MA8251 Engineering Mathematics II
Unit I Matrices

Problem 1. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution:

The characteristic equation is $| A - \lambda I | = 0$.

$$\text{i.e.,} \quad \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

i.e., $(-2 - \lambda)\,[-\lambda(1 - \lambda) -12] - 2[-2\lambda - 6] -3[-4 + 1 - \lambda] = 0$

i.e., $(-2 - \lambda)\,[\lambda^2 - \lambda -12] + 4\lambda + 12 + 9 + 3\lambda = 0$

i.e., $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ (1)

Now, $(-3)^3 + (-3)^2 - 21(-3) - 45 = -27 + 9 + 63 - 45 = 0$

$\therefore$ -3 is a root of equation (1).

Dividing $\lambda^3 + \lambda^2 - 21\lambda - 45$ by $\lambda + 3$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

Remaining roots are given by

$\lambda^2 - 2\lambda - 15 = 0$

i.e., $(\lambda + 3)(\lambda - 5) = 0$

i.e., $\lambda = -3, 5$.

$\therefore$ The eigen values are -3, -3, 5

The eigen vectors of A are given by $\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case 1 $\lambda = -3$

$$\text{Now} \begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_1 + 2x_2 - 3x_3 = 0$

Put $\quad x_2 = k_1,\ x_3 = k_2$

Then $\quad x_1 = 3k_2 - 2k_1$

$\therefore$ The general eigen vectors corresponding to $\lambda = -3$ is $\begin{bmatrix} 3k_2 - 2k_1 \\ k_1 \\ k_2 \end{bmatrix}$

When $k_1 = 0,\ k_2 = 1$, we get the eigen vector $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

When $k_1 = 1,\ k_2 = 0$, we get the eigen vector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Hence the two eigen vectors corresponding to $\lambda = -3$ are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

These two eigen vectors corresponding to $\lambda = -3$ are linearly independent.

Case 2 $\lambda = 5$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad -x_1 - 2x_2 - 5x_3 = 0$

$\qquad -8x_2 - 16x_3 = 0$

A solution is $x_3 = 1,\ x_2 = -2,\ x_1 = -1$

$\therefore$ Eigen vector corresponding to $\lambda = 5$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

Problem 2. Find the characteristic equation of $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$ and verify Cayley-Hamilton Theorem. Hence find the inverse of the matrix.

Solution: Let $A=\begin{bmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{bmatrix}$ $\therefore$ Characteristic eqn. of A is

$\lambda^3-\lambda^2[1+1-3]+\lambda[-9-9-1]+26=0$

i.e $\lambda^3+\lambda^2-19\lambda+26=0$

By **Cayley-Hamilton theorem** $\therefore A^3+A^2-19A+26I=0$.

Verification:

$$\therefore A^2=A.A=\begin{pmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{pmatrix}\begin{pmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{pmatrix}=\begin{pmatrix}9 & 2 & -7\\ 5 & 9 & -10\\ -10 & -7 & 21\end{pmatrix}$$

$$\therefore A^3=A^2.A=\begin{pmatrix}9 & 2 & -7\\ 5 & 9 & -10\\ -10 & -7 & 21\end{pmatrix}\begin{pmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{pmatrix}=\begin{pmatrix}-16 & -21 & 45\\ -43 & -16 & 67\\ 67 & 45 & -104\end{pmatrix}$$

Substituting in the characteristic equation

$$\begin{pmatrix}-16 & -21 & 45\\ -43 & -16 & 67\\ 67 & 45 & -104\end{pmatrix}+\begin{pmatrix}9 & 2 & -7\\ 5 & 9 & -10\\ -10 & -7 & 21\end{pmatrix}-\begin{pmatrix}19 & -19 & 38\\ -38 & 19 & 57\\ 57 & 38 & -57\end{pmatrix}+\begin{pmatrix}26 & 0 & 0\\ 0 & 26 & 0\\ 0 & 0 & 26\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Hence verified.

