

NE 255, Fa16
Simplifying the Transport Equation
September 15, 2016

Recall that last time we created a full version of the transport equation that looked like:

$$\begin{aligned} \frac{1}{v} \frac{\partial \psi}{\partial t}(\vec{r}, E, \hat{\Omega}, t) + \hat{\Omega} \cdot \nabla \psi(\vec{r}, E, \hat{\Omega}, t) + \Sigma_t(\vec{r}, E) \psi(\vec{r}, E, \hat{\Omega}, t) \\ = \int_0^\infty \int_{4\pi} \Sigma_s(\vec{r}, E' \rightarrow E, \hat{\Omega}' \rightarrow \hat{\Omega}) \psi(\vec{r}, E', \hat{\Omega}', t) d\hat{\Omega}' dE' \\ + \frac{\chi_p(E)}{4\pi} \int_0^\infty \int_{4\pi} \nu(E') \Sigma_f(\vec{r}, E') \psi(\vec{r}, E', \hat{\Omega}', t) d\hat{\Omega}' dE' \\ + S(\vec{r}, E, \hat{\Omega}, t). \end{aligned} \quad (1)$$

What does each term describe?

We talked a little bit about boundary and initial conditions, and I was teaching from a non-updated version of the notes. Here's what I missed: (see notes posted online, which are now up-to-date)...

Simplified Forms

At the end of class we went briefly into simplifications; we're going to spend more time on that in this class.

Time Independent

We'll spend most of our time on this version of the equation. We assume that losses and sources are balanced, and thus there is no rate of change with time:

$$\frac{\partial \psi}{\partial t} = 0$$

We can then remove the time dependence from all terms in our equation. As noted, for reactors we often

- solve a steady-state form of the equation,

- perform depletion calculations that characterize material evolution (using the Batemann equations, which we're skipping for now)
- solve a new steady-state calculation with updated material specifications.

One Speed

Assume all particles are at the same speed, $\vec{v} = v_0 \cdot \hat{\Omega}$. Thus

$$n(\vec{r}, v, \hat{\Omega}, t) = n(\vec{r}, \hat{\Omega}, t) \delta(v - v_0)$$

$$\Sigma_s(E' \rightarrow E, \hat{\Omega}' \rightarrow \hat{\Omega}) = \Sigma_s(E, \hat{\Omega}' \rightarrow \hat{\Omega}) \delta(E' - E)$$

Now we can remove E integration and E dependence:

$$\frac{1}{v} \frac{\partial \psi(\vec{r}, \hat{\Omega}, t)}{\partial t} + \hat{\Omega} \cdot \nabla \psi(\vec{r}, \hat{\Omega}, t) + \Sigma_t \psi(\vec{r}, \hat{\Omega}, t) =$$

$$\int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(\vec{r}, \hat{\Omega}', t) + \frac{\nu \Sigma_f}{4\pi} \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, \hat{\Omega}', t) + S(\vec{r}, \hat{\Omega}, t)$$

Isotropic Source

It is often the case that an external source is (or we can approximate it as) isotropic:

$$S(\vec{r}, E, \hat{\Omega}, t) = \frac{1}{4\pi} S(\vec{r}, E, t)$$

One Dimensional and Rotational (Azimuthal) Symmetry

We also often simplify by only worrying about fewer dimensions. Here we'll look at just one: we'll get rid of x and y . Let's remind ourselves of how phase-space looks (see Figure 1):

We first assume that **things only vary in the z dimension**, so $\vec{r} \rightarrow z$, e.g.

$$\psi(\vec{r}, E, \hat{\Omega}, t) \rightarrow \psi(z, E, \hat{\Omega}, t)$$

We next assume (which is often true) that our systems is **azimuthally symmetric** (see Figure 2), which means that the scattering only depends on the angle of the scattering cosine, $\mu = \hat{\Omega}' \cdot \hat{\Omega}$, because it is symmetric about the azimuthal direction (φ). We can use this assumption to simplify two things.

