NE 250, F17

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Duderstadt Chp. 5.II.A

Recall the one-group diffusion equation:

$$\frac{1}{v_1}\frac{\partial \phi_1(\vec{r},t)}{\partial t} = S_1(\vec{r},t) - \Sigma_{a,1}(\vec{r})\phi_1(\vec{r},t) + \nabla \cdot [D_1(\vec{r})\nabla\phi_1(\vec{r},t)]$$

Now, *assume that space and energy dependence of the flux can be separated*:

$$\phi(\vec{r},E,t) = \phi(\vec{r},t)\xi(E), \text{ where } \xi(E) \text{ is the neutron spectrum and } \int_0^\infty dE\,\psi(E) = 1.$$

Focusing on the *spatial dependence* of the flux, we'll assume a homogeneous, steady-state, one-group system.

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_s(\vec{r})\phi(\vec{r},t) + \nabla \cdot [D(\vec{r})\nabla\phi(\vec{r},t)]$$

Steady-state: $\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} = 0$

Homogeneous: no material dependence on position; $\Sigma_a(\vec{r}) \to \Sigma_a$, $D(\vec{r}) \to D$

This gives us

$$\nabla^2\phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \text{ where } L = \sqrt{\frac{D}{\Sigma_a}} = \text{diffusion length.}$$

Now, consider a plane source of strength S_0 in an infinitely absorbing medium.

$$\phi(\vec{r}) = \phi(x)$$
$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S_0\delta(x)}{D}$$

The boundary conditions we'd like to enforce are that the source is zero other than at the plane, so within $\epsilon \to 0$:

$$-D\frac{d\phi}{dx}|_{+\epsilon} + D\frac{d\phi}{dx}|_{-\epsilon} = J_x(0^+) - J_x(0^-) = S_0\,.$$

We can use this to examine two sets of equations and boundary conditions.

(1) For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.
Boundary conditions:

$$\lim_{x \to 0^+} \vec{J}(x) = \frac{S_0}{2},$$
$$\lim_{x \to +\infty} |\phi(x)| < \infty,\ \phi(x) \geq 0$$

(2) For $x < 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.
Boundary conditions:

$$\lim_{x \to 0^-} \vec{J}(x) = -\frac{S_0}{2},$$
$$\lim_{x \to -\infty} |\phi(x)| < \infty, \phi(x) \geq 0$$

With the above equations and boundary conditions, we have a general solution form of (which you'll recall from your differential equations class):

$$\phi(x) = c_1 e^{-|x|/L} + c_2 e^{|x|/L}.$$

From the finite flux condition, $c_2 = 0$. Then, $\phi(x) = c_1 e^{-|x|/L}$.

$$\lim_{x \to 0^+} \vec{J}(x) = \lim_{x \to 0^+} \left(\frac{D}{L} c_1 e^{-|x|/L} \right) = \frac{D}{L} c_1 = \frac{S_0}{2}$$
$$c_1 = \frac{S_0 L}{2D}$$
$$\phi(x) = \boxed{\frac{S_0 L}{2D} e^{-|x|/L}}$$

The neutron flux falls off exponentially as one moves away from the source plane with a characteristic decay length of L. This holds fairly well as long as we're not too near the source and the medium is not too strongly absorbing.

Now, consider an infinite plane centered in a slab of finite thickness a, surrounded by a vacuum.

(1) For $x > 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.

Boundary conditions:

$$\lim_{x\to 0^+} \vec{J}(x) = \frac{S_0}{2} \ ,$$
$$\phi(\tfrac{\tilde{a}}{2}) = 0$$

(2) For $x < 0$, $\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0$.
Boundary conditions:

$$\lim_{x\to 0^-} \vec{J}(x) = -\frac{S_0}{2} \ ,$$
$$\phi(-\tfrac{\tilde{a}}{2}) = 0$$

Now, we can again write the equations as

$$\phi(x) = c_1 e^{-|x|/L} + c_2 e^{|x|/L}$$

Or, with equal validity (which we could have also done before)

$$\phi(x) = c_1 \cosh(\tfrac{|x|}{L}) + c_2 \sinh(\tfrac{|x|}{L})$$

You can use the boundary conditions to reach the full solution.

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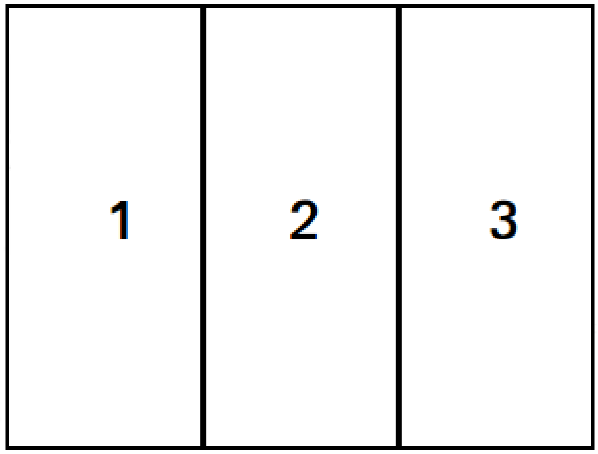
Now, consider a uniformly distributed source of strength $S_0 \frac{n}{cm^3 s}$ within a finite slab of width a with vacuum boundaries.

