2.13 Inverse Error Function and Inverse Complementary Error Function

A. Purpose

Given x, this pair of subprograms computes the value t such that

$$x = \text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds$$

for the Inverse Error Function, or

$$x = \text{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-s^2} ds$$

for the Inverse Complementary Error Function.

Procedures in Chapter 15.2 may be used to calculate the inverse of the Gaussian or normal probability integral.

Reference [1] provides further discussion of the properties of the error function.

B. Usage

B.1 Program Prototype, Single Precision

REAL X, SERFI, SERFCI, T

Assign a value to X and use one of the following function references.

To compute the Inverse Error Function:

$$\boxed{\textbf{T = SERFI(X)}}$$

To compute the Inverse Complementary Error Function:

$$\boxed{\textbf{T = SERFCI(X)}}$$

B.2 Argument Definitions

X [in] Argument of function. Require $-1 < \text{X} < 1$ for SERFI, $0 < \text{X} < 2$ for SERFCI.

B.3 Modification for Double Precision

For double precision usage change the REAL type statement to DOUBLE PRECISION and change the function names to DERFI and DERFCI respectively.

C. Examples and Remarks

See DRDERFI and ODDERFI for an example of the usage of these subprograms.

$|\frac{\text{d}}{\text{dx}}\text{erf}^{-1}(x)|$ and $|\frac{\text{d}}{\text{dx}}\text{erfc}^{-1}(x)|$ are $\frac{\sqrt{\pi}}{2}e^{x^2}$. Therefore, the relative accuracy to be expected in evaluating these functions decreases rapidly as x increases. For example, when using IEEE single-precision arithmetic, one should expect no more than one digit of the result to be correct when $x >\approx 3.6$

D. Functional Description

The computer approximations for these functions were developed by A. Strecok, [2], using Chebyshev polynomial expansions.

These subprograms were tested on the IBM PC/AT, which uses IEEE arithmetic with precision, $\rho \approx 1.19 \times 10^{-7}$ in single precision and $\rho \approx 2.22 \times 10^{-16}$ in double precision. We checked how well the Error Function subprograms (SERF and SERFC or DERF and DERFC) and the present subprograms are inverses. Two sets of relative error tests were performed:

$$\varepsilon_1 = |\text{erf}(\text{erfi}(x))/x - 1| \ / \ \rho, \ \text{ and}$$
$$\varepsilon_2 = |\text{erfi}(\text{erf}(x))/x - 1| \ / \ \rho$$

and similarly for the complementary functions. The latter test does not measure relative error precisely for the complementary functions, and was not carried out in the range $1.0\text{E}{-}35 \le x \le 1.0\text{E}{-}5$ ($1.0\text{E}{-}300 \le x \le 1.0\text{E}{-}5$ in double precision). The results are summarized below.

Function	Argument Interval	Max. ε_1	Max. ε_2
SERFI	[0.1E−2, 0.8]	0.956	0.993
	[0.8, 0.9975]	0.625	1.66
	[.9975, .99999]	0.0	1.50
SERFCI	[1.0E−35, 1.0E−5]	113	
	[1.E−5, 2.5E−3]	16.3	1.91E+4
	[2.5E−3, 0.2]	14.2	86.7
	[0.2, 0.999]	2.14	1.92
DERFI	[0.1E−2, 0.8]	0.896	0.973
	[0.8, 0.9975]	0.625	1.59
	[.9975, .99999]	0.0	1.50
DERFCI	[1.0E−300, 1.0E−5]	964	
	[1.E−5, 2.5E−3]	13.6	1.90E+4
	[2.5E−3, 0.2]	8.29	64.6
	[0.2, 0.999]	1.69	1.60

References

1. Milton Abramowitz and Irene A. Stegun, **Handbook of Mathematical Functions**, *Applied Mathematics Series 55*, National Bureau of Standards (1966) Chapter 7, 295–330.