Now to find the inverse of the matrix A, premultiply the characteristic equation by A^{-1}

$\therefore A^2+A-19I+26A^{-1}=0$

$$\therefore A^{-1}=\frac{1}{26}\left(19I-A-A^2\right)$$

$$=\frac{1}{26}\left[\begin{pmatrix}19 & 0 & 0\\ 0 & 19 & 0\\ 0 & 0 & 19\end{pmatrix}-\begin{pmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{pmatrix}-\begin{pmatrix}9 & 2 & -7\\ 5 & 9 & -10\\ -10 & -7 & 21\end{pmatrix}\right]=\frac{1}{26}\begin{pmatrix}9 & -5 & 5\\ -3 & 9 & 7\\ 7 & 5 & 1\end{pmatrix}$$

Problem 3. Given $A=\begin{bmatrix}1 & 0 & 3\\ 2 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}$, use Cayley-Hamilton Theorem to find the inverse of A and also find A^4

Solution:

The characteristic equation of A is

$$\begin{vmatrix}1-\lambda & 0 & 3\\ 2 & 1-\lambda & -1\\ 1 & -1 & 1-\lambda\end{vmatrix}=0$$

i.e., $(1-\lambda)\,[(1-\lambda)(1-\lambda)-1]+3[-2-(1-\lambda)]=0$

i.e., $(1 - \lambda)^3 - (1 - \lambda) - 6 - 3 + 3\lambda = 0$

i.e., $1 - 3\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda - 9 + 3\lambda = 0$

i.e., $-\lambda^3 + 3\lambda^2 + \lambda - 9 = 0$

i.e., $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

To find A^{-1}, multiplying by A^{-1}, $A^2 - 3A - I + 9A^{-1} = 0$

$\therefore \quad A^{-1} = \frac{1}{9}[-A^2 + 3A + I]$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

To find A^4:

We have $A^3 - 3A^2 - A + 9I = 0$

i.e., $A^3 = 3A^2 + A - 9I$ (1)

Multiplying (1) by A, we get,

$A^4 = 3A^3 + A^2 - 9A$ $\therefore$

$= 3(3A^2 + A - 9I) + A^2 - 9A$ using (1)

$= 10A^2 - 6A - 27I$

$$= 10\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

Problem 4. . If $A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$ express $A^6 - 25A^2 + 122A$ as a single matrix

Solution: To avoid higher powers of A like A^6 we use Cayley Hamilton Theorem.

Characteristic equation is $\lambda^3 - 4\lambda^2 + 5\lambda + 2 = 0$

By Cayley Hamilton Theorem $A^3 - 4A^2 + 5A + 2I = 0$

To find $A^6 - 25A^2 + 122A$ we will express this in terms of smaller powers of A using the characteristics equation. We know that (Divisor) X (Quotient) + Remainder = Dividend

Assuming $A^3 - 4A^2 + 5A + 2I$ as the divisor we get,

$$
\begin{array}{r|l}
 & A^3+4A^2+11A+22I \\ \cline{2-2}
A^3-4A^2+5A+2I & A^6+0A^5+0A^4-25A^2+122A+0I \\
 & A^6-4A^5+5A^4+2A^3 \\ \cline{2-2}
 & \quad 4A^5-5A^4-2A^3-25A^2+122A \\
 & \quad 4A^5-16A^4+20A^3+8A^2 \\ \cline{2-2}
 & \quad 11A^4-22A^3-33A^2+122A \\
 & \quad 11A^4-44A^3+55A^2+22A \\ \cline{2-2}
 & \qquad 22A^3-88A^2+100A \\
 & \qquad 22A^3-88A^2+110A+44I \\ \cline{2-2}
 & \qquad\qquad -10A-44I
\end{array}
$$

$$\therefore A^6-25A^2+122A=\left(A^3-4A^2+5A+2I\right)\left(A^3+4A^2+11A+22I\right)+\left(-10A-44I\right)$$

