NE 255, Fa16
Simplified P_N Equations
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In slab geometry the P_N equations can be written as a system of 1-D diffusion equations; this is not true in general geometry. This is the motivation behind the simplified P_N equations: what would happen if the P_N method in general geometry was as nice as it is in slab geometry?

Gelbard introduced the SP_N equations in a series of papers in 1962; however, they were not widely accepted as an approximate transport method because of the lack of a true theoretical foundation. For approximately 30 years, the SP_N equations were occasionally mentioned in American Nuclear Society conference talks and brief publications. It was not until the early 1990's that theoretical work was published demonstrating that the SP_N approximations have a valid mathematical foundation, and can be derived from either an asymptotic or a variational analysis.

"Heuristic" Derivation of the SP_N Equations

Consider the planar (slab) geometry P_N equations as before: for $l' = 0, 1, \ldots$, we have

$$\left(\frac{l'+1}{2l'+1}\right)\frac{d}{dx}\phi_{l'+1}(x) + \left(\frac{l'}{2l'+1}\right)\frac{d}{dx}\phi_{l'-1}(x) + \Sigma_t(x)\phi_{l'} = \Sigma_{sl'}(x)\phi_{l'}(x) + s_{l'}(x)\,,$$

with $\phi_{-1} = 0$ and

$$\phi_{N+1} = 0 \quad \text{or} \quad \frac{d}{dx}\phi_{N+1} = 0\,.$$

The second-order form of the planar geometry P_1 equations with Marshak boundary conditions is the diffusion equation

$$\begin{aligned}
&-\frac{d}{dx}D\frac{d\phi_0}{dx} + \Sigma_a(x)\phi_0(x) = s_0(x)\,, 0 < x < X\,,\\
&\frac{1}{2}\phi_0(0) - D\frac{d\phi_0}{dx}(0) = 2J^+(0),\\
&\frac{1}{2}\phi_0(X) + D\frac{d\phi_0}{dx}(X) = 2J^-(X),
\end{aligned}$$

where

$$D = \frac{1}{3\left[\Sigma_t(x) - \Sigma_{s1}(x)\right]}.$$

This can be generalized to 3-D by making the two **formal** modifications:

1. Replace the 1-D diffusion operator
$$\frac{d}{dx} D \frac{d}{dx}$$
by the 3-D diffusion operator
$$\nabla \cdot D \nabla \equiv \frac{\partial}{\partial x} D \frac{\partial}{\partial x} + \frac{\partial}{\partial y} D \frac{\partial}{\partial y} + \frac{\partial}{\partial z} D \frac{\partial}{\partial z} \, ;$$

2. In the boundary conditions, replace the derivative terms
$$\pm \frac{d}{dx}$$
by the outward normal derivative
$$\vec{n} \cdot \nabla$$

Making these formal modifications, we obtain the standard 3-D diffusion (P_1) equations

$$-\nabla \cdot D \nabla \phi_0(\vec{r}) + \Sigma_a(\vec{r}) \phi_0(\vec{r}) = s_0(\vec{r}) \, , \vec{r} \in V,$$
$$\frac{1}{2} \phi_0(\vec{r}) + D \vec{n} \cdot \nabla \phi_0 = 2 J^-(\vec{r}), \, \vec{r} \in \partial V \, .$$

These equations obviously reduce to the standard 1-D diffusion equations in planar geometry.

We carry out the same procedure for the general SP_N equations. First, for odd values of l', $\phi_{l'}$ is replaced by a vector:
$$\phi_{l'} \to \vec{\phi}_{l'} = (\phi^x_{l'}, \phi^y_{l'}, \phi^z_{l'})^t \, .$$

