	$c+d=6$	$a+b=4$	$c+d=6$
$a-b=2$	$c-d=10$	$3+b=4$	$8+d=6$
$2a=6$	$2c=16$	$b=1$	$d=-2$
$a=3$	$c=8$		

Theorem

Every square matrix can be uniquely expressed as the sum of symmetric and skew symmetric Matrix. i.e. If 'A' is any square Matrix, then $A = \frac{1}{2}\left[(A+A') + (A-A')\right]$

Express the Matrix $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ as a sum of symmetric and skew symmetric matrix

$$A' = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$$

$$A + A' = \begin{bmatrix} -2 & 9 & 6 \\ 9 & 6 & 4 \\ 6 & 4 & 10 \end{bmatrix}$$

$$A - A' = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix}$$

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

where $A + A'$ is symmetric and

$A - A'$ is skew symmetric

$$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$$

$$\therefore A = \frac{1}{2}\left[(A + A') + (A - A')\right]$$

⇒ Operations of Matrices

1. Addition
2. Substraction
3. Scalar Multiplication

⇒ $A = \begin{bmatrix} 5 & 7 \\ 3 & 2 \\ 1 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 9 & 2 \\ 5 & -8 \\ -6 & 4 \end{bmatrix}$

find $A+B$, $A-B$, $2A$, $-3B$, $4A+5B$

$$A+B = \begin{bmatrix} 14 & 9 \\ 8 & -6 \\ -5 & 1 \end{bmatrix} \qquad 4A = \begin{bmatrix} 20 & 28 \\ 12 & 8 \\ 4 & -12 \end{bmatrix}$$

$$A-B = \begin{bmatrix} -4 & 5 \\ -2 & 10 \\ 7 & -7 \end{bmatrix} \qquad 5B = \begin{bmatrix} 45 & 10 \\ 25 & -40 \\ -30 & 20 \end{bmatrix}$$

$$2A = \begin{bmatrix} 10 & 14 \\ 6 & 4 \\ 2 & -6 \end{bmatrix} \qquad 4A+5B = \begin{bmatrix} 65 & 38 \\ 37 & -32 \\ -26 & 8 \end{bmatrix}$$

$$-3B = \begin{bmatrix} -27 & -6 \\ -15 & 24 \\ 18 & -12 \end{bmatrix}$$

15. Hermitian matrix:

A square matrix 'A' is said to be Hermitian if $A = (\bar{A})^T$

e.g. $A = \begin{bmatrix} 5 & 2-3i & 7+i \\ 2+3i & 6 & i \\ 7-i & -i & 9 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 5 & 2+3i & 7-i \\ 2-3i & 6 & -i \\ 7+i & i & 9 \end{bmatrix}$

$(\bar{A})^T = \begin{bmatrix} 5 & 2-3i & 7+i \\ 2+3i & 6 & i \\ 7-i & -i & 9 \end{bmatrix}$

$\therefore A = (\bar{A})^T$

$\therefore$ A is Hermitians.

16. Skew Hermitian matrix:

A square matrix 'A' is said to Skew Hermitian matrix if $A = -(\bar{A})^T$

e.g. $A = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$

Show that: A is Hermitian & iA is skew-Hermitian

$\bar{A} = \begin{bmatrix} 2 & 3-2i & -4 \\ 3+2 & 5 & -6i \\ -4 & 6i & 3 \end{bmatrix}$

$(\bar{A})^T = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$ $A = (\bar{A})^T$

$\therefore$ A is Hermitian

$$iA = \begin{bmatrix} 2i & 3i+2i^2 & -4i \\ 3i-2i^2 & 5i & 6i^2 \\ -4i & -6i^2 & 3i \end{bmatrix}$$

$$iA = \begin{bmatrix} 2i & 3i-2 & -4i \\ 3i+2 & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

To prove skew Hermitian

$iA = -(i\bar{A})^T$

$\cancel{(i\bar{A})} =$

$$-iA = \begin{bmatrix} -2i & 3i-2 & -4i \\ 3i+2 & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

$$(i\bar{A})^T = \begin{bmatrix} -2i & 2-3i & 4i \\ -2-3i & -5i & 6 \\ 4i & -6 & -3i \end{bmatrix}$$

$$-(i\bar{A})^T = \begin{bmatrix} 2i & \cancel{2+}3i-2 & -4i \\ 2+3i & 5i & -6 \\ -4i & 6 & 3i \end{bmatrix}$$

$\therefore A = -(i\bar{A})^T$

Inverse of Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj } A$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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e.g 2×2

$\therefore A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{1-4} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

$= \frac{1}{-3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$

3×3

$\therefore A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \cdot Adj\, A$

$|A| = 1(1-4) - 2(2-4) + 2(4-2)$

$= -3 + 4 + 4$

$= 5$

$\neq 0$

$\therefore A^{-1}$ exists

$$\text{Co-factor of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1(1-4) = -3$$

$$A_{12} = (-1)^{1+2} (-2) = (-1)^3 -2 = 2$$

$$A_{13} = (-1)^{1+3} (2) = (-1)^4 2 = 2$$

$$A_{21} = (-1)^{2+1} (-2) = (-1)^3 -2 = 2$$

$$A_{22} = (-1)^{2+2} (-3) = (-1)^4 -3 = -3$$

$$A_{23} = (-1)^{2+3} (-2) = (-1)^5 -2 = 2$$

$$A_{31} = (-1)^{3+1} (2) = (-1)^4 2 = 2$$

$$A_{32} = (-1)^{3+2} (-2) = (-1)^5 -2 = 2$$

$$A_{33} = (-1)^{3+3} (-3) = (-1)^6 -3 = -3$$

$$\text{Adj } A = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\text{Adj } A = [\text{cofactor of } A]^T$$
$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

* **Orthogonal Matrix**

A square matrix 'A' is said to be orthogonal if $AA^T = I$

$$AA^{-1} = I$$
$$A^T = A^{-1}$$

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Problem

Show that the matrix is Orthogonal

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$\therefore$ A is Orthogonal

Verify whether A is orthogonal if so write A^{-1}

$$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^T = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$AA^T = \frac{1}{9}\begin{bmatrix} 9 & 0 & 2 \\ 0 & 9 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

$$\neq I$$

$\therefore$ A is not Orthogonal

* Rank of Matrix.