But $A^3-4A^2+5A+2I=0$

$$A^6-25A^2+122A=0-10A-44I$$

$$=-\left(10A+44I\right)$$

$$=-\left[\begin{pmatrix} 0 & 0 & 20 \\ 20 & 10 & 0 \\ -10 & -10 & 20 \end{pmatrix}+\begin{pmatrix} 44 & 0 & 0 \\ 0 & 44 & 0 \\ 0 & 0 & 44 \end{pmatrix}\right]$$

$$=-\begin{pmatrix} 44 & 0 & 20 \\ 20 & 54 & 0 \\ -10 & -10 & 74 \end{pmatrix}$$

$$=-\begin{pmatrix} -44 & 0 & -20 \\ -20 & -54 & 0 \\ -10 & 10 & -74 \end{pmatrix}$$

Problem 5. If λi are the eigen values of the matrix A, then prove that

i $k\lambda i$ are the eigen values of kA where ‘k’ is a nonzero scalar.

ii. λ_i^m are the eigen value of A^m and

iii. $\dfrac{1}{\lambda i}$ are the eigen values of A^{-1}.

Solution: Let λi be the eigen values of matrix A and Xi be the corresponding eigen vectors. Then by defn: $AXi=\lambda iXi......(I)$ (i.e by defn. of eigen vectors)

i. Premultiply (I) with the scalar k. Then

$$k\left(AXi\right)=k\left(\lambda iXi\right)$$

$$i.e.\left(kA\right)X_i=\left(k\lambda i\right)Xi$$

$\therefore k\lambda i$ are the eigen values of kA (comparing with (I) i.e by defn.)

ii. Premultiply (I) with A, then

$A(AXi) = A(\lambda iXi)$

$i.e. A^2 X^i = \lambda i(AXi)$

$= \lambda i(\lambda_i Xi)$ from (I)

$= (\lambda i)^2 Xi$

III^{1y} we can prove that $A^3 Xi = (\lambda_i)^3 Xi$ and so on $A^m Xi = (\lambda i)^m Xi$

$\because \lambda i^m$ are the eigen values of the A^m (comparing with (I) i.e. by defn.)

iii. Premultiply (I) with A^{-1}, then

$A^{-1}(AXi) = A^{-1}(\lambda iXi)$

$i.e. (A^{-1}A)Xi = \lambda i(A^{-1}Xi)$

$i.e.\ IXi = \lambda i(A^{-1}Xi)$

$i.e. A^{-1}Xi = \dfrac{1}{\lambda i} Xi$

$\therefore \dfrac{1}{\lambda i}$ are the eigen values of A^{-1} (comparing with (I)).

Problem 6. Find the characteristic vectors of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and verify that they are mutually orthogonal.

Solution: A = $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ Characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Solving: $\lambda = 1, 2, 3$

Consider the matrix equation $(A - \lambda I)X = 0$

Case (i) when $\lambda = 1$;

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; i.e. \; \begin{array}{l} 1x_1 + 0x_2 + 1x_3 = 0 - (1) \\ 0x_1 + 1x_2 + 0x_3 = 0 - (2) \\ 1x_1 + 0x_2 + 1x_3 = 0 - (3) \end{array}$$ equation (1) & (3) are identical.

Solving (1) and (2) using the rule of cross multiplication

$$\frac{x_1}{0-1} = \frac{x_2}{0-1} = \frac{x_3}{0-1} \; i.e. \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1} \therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Case (ii) when $\lambda = 2$;

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{i.e.} \quad \begin{matrix} 0x_1 + 0x_2 + 1x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \\ 1x_1 + 0x_2 + 0x_3 = 0 \end{matrix} \qquad \begin{matrix} x_3 = 0 \\ \text{i.e. } x_2 \text{ is arbitrary } say\ k \\ x_1 = 0 \end{matrix}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} i.e \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Case (ii) when $\lambda = 3$;

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{i.e.} \quad \begin{matrix} -x_1 + 0x_2 + 1x_3 = 0 \\ 0x_1 + 1x_2 + 0x_3 = 0 \\ 1x_1 + 0x_2 + 1x_3 = 0 \end{matrix} \qquad \text{Solving (1) and (2)}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1} \therefore X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus the eigen values are 1,2,3 and the correspondent eigen vectors are

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \quad \text{To check orthogonallity,} \quad X_1^T X_2 = 0$$

$X_2^T X_3 = 0$

$X_1^T X_3 = 0$

$\therefore X_1, X_2, X_3$

are mutually orthogonal.

