**NE 250, F17**

**September 22, 2017**

Recall the one-group diffusion equation:

$$\frac{1}{v_1}\frac{\partial \phi_1(\vec{r},t)}{\partial t} = S_1(\vec{r},t) - \Sigma_{a,1}(\vec{r})\phi_1(\vec{r},t) + \nabla \cdot [D_1(\vec{r})\nabla\phi_1(\vec{r},t)]$$

Now, *assume that space and energy dependence of the flux can be separated*:

$$\phi(\vec{r},E,t) = \phi(\vec{r},t)\xi(E), \text{ where } \xi(E) \text{ is the neutron spectrum and } \int_0^\infty dE\, \psi(E) = 1.$$

Focusing on the *spatial dependence* of the flux, we'll assume a homogeneous, steady-state, one-group system.

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_s(\vec{r})\phi(\vec{r},t) + \nabla \cdot [D(\vec{r})\nabla\phi(\vec{r},t)]$$

Steady-state: $\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = 0$

Homogeneous: no material dependence on position; $\Sigma_a(\vec{r}) \to \Sigma_a$, $D(\vec{r}) \to D$

This gives us

$$\nabla^2\phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \text{ where } L = \sqrt{\frac{D}{\Sigma_a}} = \text{diffusion length.}$$

Now, consider a plane source of strength $S_0$ in an infinitely absorbing medium.

$$\phi(\vec{r}) = \phi(x)$$
$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S_0\delta(x)}{D}$$

The boundary conditions we'd like to enforce are that the source is zero other than at the plane, so within $\epsilon \to 0$:

$$-D\frac{d\phi}{dx}|_{+\epsilon} + D\frac{d\phi}{dx}|_{-\epsilon} = J_x(0^+) - J_x(0^-) = S_0\ .$$

We can use this to examine two sets of equations and boundary conditions.

(1) For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions:

$$\lim_{x \to 0^+} \vec{J}(x) = \frac{S_0}{2},$$
$$\lim_{x \to +\infty} |\phi(x)| < \infty,\ \phi(x) \geq 0$$

(2) For $x < 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.
Boundary conditions:

$$\lim_{x \to 0^-} \vec{J}(x) = -\frac{S_0}{2}\ ,$$
$$\lim_{x \to -\infty} |\phi(x)| < \infty, \phi(x) \geq 0$$

With the above equations and boundary conditions, we have a general solution form of (which you'll recall from your differential equations class):

$$\phi(x) = c_1 e^{-|x|/L} + c_2 e^{|x|/L}.$$

From the finite flux condition, $c_2 = 0$. Then, $\phi(x) = c_1 e^{-|x|/L}$.

$$\lim_{x \to 0^+} \vec{J}(x) = \lim_{x \to 0^+} \left( \frac{D}{L} c_1 e^{-|x|/L} \right) = \frac{D}{L} c_1 = \frac{S_0}{2}$$
$$c_1 = \frac{S_0 L}{2D}$$
$$\phi(x) = \boxed{\frac{S_0 L}{2D} e^{-|x|/L}}$$

The neutron flux falls off exponentially as one moves away from the source plane with a characteristic decay length of $L$. This holds fairly well as long as we're not too near the source and the medium is not too strongly absorbing.

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Now, consider an infinite plane centered in a slab of finite thickness $a$, surrounded by a vacuum.

(1) For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions:

$$\lim_{x \to 0^+} \vec{J}(x) = \frac{S_0}{2} ,$$
$$\phi(\tfrac{\tilde{a}}{2}) = 0$$

(2) For $x < 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.
Boundary conditions:

$$\lim_{x \to 0^-} \vec{J}(x) = -\frac{S_0}{2} ,$$
$$\phi(-\tfrac{\tilde{a}}{2}) = 0$$

Now, we can again write the equations as

$$\phi(x) = c_1 e^{-|x|/L} + c_2 e^{|x|/L}$$

Or, with equal validity (which we could have also done before)

$$\phi(x) = c_1 \cosh(\tfrac{|x|}{L}) + c_2 \sinh(\tfrac{|x|}{L})$$

You can use the boundary conditions to reach the full solution.

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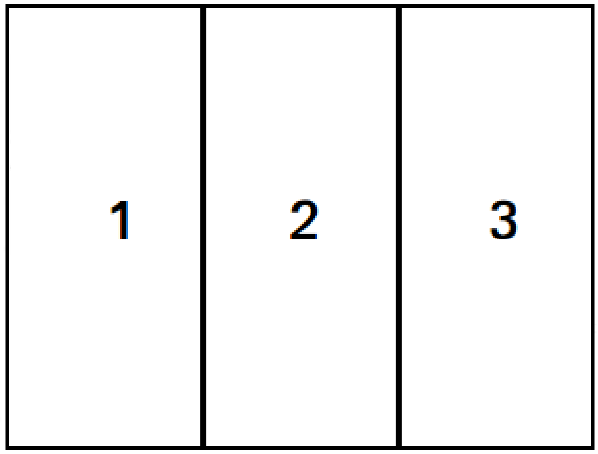
Now, consider a uniformly distributed source of strength $S_0 \frac{n}{cm^3 s}$ within a finite slab of width $a$ with vacuum boundaries.