A matrix is said to be of rank R if:

i. Atleast 1 minor (M) of order R is non-vanishing (non-zero)

ii. Every minor of order R+1 vanishes (if zero).

e.g. if non-zero

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$

$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}_{2\times 2}$

$= 4 - 6$

$= -2$

$\neq 0$

$\boxed{\rho(A)} = 2$

Rank of A

if zero

2. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times 2}$

$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

$= 4 - 4$

$= 0$

minor of 1 is $|4|_{1\times 1}$

$= 4$

$\neq 0$

$\therefore \rho(A) = 1$

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3. $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$

$= 4 - (-3)$

$= 4 + 3$

$= 7$

$\neq 0$

$\therefore \rho(A) = 2$

4. $A = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -2 \\ 3 & -3 \end{vmatrix}$

$= -6 - (-6)$

$= -6 + 6$

$= 0$

minor of order 1 is non-zero

$\therefore \rho(A) = 1$

Note: 1. The rank of every non-zero matrix is greater than or equal to one.

2. The rank of null matrix is zero.

5) $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 3 & -1 \\ 0 & 2 & 5 \end{bmatrix}$

$A = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 3 & -1 \\ 0 & 2 & 5 \end{vmatrix}$

$|A| = 3(15+2) - 1(5-0) + 2(2-0)$

$= 3(17) - 5 + 4$

$= 51 - 1$

$|A| = 50$

$\neq 0$ $\quad \therefore \rho(A) = 3$

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6) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 2 & 2 & 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 2 & 2 & 3 \end{vmatrix}$

$= 1(3+4) - 2(0+4) - 1(0-2)$

$= 7 - 8 + 2$

$= \cancel{8}^{-1} + 2$

$= 1$

$\neq 0$

$\therefore \rho(A) = 3$

7) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \\ 2 & 4 & 6 \end{bmatrix}$

$|A| = 1(-6-0) - 2(24-0) + 3(16+2)$

$= -6 - 48 + 54$

$= 0$

minor of $1 = \begin{vmatrix} -1 & 0 \\ 4 & 6 \end{vmatrix}$

$= -6 - 0$

$= -6$

$\neq 0$

$\therefore \rho(A) = 2$

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⇒ Echelon or Normal or Conorical

The Echelon form or Normal form of a Matrix A is a row equivalent matrix of rank R that holds the following properties.

i. One or more element in the first R rows are non-zero and remaining rows are complete zero rows.

ii. In the non-zero R rows the leading element in each row is one

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \\ 2 & 4 & 6 \end{bmatrix}$

$R_3 - 2R_1 \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 - 4R_1 \begin{bmatrix} 1 & 2 & 3 \\ 0 & -9 & -12 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 / -9 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/3 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$

⇒ Reduce A to normal form & find its rank.

$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 0 \end{bmatrix}$

$R_2 - 3R_1 \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{bmatrix}$

$R_3 - R_1$ $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

$R_2 \Leftrightarrow R_3$ $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\frac{R_2}{3}$ $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 6 & 0 \end{bmatrix}$

This is in normal form

Since there are 2 non-zero rows

$\therefore \rho(A) = 2$

2) $B = \begin{bmatrix} 3 & 1 & 0 & 3 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

$R_1 \Leftrightarrow R_2$ $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 1 & 0 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

$R_3 - R_1$ $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 1 & 0 & 3 \\ 0 & 2 & -3 & 0 \end{bmatrix}$

$R_2 - 3R_1$ $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 4 & -6 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix}$

$R_2 - 2R_3$ $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\frac{R_2}{2}$ $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\underline{\rho(B) = 2}$

3) $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$R_2 - R_4$ $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$R_4 + R_1$ $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 4 \\ 1 & 0 & 2 & -1 \end{bmatrix}$

$R_2/4$
$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & -3 \\ 0 & 3 & 1 & 4 \\ 1 & 0 & 2 & -1 \end{bmatrix}$

$R_4 - R_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & -3 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$R_2 - R_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$R_3 - 3R_4$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$R_4 + R_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

$R_2/-1$
$R_4 - 2R_3$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$R_4/3$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

This is in normal form
$\therefore$ Here are 4 non-zero rows
$\therefore \rho(A) = 4$

Linear Equation

Consider a system of linear equations.

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = d_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = d_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = d_n$

m linear equation
n unknowns.

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$AX = D \quad -①$$

Here A is coefficient matrix
X is matrix of unknowns
and D is solution matrix

Equation ① is known as non-homogeneous equation.

If $D = 0$, then $AX = 0$ is homogeneous equation

This system of equations is consistent if the coefficient matrix A and augmented matrix $[A/D]$ have the same rank.

$$\text{Augmented matrix } [A/D] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & d_1 \\ a_{21} & a_{22} & & a_{2n} & d_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & d_m \end{array}\right]$$

Consistency of non-Homogenious system of equation.

1. The system $AX=D$ is said to be inconsistent if $\rho(A) \neq \rho(A/D)$ i.e. the system has no solution.

2. If $\rho(A)=\rho(A/D)=r$ the system is consistent & one of the following case is possible.

Case-I If $r=n$ then system has unique solution

Case-II If $r<n$ then system has infinitely many solution.

e.g.
Examine the consistency & solve the following equations.

1)
$x+y+z=3$
$2x-y+3z=1$
$4x+y+5z=2$
$3x-2y+z=4$

The equation can be written in matrix form as $AX=D$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 5 \\ 3 & -2 & 1 \end{bmatrix}_{4\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}_{4\times1}$$

Augmented matrix

$$[A/D] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 1 \\ 4 & 1 & 5 & 2 \\ 3 & -2 & 1 & 4 \end{array}\right]$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 4R_1 \\ R_4 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & -3 & 1 & -10 \\ 0 & -5 & -2 & -5 \end{array}\right]$$

$$\begin{array}{l} R_3 - R_2 \\ R_4 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & -5 \\ 0 & -2 & -3 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2 - R_3 \\ \dfrac{R_3}{-5} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & -3 & 0 \end{array}\right]$$

$$R_3 \Leftrightarrow R_4 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 - 2R_3 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 7 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 + 2R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{R_3}{11}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This is in normal form

$\rho(A/D) = 4$

$\rho(A) = 3$

$\rho(A/D) \neq \rho(A)$

System is inconsistent.

