NE 255, Fa16
Simplified P$_N$ Equations
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In slab geometry the P$_N$ equations can be written as a system of 1-D diffusion equations; this is not true in general geometry. This if the motivation behind the simplified P$_N$ equations: what would happen if the P$_N$ method in general geometry was as nice as it is in slab geometry?

Gelbard introduced the SP$_N$ equations in a series of papers in 1962; however, they were not widely accepted as an approximate transport method because of the lack of a true theoretical foundation. For approximately 30 years, the SPN equations were occasionally mentioned in American Nuclear Society conference talks and brief publications. It was not until the early 1990s that theoretical work was published demonstrating that the SP$_N$ approximations have a valid mathematical foundation, and can be derived from either an asymptotic or a variational analysis.

"Heuristic" Derivation of the SP$_N$ Equations

Consider the planar (slab) geometry P$_N$ equations as before: for $l' = 0, 1, \ldots$, we have

$$\left(\frac{l'+1}{2l'+1}\right)\frac{d}{dx}\phi_{l'+1}(x) + \left(\frac{l'}{2l'+1}\right)\frac{d}{dx}\phi_{l'-1}(x) + \Sigma_t(x)\phi_{l'} = \Sigma_{sl'}(x)\phi_{l'}(x) + s_{l'}(x)\,,$$

with $\phi_{-1} = 0$ and

$$\phi_{N+1} = 0 \quad \text{or} \quad \frac{d}{dx}\phi_{N+1} = 0\,.$$

The second-order form of the planar geometry P$_1$ equations with Marshak boundary conditions is the diffusion equation

$$\begin{aligned}
&-\frac{d}{dx}D\frac{d\phi_0}{dx} + \Sigma_a(x)\phi_0(x) = s_0(x)\,, 0 < x < X\,,\\
&\frac{1}{2}\phi_0(0) - D\frac{d\phi_0}{dx}(0) = 2J^+(0),\\
&\frac{1}{2}\phi_0(X) + D\frac{d\phi_0}{dx}(X) = 2J^-(X),
\end{aligned}$$

where

$$D = \frac{1}{3\left[\Sigma_t(x) - \Sigma_{s1}(x)\right]}.$$

This can be generalized to 3-D by making the two **formal** modifications:

1. Replace the 1-D diffusion operator
$$\frac{d}{dx}D\frac{d}{dx}$$
by the 3-D diffusion operator
$$\nabla\cdot D\nabla \equiv \frac{\partial}{\partial x}D\frac{\partial}{\partial x}+\frac{\partial}{\partial y}D\frac{\partial}{\partial y}+\frac{\partial}{\partial z}D\frac{\partial}{\partial z}\,;$$

2. In the boundary conditions, replace the derivative terms
$$\pm\frac{d}{dx}$$
by the outward normal derivative
$$\vec{n}\cdot\nabla$$

Making these formal modifications, we obtain the standard 3-D diffusion (P_1) equations

$$\begin{aligned}&-\nabla\cdot D\nabla\phi_0(\vec{r})+\Sigma_a(\vec{r})\phi_0(\vec{r})=s_0(\vec{r})\,,\vec{r}\in V,\\ &\frac{1}{2}\phi_0(\vec{r})+D\vec{n}\cdot\nabla\phi_0=2J^-(\vec{r}),\,\vec{r}\in\partial V\,.\end{aligned}$$

These equations obviously reduce to the standard 1-D diffusion equations in planar geometry.