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S_0}{D}$$
$$\phi(x) = c_1 e^{-|x|/L} + c_2 e^{|x|/L} + \frac{S_0 L^2}{D}$$

Boundary conditions: $\phi(\pm\frac{\tilde{a}}{2}) = 0$

Alternative boundary condition: $\left.\frac{d\phi(x)}{dx}\right|_{x=0} = 0$ (symmetry)

Now, consider a uniform source in a reflected slab, where region 2 contains the source and 1 and 3 are reflectors (high scattering compared to absorption). The center of region 2 is located at $x = 0$ and the edges of region two are at $x = \pm\frac{a}{2}$.




We start by using an equation containing the source in region 2, designating the solution $\phi_2$:

$$\frac{d^2\phi_2(x)}{dx_2^2} - \frac{1}{L_2^2}\phi_2(x) = -\frac{S_0}{D_2}, \quad -\tfrac{a}{2} < x < \tfrac{a}{2}$$

We need a collection of boundary conditions (where subscript indicates region number) that relate region 2 to the other regions:

$$\begin{aligned} \phi_2(\tfrac{a}{2}) &= \phi_3(\tfrac{a}{2}) \\ \vec{J}_2(\tfrac{a}{2}) &= \vec{J}_3(\tfrac{a}{2}) \\ \phi_1(-\tfrac{a}{2}) &= \phi_2(-\tfrac{a}{2}) \\ \vec{J}_1(-\tfrac{a}{2}) &= \vec{J}_2(-\tfrac{a}{2}) \end{aligned}$$

We also need equations for regions 1 and 3, which do not have sources:

$$\begin{aligned} \frac{d^2\phi_1(x)}{dx_1^2} - \frac{1}{L_1^2}\phi_1(x) = 0, \quad -\infty < x < -\tfrac{a}{2} \\ \frac{d^2\phi_3(x)}{dx_3^2} - \frac{1}{L_3^2}\phi_3(x) = 0, \quad \tfrac{a}{2} < x < \infty \end{aligned}$$

These regions have boundary conditions as well: $\lim_{x\to-\infty} |\phi_1(x)| < \infty, \quad \lim_{x\to\infty} |\phi_3(x)| < \infty$

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What if we want to solve a **generic diffusion problem**?

$$D\nabla^2\phi(\vec{r}) - \Sigma_a\phi(\vec{r}) = -S(\vec{r})$$

Working in 1D:

$$M\phi(x) = f(x)$$

where $M$ is non-homogenous differential operator of order *n* on a differential equation:

$$a_0(x)\phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \ldots + a_n(x)\phi(x) = f(x)$$

for example

$$M = \frac{d^2}{dx^2} - \frac{1}{L}$$

One solution method is *variation of constants / parameters*, where we break the solution into homogeneous and particular portions:

$$M\phi_{homog}(x) = 0$$
$$M\phi_{part} = rhs$$
$$\phi(x) = \phi_{homog}(x) + \phi_{part}(x)$$

We solve the homogeneous part to yield

$$\phi_{i,h}(x),\ i = 1, \ldots, n \ \text{ independent solutions}$$

and we also know

$$\phi_{part}(x) = \sum_{i=1}^{n} c_i \phi_{i,h}(x)$$

where $c_i = c_i(x)$ and are differentiable functions that satisfy the conditions

$$\sum_{i=1}^{n} c_i'(x)\phi_{i,h}^{(j)}(x) = 0, \quad j = 0, \ldots, n-1 \,.$$

If

$$\sum_{i=1}^{n} c_i' \phi_{i,h}(x) = 0$$
$$\sum_{i=1}^{n} c_i' \phi_{i,h}^{(1)}(x) = 0$$
$$\vdots$$
$$\sum_{i=1}^{n} c_i' \phi_{i,h}^{(n-2)}(x) = 0$$

then

$$\sum_{i=1}^{n} c_i' \phi_{i,h}^{(n-1)}(x) = \frac{f(x)}{a_0(x)}$$

which indicates

$$c_i(x) = \int dx\ c_i'(x)\ .$$

For n = 2:

$$\phi_1(x)c_1'(x) + \phi_2(x)c_2'(x) = 0$$
$$\phi_1^{(1)}(x)c_1'(x) + \phi_2^{(1)}(x)c_2'(x) = \frac{f(x)}{a_0(x)}$$
$$\phi_{part} = c_1(x)\phi_1(x) + c_2(x)\phi_2(x)$$
$$a_0(x)\phi^{(2)}(x) + a_1(x)\phi^{(1)}(x) + a_2(x)\phi(x) = 0$$
$$\phi^{(1)}(x) = c_1'(x)\phi_1(x) + c_1(x)\phi_1'(x) + c_2'(x)\phi_2(x) + c_2(x)\phi_2'(x)$$

$$c_i(x) = \int dx c_i'(x) = A_i + \psi_i(x)$$

$$\phi(x) = \sum_{i=1}^{n} A_i\phi_i(x) + \sum_{i=1}^{n} \psi_i(x)\phi_i(x)$$

For n = 2: $\phi(x) = A_1\phi_1(x) + A_2\phi_2(x) + \psi_1(x)\phi_1(x) + \psi_2(x)\phi_2(x)$