2) $x - y - z = 2$

$x + 2y + z = 2$

$4x - 7y - 5z = 2$

System of equation in matrix is

$$AX = D$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Augment matrix

$$[A/D] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & 5 & 2 \end{array}\right]$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 4R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & -3 & -1 & -6 \end{array}\right]$$

$$R_3 + R_2 \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

$$\frac{R_2}{3} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

This is in normal form

$\rho(A/D) = 3 = \rho(A) =$ no of unknowns.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

$x - y - z = 2$ — ①

$y + \frac{2}{3} z = 0$ — ②

$z = -6$ — ③

$y + \frac{2}{\cancel{3}_1}(\cancel{-6}^{2}) = 0$

$y - 4 = 0$

$y = 4$

$x - 4 + 6 = 2$

$x + 2 = 2$

$x = 0$

Linear dependence & independence of vectors :

A vector X which can be expressed as $X = C_1X_1 + C_2X_2 + \dots + C_nX_n$ where $C_1, C_2, \dots C_n$ are any numbers is said to be linear combination of vectors of sequence $X_1 X_2 \dots X_n$

The set of vector $X_1 X_2 \dots X_n$ are said to be linearly dependent (LD) if there exists n scalar $C_1, C_2, \dots C_n \neq 0$ not all zero such that $C_1X_1 + C_2X_2 \dots + C_nX_n = 0$, where 0 is a null vector.

The vector are linearly independent if vectors are all $C_1 = C_2 = \dots = C_n = 0$.

Examine for linear dependence.

1) $X_1 = [1 \ 2 \ 4]$
$X_2 = [2 \ -1 \ 3]$
$X_3 = [0 \ 1 \ 2]$
$X_4 = [-3 \ 7 \ 2]$

Take scalars
C_1, C_2, C_3, C_4, show that
$C_1X_1 + C_2X_2 + C_3X_3 + C_4X_4 = 0$

$$C_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + C_4 \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$C_1 + 2C_2 + 0C_3 - 3C_4 = 0$
$2C_1 - C_2 + C_3 + 7C_4 = 0$
$4C_1 + 3C_2 + 2C_3 + 2C_4 = 0$

In Matrix form

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}_{3\times4} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}_{4\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1}$$

$$[A/D] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 4 & 3 & 2 & 2 & 0 \end{array}\right]$$

$$R_3 - 2R_2 \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 0 & 5 & 0 & -12 & 0 \end{array}\right]$$

$$R_2 - 2R_1 \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 0 & -5 & 1 & 13 & 0 \\ 0 & 5 & 0 & -12 & 0 \end{array}\right]$$

$$R_3 + R_2 \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 0 & -5 & 1 & 13 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\frac{R_2}{-5} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 0 & 1 & -1/5 & -13/5 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

This is in normal form

$\rho(A/D) = 3 \quad \rho(A) = 3$

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$$C_1 + 2C_2 - 3C_4 = 0 \quad \text{—(1)}$$
$$C_2 - \tfrac{1}{5}C_3 - \tfrac{13}{5}C_4 = 0 \quad \text{—(2)}$$
$$C_3 + C_4 = 0 \quad \text{—(3)}$$
$$\therefore \; C_3 = C_4 \quad \text{—(3)}$$
$$\& \quad C_2 + \tfrac{1}{5}C_4 - \tfrac{13}{5}C_4 = 0$$
$$C_2 - \tfrac{12}{5}C_4 = 0$$
$$C_2 = \tfrac{12}{5}C_4$$
$$C_1 + \tfrac{24}{5}C_3 - 3C_4 = 0$$
$$C_1 + \tfrac{24-15}{5}C_4 = 0$$
$$C_1 = \frac{-9}{5}C_4$$

$\therefore$ for different values of C_4 we get infinite solution.

Check whether following set of vectors are linearly dependent or independent.
If they are LD find the relation between them.

1) (1 2 3) (3 5 0) (1 0 5)

Let $X_1 = (1\ 2\ 3)$
$X_2 = (3\ 5\ 0)$
$X_3 = (1\ 0\ 5)$

Let C_1, C_2, C_3 be numbers
show that $C_1X_1 + C_2X_2 + C_3X_3 = 0$
$C_1(1\ 2\ 3) + C_2(3\ 5\ 0) + C_3(1\ 0\ 5) = (0\ 0\ 0)$

$$C_1 + 3C_2 + C_3 = 0$$
$$2C_1 + 5C_2 + 0C_3 = 0$$
$$3C_1 + 0C_2 + 5C_3 = 0$$

In Matrix form

$AX = 0$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -2 \\ 0 & -9 & 2 \end{bmatrix}$$

$$-R_2 \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & -9 & 2 \end{bmatrix}$$

$$R_3 + 9R_2 \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 20 \end{bmatrix}$$

$$R_3/20 \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

This is normal form

$\rho(A) = 3 =$ no. of unknowns

System is consistent & has unique solution

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1+3C_2+C_3=0$$
$$C_2+2C_3=0$$
$$C_3=0$$
$$C_1=C_2=C_3=0$$

$\therefore X_1, X_2, X_3$ are linearly dependent.

