NE 255, Fa16
Simplified P$_N$ Equations - part 2
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There are at least two ways that asymptotic analysis can be used to derive the SP$_N$ equations, both dating back to the early 1990's. (I) Pomraning presented a derivation demonstrating that, if the transport solution is locally 1-D, the SP$_N$ solution will asymptotically agree with the transport solution. (II) Larsen, Morel, and McGhee derived the SP$_N$ equations for anisotropic scattering that is not highly forward peaked and included energy dependence through the multigroup method. Neither of these derivations includes asymptotic boundary conditions.

Asymptotic Derivation of the SP$_N$ Equations

We will study the second approach for isotropic scattering in the one-speed case. The derivation when there is anisotropic scattering requires some tensor analysis that is beyond the scope of this class. The transport equation is written as

$$\hat{\Omega} \cdot \nabla \psi(\vec{r}, \hat{\Omega}) + \Sigma_t(\vec{r}) \psi(\vec{r}, \hat{\Omega}) = \frac{\Sigma_s(\vec{r})}{4\pi} \phi(\vec{r}) + \frac{Q(\vec{r})}{4\pi}, \tag{1}$$

where $\phi = \int_{4\pi} \psi d\Omega = \mathrm{P}_0$ moment of ψ. Defining $0 < \varepsilon \ll 1$, we scale the parameters in order to consider a *diffusive* system:

$$\begin{aligned} \Sigma_t(\vec{r}) &= \frac{\sigma_t(\vec{r})}{\varepsilon}, \\ \Sigma_t(\vec{r}) - \Sigma_s(\vec{r}) = \Sigma_a(\vec{r}) &= \varepsilon \sigma_a(\vec{r}), \\ Q(\vec{r}) &= \varepsilon q(\vec{r}), \end{aligned}$$

where σ_t, σ_a, and q are $O(1)$. The physics implied by this scaling is as follows:

1. The system is optically thick ($\Sigma_t >> 1$)
2. The rates of absorption and production due to interior sources are comparable and weak: $\Sigma_a = O(\varepsilon)$ and $Q = O(\varepsilon)$
3. The infinite medium solution $\phi = Q/\Sigma_a = q/\sigma_a$ is $O(1)$
4. The diffusion length $L = (3\Sigma_t \Sigma_a)^{-1/2} = (3\sigma_t \sigma_a)^{-1/2}$ is $O(1)$

5. If one introduces this scaling into the standard diffusion approximation, the resulting equation is independent of ε. In other words, the standard diffusion equation is invariant under this scaling.

Under this scaling, Eq. (1) becomes

$$\left(1 + \frac{\varepsilon}{\sigma_t}\hat{\Omega} \cdot \nabla\right)\psi = \frac{1 - \varepsilon^2\sigma_a/\sigma_t}{4\pi}\phi + \frac{\varepsilon^2 q}{4\pi\sigma_t}.$$

We invert the operator on the left-hand side of this equation to obain and expression for ψ in terms of ϕ and q:

$$\psi = \left(1 + \frac{\varepsilon}{\sigma_t}\hat{\Omega} \cdot \nabla\right)^{-1}\left[\frac{1 - \varepsilon^2\sigma_a/\sigma_t}{4\pi}\phi + \frac{\varepsilon^2 q}{4\pi\sigma_t}\right].$$

Since ε is small, we expand the inverse operator in a power series:

$$\psi = \left(1 - \varepsilon\left(\frac{1}{\sigma_t}\hat{\Omega} \cdot \nabla\right) + \varepsilon^2\left(\frac{1}{\sigma_t}\hat{\Omega} \cdot \nabla\right)^2 - \varepsilon^3\left(\frac{1}{\sigma_t}\hat{\Omega} \cdot \nabla\right)^3 + \varepsilon^4\left(\frac{1}{\sigma_t}\hat{\Omega} \cdot \nabla\right)^4 - \right. \tag{2}$$
$$\left. \varepsilon^5\left(\frac{1}{\sigma_t}\hat{\Omega} \cdot \nabla\right)^5 + \varepsilon^6\left(\frac{1}{\sigma_t}\hat{\Omega} \cdot \nabla\right)^6 + O(\varepsilon^7)\right)\left[\frac{1 - \varepsilon^2\sigma_a/\sigma_t}{4\pi}\phi + \frac{\varepsilon^2 q}{4\pi\sigma_t}\right].$$