Then, in the even l' equations the derivative in x is replaced by a divergence:

$$\frac{d}{dx} \to \nabla \cdot \, ;$$

and in the odd l' equations the x derivative is changed to a gradient:

$$\frac{d}{dx} \to \nabla$$

This allows us to write the first-order form of the SP$_N$ equations as

$$\begin{aligned}
&\nabla \cdot \vec{\phi}_1 + \Sigma_a \phi_0 = s_0 \,, \\
&\left(\frac{l'+1}{2l'+1}\right) \nabla \phi_{l'+1} + \left(\frac{l'}{2l'+1}\right) \nabla \phi_{l'-1} + \Sigma_t \vec{\phi}_{l'} = \Sigma_{sl'} \vec{\phi}_{l'} + s_{l'} \,, \qquad \text{for odd } l', \\
&\left(\frac{l'+1}{2l'+1}\right) \nabla \cdot \vec{\phi}_{l'+1} + \left(\frac{l'}{2l'+1}\right) \nabla \cdot \vec{\phi}_{l'-1} + \Sigma_t \phi_{l'} = \Sigma_{sl'} \phi_{l'} + s_{l'} \,, \qquad \text{for even } l' > 0.
\end{aligned}$$

The boundary conditions for the SP$_N$ equations can be obtained from the P$_N$ Marshak boundary conditions by replacing $\phi_{l'}$ with the SP$_N$ unknowns and μ with $\vec{n} \cdot \hat{\Omega}$, where $\vec{n}$ is the unit inward normal to the boundary.

The SP$_3$ Equations

Assuming an isotropic source, the SP$_3$ equations in their first-order form are

$$\begin{aligned}
\nabla \cdot \vec{\phi}_1 + \Sigma_a \phi_0 &= s_0 \,, \\
\frac{1}{3} \nabla \phi_0 + \frac{2}{3} \nabla \phi_2 + [\Sigma_t - \Sigma_{s1}] \vec{\phi}_1 &= 0 \,, \\
\frac{2}{5} \nabla \cdot \vec{\phi}_1 + \frac{3}{5} \nabla \cdot \vec{\phi}_3 + [\Sigma_t - \Sigma_{s2}] \phi_2 &= 0 \,, \\
\frac{3}{7} \nabla \phi_2 + [\Sigma_t - \Sigma_{s3}] \vec{\phi}_3 &= 0 \,.
\end{aligned}$$

We can rewrite them in their second-order form by using the relation

$$\vec{\phi}_{l'} = -\frac{1}{\Sigma_t - \Sigma_{sl'}} \left(\frac{l'}{2l'+1} \nabla \phi_{l'-1} + \frac{l'+1}{2l'+1} \nabla \phi_{l'+1} \right) \,,$$

yielding

$$\begin{aligned}
&- \nabla \cdot \frac{1}{3[\Sigma_t - \Sigma_{s1}]} \nabla \phi_0 - \nabla \cdot \frac{2}{3[\Sigma_t - \Sigma_{s1}]} \nabla \phi_2 + \Sigma_a \phi_0 = s_0 \,, \\
&- \nabla \cdot \frac{2}{15[\Sigma_t - \Sigma_{s1}]} \nabla \phi_0 - \nabla \cdot \left(\frac{4}{15[\Sigma_t - \Sigma_{s1}]} + \frac{9}{35[\Sigma_t - \Sigma_{s3}]} \right) \nabla \phi_2 + [\Sigma_t - \Sigma_{s2}] \phi_2 = 0 \,.
\end{aligned}$$

The second-order form is useful because it makes the SP$_N$ equations look like a set of coupled diffusion equations.

The SP$_3$ equations can be manipulated into a form that resembles a two group diffusion equation by defining $\hat{\phi}_0 = \phi_0 + 2\phi_2$. Using this new variable, we can write

$$
\begin{aligned}
&-\nabla \cdot \frac{1}{3[\Sigma_t - \Sigma_{s1}]} \nabla \hat{\phi}_0 + \Sigma_a \hat{\phi}_0 = 2\Sigma_a \phi_2 + s_0 \, , \\
&-\nabla \cdot \frac{9}{35[\Sigma_t - \Sigma_{s3}]} \nabla \phi_2 + \left([\Sigma_t - \Sigma_{s2}] + \frac{4}{5}\Sigma_a \right) \phi_2 + = \frac{2}{5} \left[\Sigma_a \hat{\phi}_0 - s_0 \right] \, .
\end{aligned}
$$

These equations can be solved with a two-group diffusion code by properly setting the diffusion coefficients and cross-sections or with a one-group diffusion code utilizing an iteration strategy for the coupling terms (FLIP).