2) $(2,4,6)^T, (3,-2,1)^T, (1,-6,-5)^T$

Let $X_1=(2,4,6)^T=\begin{bmatrix}2\\4\\6\end{bmatrix}$

$$X_2=\begin{bmatrix}3\\-2\\1\end{bmatrix}$$

$$X_3=\begin{bmatrix}1\\-6\\-5\end{bmatrix}$$

Let C_1, C_2, C_3 be numbers such that $C_1X_1+C_2X_2+C_3X_3$

$$\cancel{2C_1+3C}\ C_1\begin{bmatrix}2\\4\\6\end{bmatrix}+C_2\begin{bmatrix}3\\-2\\1\end{bmatrix}+C_3\begin{bmatrix}1\\-6\\-5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$2C_1+3C_2+C_3=0$$
$$4C_1-2C_2-6C_3=0$$
$$6C_1+C_2-5C_3=0$$

In Matrix form

$AX = 0$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -2 & -6 \\ 6 & 1 & 5 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1/2 \quad \begin{bmatrix} 1 & 3/2 & 1/2 \\ 4 & -2 & -6 \\ 6 & 1 & -5 \end{bmatrix}$$

$$\begin{matrix} R_2 - 4R_1 \\ R_3 - 6R_1 \end{matrix} \quad \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & -8 & -8 \\ 0 & -8 & -8 \end{bmatrix}$$

$$R_3 - R_2 \quad \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2/-8 \quad \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 <$ no. of unknowns

The system is inconsistent & has infinitely ~~ma~~ solution

$C_1 + \frac{3}{2}C_2 + \frac{1}{2}C_3 = 0$ — 1

$C_2 + C_3 = 0$ — 2

$C_2 = -C_3$

Let $C_3 = 1$

$C_2 = -1$

$$C_1 + \frac{3}{2}(-1) + \frac{1}{2}(1) = 1$$

$$C_1 - \frac{3}{2} + \frac{1}{2} = 0$$

$$C_1 - 1 = 0$$

$$C_1 = 1$$

3) $(1\ 3\ 5)^T$ $(4\ 8\ 11)^T$

$$C_1 = (1\ \ 3\ \ 5)^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$C_2 = (4\ \ 8\ \ 11)^T = \begin{bmatrix} 4 \\ 8 \\ 11 \end{bmatrix}$$

$$C_1 X_1 + C_2 X_2 = 0$$

$$C_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + C_2 \begin{bmatrix} 4 \\ 8 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1C_1 + 4C_2 = 0$$

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Determinant

A matrix is an array of Numbers corresponding to this array is called determinant and it exists only for square matrices.

Determinent place a very important role in solving system of linear equation.

Determinent

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a square matrix then the number $ad-bc$ is called determinent of A denoted by $|A|$ or $\det(A)$

e.g. $a = (1,2,3,4)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|A| = 4-6$$
$$= -2$$

$$A = [2]$$

⇒ Determinent of singleton matrix is a same element itself.

⇒ Determinent of 3X3 matrix is the sum of products of element of first row with their corresponding co-factor

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$|A| = a_1\begin{bmatrix} b_2 & c_2 \\ c_3 & c_3 \end{bmatrix} - b_1\begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} + c_1\begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 7 \\ -4 & 6 & 3 \\ 5 & -2 & 6 \end{bmatrix}$$

$$|A| = 1\begin{bmatrix} 6 & 3 \\ -2 & 6 \end{bmatrix} - 3\begin{bmatrix} -4 & 3 \\ 5 & 6 \end{bmatrix} + 7\begin{bmatrix} -4 & 6 \\ 5 & -2 \end{bmatrix}$$

$$= 1(36-(-6)) - 3(-20-15) + 7(8-30)$$

$$= 42 + 105 - 154$$

$$= 147 - 154$$

$$\therefore |A| = 7$$

* Properties of determinents

1. Property 1:

If rows and colums in a square matrix are interchange then the value of determinent remain unaltered

i.e.

$$\det(A) = \det(A^T)$$

2. Property 2:

The determinent of square matrix changes sign when any 2 rows or columns are interchange

e.g.

$$B = \begin{bmatrix} 1 & 3 & 7 \\ -4 & 6 & 3 \\ 5 & -2 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & 6 & 3 \\ 1 & 3 & 7 \\ 5 & -2 & 6 \end{bmatrix}$$

$$= -4\begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix} - 6\begin{bmatrix} 1 & 7 \\ 5 & 6 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix}$$

$= -5$

property 3:

If a row or column of a square matrix is identical then the value of determinent is 0.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$|A| = (12-10) - 2(9-5) + 3(6-4)$

$= 2 - 8 + 6$

$= 0$

property 4:

The determinent of square with all elements ~~zero~~ of row or column is zero.

If row/column with all element '0' then determinant is 0.

Property 5:

If all element of row or column is multiply by number is equal to k. K, then the determinent of resultant matrix times of original matrix.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$|A| = 4 - 6$
$= -2$

$B = A = 3 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$
$= \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$
$= 12 - 18$
$= -6$

Property 6:
If the element of a row or column of a square matrix are k-times the element of another row or column then its determenent value is zero.

i.e.
$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$
$= 3 \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$|A| = 0$
$|A| = 0$

Property-7:
If each element in a row or column is sum of 2 teams then its determinant can be expressed as sum of 2 determinants of 2 square matrices of same order

Property : 8

If element of a row or a column of a square matrix are added or subtracted with k times, corresponding element of any other row, then the value of determinant of resulting matrix is unaltered.

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$B = \begin{bmatrix} a_1 \pm ka_2 & b_1 \pm kb_2 & c_1 \pm kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$|B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 & kb_2 & kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$|A| + k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= A + k(0)$

$|B| = |A|$

Property 9 :

If all the element on one side or on both side of diagonal are zero then the value of determinent is equal to product of element in diagonal.

Let $A = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $|A| = a_1(b_2c_3)$

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$$B = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}$$

$$|B| = a_1(b_2c_3 - 0)$$
$$= a_1b_2c_3$$

$$C = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

$$|C| = a_1b_2c_3$$

Property 10:

The determinant of identity matrix of any order is equal to 1.

i.e. $|I_1| = |I_2| = |I_3| \ldots = 1$

* Using the full property evaluate.

1) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$R_2 - R_1 \quad R_3 - R_1$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix}$$

Taking 2 common from R_3

$$= 2\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & a & a+b+c \\ 1 & b & c+a+b \\ 1 & c & a+b+c \end{vmatrix}$$

Taking a+b+c common from C_3

$$=(a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$$= 0 \quad [\because C_1 = C_3]$$

3) $$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$$

$C_2 - 4C_1$
$C_3 - 9C_1$
$C_4 - 16C_1$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & -7 & -20 & -39 \\ 9 & -20 & -56 & -108 \\ 16 & -39 & -108 & -207 \end{vmatrix}$$

Expanding along first row

$$= 1\begin{vmatrix} -7 & -20 & -39 \\ -20 & -56 & -108 \\ \cancel{16}-39 & -108 & -207 \end{vmatrix}$$

taking '-1' common from ~~an~~ all 3 rows.

$$= (-1)^3 \begin{vmatrix} 7 & 20 & 39 \\ 20 & 56 & 108 \\ 39 & 108 & 207 \end{vmatrix}$$

$C_2 - 3C_1$

$$= (-1) \begin{vmatrix} 7 & -1 & 39 \\ 20 & -4 & 108 \\ 39 & -9 & 207 \end{vmatrix}$$

~~$-1(7(-72)) - 72$~~
~~-20~~
~~$+39(-29))$~~

$C_3 - 2C_2$, $C_2 - 3C_1$

$$(-1) \begin{vmatrix} 7 & -1 & -1 \\ 20 & -4 & -4 \\ 39 & -9 & -9 \end{vmatrix}$$

$= 0$ $[\because C_2 = C_3]$

* Show that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

$$L.H.S = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

$C_1+C_2+C_3$

$$\begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

$= -(-504+1440-936)$

$= -(0)$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

taking $a+b+c$ common from R_2 & R_3

$$2(a+b+c)^3\begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 2(a+b+c)^3[1(1-0)-a(0-0)+b(0-0)]$

$= 2(a+b+c)^3$

$= RHS$

* Q $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

R_2-R_1, R_3-R_1

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

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$$=(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$=(b-a)(c-a)[1(c+a)-(b+c))-a(0-0)+a^2(0-0)]$$

$$=(b-a)(c-a)(c-b)$$
$$=-1(a-b)(a-c)(b-c)$$
$$=(a-b)(b-c)(c-a)$$
$$=RHS$$

* Solving Non Homogenious linear equation

* Consider Non Homogenious linear equations

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}$$

* Solve by Cramer's Rule

$x+y+z=9$
$2x+5y+7z=52$
$2x+y-z=0$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(-5-7) - 1(-2-14) + 1(2-10)$
$= -12 + 16 - 8$
$= -4$

$$D_x = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 9(-5-7) - 1(-52-0) + 1(52-0)$
$= -108 + 52 + 52$
$= -4$

$$D_y = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$$

$= 1(-52-0) - 9(-2-14) + 1(0-104)$
$= -52 + 144 - 104$
$= -12$

$$D_z = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 1(0-52) - 1(0-10) + 9(2-10)$$

$$= -52 + 104 - 72$$

$$= -20$$

$$x = \frac{D_x}{D} = \frac{-4}{-4} = 1$$

$$y = \frac{D_y}{D} = \frac{-12}{-4} = 3$$

$$z = \frac{D_z}{\cancel{D_z}\,D} = \frac{-20}{-4} = 5$$

UNIT-2

Characteristics roots & vectors

Consider linear transformation

$Y = AX$ — ①

This transforms from vector X to vector Y when

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Consider another transformation

$Y = \lambda X$ — ②

i.e. vector X is transformed into vector Y which has some direction as X but different magnitude.

$\therefore AX = \lambda X$ [from ① & ②]

$AX - \lambda X = 0$

$[A - \lambda I] X = 0$

where, I is identity matrix of order n then,

$|A - \lambda I| = 0$ is called characteristics equation of A.

The roots of this equation $\lambda_1, \lambda_2, \ldots \lambda_n$ are known as characteristic values or eigen values or latent roots of matrix A.

Corresponding to n characteristics roots we get n values of column vector X obtained by solving $AX - \lambda X$ this column vectors are known as characteristics vectors or Eigen vectors of matrix A.

* Find Eigen values & Eigen vectors of the following

1) $A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}$

Consider characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -3 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(5-\lambda) - 12 = 0$$

$$5 - 1\lambda - 5\lambda + \lambda^2 - 12 = 0$$

$$\therefore \lambda^2 - 6\lambda - 7 = 0$$

$$\lambda^2 - 7\lambda + \lambda - 7 = 0$$

$$(\lambda - 7)(\lambda + 1) = 0$$

$$\lambda = 7, -1$$

7, -1 are ~~eigi~~ eigen values of A.

For $\lambda_1 = 7$ consider $[A - \lambda_1 I] X_1 = 0$

$$[A - \lambda I] X_1 = 0$$

$$\begin{bmatrix} 1-7 & -3 \\ -4 & 5-7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & -3 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{R_1}{-3}, \frac{R_2}{2} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$2x_1 + x_2 = 0$

$x_2 = -2x_1$

Take $x_1 = k, k \in R$

$x_2 = -2k$

$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -2k \end{bmatrix}$

For $\lambda_2 = -1$ consider

$[A - \lambda_1 I] X_1 = 0$

$\begin{bmatrix} 1+1 & -3 \\ -4 & 5+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x_1 - 3x_2 = 0$

$-4x_1 + 6x_2 = 0$

$2x - 3x_2 = 0$ (Divide by -2)

$2x_1 - 3x_2 = 0$

$x_1 = \frac{3}{2} x_2$

Take $x_2 = 2k$,

$x_1 = \frac{3}{\cancel{2}} \cancel{2}k =$

$\therefore x_1 = 3k$

$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3k \\ 2k \end{bmatrix}$

$\therefore X_1$ & X_2 are required eigen vectors.

2) $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$

Consider characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 5 & 7-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(7-\lambda) - 15 = 0$$

$$7 - 1\lambda - 7\lambda + \lambda^2 - 15 = 0$$

$$\lambda^2 - 8\lambda - 8 = 0$$

$$\frac{-8 \pm \sqrt{96}}{2} = \lambda$$

$\frac{-8 \pm \sqrt{96}}{2}$ are eigen values of A.

3) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

Consider characteristics equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - 1\lambda - 3\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda^2 - 5\lambda + \lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\lambda = 5, \lambda = -1$$

5, -1 are eigen values of A

For $\lambda_1 = 5, \lambda_2 = -1$

for $\lambda_1 = 5$ consider,

$$[A - \lambda_1 I] X_1 = 0$$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + 2x_1 = 0$$
$$4x_1 - 2x_2 = 0$$

$$-4x_1 + 2x_2 = 0$$
$$2x_2 = 4x_1$$
$$x_2 = \frac{\cancel{4}^{2} x_1}{\cancel{2}}$$
$$x_2 = 2x_1$$

Take $x_1 = k$

$$x_2 = 2k$$
$$X_1 = \begin{bmatrix} k \\ 2k \end{bmatrix}$$

For $\lambda = -1$ consider,

$$[A - \lambda I] X_1 = 0$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{R_1}{2}, \frac{R_2}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

Take $x_1 = k$,
$x_2 = -k$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix}$$

4) $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Consider characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix}$$

$\lambda^3 -$ (sum of diagonal element of A) $\lambda^2 +$ (sum of minors of diagonal elements of A) $- \lambda - |A| = 0$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$
$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$
$$\lambda = 1, 2, 3$$

For $\lambda_1 = 1$

$$[A - \lambda_1 I] X_1 = 0$$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}$$

$$R_3 - R_2 \begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ -1 & 0 & 0 \end{bmatrix}$$

$$R_1 - 2R_2 \begin{bmatrix} -1 & 0 & 2 \\ 4 & -4 & -2 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_3 = 0$$
$$4x_1 - 4x_2 - 2x_3 = 0$$
$$-x_1 + 2x_3 = 0$$

from ① $x_1 = 2x_3$

put $x_3 = k$

$\therefore x_1 = 2k$

$$8k - 4x_2 - 2k = 0$$
$$6k = 4x_2$$
$$\therefore x_2 = 3k/2$$

for $\lambda_2 = 2$

$$[A - \lambda_2 I] X_1 = 0$$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$

$$\begin{vmatrix} 0 & 0 & 0 \\ 4 & -5 & 2 \\ 3 & -4 & -1 \end{vmatrix}$$

$\cancel{R_2} \cdot R_2 - R_3$

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ 3 & -4 & -1 \end{vmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{vmatrix} 3 & -4 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$3x_1 - 4x_2 - x_3 = 0$$

$$x_1 - (x_2 - x_3) = 0$$

$$x_1 - x_3 = x_2$$

$$x_1 = x_2 + x_3$$

$$3(x_2 + x_3) - 4x_2 - x_3 = 0$$

$$3x_2 + 3x_3 - 4x_2 - x_3 = 0$$

$$-1x_2 - 2x_3 = 0$$

$$x_1 = 4x_2 + kx_3 \quad -(1)$$

$$-x_2 = 2x_3$$

$$-x_2 = kx_3 \quad -(2)$$

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$\lambda = 3$

$$\begin{vmatrix} 8-3 & -8 & -2 \\ 4 & -3-3 & -2 \\ 3 & -4 & 1-3 \end{vmatrix}$$

Cayley Hamilton Theorem

Statement:

Every square matrix satisfies its own characteristic equation.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

Characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) = 0$$

$$6 - 3\lambda - 2\lambda + \lambda^2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \quad \text{— (1)}$$

Put $\lambda = A$ in (1)

To verify:

$$A^2 - 5A + 6I = 0 \quad \text{— (2)}$$

$$A^2 = A.A$$

$$= \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 \\ -5 & 4 \end{bmatrix}$$

$$5A = 5 \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \qquad 6I = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 0 \\ -5 & 10 \end{bmatrix}$$

Consider LHS of ②,

$$A^2 - 5A = 6I =$$

$$\begin{bmatrix} 9 & 0 \\ -5 & 4 \end{bmatrix} - \begin{bmatrix} 15 & 0 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence Cayley Hamilton theorem is verified.

Q $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ verify Cayley Hamilton theorem & hence find A^{-1}

Char equation.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) + 1 = 0$$

$$6 - 2\lambda - 3\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 5\lambda + 7 = 0 \quad \text{—①}$$

put $A = \lambda$ in ①

To verify:

$$A^2 - 5A + 7I = 0 \quad \text{—②}$$

$A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$$

$$5A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \times 5 = \begin{bmatrix} 10 & -5 \\ 5 & 15 \end{bmatrix}$$

$$7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Consider LHS of (2)

$$A^2 - 5A + 7I = \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix} - \begin{bmatrix} 10 & -5 \\ 5 & 15 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Hence cayley Hamiton theorem.

To find A^{-1}:

Consider $A^2 - 5A + 7I = 0$

Post multiply by A^{-1},

$$A^2 \cdot A^{-1} - 5A \cdot A^{-1} + 7I \cdot A^{-1} = 0 \cdot A^{-1}$$

$$A \cdot A \cdot A^{-1} - 5(A \cdot A^{-1}) + 7(I \cdot A^{-1}) = 0$$

$$AI - 5I + 7I \cdot A^{-1} = 0 \qquad (\because A A^{-1} = I)$$

$$7A^{-1} = 5I - A$$

$$A^{-1} = \frac{1}{7}(5I - A)$$

Q Verify Cayley Hamiton for $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

Put $\lambda = A$ in ①,

To verify:

$$A^3 - 5A^2 + 9A - I = 0 \quad \text{—②}$$

$$A^2 = A.A$$

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

* multiply 1r 1c, & 1r 2c, 1r 3c, 2r 1c, 3r 1c, 2r 2c, 3r 2c, 2r 3c, 3r 3c

$$= \begin{bmatrix} 1-2+0 & 2+6+4 & -2+0-2 \\ -1-3+0 & -2+9+0 & 2+0+0 \\ 0+2+0 & 0-6-2 & 0+0+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2 . A$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-12+0 & -2+36+8 & 2+0-4 \\ -4-7+0 & -8+21-4 & 8+0+2 \\ 2+8+0 & 4-24-2 & -4+0+1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -3 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

$$5A^2 = 5 \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$5A^2 = \begin{bmatrix} -5 & 60 & -20 \\ -20 & 35 & 10 \\ 10 & -40 & 5 \end{bmatrix}$$

$$9A = 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$9A = \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix}$$

Consider LHS of ②

$A^3 - 5A^2 - 9A - I =$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - \begin{bmatrix} -5 & 60 & -20 \\ -20 & 35 & 10 \\ 10 & -40 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix}$$

$$= \qquad - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Properties of Eigen values.