Next, we will integrate Eq. (2) upon the unit sphere and divide it by 4π. We need the following identity:

$$\frac{1}{4\pi}\int_{4\pi}\left(\frac{1}{\sigma_t}\hat{\Omega} \cdot \nabla\right)^n d\Omega = \frac{1 + (-1)^n}{2}\frac{1}{n+1}\left(\frac{1}{\sigma_t}\nabla\right)^n,$$

for $n = 0, 1, 2, \ldots$. Thus, Eq. (2) becomes

$$\frac{\phi}{4\pi} = \left(1 + \frac{\varepsilon^2}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2 + \frac{\varepsilon^4}{5}\left(\frac{1}{\sigma_t}\nabla\right)^4 + \frac{\varepsilon^6}{7}\left(\frac{1}{\sigma_t}\nabla\right)^6 + O(\varepsilon^8)\right)\left[\frac{1 - \varepsilon^2\sigma_a/\sigma_t}{4\pi}\phi + \frac{\varepsilon^2 q}{4\pi\sigma_t}\right]. \tag{3}$$

If there are non-vacuum boundary conditions, then extra terms occur in Eq. (3). However, these are $O(e^{-\rho/\varepsilon})$, where ρ is the optical distance to the boundary. Thus, in the interior of the system these terms are exponentially small and we will ignore them.

Now we invert the operator on the right-hand side of this equation and once again expand it in a power series:

$$\left(1+\frac{\varepsilon^2}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2+\frac{\varepsilon^4}{5}\left(\frac{1}{\sigma_t}\nabla\right)^4+\frac{\varepsilon^6}{7}\left(\frac{1}{\sigma_t}\nabla\right)^6+O(\varepsilon^8)\right)^{-1} =$$

$$1-\left(\frac{\varepsilon^2}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2+\frac{\varepsilon^4}{5}\left(\frac{1}{\sigma_t}\nabla\right)^4+\frac{\varepsilon^6}{7}\left(\frac{1}{\sigma_t}\nabla\right)^6+O(\varepsilon^8)\right)+$$

$$\left(\frac{\varepsilon^4}{9}\left(\frac{1}{\sigma_t}\nabla\right)^4+\frac{2\varepsilon^6}{15}\left(\frac{1}{\sigma_t}\nabla\right)^6+O(\varepsilon^8)\right)-$$

$$\left(\frac{\varepsilon^6}{27}\left(\frac{1}{\sigma_t}\nabla\right)^6+O(\varepsilon^8)\right)=$$

$$=1-\frac{\varepsilon^2}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2-\frac{4\varepsilon^4}{45}\left(\frac{1}{\sigma_t}\nabla\right)^4-\frac{44\varepsilon^6}{945}\left(\frac{1}{\sigma_t}\nabla\right)^6+O(\varepsilon^8)\,.$$

Thus, Eq. (3) becomes

$$\left(1-\frac{\varepsilon^2}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2-\frac{4\varepsilon^4}{45}\left(\frac{1}{\sigma_t}\nabla\right)^4-\frac{44\varepsilon^6}{945}\left(\frac{1}{\sigma_t}\nabla\right)^6+O(\varepsilon^8)\right)\phi=\left(1-\varepsilon^2\sigma_a/\sigma_t\right)\phi+\frac{\varepsilon^2 q}{\sigma_t}\,,$$

or

$$-\sigma_t\left(\frac{1}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2+\frac{4\varepsilon^2}{45}\left(\frac{1}{\sigma_t}\nabla\right)^4+\frac{44\varepsilon^4}{945}\left(\frac{1}{\sigma_t}\nabla\right)^6+O(\varepsilon^6)\right)\phi+\sigma_a\phi=q\,. \tag{4}$$