Problem 7. Find the latent vectors of $\begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$

Solution: Characteristic equation is $(\lambda + 1)^3 = 0 \therefore \lambda = -1, -1, -1$

When $\lambda = -1$ (repeated 3 times) $\therefore$ we have to find 3 corresponding latent vectors.

$$\begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{i.e.} \quad \begin{matrix} 7x_1 + 6x_2 + 5x_3 = 0 \\ 14x_1 - 12x_2 + 10x_3 = 0 \\ 7x_1 + 6x_2 + 5x_3 = 0 \end{matrix} \qquad \text{All three equation are identical}$$

.i.e. we get only one equation, but we have to find three vectors that are linearly independent.

$$\therefore \text{Assume } x_1 = 0 \Rightarrow -6x_2 + 5x_3 = 0 \, i.e. -6x_2 = -5x_3 \ i.e. \frac{x_2}{5} = \frac{x_3}{6} \therefore X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

Assume $x_2 = 0 \Rightarrow -7x_2 + 5x_3 = 0$ $i.e.\ 7x_1 = -5x_3 i.e.. \dfrac{x_1}{-5} = \dfrac{x_3}{7} \therefore X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}$

And assume $x_2 = 0 \Rightarrow 7x_2 - 6x_3 = 0$ $i.e.\ 7x_1 = 6x_2\ 0 i.e.. \dfrac{x_1}{6} = \dfrac{x_2}{7} \therefore X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$

X_1, X_2 and X_3 are linearly independent.

Problem 8. Find the eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

Solution:

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$

$$\begin{aligned}
\text{i.e., } (1-\lambda)[(2-\lambda)(3-\lambda) - 4] - 1[0+4] + 1[0 + 4(2-\lambda)] &= 0 \\
\text{i.e., } (1-\lambda)(\lambda^2 - 5\lambda + 6 - 4) - 4 + 8 - 4\lambda &= 0 \\
\text{i.e., } (1-\lambda)(\lambda^2 - 5\lambda + 2) + 4 - 4\lambda &= 0 \\
\text{i.e., } (1-\lambda)(\lambda^2 - 5\lambda + 2 + 4) &= 0 \\
\text{i.e., } (\lambda - 1)(\lambda^2 - 5\lambda + 6) &= 0 \\
\text{i.e., } (\lambda - 1)(\lambda - 2)(\lambda - 3) &= 0
\end{aligned}$$

$\therefore$ The eigen values of A are $\lambda = 1, 2, 3$.

The eigen vectors are given by $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case 1 $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} -4 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-4x_1 + 4x_2 + 2x_3 = 0$$
$$x_2 + x_3 = 0$$

A solution is, $x_3 = 2$, $x_2 = -2$, $x_1 = -1$

$\therefore$ Eigen vector $X_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$

Case 2 $\lambda = 2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0$$
$$x_3 = 0$$

A solution is, $x_3 = 0$, $x_2 = 1$, $x_1 = 1$

$\therefore$ Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Case 3 $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2x_1 + x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$

A solution is, $x_3 = 1$, $x_2 = 1$, $x_1 = 1$

$\therefore$ Eigen vector $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Problem 9. Diagonalise the matrix $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ using orthogonal transformation.

Solution: Characteristic equation is $\lambda^3 - 10\lambda^2 + 27 - 18 = 0$

Solving we get the eigen value as $\lambda = 1, 3, 6$

When $\lambda = 1, X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$; When $\lambda = 3, X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; When $\lambda = 6, X_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Normalizing each vector, we get $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}$

$\therefore$ Normalized Modal Matrix, $N=\begin{pmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix}$. $N'=N^T=\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}$,

Then by the orthogonal transformation,

$N'AN=\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 0 & 2/\sqrt{5} \\ 1/\sqrt{5} & 1 & 0 \end{pmatrix}$. On simplifying, we get

$N'AN=D(\lambda_1,\lambda_2,\lambda_3)$

$=D(1,3,6)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ which is diagonal matrix with eigen values along the diagonal (in order).