We carry out the same procedure for the general SP_N equations. First, for odd values of l', $\phi_{l'}$ is replaced by a vector:
$$\phi_{l'}\rightarrow\vec{\phi}_{l'}=(\phi_{l'}^x,\phi_{l'}^y,\phi_{l'}^z)^t\,.$$

Then, in the even l' equations the derivative in x is replaced by a divergence:

$$\frac{d}{dx}\rightarrow\nabla\cdot\,;$$

and in the odd l' equations the x derivative is changed to a gradient:

$$\frac{d}{dx}\rightarrow\nabla$$

This allows us to write the first-order form of the SP$_N$ equations as

$$
\begin{aligned}
&\nabla \cdot \vec{\phi}_1 + \Sigma_a \phi_0 = s_0\,, \\
&\left(\frac{l'+1}{2l'+1}\right)\nabla\phi_{l'+1} + \left(\frac{l'}{2l'+1}\right)\nabla\phi_{l'-1} + \Sigma_t\vec{\phi}_{l'} = \Sigma_{sl'}\vec{\phi}_{l'} + s_{l'}\,, \qquad \text{for odd } l', \\
&\left(\frac{l'+1}{2l'+1}\right)\nabla\cdot\vec{\phi}_{l'+1} + \left(\frac{l'}{2l'+1}\right)\nabla\cdot\vec{\phi}_{l'-1} + \Sigma_t\phi_{l'} = \Sigma_{sl'}\phi_{l'} + s_{l'}\,, \qquad \text{for even } l' > 0.
\end{aligned}
$$

The boundary conditions for the SP$_N$ equations can be obtained from the P$_N$ Marshak boundary conditions by replacing $\phi_{l'}$ with the SP$_N$ unknowns and μ with $\vec{n}\cdot\hat{\Omega}$, where $\vec{n}$ is the unit inward normal to the boundary.

The SP$_3$ Equations

Assuming an isotropic source, the SP$_3$ equations in their first-order form are

$$
\begin{aligned}
\nabla \cdot \vec{\phi}_1 + \Sigma_a \phi_0 &= s_0\,, \\
\frac{1}{3}\nabla\phi_0 + \frac{2}{3}\nabla\phi_2 + [\Sigma_t - \Sigma_{s1}]\vec{\phi}_1 &= 0\,, \\
\frac{2}{5}\nabla\cdot\vec{\phi}_1 + \frac{3}{5}\nabla\cdot\vec{\phi}_3 + [\Sigma_t - \Sigma_{s2}]\phi_2 &= 0\,, \\
\frac{3}{7}\nabla\phi_2 + [\Sigma_t - \Sigma_{s3}]\vec{\phi}_3 &= 0\,.
\end{aligned}
$$

We can rewrite them in their second-order form by using the relation

$$
\vec{\phi}_{l'} = -\frac{1}{\Sigma_t - \Sigma_{sl'}}\left(\frac{l'}{2l'+1}\nabla\phi_{l'-1} + \frac{l'+1}{2l'+1}\nabla\phi_{l'+1}\right)\,,
$$

yielding

$$
\begin{aligned}
&-\nabla\cdot\frac{1}{3[\Sigma_t - \Sigma_{s1}]}\nabla\phi_0 - \nabla\cdot\frac{2}{3[\Sigma_t - \Sigma_{s1}]}\nabla\phi_2 + \Sigma_a\phi_0 = s_0\,, \\
&-\nabla\cdot\frac{2}{15[\Sigma_t - \Sigma_{s1}]}\nabla\phi_0 - \nabla\cdot\left(\frac{4}{15[\Sigma_t - \Sigma_{s1}]} + \frac{9}{35[\Sigma_t - \Sigma_{s3}]}\right)\nabla\phi_2 + [\Sigma_t - \Sigma_{s2}]\phi_2 = 0\,.
\end{aligned}
$$

The second-order form is useful because it makes the SP$_N$ equations look like a set of coupled diffusion equations.

The SP$_3$ equations can be manipulated into a form that resembles a two group diffusion equation by defining $\hat{\phi}_0 = \phi_0 + 2\phi_2$. Using this new variable, we can write

$$
\begin{aligned}
&-\nabla \cdot \frac{1}{3[\Sigma_t - \Sigma_{s1}]}\nabla\hat{\phi}_0 + \Sigma_a\hat{\phi}_0 = 2\Sigma_a\phi_2 + s_0 \,, \\
&-\nabla \cdot \frac{9}{35[\Sigma_t - \Sigma_{s3}]}\nabla\phi_2 + \left([\Sigma_t - \Sigma_{s2}] + \frac{4}{5}\Sigma_a\right)\phi_2 + = \frac{2}{5}\left[\Sigma_a\hat{\phi}_0 - s_0\right] \,.
\end{aligned}
$$

These equations can be solved with a two-group diffusion code by properly setting the diffusion coefficients and cross-sections or with a one-group diffusion code utilizing an iteration strategy for the coupling terms (FLIP).