If we now retain terms of $O(\varepsilon^{2n})$ but discard all higher order terms, we obtain a partial differential equation for ϕ of order $2n$. This equation is an asymptotic approximation to Eq. (3), but it is *not* any of the simplifieded P_N approximations. To derive these approximations, we must rewrite the equation obtained from Eq. (4) in an asymptotically equivalent form as either a single second-order equation or as a coupled system of second-order equations. We will do that now for SP_1, SP_2, and SP_3.

Diffusion Equation (P_1)

We delete terms of $O(\varepsilon^2)$ and higher in Eq. (4) to get

$$-\nabla\cdot\frac{1}{3\sigma_t}\nabla\phi+\sigma_a\phi=q.$$

Multiplying this by ε and going back to the original unscaled parameters, we get the diffusion equation (P_1 or SP_1):

$$-\nabla \cdot \frac{1}{3\Sigma_t(\vec{r})}\nabla\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = Q(\vec{r}).$$

SP_2 Equation

We delete terms of $O(\varepsilon^4)$ and higher in Eq. (4) and rearrange the terms:

$$\left(I + \frac{4\varepsilon^2}{15}\left(\frac{1}{\sigma_t}\nabla\right)^2\right)\frac{1}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2\phi = \frac{\sigma_a\phi - q}{\sigma_t}.$$

Then we operate on this equation by $\left(I - \frac{4\varepsilon^2}{15}\left(\frac{1}{\sigma_t}\nabla\right)^2\right)$ and discard terms of $O(\varepsilon^4)$ to get

$$\frac{1}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2\phi = \left(I - \frac{4\varepsilon^2}{15}\left(\frac{1}{\sigma_t}\nabla\right)^2\right)\frac{\sigma_a\phi - q}{\sigma_t},$$

which simplifies to

$$-\nabla \cdot \frac{1}{3\sigma_t}\nabla\left(\phi + \frac{4\varepsilon^2}{5}\frac{\sigma_a\phi - q}{\sigma_t}\right) + \sigma_a\phi = q.$$

Multiplying this by ε and going back to the original unscaled parameters, we get the simplified P_2 equation:

$$-\nabla \cdot \frac{1}{3\Sigma_t(\vec{r})}\nabla\left(\phi(\vec{r}) + \frac{4}{5}\frac{\Sigma_a(\vec{r})\phi(\vec{r}) - Q(\vec{r})}{\Sigma_t(\vec{r})}\right) + \Sigma_a(\vec{r})\phi(\vec{r}) = Q(\vec{r}).$$

SP_3 Equations

We delete terms of $O(\varepsilon^6)$ and higher in Eq. (4) and rearrange the terms:

$$-\sigma_t\frac{1}{3}\left(\frac{1}{\sigma_t}\nabla\right)^2\left(\phi + \frac{4\varepsilon^2}{15}\left(\frac{1}{\sigma_t}\nabla\right)^2\phi + \frac{44\varepsilon^4}{315}\left(\frac{1}{\sigma_t}\nabla\right)^4\phi\right) + \sigma_a\phi = q. \tag{5}$$

Now we define

$$\phi_2 = \frac{2\varepsilon^2}{15}\left(\frac{1}{\sigma_t}\nabla\right)^2\phi + \frac{22\varepsilon^4}{315}\left(\frac{1}{\sigma_t}\nabla\right)^4\phi = \left(I + \frac{11\varepsilon^2}{21}\left(\frac{1}{\sigma_t}\nabla\right)^2\right)\frac{2\varepsilon^2}{15}\left(\frac{1}{\sigma_t}\nabla\right)^2\phi \qquad (6)$$

and rewrite Eq. (5) as

$$-\nabla\cdot\frac{1}{3\sigma_t}\nabla\left(\phi + 2\phi_2\right) + \sigma_a\phi = q. \qquad (7)$$