Problem 10. Reduce $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ to a diagonal matrix by orthogonal reduction.

Solution: Characteristic equation is $\lambda^3-12\lambda^2+36\lambda-32=0 \therefore \lambda=8,2,2$

When $\lambda=8$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e
$$-2x_1+2x_2+2x_3=0$$
$$-2x_1-5x_2+1x_3=0$$
$$2x_1-1x_2+5x_3=0$$

Solving any two equations $\frac{x_1}{2}=\frac{x_2}{-1}=\frac{x_3}{1} \therefore X_1=\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

When $\lambda=2$ (repeated twice)

$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ i.e $-2x_1+2x_2+2x_3=0$. All the equations are identical.

To get one of the vectors, assume $x_1 = 0 \Rightarrow x_2 - x_3 = 0$ $i.e. \frac{x_2}{1} = \frac{x_3}{1} \therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$X_1^T X_2 = 0$. Therefore X_1 and X_2 are orthogonal. Now assume $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to be mutually orthogonal with X_1 and X_2.

$$\left.\begin{array}{l} X_1^T X_3 = 0 \;\; i.e. \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \;\; i.e. 2a - b + c = 0 \\ and \; X_2^T X_3 = 0 \;\; i.e. \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \;\; i.e. 0a - b + c = 0 \end{array}\right\} i.e \frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

After normalizing these 3 mutually orthogonal vectors, we get the normalized Modal

Matrix $N = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix}$

Diagonalizing we get

$$D = N^T A N = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

on simplifying we get $D = D(\lambda_1, \lambda_2, \lambda_3)$

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= D(8, \;\; 2, \;\; 2)$$

Problem 11. Diagonalise the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Solution:

The characteristic equation of A is $\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} = 0$

i.e., $(\lambda-1)(\lambda^2 - 8\lambda + 16) = 0$

$\therefore$ The eigen values of A are $\lambda = 1, 4, 4$.

The eigen vectors are given by $\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case 1 $\lambda = 1$

Eigen vector $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Case 2 $\lambda = 4$

Eigen vector $X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now assume $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to be mutually orthogonal with X_1 and X_2.

$$\left.\begin{aligned} X_1^T X_3 = 0 \;\; i.e. -a+b+c=0 \\ and\; X_2^T X_3 = 0 \;\; i.e. -b+c=0 \end{aligned}\right\} \quad i.e \frac{a}{2} = \frac{b}{1} = \frac{c}{1}$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Hence the modal matrix $M = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

The Normalized Modal Matrix is $N = \begin{pmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$

Diagonalizing, we get

$$D = N^T AN = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \text{D}(1, 4, 4)$$

Problem 12. Reduce the Quadratic From $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ into canonical form by orthogonal reduction. Hence find the nature, rank, index and the signature of the Q.F. Find also a nonzero set of values of X which will make the Q.F. vanish.

Solution: Matrix of the given Q.F. is $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & -5 \end{pmatrix}$, which is a real and symmetric matrix. The characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

Solving, we get $\lambda = 0, \quad 3, \quad 14$

When $\lambda = 0, X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$; When $\lambda = 3, X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; When $\lambda = 14, X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

and X_1, X_2, X_3 are mutually orthogonal since $X_1^T, X_2 = 0, X_2^T X_3 = 0$ $and X_3^T X_1 = 0$

Normalizing these vectors we get the normalized model matrix

$$N = \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{pmatrix}$$

Diagonalising we get $D = N^T AN$

$= D(\lambda_1 \lambda_2, \lambda_3)$ in order

$= D(0, \quad 3, \quad 14)$

i.e $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ (i.e. the eigen values in order along the principal diagonal).