General Properties of the SP$_N$ Equations

The SP$_N$ equations can be understood as a "super" diffusion theory. The structure of the SP$_N$ equations is that of a coupled system of diffusion equations, and the class of problems for which the SP$_N$ equations are accurate encompasses the class of problems for which diffusion theory is accurate.

1. In 1-D planar geometry, SP$_N$ and P$_N$ are identical

2. In multidimensional problems, SP$_N$ form a system of (N + 1) equations; P$_N$ form a much larger system of $(N + 1)^2$ equations

3. The SP$_N$ equations have the same "diffusion" (elliptic) structure as the P$_1$ equations; the P$_N$ equations have a more complicated (hyperbolic) mathematical structure.

4. The above derivation of the SP$_N$ equations assumes as its starting point a 1-group transport problem. However, applying the same procedures to a multigroup transport problem is straightforward. The only complication is that the diffusion coefficients can become non-diagonal matrices. Thus, unlike standard multigroup diffusion theory (but like standard multigroup P$_1$ theory), the multigroup SP$_N$ equations generally have non-diagonal matrix diffusion coefficients.

5. In principle, the 2-D or 3-D SP$_N$ equations can be implemented in a 2-D or 3-D diffusion code without fundamentally rewriting the code. This is not the case for the P$_N$ equations.

6. The SP$_N$ equations contain more "transport physics" than the diffusion equations. For this reason, solutions of the SP$_N$ equations can contain boundary layers that are not present in P$_1$ solutions. In order to properly resolve these boundary layers, it may be necessary to use a finer spatial grid for the SP$_N$ equations than for the diffusion equation. Alternatively, the

use of nodal methods with extra expansion terms capable of expressing the boundary layer effects may be required.

