The Adjoint Equation in **Multiplying Media**

We will call the fission operator G, where

$$G\zeta = \frac{\chi(E)}{4\pi} \int_0^\infty dE' \, \nu(E')\Sigma_f(E') \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, \hat{\Omega}', E')$$

To get the adjoint version, we can take the same approach as with the scattering operator to get

$$G^\dagger \zeta^\dagger = \frac{\nu(E)\Sigma_f(E)}{4\pi} \int_0^\infty dE' \, \chi(E') \int_{4\pi} d\hat{\Omega}' \psi^\dagger(\vec{r}, \hat{\Omega}', E')$$

We can combine this operator with H to use the techniques in the preceding section to look at subcritical systems.

We replace H and $H^\dagger$ with $H - G$ and $H^\dagger - G^\dagger$, respectively.

To look at the *eigenvalue* form of the problem:

$$H\psi - \frac{1}{k}G\psi = 0$$

where ψ is the fundamental solution mode, $\psi > 0$ for $\vec{r} \in V$, $\psi(\vec{r}_s, \hat{\Omega}, E) = 0$ for $\hat{n} \cdot \hat{\Omega} < 0$ with $\vec{r}_s \in \Gamma$.

$$H^\dagger \psi^\dagger - \frac{1}{k^\dagger}G^\dagger \psi^\dagger = 0$$

Now $\psi^\dagger$ is the fundamental mode, $\psi^\dagger > 0$ for $\vec{r} \in V$, $\psi^\dagger(\vec{r}_s, \hat{\Omega}, E) = 0$ for $\hat{n} \cdot \hat{\Omega} > 0$ with $\vec{r}_s \in \Gamma$.

Next, compare the two equations the way we did for response: multiply the forward equation by $\psi^\dagger$, integrate both over volume, and subtract for the adjoint equation multiplied by ψ and integrated:

$$\langle \psi^\dagger, H\psi \rangle - \langle \psi, H^\dagger \psi^\dagger \rangle - \frac{1}{k}\langle \psi^\dagger, G\psi \rangle + \frac{1}{k^\dagger}\langle \psi, G^\dagger \psi^\dagger \rangle = 0$$

$$\left(\frac{1}{k} - \frac{1}{k^\dagger}\right)\langle \psi^\dagger, G\psi \rangle = 0$$

$$k = k^\dagger \quad \text{because } G\psi > 0$$

Cool. What do we do with that? We can figure out the effect of small changes to critical systems,

which is super useful in NE!

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Reactivity of a Perturbed System

Recall that reactivity is the measure of a system's deviation from critical: $\rho = \frac{k-1}{k}$.
We are interested in how small changes (perturbations) in a system will cause reactivity to change.
We can use the adjoint equation to help us.

We will denote small changes with a preceding δ and unperturbed values with a subscript 0. Thus, for small changes in k we can define

$$\delta\rho = \frac{\delta k}{k_0^2}$$

For an unperturbed reactor we have these fundamental mode equations:

$$H_0\psi_0 - \frac{1}{k_0}G_0\psi_0 = 0$$

$$H_0^\dagger\psi_0^\dagger - \frac{1}{k_0^\dagger}G_0^\dagger\psi_0^\dagger = 0$$

We can represent system perturbation by adding small quantities to the operators and fluxes (very small compared to things themselves). For the forward system we get

$$(H_0 + \delta H)(\psi_0 + \delta\psi) - \frac{1}{k_0 + \delta k}(G_0 + \delta G)(\psi_0 + \delta\psi) = 0$$

Because perturbations are small, we can ignore second order terms. Then

$$H_0\psi_0 + \delta H\psi_0 + H_0\delta\psi - \left(\frac{1}{k_0 + \delta k}\right)\left(G_0\psi_0 + \delta G\psi_0 + G_0\delta\psi\right) \approx 0$$

$$\frac{1}{k_0 + \delta k} = \frac{1}{k_0(1 + \delta k/k_0)} \approx \frac{1}{k_0}\left(1 - \frac{\delta k}{k_0}\right)$$

$$-\frac{\delta k}{k_0^2}G_0\psi_0 \approx \underbrace{\left(H_0 - \frac{1}{k_0}G_0\right)\psi_0}_{\text{fundamental mode}=0} + \left(\delta H - \frac{1}{k_0}\delta G\right)\psi_0 + \left(H_0 - \frac{1}{k_0}G_0\right)\delta\psi$$