Operating on Eq. (6) by

$$\left(I - \frac{11\varepsilon^2}{21}\left(\frac{1}{\sigma_t}\nabla\right)^2\right)$$

and discarding terms of $O(\varepsilon^6)$, we obtain

$$\left(I - \frac{11\varepsilon^2}{21}\left(\frac{1}{\sigma_t}\nabla\right)^2\right)\phi_2 = \frac{2\varepsilon^2}{15}\left(\frac{1}{\sigma_t}\nabla\right)^2\phi,$$

which simplifies to

$$-\frac{1}{\sigma_t}\nabla\cdot\frac{1}{3\sigma_t}\nabla\left(\frac{11\varepsilon^2}{7}\phi_2 + \frac{2\varepsilon^2}{5}\phi\right) + \phi_2 = 0. \qquad (8)$$

Multiplying Eq. (7) by ε and going back to the original unscaled parameters, we get

$$-\nabla\cdot\frac{1}{3\Sigma_t(\vec{r})}\nabla\left(\phi(\vec{r}) + 2\phi_2(\vec{r})\right) + \Sigma_a(\vec{r})\phi(\vec{r}) = Q(\vec{r});$$

similarly, multiplying Eq. (8) by σ_t/ε and going back to the original unscaled parameters, we get

$$-\nabla\cdot\frac{1}{3\Sigma_t(\vec{r})}\nabla\left(\frac{11}{7}\phi_2(\vec{r}) + \frac{2}{5}\phi(\vec{r})\right) + \Sigma_t(\vec{r})\phi_2(\vec{r}) = 0.$$

These are the simplified P_3 equations for Eq. (1).

NOTE: In the previous class we had obtained the following SP_3 equations for a problem with an isotropic source:

$$-\nabla\cdot\frac{1}{3[\Sigma_t - \Sigma_{s1}]}\nabla\phi_0 - \nabla\cdot\frac{2}{3[\Sigma_t - \Sigma_{s1}]}\nabla\phi_2 + \Sigma_a\phi_0 = s_0\,,$$

$$-\nabla\cdot\frac{2}{15[\Sigma_t - \Sigma_{s1}]}\nabla\phi_0 - \nabla\cdot\left(\frac{4}{15[\Sigma_t - \Sigma_{s1}]} + \frac{9}{35[\Sigma_t - \Sigma_{s3}]}\right)\nabla\phi_2 + [\Sigma_t - \Sigma_{s2}]\phi_2 = 0\,.$$

If scattering is isotropic, $\Sigma_{s1} = \Sigma_{s2} = \Sigma_{s3} = 0$, and these equations simplify to

$$
\begin{aligned}
&-\nabla \cdot \frac{1}{3\Sigma_t}\nabla\phi_0 - \nabla \cdot \frac{2}{3\Sigma_t}\nabla\phi_2 + \Sigma_a\phi_0 = s_0\,, \\
&-\nabla \cdot \frac{2}{15\Sigma_t}\nabla\phi_0 - \nabla \cdot \left(\frac{4}{15\Sigma_t} + \frac{9}{35\Sigma_t}\right)\nabla\phi_2 + \Sigma_t\phi_2 = 0\,,
\end{aligned}
$$

reducing to the SP_3 equations derived above.

These notes are derived from Ryan McClarren's review paper on the SP_N equations: "Theoretical Aspects of the Simplified P_n Equations", Transport Theory and Statistical Physics 39: 73–109, 2011; and from the conference paper by Larsen, Morel, and McGhee: "Asymptotic Derivation of the Simplified P_N Equations", Proceedings of the ANS - M&C Topical Meeting in Karlsruhe, Germany (1993).