2. A. Strecok, *On the calculation of the inverse of the error function*, **Math. of Comp. 22** (1968) 144–158.

E.   Error Procedures and Restrictions

These subprograms issue an error message and terminate execution if

 (a)   X ≤ −1 or X ≥ 1 for SERFI, or

 (b)   X ≤ 0 or X ≥ 2 for SERFCI.

The error message is issued by way of the error message processor at level 0, and the returned function value is 0.0. If it is desired, in the event an error occurs, to issue the error message and then terminate program execution

   CALL ERMSET (2)

before invoking SERFI or SERFCI.

See Chapter 19.2 for further description of the error message processor.

F.   Supporting Information

The source language is ANSI Fortran 77.

Subprograms designed and developed by W. V. Snyder, JPL, 1986.

Entry	Required Files
DERFCI	AMACH, DERFI, DERM1, DERV1, ERFIN, ERMSG
DERFI	AMACH, DERFI, DERM1, DERV1, ERFIN, ERMSG
SERFCI	AMACH, ERFIN, ERMSG, SERFI, SERM1, SERV1
SERFI	AMACH, ERFIN, ERMSG, SERFI, SERM1, SERV1

DRDERFI

```
c       program DRDERFI
c>> 1996−06−27 DRDERFI  Krogh Minor change in formats for C conversion.
c>> 1996−05−28 DRDERFI  Krogh Added external statement.
c>> 1994−10−19 DRDERFI  Krogh   Changes to use M77CON
c>> 1992−04−21 DRDERFI   CLL
c>> 1991−11−19 CLL
c>> 1987−12−09 DRDERFI  Snyder   Initial Code.
c
c       Demonstration driver for DERFI and DERFCI.
c       ————————————————————————————————————————————————————————————
        external D1MACH, DERFI, DERFCI
        double precision DERFI, DERFCI, R, X, D1MACH, FLOOR
        integer I
c       ————————————————————————————————————————————————————————————
c——D replaces ”?”: DR?ERFI, ?ERFI, ?ERFCI
c       ————————————————————————————————————————————————————————————
   90   format (11x,1p,g12.5,g29.14)
        print '(/11x,''DRDERFI.. Demo driver for DERFI and DERFCI''//
     *    16x,''X'',13x,''DERFI(X) = DERFCI(1−X)'')'
        R = 1.0d−5
        do 10 I = 1,4
           X = 1.0d0 − R
           R = 10.0d0 * R
           print 90, X, DERFI(x)
   10   continue
        X = 0.9d0
        do 20 I = 1,9
           print 90, X, DERFI(x)
           X = X − 0.1d0
   20   continue
        print 90, 1.0d−10, DERFI(1.0d−10)
        print 90, 0.0d0, DERFI(0.0d0)
c
        print '(/16x,''X'',13x,''DERFCI(X) = DERFI(1−X)'')'
        X = 1.0d−10
        FLOOR = D1MACH(1) * 1.0d11
        do 30 I = 1,10
           print 90, X, DERFCI(X)
           if (X .lt. FLOOR) go to 40
```

```
      X = 1.0d−10 * X
30 continue
40 continue
   stop
   end
```

ODDERFI

DRDERFI.. Demo driver for DERFI and DERFCI

X	DERFI(X) = DERFCI(1−X)
0.99999	3.1234132743416
0.99990	2.7510639057121
0.99900	2.3267537655135
0.99000	1.8213863677184
0.90000	1.1630871536767
0.80000	0.90619380243682
0.70000	0.73286907795922
0.60000	0.59511608144999
0.50000	0.47693627620447
0.40000	0.37080715859356
0.30000	0.27246271472675
0.20000	0.17914345462129
0.10000	8.88559904942578E−02
1.00000E−10	8.86226925452758E−11
0.0000	0.0000000000000

X	DERFCI(X) = DERFI(1−X)
1.00000E−10	4.5728249673895
1.00000E−20	6.6015806223551
1.00000E−30	8.1486162231699
1.00000E−40	9.4487897667209
1.00000E−50	10.592090169527
1.00000E−60	11.624369672300
1.00000E−70	12.572745194633
1.00000E−80	13.454826697933
1.00000E−90	14.282814856163
1.00000−100	15.065574702593