General Properties of the SP$_N$ Equations

The SP$_N$ equations can be understood as a "super" diffusion theory. The structure of the SP$_N$ equations is that of a coupled system of diffusion equations, and the class of problems for which the SP$_N$ equations are accurate encompasses the class of problems for which diffusion theory is accurate.

1. In 1-D planar geometry, SP$_N$ and P$_N$ are identical

2. In multidimensional problems, SP$_N$ form a system of (N + 1) equations; P$_N$ form a much larger system of (N + 1)^2 equations

3. The SP$_N$ equations have the same "diffusion" (elliptic) structure as the P$_1$ equations; the P$_N$ equations have a more complicated (hyperbolic) mathematical structure.

4. The above derivation of the SP$_N$ equations assumes as its starting point a 1-group transport problem. However, applying the same procedures to a multigroup transport problem is straightforward. The only complication is that the diffusion coefficients can become non-diagonal matrices. Thus, unlike standard multigroup diffusion theory (but like standard multigroup P$_1$ theory), the multigroup SP$_N$ equations generally have non-diagonal matrix diffusion coefficients.

5. In principle, the 2-D or 3-D SP$_N$ equations can be implemented in a 2-D or 3-D diffusion code without fundamentally rewriting the code. This is not the case for the P$_N$ equations.

6. The SP$_N$ equations contain more "transport physics" than the diffusion equations. For this reason, solutions of the SP$_N$ equations can contain boundary layers that are not present in P$_1$ solutions. In order to properly resolve these boundary layers, it may be necessary to use a finer spatial grid for the SP$_N$ equations than for the diffusion equation. Alternatively, the

use of nodal methods with extra expansion terms capable of expressing the boundary layer effects may be required.