1. Any square matrix A & its transpose A^T have same Eigen values.
2. The Eigen values of a triangular matrix are just diagonal elements of the matrix.
3. The sum of Eigen values of a Matrix is the sum of elements of principle diagonal.
4. The product of Eigen values of a matrix A is equal to its determinents.
5. If λ is Eigen value of a Matrix 'A', then λ^2 is Eigen value of A^2, λ^3 is Eigen value of A^3.
   Also λ^n is Eigen value of A^n,
   $1/\lambda$ is eigen value of A^{-1}

Reduction of a Matrix to a diagonal matrix.

If A & B are two square matrices of order n then B is said to be ~~a sing~~ similar to A if there exists invertible matrix P, such that $B = P^{-1}AP$

The process of reduction of a matrix to a diagonal matrix whose diagonal elements are its characteristic values is known as diagonalisation.

A matrix 'A' of order n is said to be diagonalizable if there exist non-singular matrix P such that $P^{-1}AP = D$, where P is formed by characteristic vectors of matrix 'A' ~~is~~ and is known as Modal matrix.
D is diagonal matrix where $\lambda_1, \lambda_2, \lambda_3$ are Eigen values or characteristics values of A.

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Q Diagonalize matrix $A=\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}$ & hence find A^4

Characteristic equation is

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 3\\ 3 & 1-\lambda\end{vmatrix}=0$$

$$(1-\lambda)(1-\lambda)-9=0$$

$$1-2\lambda+\lambda^2-9=0$$

$$\lambda^2-2\lambda-8=0$$

$$\lambda^2-4\lambda+2\lambda-8=0$$

$$(\lambda-4)(\lambda+2)=0$$

$$\lambda=4,-2$$

for $\lambda_1=4$

$$[A-\lambda_1 I]X_1=0$$

$$\begin{bmatrix}-3 & 3\\ 3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$3x_1+3x_2=0$$

Divide throughout by 3

$$x_1+x_2=0$$

$$x_2=-x_1$$

Take $x_1=1$

$$\therefore x_2=-1$$

$$X_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$P = [X_1 X_2]$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \cdot adj\ P.$$

$$P^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Consider $P^{-1}AP$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 4 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 8 & 0 \\ 0 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$
$$= D$$

Hence A is diagonalize

Consider

$$P^{-1}AP = D$$

Premultiply by P & post multiply by P^{-1}

$$(PP^{-1}) A (PP^{-1}) = PDP^{-1}$$
$$IAI = PDP^{-1}$$
$$A = PDP^{-1}$$

$$A^n = PD^nP^{-1}$$

$$A^4 = PD^4P^{-1}$$

$$A^4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 16 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 256 & 16 \\ 256 & -16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 272 & 240 \\ 240 & 272 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 136 & 120 \\ 120 & 136 \end{bmatrix}$$

Q) Diagonalise A and hence find A^3, for $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$$

Characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 5 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - 10 = 0$$

$$4 - 4\lambda - \lambda + \lambda^2 - 10 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$\lambda^2 - 6\lambda + \lambda - 6 = 0$$

$$(\lambda - 6)(\lambda + 1) = 0$$

$$\lambda = 6, -1$$

for $\lambda_1 = 6$

$$[A - \lambda_1 I] X_1 = 0$$

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$-2x_1 + 2x_2 = 0$ — (1)

$5x_1 \cancel{+} -5x_2 = 0$ — (2)

$\frac{(1)}{2}, \frac{(2)}{5}$

$$\left.\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_2$$

Let $x_1 = 1 = x_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -1$

$$[A - \lambda_2 I] X_2 = 0$$

$$\begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5x_1 + 2x_2 = 0$$

$$5x_1 = -2x_2$$

$$x_1 = -\frac{2}{5} x_2$$

Let $x_2 = 5$

$$x_1 = -\frac{2}{\cancel{5}} \times \cancel{5}$$

$$x_1 = -2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$P = [X_1 \ X_2]$$

$$P = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$$

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$$P^{-1} = \frac{1}{5-(-2)} \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{7} \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$$

Consider

$$P^{-1}AP$$

$$= \frac{1}{7} \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 30 & 12 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 42 & 0 \\ 0 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} = D$$

Hence A is diagonalize

Consider

$$P^{-1}AP = D$$

Premultiply by P & post multiply by P^{-1}

$$(PP^{-1}) A (PP^{-1}) = PDP^{-1}$$

$$IAI = PDP^{-1}$$

$$A = PDP^{-1}$$

$$A^n = PD^nP^{-1}$$

$$A^3 = PD^3P^{-1}$$

$$A^3 = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 216 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 216 & 2 \\ 216 & -5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 1078 & 434 \\ 1085 & 427 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 154 & 62 \\ 155 & 61 \end{bmatrix}$$

Q Verify Cayley Hamilton Theorem for $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ and hence find $A^6 - 2A^5 + 3A^4 - 6A^3 + 4A^2 - 3A + 2I$

Characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(1-\lambda) - 8 = 0$$

$$3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0 \text{ —— } (1)$$

Put $A = \lambda$

To verify:

$$A^2 - 4A - 5I = 0 \text{ —— } (2)$$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 17 & 8 \\ 16 & 9 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 8 \\ 16 & 4 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Consider LHS

$$A^2 - 4A - 5I = 0$$

$$\begin{bmatrix} 17 & 8 \\ 16 & 9 \end{bmatrix} - \begin{bmatrix} 12 & 8 \\ 16 & 4 \end{bmatrix} - \begin{bmatrix} 5 & \cancel{3}0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence Cayley Hamilton theorem