To use inner product identities, we (as before) multiply by $\psi_0^\dagger$ and integrate over phase space to get

$$-\frac{\delta k}{k_0^2}\langle\psi_0^\dagger, G_0\psi_0\rangle \approx \langle\psi_0^\dagger, \left(\delta H - \frac{1}{k_0}\delta G\right)\psi_0\rangle + \langle\psi_0^\dagger, \left(H_0 - \frac{1}{k_0}G_0\right)\delta\psi\rangle \qquad (1)$$

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Next, we can use the adjoint identity (using a modified operator) and let $\zeta = \delta\psi$ and $\zeta^\dagger = \psi_0^\dagger$ we can see:

$$\langle \zeta^\dagger, H\zeta \rangle = \langle \zeta, H^\dagger \zeta^\dagger \rangle$$

$$\langle \psi_0^\dagger, \left(H_0 - \frac{1}{k_0}G_0\right)\delta\psi \rangle = \langle \delta\psi, \left(H_0^\dagger - \frac{1}{k_0^\dagger}G_0^\dagger\right)\psi_0^\dagger \rangle$$

We know that the second part of the RHS is 0 because of the definition of the adjoint equation. This means the LHS is also 0, so we can can remove it from Eqn. **??**.

With all of this, we can now say

$$\frac{\delta k}{k_0^2} \approx -\frac{\langle \psi_0^\dagger, \left(\delta H - \frac{1}{k_0}\delta G\right)\psi_0 \rangle}{\langle \psi_0^\dagger, G_0\psi_0 \rangle}$$

This means that if we know the two unperturbed solutions $\psi_0^\dagger$ and ψ_0 we can find the reactivity change for any perturbations.

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Other **Boundary Conditions** for the adjoint operator

For reflected and periodic boundaries:

- Take the reflected BC configuration and unfold it to the full extent (write explicitly in each region).

- Then, take the adjoint of this with zero outgoing flux at the boundaries.

- Apply the same argument as in the forward case, but now we know what's going in instead of what's going out. This preserves the form of reflected and periodic BCs.

For a fixed incoming flux:

$$\psi(\vec{r}_s, \hat{\Omega}, E) = \psi_{in}(\vec{r}, \hat{\Omega}, E) \quad \hat{n} \cdot \hat{\Omega} < 0$$
$$\psi^\dagger(\vec{r}_s, \hat{\Omega}, E) = \psi_{out}^\dagger(\vec{r}, \hat{\Omega}, E) \quad \hat{n} \cdot \hat{\Omega} > 0$$

Now, we take the inner product of the boundary conditions with the operator as the angle-normal

dot product:

$$\int_\Gamma dS \int_0^\infty dE \Big[\int_{\hat{n}\cdot\hat{\Omega}<0} d\hat{\Omega} \, \hat{n}\cdot\hat{\Omega}\psi_{in}(\vec{r}_s,\hat{\Omega},E)\psi_{out}^\dagger(\vec{r}_s,\hat{\Omega},E)+$$
$$\int_{\hat{n}\cdot\hat{\Omega}>0} d\hat{\Omega} \, \hat{n}\cdot\hat{\Omega}\psi_{in}(\vec{r}_s,\hat{\Omega},E)\psi_{out}^\dagger(\vec{r}_s,\hat{\Omega},E)\Big]$$

We need this to vanish, thus

$$\int_\Gamma dS \int_0^\infty dE \int_{\hat{n}\cdot\hat{\Omega}>0} d\hat{\Omega} \, \big[-|\hat{n}\cdot\hat{\Omega}|\psi_{in}(\vec{r}_s,|\hat{\Omega}|,E)\psi_{out}^\dagger(\vec{r}_s,|\hat{\Omega}|,E)+$$
$$\hat{n}\cdot\hat{\Omega}\psi_{in}(\vec{r}_s,\hat{\Omega},E)\psi_{out}^\dagger(\vec{r}_s,\hat{\Omega},E)\big] = 0$$

Some notes on this:

- The boundary condition has an integral nature for $\psi_{out}^\dagger$.

- This balance requires both $\psi_{out}^\dagger$ and ψ_{in}. This means we have to solve both the forward and adjoint TEs to determine these values and therefore the BCs.

- the contribution of outgoing "adjoint neutrons" to detector response must equal the contribution of incoming "forward neutrons": $R = \langle \psi^\dagger, q_{ex}\rangle$.