NE 255, Class 2, Fa16
Types of Equations in the Engineering Fields
August 30, 2016

Introduction

In science and engineering in general, and nuclear engineering and reactor analysis in specific, we encounter a wide range of mathematical physics equations. In today's lecture we will introduce some of them.

- Ordinary differential equations (ODEs)
- Partial differential equations (PDEs)
  - Elliptic PDEs
  - Parabolic PDEs
  - Hyperbolic PDEs
- Integro-differential equations
- Integral equations

1 ODEs

The most general form of an n$^{\text{th}}$ order linear ordinary differential eqn. is

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_2(x)y^{(2)}(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

where

- a_n are coefficients
- $y^{(n)}$ is the n$^{\text{th}}$ derivative of y.

Boundary conditions:

1. Initial Value Problem (**IVP**): if y and its derivatives are given at one end of the domain (e.g. time zero if there's time or spatial starting point if there's only space, etc.)

2. Boundary Value Problem (**BVP**): if y and/or its derivatives are given at each end of the interval

Linear 1st order ODE's

Reminders

- 1st order means that $n = 1$. The coefficients a_1 and a_0 may depend on y or y'.
- Linear means each coefficient only depends on x (i.e., not on y or derivatives of y).

Linear 1st order ODE Example:

$$\frac{dy}{dx} + 3y(x) = \sin(x) \qquad x \in [0, 1]$$

- IVP if boundary conditions are y(0) = 1; y'(0) = 2
- BVP if boundary conditions are y(0)=-1, y(1) = 3

In this case the general solution is obtained through the use of an integrating factor.

2nd order ODE Example:

$$-\frac{d}{dx}p(x)\frac{d}{dx}\phi(x) + q(x)\phi(x) = S(x)$$

defined for $a \leq x \leq b$

BC: $\alpha(x)\frac{d\phi}{dx} + \gamma(x)\phi(x) = \sigma$ at $x = a$ and $x = b$ (BVP)

This has

- **Neumann** BCs if $\gamma = 0$ (specifies the values that the *derivative* of a solution is to take on the boundary of the domain)
- **Dirichet** BCs: if $\alpha = 0$ (specifies the values that a *solution* is to take on the boundary of the domain)
- **Mixed** BCs if [$\gamma \neq 0$ and $\alpha = 0$ at $x = a$] and [$\alpha \neq 0$ and $\gamma = 0$ at $x = b$] (the solution is required to satisfy a Dirichlet or a Neumann boundary condition in a mutually exclusive way on disjoint parts of the boundary)

- If $S(x)$ is nonzero at least somewhere over the physical range, a unique solution exists.

2 PDEs

A partial differential equation is an equation containing an unknown function of two or more variables and its derivatives with respect to those variables.

If the PDE is linear in u and all derivatives of u, then we say that the PDE is linear.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu(x, y) = G$$

This equation is a 2nd order PDE in two variables. It is linear if A through G do not depend on u (they may depend on x and/or y).

Classification of PDEs:

Just as one classifies conic sections and quadratic forms into parabolic, hyperbolic, and elliptic based on the discriminant $B^2 - 4AC$, the same can be done for a second-order PDE at a given point.

[To think about classification, think about replacing ∂x by x and ∂y by y (formally this is done via Fourier transform). This converts the PDE into a polynomial of the same degree.]

Note: these classifications only apply to second order PDEs.

The reason we care about this in the context of the Transport Equation:

- In a void, the transport equation is like a hyperbolic wave equation.
- For highly-scattering regions where Σ_s is close to Σ, the equation becomes elliptic for the steady-state case.
- If the scattering is forward-peaked then the equation is parabolic.
- **Elliptic** if $B^2 - 4AC < 0$.

Some famous elliptic PDEs:

$$\nabla^2 u = 0 \qquad \text{Laplace's eqn.}$$
$$\nabla^2 u = f(x) \qquad \text{Poisson's eqn.}$$
$$-\frac{\partial}{\partial x}D(x,y)\frac{\partial}{\partial x}\phi(x,y) - \frac{\partial}{\partial y}D(x,y)\frac{\partial}{\partial y}\phi(x,y) + \big(\Sigma_a(x,y) - \frac{1}{k}\nu\Sigma_f(x,y)\big)\phi(x,y) = 0$$

For each of these there's no B term, so $-4AC < 0$ (in the diffusion equation case since $D(x,y)$ is positive).

One property of constant coefficient elliptic equations is that their solutions can be studied using the Fourier transform.