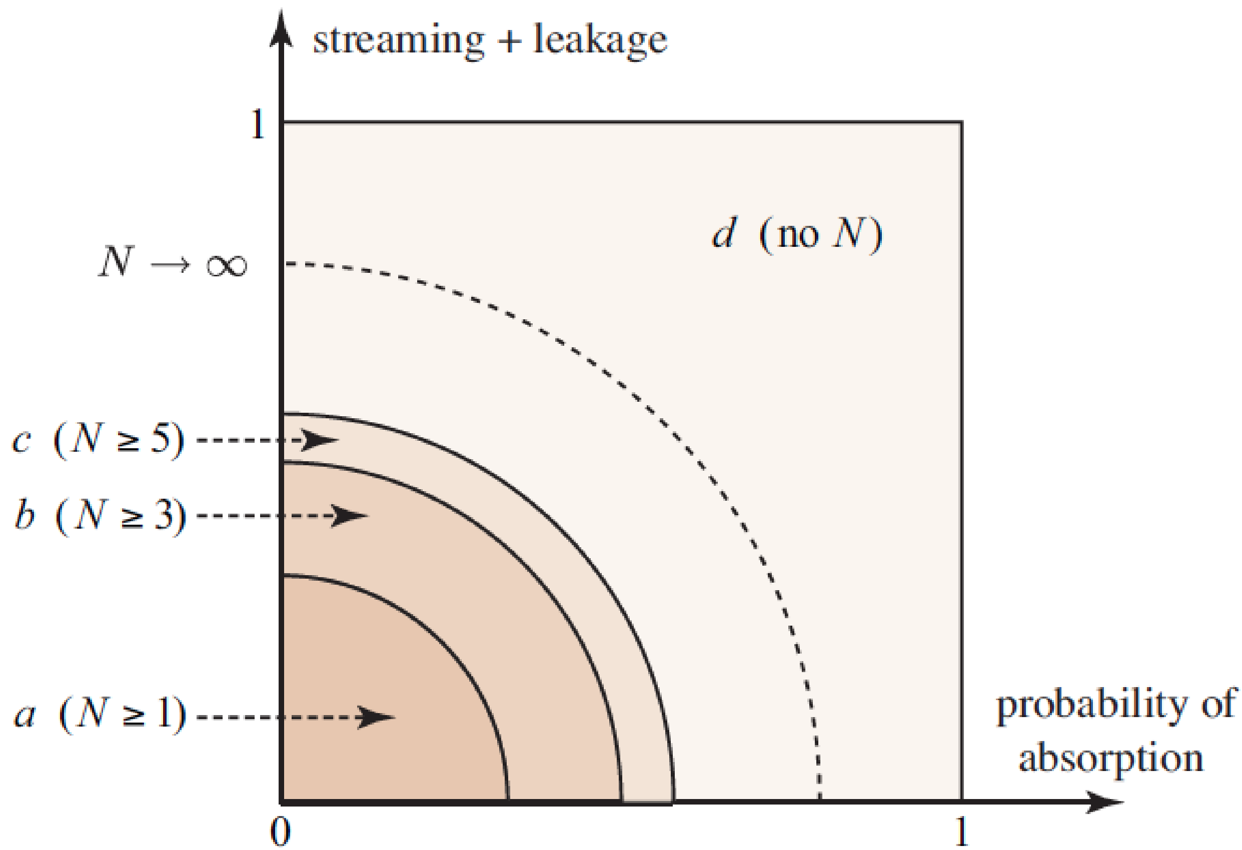
7. The multigroup SP_3 equations are about twice as costly to solve as the multigroup P_1 equations. However, SP_3 solutions are usually much more accurate (transport-like) than P_1 solutions.

8. In the limit as $N \to \infty$, the P_N solutions converge to the transport solution.

9. In the limit as $N \to \infty$, the SP_N solutions do not generally converge to the transport solution–unless the underlying problem is 1-D. Therefore, high-order SP_N equations cannot be used to obtain arbitrarily accurate solutions of neutron transport problems in 2 or 3 dimensions.

10. For 3-D problems, the system of P_N equations is much more complicated in structure and greater in number than the system of SP_N equations. Also, for problems having 1D symmetry, the P_N and SP_N equations become identical. For these reasons, it is widely believed that the 3-D SP_N equations can be derived by discarding the proper terms (and equations) from the 3D P_N equations. However, this has never been shown. In fact, the precise relationship between the 3D P_N and the 3D SP_N equations is not known.

11. For problems in which the P_1 solution is accurate, the SP_3 solution is generally much more accurate. As problems become less "diffusive" (absorption, streaming, or leakage become increasingly important), the P_1 and SP_3 solutions both degrade in accuracy. However, the P_1 solutions degrade more rapidly, and the SP_3 solutions can remain accurate well into the range in which P_1 solutions are not accurate. When the problem becomes sufficiently "difficult", the P_1 and SP_N solutions both become inaccurate (see figure in the next page).



This figure shows the (qualitative) range of validity of the SP_N equations. The amounts of absorption and streaming/leakage are indicated on arbitrary scales ranging from 0 to 1. In region a, where streaming, leakage, and absorption are weak, the P_1 and all SP_N solutions are accurate. As absorption or streaming increase (region b), P_1 becomes inaccurate but SP_N with $N \geq 3$ is still accurate. As absorption or streaming increase further (region c), P_1 and SP_3 are inaccurate but SP_N with $N \geq 5$ is still accurate. In region d, no SP_N solution is accurate.

These notes are derived from Edward Larsen's class notes for NE 644 at the University of Michigan, and from Ryan McClarren's review paper on the SP_N equations: "Theoretical Aspects of the Simplified P_n Equations", Transport Theory and Statistical Physics 39: 73–109, 2011.