$$A^4 + 2A^3 + 16A^2 + 68A + 356I$$

$$A^2 - 4A - 5I \,\big)\, A^6 - 2A^5 + 3A^4 - 6A^3 + 4A^2 - 3A + 2I$$

$$(-)\ \cancel{A^6} \mp 4A^5 \mp 5A^4$$

$$\cancel{2A^5} + 8A^4 - 6A^3$$

$$(-)\ \cancel{2A^5} \mp 8A^4 \mp 10A^3$$

$$\cancel{16A^4} + 4A^3 + 4A^2$$

$$(-)\ \cancel{16A^4} \mp 64A^3 \mp 80A^2$$

$$\cancel{68A^3} + 84A^2 - 3A$$

$$(-)\ \cancel{68A^3} \mp 272A^2 \mp 80A^2$$

$$\cancel{356A^2} + 337A + 2I$$

$$(-)\ \cancel{356A^2} \mp 1424A \mp 1780I$$

$$1761A + 1782I$$

$$R = 176\cancel{3}1A + 1782I$$

$$A^6 - 2A^5 + 3A^4 - 6A^3 + 4A^2 - 3A + 2I =$$

$$A^2 - 4A - 5I \times A^4 + 2A^3 + 16A^2 + 62A + 336I$$

$$+ 1761A + 1782I$$

$$0 + (1761A + 1782I)$$

$$1761 \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} + 1782 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5283 & 3362 \\ 7044 & 1761 \end{bmatrix} + \begin{bmatrix} 1782 & 0 \\ 0 & 1782 \end{bmatrix}$$

$$\begin{bmatrix} 7065 & 3522 \\ 7044 & 3543 \end{bmatrix}$$

2. Counting Principle

1. Product rule:

If a Procedure can be broken down into a sequence of task and if there are $n_1, n_2, n_3 \ldots n_k$ ways to do this k task then there are $n_1 \times n_2 \times n_3 \ldots \times n_k$ ways, to complete entire procedure.

e.g. Q There are 55 students in tyit and 34 students in tycs in a college how many ways are there two select two representative so that one of them is from tyit and other from tycs.

sol: One representative can be selected in 1870 ways

2 Sum rule:

If a procedure can ~~choose~~ be compled by performing any of k task, each of this task ~~can be~~ is done in $n_1, n_2, \ldots n_k$ ways. Then the procedure can be completed in $n_1 + n_2 + n_3 \ldots + n_k$ ways.

Q An MBA student can choose a project from one of the three categories HR, Finance, Marketing. How many possible project are there to choose from?

sol: The students can select the project from any one of the three categories HR, finance or Marketing.

$\therefore$ By sum rule.

No. of ways to do so is $37 + 52 + 47 = 136$ ways

* Problems

Q How many different 3 letter intials can people also find how many of them have no repetation in their initials And how many of them begin with the alphabet A.

sol→ $\overline{A-Z}$ $\overline{A-Z}$ $\overline{A-Z}$
26 26 26

In first place a letter can be selected in 26 ways. in second and third place also in 26 ways

~~= 26 × 26 × 26~~

∴ By Product rule.

$= 26 \times 26 \times 26$

$= 17576$ ways

If the repetation of letters are not allowed in first place a letter can be selected in 26 ways, second place 25 way & third place 24 ways

∴ By Product rule.

$\overline{26} \times \overline{25} \times \overline{24}$

$= 1560$ ways

If the first initial is A then it can be selected in one way, 2^{nd} place 26 ways & 3^{rd} place 26 ways

∴ By Product rule

$= \overline{1} \times \overline{26} \times \overline{26}$

$= 676$ ways

Q How many bit strings are there of length 8 also find how many of them begin with 1. How many of them end with 2-bits 00.

→ A bit string of length 8 can be constructed in $2^8 = 256$

∵ Every bit can be choosen in two ways either 0 or 1.

_ _ _ _ _ _ _ _

∴ By Product rule

no. of strings $= 2^8$

$= 256$ ways

1 _ _ _ _ _ _ _

first bit is choosen in one way, whereas each of 7-bit can be choosen in 2 ways

$\therefore$ By Product rule

Total no. of ways $= 1 \times 2^7$

$= 128$ ways

_ _ _ _ _ _ 0 0

Last two bit are choosen in one way whereas remaining each of 6 bits can be choosen in two ways

$\therefore$ By Product rule

Total no. of ways $= 2^6 \times 1$

$= 64$ ways

Q How many strings of three ASCII characters contain the character # atleast once (There are total 128 characters)

→ There are 128 different ASCII characters

$\therefore$ Total no. of strings that can be formed of length $3 = 128^3$

_ _ _

128 128 128

Out of this no. of strings without # character $= 127^3$

_ _ _

127 127 127

$\therefore$ No. of string containing atleast 1 # char $= 128^3 - 127^3$

$= 2097152 - 2048383$

$= 48,769$ ways

Q. How many strings of three decimal digits.
i. do not contain same digit 3 times
ii. begin with odd digit
iii. have exactly two digits that are 7.

→ A string of 3 digit can be formed using (0-9)
total no. of string possible $= 10^3 = 1000$

i. No. of string which do not contain same digit 3 times $= 1000 - 10$
$= 990$ ways

ii. There are 5 odd digits 1, 3, 5, 7, 9
∴ 1st ~~digit~~ place in 10 ways each
∴ By Product rule
No. of string that begin with odd digit $= 5 \times 10 \times 10$
$= 500$

iii. 7 7 _
7 _ 7
_ 7 7

The blank place can be filled in by any of 9 digits excluding 7
1st case can be done in 9, second case in 9 ways and 3rd ways in 9 way
∴ By sum rule.
Total no. of string having two no. 7 $= 9 + 9 + 9$
$= 27$

Pigeonhole Principle

⇒ If n pigeon are assigned to m pigeonhole and $m<n$ then atleast one pigeonhole contains two or more pigeons.

1) Show that if 7 members from 1-12 are choosen then 2 of them will add upto 13

⇒ Construct 6 different set each containing two number that add upto 13

$A_1 = \{1, 12\}$ $A_4 = \{4, 9\}$
$A_2 = \{2, 11\}$ $A_5 = \{5, 8\}$
$A_3 = \{3, 10\}$ $A_6 = \{6, 7\}$

each of the 7 numbers choosen must belong to one of this 6 sets let this 7 number be pigeon and 6 sets be pigeonholes.

$\therefore n = 7, m = 6$ $m < n$

$\therefore$ By Pigeon hole two of the choosen numbers belong to the same set this number add upto 13.