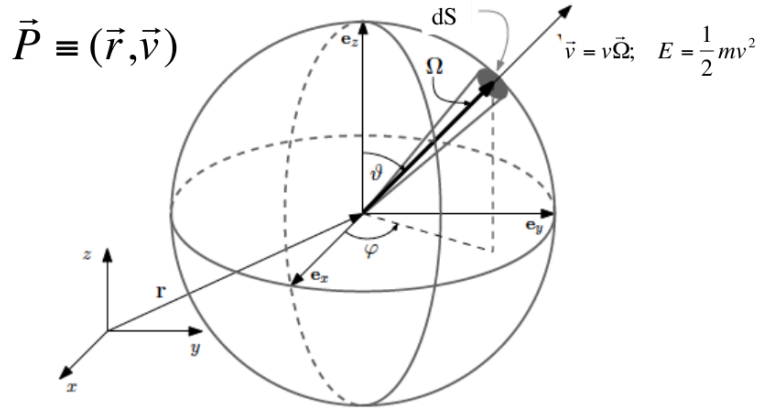


Figure 1: Schematic of Phase Space

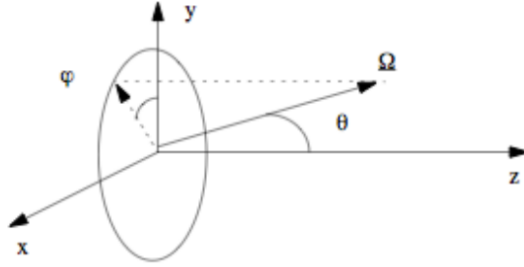


Figure 2: Azimuthal symmetry

We will use these terms in our simplifications:

$$d\hat{\Omega} = \sin(\theta)d\theta d\varphi = d\mu d\varphi; \quad \mu = \cos(\theta) \text{ so } d\mu = -\sin(\theta)d\theta; \quad \Omega_x = \cos(\theta) = \mu.$$

$$\int_{4\pi} d\hat{\Omega} = \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} \sin(\theta)d\theta = \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu = 4\pi$$

$$\psi(z, \hat{\Omega}, E, t)d\hat{\Omega} = \psi(z, \varphi, \mu, E, t)d\varphi d\mu = \psi(z, \mu, E, t)d\varphi d\mu$$

With azimuthal symmetry we can rewrite the scattering cross section since it is no longer a function of φ .

$$\Sigma_s(z, \hat{\Omega}' \rightarrow \hat{\Omega}) \rightarrow \Sigma_s(z, \hat{\Omega}' \cdot \hat{\Omega}) = \Sigma_s(z, \mu).$$

Further, we can execute the angular integration over φ since nothing depends on it anymore. For

example:

$$\begin{aligned}
\int_{4\pi} d\hat{\Omega} \psi(z, \hat{\Omega}, E, t) &= \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu \psi(z, \hat{\Omega}, E, t) \\
&= 2\pi \int_{-1}^1 d\mu \psi(z, \hat{\Omega}, E, t) \\
\psi(z, \mu, E, t) &= \frac{1}{2\pi} \psi(z, \hat{\Omega}, E, t) .
\end{aligned}$$

The next simplification is in the streaming term:

$$\hat{\Omega} \cdot \nabla \psi(\vec{r}, \hat{\Omega}, E, t) \rightarrow \Omega_z \frac{\partial \psi(z, \hat{\Omega}, E, t)}{\partial z} = \mu \frac{\partial \psi(z, \hat{\Omega}, E, t)}{\partial z}$$