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S_0}{D}$$
$$\phi(x) = c_1 e^{-|x|/L} + c_2 e^{|x|/L} + \frac{S_0 L^2}{D}$$

Boundary conditions: $\phi(\pm\frac{\tilde{a}}{2}) = 0$

Alternative boundary condition: $\left.\frac{d\phi(x)}{dx}\right|_{x=0} = 0$ (symmetry)

Now, consider a uniform source in a reflected slab, where region 2 contains the source and 1 and 3 are reflectors (high scattering compared to absorption). The center of region 2 is located at $x = 0$ and the edges of region two are at $x = \pm\frac{a}{2}$.



We start by using an equation containing the source in region 2, designating the solution ϕ_2:

$$\frac{d^2\phi_2(x)}{dx_2^2} - \frac{1}{L_2^2}\phi_2(x) = -\frac{S_0}{D_2}, \quad -\tfrac{a}{2} < x < \tfrac{a}{2}$$

We need a collection of boundary conditions (where subscript indicates region number) that relate region 2 to the other regions:

$$\begin{aligned}\phi_2(\tfrac{a}{2}) &= \phi_3(\tfrac{a}{2})\\ \vec{J}_2(\tfrac{a}{2}) &= \vec{J}_3(\tfrac{a}{2})\\ \phi_1(-\tfrac{a}{2}) &= \phi_2(-\tfrac{a}{2})\\ \vec{J}_1(-\tfrac{a}{2}) &= \vec{J}_2(-\tfrac{a}{2})\end{aligned}$$

We also need equations for regions 1 and 3, which do not have sources:

$$\begin{aligned}\frac{d^2\phi_1(x)}{dx_1^2} - \frac{1}{L_1^2}\phi_1(x) = 0, &\quad -\infty < x < -\tfrac{a}{2}\\ \frac{d^2\phi_3(x)}{dx_3^2} - \frac{1}{L_3^2}\phi_3(x) = 0, &\quad \tfrac{a}{2} < x < \infty\end{aligned}$$

These regions have boundary conditions as well: $\lim_{x\to-\infty} |\phi_1(x)| < \infty, \quad \lim_{x\to\infty} |\phi_3(x)| < \infty$

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What if we want to solve a **generic diffusion problem**?

$$D\nabla^2\phi(\vec{r}) - \Sigma_a\phi(\vec{r}) = -S(\vec{r})$$

Working in 1D:

$$M\phi(x) = f(x)$$

where M is non-homogenous differential operator of order *n* on a differential equation:

$$a_0(x)\phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \ldots + a_n(x)\phi(x) = f(x)$$

for example

$$M = \frac{d^2}{dx^2} - \frac{1}{L}$$

One solution method is *variation of constants / parameters*, where we break the solution into homogeneous and particular portions:

$$\begin{aligned} M\phi_{homog}(x) &= 0 \\ M\phi_{part} &= rhs \\ \phi(x) &= \phi_{homog}(x) + \phi_{part}(x) \end{aligned}$$

We solve the homogeneous part to yield

$$\phi_{i,h}(x),\ i = 1, \ldots, n \ \text{ independent solutions}$$

and we also know

$$\phi_{part}(x) = \sum_{i=1}^{n} c_i \phi_{i,h}(x)$$

where $c_i = c_i(x)$ and are differentiable functions that satisfy the conditions

$$\sum_{i=1}^{n} c_i'(x)\phi_{i,h}^{(j)}(x) = 0, \quad j = 0, \ldots, n-1\ .$$

If

$$\sum_{i=1}^{n} c_i' \phi_{i,h}(x) = 0$$
$$\sum_{i=1}^{n} c_i' \phi_{i,h}^{(1)}(x) = 0$$
$$\vdots$$
$$\sum_{i=1}^{n} c_i' \phi_{i,h}^{(n-2)}(x) = 0$$

then

$$\sum_{i=1}^{n} c_i' \phi_{i,h}^{(n-1)}(x) = \frac{f(x)}{a_0(x)}$$

which indicates

$$c_i(x) = \int dx \, c_i'(x) \, .$$

For n = 2:

$$\phi_1(x)c_1'(x) + \phi_2(x)c_2'(x) = 0$$
$$\phi_1^{(1)}(x)c_1'(x) + \phi_2^{(1)}(x)c_2'(x) = \frac{f(x)}{a_0(x)}$$
$$\phi_{part} = c_1(x)\phi_1(x) + c_2(x)\phi_2(x)$$
$$a_0(x)\phi^{(2)}(x) + a_1(x)\phi^{(1)}(x) + a_2(x)\phi(x) = 0$$
$$\phi^{(1)}(x) = c_1'(x)\phi_1(x) + c_1(x)\phi_1'(x) + c_2'(x)\phi_2(x) + c_2(x)\phi_2'(x)$$

$$c_i(x) = \int dx c_i'(x) = A_i + \psi_i(x)$$

$$\phi(x) = \sum_{i=1}^{n} A_i \phi_i(x) + \sum_{i=1}^{n} \psi_i(x)\phi_i(x)$$

For n = 2: $\phi(x) = A_1\phi_1(x) + A_2\phi_2(x) + \psi_1(x)\phi_1(x) + \psi_2(x)\phi_2(x)$