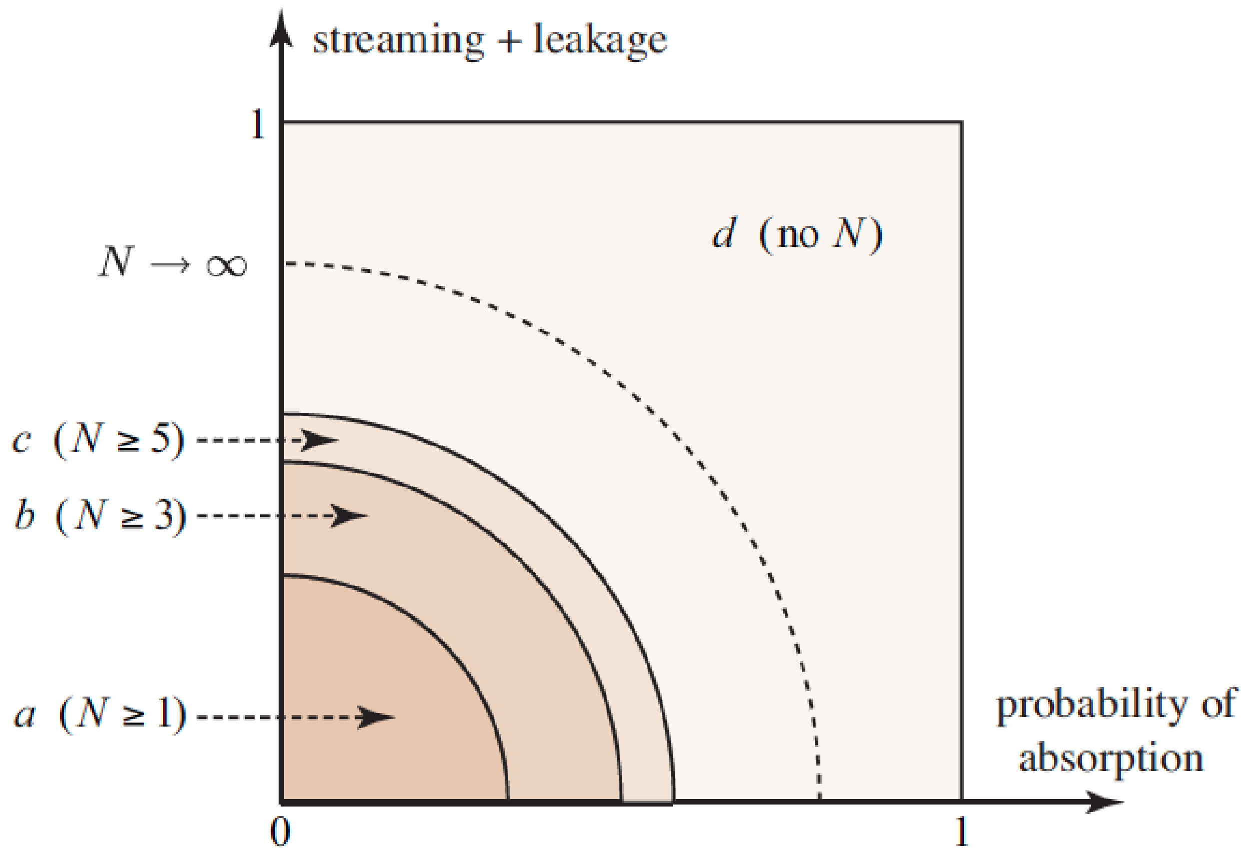
7. The multigroup SP_3 equations are about twice as costly to solve as the multigroup P_1 equations. However, SP_3 solutions are usually much more accurate (transport-like) than P_1 solutions.

8. In the limit as $N \to \infty$, the P_N solutions converge to the transport solution.

9. In the limit as $N \to \infty$, the SP_N solutions do not generally converge to the transport solution–unless the underlying problem is 1-D. Therefore, high-order SP_N equations cannot be used to obtain arbitrarily accurate solutions of neutron transport problems in 2 or 3 dimensions.

10. For 3-D problems, the system of P_N equations is much more complicated in structure and greater in number than the system of SP_N equations. Also, for problems having 1-D symmetry, the P_N and SP_N equations become identical. For these reasons, it is widely believed that the 3-D SP_N equations can be derived by discarding the proper terms (and equations) from the 3-D P_N equations. However, this has never been shown. In fact, the precise relationship between the 3-D P_N and the 3-D SP_N equations is not known.

11. For problems in which the P_1 solution is accurate, the SP_3 solution is generally much more accurate. As problems become less "diffusive" (absorption, streaming, or leakage become increasingly important), the P_1 and SP_3 solutions both degrade in accuracy. However, the P_1 solutions degrade more rapidly, and the SP_3 solutions can remain accurate well into the range in which P_1 solutions are not accurate. When the problem becomes sufficiently "difficult", the P_1 and SP_N solutions both become inaccurate (see figure in the next page).



This figure shows the (qualitative) range of validity of the SP_N equations. The amounts of absorption and streaming/leakage are indicated on arbitrary scales ranging from 0 to 1. In region a, where streaming, leakage, and absorption are weak, the P_1 and all SP_N solutions are accurate. As absorption or streaming increase (region b), P_1 becomes inaccurate but SP_N with $N \geq 3$ is still accurate. As absorption or streaming increase further (region c), P_1 and SP_3 are inaccurate but SP_N with $N \geq 5$ is still accurate. In region d, no SP_N solution is accurate.

These notes are derived from Edward Larsen's class notes for NE 644 at the University of Michigan, and from Ryan McClarren's review paper on the SP_N equations: "Theoretical Aspects of the Simplified P_n Equations", Transport Theory and Statistical Physics 39: 73–109, 2011.