We can combine all of this into a 1-D equation, using $\mu_0 = \hat{\Omega} \cdot \hat{\Omega}'$:

$$\begin{aligned}
\frac{1}{v} \frac{\partial \psi}{\partial t}(z, E, \hat{\Omega}, t) &+ \Omega_z \frac{\partial \psi(z, \hat{\Omega}, E, t)}{\partial z} + \Sigma_t(z) \psi(z, \hat{\Omega}, E, t) \\
&= \frac{\chi(E)}{4\pi} \int_0^\infty dE' \int_{4\pi} d\hat{\Omega}' \nu(E') \Sigma_f(z, E', t) \psi(z, \hat{\Omega}', E', t) \\
&+ \int_0^\infty dE' \int_{4\pi} d\hat{\Omega}' \Sigma_s(z, \mu_0) \psi(z, \hat{\Omega}', E', t) + S(z, E, \hat{\Omega}, t) ,
\end{aligned}$$

which we can then integrate over the φ component of angle:

$$\begin{aligned}
\frac{1}{v} \frac{\partial \psi}{\partial t}(z, E, \hat{\Omega}, t) &+ \mu \frac{\partial \psi(z, \mu, E, t)}{\partial z} + \Sigma_t(z) \psi(z, \mu, E, t) \\
&= 2\pi \frac{\chi(E)}{4\pi} \int_0^\infty dE' \nu(E') \Sigma_f(z, E', t) \int_{-1}^1 d\mu' \psi(z, \mu', E', t) \\
&+ 2\pi \int_0^\infty dE' \int_{-1}^1 d\mu' \Sigma_s(z, \mu_0) \psi(z, \mu', E', t) + 2\pi S(z, E, \mu, t)
\end{aligned}$$

The fission terms becomes

$$\frac{\chi(E)}{2} \int_0^\infty dE' \nu(E') \Sigma_f(z, E', t) \underbrace{\phi(z, E', t)}_{\text{scalar flux}} .$$

And, if we have an isotropic source

$$2\pi S(z, E, \mu, t) \rightarrow \frac{S(z, E, t)}{2} .$$

Combination

If we combine one-speed, time-independent, isotropic source, one-dimensional, and azimuthally-symmetric, we get

$$\mu \frac{\partial \psi(z, \mu)}{\partial z} + \Sigma_t(z) \psi(z, \mu) = \frac{\nu \Sigma_f(z)}{2} \phi(z) + 2\pi \int_{-1}^1 d\mu' \Sigma_s(z, \mu_0) \psi(z, \mu') + \frac{S(z)}{2}.$$

Scattering

Our favorite friend, double differential scattering, can be simplified. Let's start by recalling

$$\Sigma_s(\vec{r}, E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) = \Sigma_s(\vec{r}, E') f_s(E', \hat{\Omega}' \rightarrow E, \hat{\Omega}).$$

We can represent the fractional probability as a product of two independent fractional probabilities (Which is technically an assumption, but is well-supported by reality):

$$f_s(E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) = f_{sE}(E' \rightarrow E) f_{s\Omega}(\hat{\Omega}' \rightarrow \hat{\Omega}).$$

In the **monoenergetic** case, $f_{sE}(E \rightarrow E') = 1$ and with **isotropic scattering**, $f_{s\Omega}(\hat{\Omega}' \rightarrow \hat{\Omega}) = \frac{1}{4\pi}$, which would combine to give $\Sigma_s(\vec{r}, E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) = \frac{\Sigma_s(\vec{r})}{4\pi}$.

If **scattering is not isotropic**, what do we do? That is, when we can't simplify so easily, we at least need to represent it in a way we can numerically manage. There are several approaches, but this one makes the most sense to me (different from Duderstadt and Hamilton).

We expand the scattering cross section in Legendre Polynomials, which are a sequence of orthogonal polynomials:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n],$$

as follows

$$\Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \Sigma_{sl} P_l(\mu_0) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \Sigma_{sl} P_l(\hat{\Omega}') P_l(\hat{\Omega}).$$

(Note that we used the Legendre Addition Theorem, which uses spherical harmonics. You can learn more here:

https://en.wikipedia.org/wiki/Legendre_polynomials

https://en.wikipedia.org/wiki/Spherical_harmonics,
but for now just take my word for the fact that this works).

