Statistics

The "true" mean value, μ, of any PDF is the expected value, $E(x)$

$$\mu = E(x) = \int xf(x)dx$$

Because we can't usually do this, we use random samples and estimate the true mean from the "sample" mean, $\bar{x}$

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad \lim_{N\to\infty} \bar{x} \to \mu \; .$$

The variance of a PDF is the measure of spread in that PDF

$$\sigma^2 = E[(x-\mu)^2] = \int (x-\mu)^2 f(x)dx$$
$$= \int x^2 f(x)dx - 2\mu \int xf(x)dx + \mu^2 \int f(x)dx$$
$$= E(x^2) - \mu^2$$

However, we don't know the PDF so we use the samples to get the sample variances

$$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$
$$= \frac{1}{N-1}\left[\sum_{i=1}^{N} x_i^2 - 2\bar{x}\sum_{i=1}^{N} x_i + \bar{x}^2 \sum_{i=1}^{N} 1\right]$$
$$\approx \bar{x^2} - \bar{x}^2$$

The <u>Central Limit Theorem</u> states that For N *independent* random variables, x_i, sampled from *identical distributions*, their mean follows a Normal (Gaussian) distribution.

(Note: this is the *IID* requirement for MC)

We can use this information to define *confidence intervals*

$$\bar{x} - S_{\bar{x}} < E(x) < \bar{x} + S_{\bar{x}} \quad \text{about 68\% of the time}$$
$$\bar{x} - 2S_{\bar{x}} < E(x) < \bar{x} + 2S_{\bar{x}} \quad \text{about 95\% of the time}$$

The **standard deviation** of the mean is a measure of the error in the result

$$S_{\bar{x}}^2 = E[(\bar{x} - \mu)^2] = E\left[\left(\frac{1}{N}\sum_{i=1}^{N} x_i - \mu\right)^2\right] = E\left[\left(\frac{1}{N^2}\sum_{i=1}^{N}(x_i - \mu)\right)^2\right]$$

$$= \frac{1}{N^2}E\left[\sum_{i=1}^{N}(x_i - \mu)\sum_{j=1}^{N}(x_j - \mu)\right] = \frac{1}{N^2}E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - \mu)(x_j - \mu)\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left[(x_i - \mu)(x_j - \mu)\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}S_x^2\delta_{ij} = \frac{1}{N^2}\sum_{i=1}^{N}S_x^2$$

$$= \frac{NS_x^2}{N^2} = \boxed{\frac{S_x^2}{N}}$$

The error in the results decreases with the square of increasing the number of histories.

Relative Error is

$$R = \frac{S_{\bar{x}}}{\bar{x}} = \sqrt{\frac{\sum_{i=1}^{N} x_i^2}{\left(\sum_{i=1}^{N} x_i\right)^2} - \frac{1}{N}}$$

If x_i are equal and non-zero, R=0.

Thus, we can reduce the error by reducing the spread in x_i.

Accuracy vs. Precision

The distinction between accuracy and precision can be seen in Fig. 1.

Accuracy is the degree of closeness of measurements of a quantity to that quantity's true value.

The *precision* of a measurement system, related to reproducibility and repeatability, is the degree to which repeated measurements under unchanged conditions show the same results.

Accuracy can be affected by systematic errors in simulation: physical and mathematical models; errors in geometry or source model; incorrect code use by user.

Usually unknown.

Conversely, precision can usually be improved: run more histories; use variance reduction; adjust
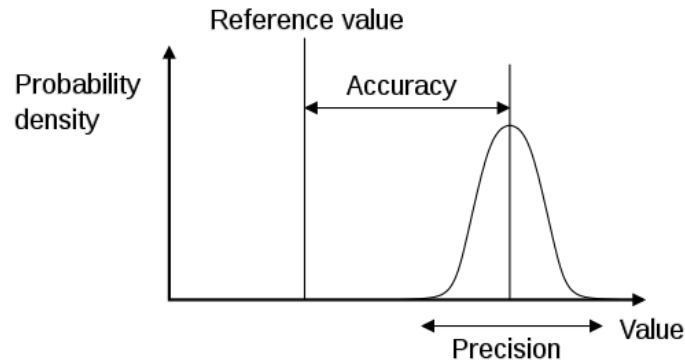
Figure 1: "Accuracy and precision" by Pekaje at English Wikipedia - Transferred from en.wikipedia to Commons.. Licensed under GFDL via Commons - https://commons.wikimedia.org/wiki/File:Accuracy_and_precision.svg#/media/File:Accuracy_and_precision.svg

your measurement (fewer scoring bins).

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Variance Reduction

What we have talked about so far is *Analog* Monte Carlo:

- Natural laws are **preserved**

- The game is the "analog" of the physical problem of interest (the history of each particle is simulated exactly)

- No alteration of PDFs

- At collision, particle is killed if absorption

- Particle is born with weight 1

- weight unchanged throughout history

- Score when tallying events is 1

We often, instead, want to do *Non-Analog* Monte Carlo:

- To reduce computation time, the strict analog simulation of particles is abandoned

- Variance Reduction techniques: Absorption suppression, Russian Roulette (history termination), Splitting (history propagation), Forced collisions, Source biasing, Hybrid methods

- Alter PDFs to favor events of interest

- Particle can have different birth weight

- Weight is altered if biased PDF is used

- Particle survives "absorption" and weight is changed

- Splitting and RR can change weight

- Score current weight when tallying

We'll talk about implicit capture (a.k.a. survival biasing), roulette, splitting, and weight window maps.

The first thing to think about is how to measure success. How do we know if a calculation is "better"?

We use the figure of merit

$$FOM = \frac{1}{R^2 t} \, ,$$

where R is the relative error and t is the particle tracking time.

What we really want is to reduce both of these.

Why are they related to one another this way? Recall that $S_x \propto \sqrt{\frac{1}{N}}$.

It's clear that without variance reduction techniques to reduce error by a factor of two you need to increase particle count (and hence time) by a factor of four.

FOM measures if we're really winning.

The idea of VR is to track particles that will contribute meaningfully to the desired results and to avoid tracking those that will not while maintaining a fair game.