- **Parabolic** if $B^2 - 4AC = 0$, e.g.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{1-D heat eqn.}$$
$$\frac{1}{v}\frac{\partial\phi(x,t)}{\partial t} = \frac{\partial}{\partial x}D(x,t)\frac{\partial}{\partial x}\phi(x,t) + \big(\nu\Sigma_f(x,t) \ - \Sigma_a(x,t)\big)\phi(x,t) + S(x,t)$$

There aren't B or C terms, so $-4AC = 0$

Equations that are parabolic at every point can be transformed into a form analogous to the heat equation by a change of independent variables. Solutions smooth out as the transformed time variable increases.

A perturbation of the initial (or boundary) data of an *elliptic or parabolic* equation is felt at once by essentially all points in the domain.

- **Hyperbolic** if $B^2 - 4AC > 0$, e.g.

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{1-D wave eqn.}$$

There's no B term, and the C term is negative so $-4AC > 0$

  - if u and its first t derivative are arbitrarily specified with initial data on the initial line $t = 0$ (with sufficient smoothness properties), then there exists a solution for all of t.
  - The solutions of hyperbolic equations are "wave-like." If a disturbance is made in the initial data of a hyperbolic differential equation, then not every point of space feels the disturbance at once.

– Relative to a fixed time coordinate, disturbances have a finite propagation speed. They travel along the characteristics of the equation.

higher order PDE classification

If there are n independent variables $x_1, x_2, \ldots, x_n$, a general linear partial differential equation of second order has the form

$$Lu = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \frac{\partial^2 u}{\partial x_i x_j} + \text{ Lower Order Terms } = 0$$

The classification depends upon the signature of the eigenvalues of the coefficient matrix $a_{i,j}$.

1. Elliptic: The eigenvalues are all positive or all negative.
2. Parabolic: The eigenvalues are all positive or all negative, save one that is zero.
3. Hyperbolic: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.
4. Ultrahyperbolic: There is more than one positive eigenvalue and more than one negative eigenvalue, and there are no zero eigenvalues. There is only limited theory for ultrahyperbolic equations (Courant and Hilbert, 1962).

Integro-Differential Equations

...are equations that involves both integrals and derivatives of a function. The general first-order, linear (only with respect to the term involving the derivative) integro-differential equation is of the form

$$\frac{d}{dx}u(x) + \int_{x_0}^{x} f(t, u(t))dt = g(x, u(x)) \, , \qquad u(x_0) = u_0 \, , \qquad x_0 \geq 0 \, .$$

This is the equation type we will likely deal with the most.

Nuclear Example:

One-dimensional in space, one-dimensional in angle, time-independent, monoenergetic neutron transport equation:

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \Sigma_t \psi(x,\mu) = \frac{\Sigma_s}{2} \int_{-1}^{1} d\mu' \psi(x,\mu') + S(x,\mu)$$

where the angular neutron flux is a function of one spatial variable (x) and one anglular variable

($\mu = \cos(\theta)$).

Integral Equations

...are equations in which an unknown function appears under an integral sign. Integral equations are classified according to three different dichotomies, creating eight different kinds:

1. Limits of integration

   (a) both fixed: Fredholm equation

   $$f(x) = \int_a^b K(x,t)\,\varphi(t)\,dt$$

   (b) one variable: Volterra equation

   $$f(x) = \int_a^x K(x,t)\,\varphi(t)\,dt$$

2. Placement of unknown function

   (a) only inside integral: first kind (both above examples)

   (b) both inside and outside integral: second kind

   $$\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\,\varphi(t)\,dt$$

3. Nature of known function, f

   (a) identically zero: homogeneous

   (b) not identically zero: inhomogeneous

Both Fredholm and Volterra equations are linear integral equations, due to the linear behaviour of $\phi(x)$ under the integral. A nonlinear Volterra integral equation has the general form:

$$\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\,F(x,t,\varphi(t))\,dt$$

where F is a known function.

The integral form of the Neutron Transport Equation is

$$\psi(\vec{r}, \hat{\Omega}, E) = \int_0^\infty d\rho' \ \exp[-\int_0^{\rho'} d\rho'' \ \Sigma_t(\vec{r} - \rho''\hat{\Omega}, E)] q(\vec{r} - \rho'\hat{\Omega}, \hat{\Omega}, E)$$

where q contains fixed, inscattering, and fission sources. That is, $q = f(\psi)$.