Legendre polynomials are solutions to Legendre's differential equations,

$$\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} P_n(x) \right] + n(n+1)P_n(x) = 0 ,$$

and are orthogonal on $[-1, 1]$:

$$\int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{n,m} = \frac{2}{2n+1} \text{ if } n = m, \quad 0 \text{ if } n \neq m.$$

$$\therefore P_0(\mu) = 1$$

$$P_1(\mu) = \mu .$$

We can see the first several polynomials in Figure 3.

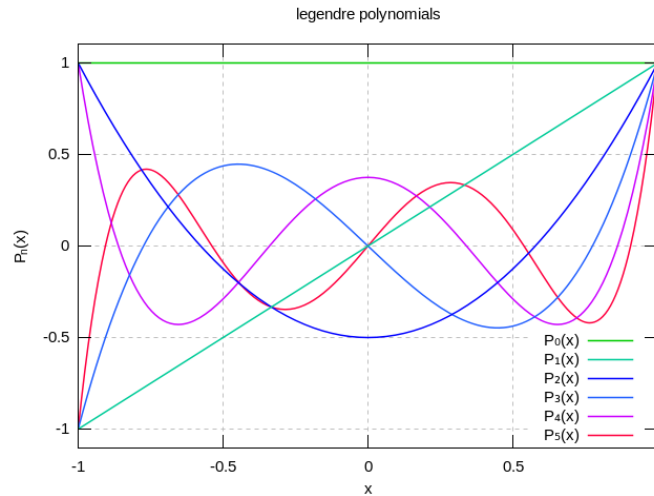


Figure 3: Legendre polynomials

We get the scattering expansion coefficients from the orthogonality of Legendre Polynomials. Any function can be expanded on the interval $-1 \leq x \leq 1$

$$f(x) = \sum_{l=0}^{\infty} \frac{2n+1}{2} f_n P_n(x)$$

To find f_n , multiply by P_l and integrate over $(-1, 1)$:

$$\begin{aligned}\int_{-1}^1 dx f(x) P_l(x) &= \sum_{l=0}^{\infty} \frac{2n+1}{2} f_n \int_{-1}^1 dx P_n(x) P_l(x) \\ \therefore f_l &= \int_{-1}^1 dx f(x) P_l(x) \\ \text{and } \Sigma_{sl} &= 2\pi \int_{-1}^1 d\mu_0 P_l(\mu_0) \Sigma_s(\mu_0) .\end{aligned}$$

All of that was to say, we can actually deal with scattering when it's *not* isotropic by expressing the scattering cross section as an expansion in Legendre polynomials. In our simplified transport equation, this looks like:

$$\begin{aligned}\mu \frac{\partial \psi(z, \mu)}{\partial z} + \Sigma_t(z) \psi(z, \mu) &= \\ \frac{\nu \Sigma_f(z)}{2} \phi(z) + \sum_{l=0}^{\infty} \frac{2l+1}{2} \Sigma_{sl}(z) \int_{-1}^1 d\mu' P_l(\mu_0) \psi(z, \mu') + \frac{S(z)}{2} .\end{aligned}$$

In reality, we truncate the expansion at some point. Some common truncations:

$$\begin{aligned}l = 0 \text{ is isotropic; (note } P_0(\hat{\Omega}) &= 1) \\ \Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) &\cong \frac{1}{4\pi} \Sigma_{s0} \\ l = 1 \text{ is linearly isotropic; (note } P_1(\hat{\Omega}) &= \hat{\Omega}) \\ \Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) &\cong \frac{1}{4\pi} (\Sigma_{s0} + 3\hat{\Omega}' \cdot \hat{\Omega} \Sigma_{s1})\end{aligned}$$

In describing representations of the TE, people commonly describe call out the scattering by saying the P_n expansion of scattering.