Now to reduce the Q.F to C.F (.i.e Canonical form)

Consider the orthogonal transformation X = NY where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Then the Q.F. $X^T AX$ becomes $(NY)^T A(NY) = Y^T (N^T AN) Y$

$= Y^T DY$ since $N^T AN = D$

$$= (y_1 y_2 y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0y_1^2 + 3y_2^2 + 14y_3^2$$

Thus = $0y_1^2 + 3y_2^2 + 14y_3^2$ is the Canonical form of the given Q.F. And the equations of this transformation are got from X= NY.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = NY = \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\therefore x_1 = \frac{1}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 - \frac{3}{\sqrt{14}} y_3$$

$$x_2 = -\frac{5}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 + \frac{3}{\sqrt{14}} y_3$$

$$x_3 = \frac{4}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 - \frac{3}{\sqrt{14}} y_3$$

To get the non-zero set of values of x which make the Q.F zero we assume values for y_1, y_2 and y_3 such that the C.F. vanishes.

i.e $0y_1^{\,2}+3y_2^{\,2}+14y_3^{\,2}$ will vanish if $y_2=0, y_3=0$ and y_1 is any arbitrary value (for simplicity sake, assume y_1 as the denominator of the coeff. of y_1 in the equations) let $y_1=\sqrt{42}$

$$\therefore x_1=\frac{1}{\sqrt{42}}\left(\sqrt{42}\right)+\frac{1}{\sqrt{3}}(0)-\frac{3}{\sqrt{14}}(0)$$

i.e. $x_1=1+0-0=1$

III^{1y} $x_2=-5+0+0=-5$

and $x_3=4+0-0=4$

Thus the set of values of x $i.e(1,\quad -5,\quad 4)$ will reduce the given Q.F. to zero.

To find the rank, index, signature and nature using canonical form:

C.F. is $0y_1^{\,2}+3y_2^{\,2}+14y_3^{\,2}$

$\therefore$ rank is 2 (no. of terms in C.F)
Index is 2 (no. of positive terms)
Signature of Q.F. = (no. of positive terms) – (no. of negative terms) = 2
Nature of the Q.F. is positive semi definite.

Problem 13. Reduce the Q.F. $2xy+2yz+2zx$ into a form of sum of squares. Find the rank, index and signature of it. Find also the nature of the Q.F.

Solution: Matrix of the Q.F. is $A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$

Characteristic equation is $\lambda^3-3\lambda-2=0$ solving $\lambda=2,-1,-1$

When $\lambda=2, X_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$

When $\lambda=-1$ (repeated twice) we get identical equations as $x_1+x_2+x_3=0$

$$x_1=0\Rightarrow x_2+x_3=0\ \ i.e.\ x_2=-x_3\ i.e.\frac{x_2}{-1}=\frac{x_3}{1}$$

Assume

$$\therefore X_2\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

which is orthogonal with X_1.

Now to find X_3 orthogonal with both X_1 and X_2 assume $X_3=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\left.\begin{array}{ll} if\ X_2^T X_3 = 0, & a+b+c=0 \\ if\ X_2^T X_3 = 0, & 0a-b+c=0 \end{array}\right\}$$

$$i.e.\ \frac{a}{2} = \frac{b}{-1} = \frac{c}{-1}$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad i.e. \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

which is orthogonal with X_1 and X_2.

Normalising these vectors we get $N = \begin{pmatrix} 1/\sqrt{3} & 0/\sqrt{2} & -3/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 2/\sqrt{6} \end{pmatrix}$ and $D = N'AN$

$= D(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.Consider the orthonormal transformation X = NY such that Q.F.is reduced to C.F.

The Q.F. is reduced as

$$\begin{aligned} X^T AX &= (NY)^T A(NY) \\ &= Y^T (N^T AN)Y \\ &= Y^T DY \\ &= (y_1, y_2, y_3,) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \end{aligned}$$

$\therefore$ The C.F. is $2y_1^2 - y_2^2 - y_3^2$

rank of Q.F.is = no. of terms in C.F=3

index of Q.F. = no. of positive terms in C.F. = 1

signature of Q.F. = (no. of positive terms) – (no. of negative terms)

= 1-2 = -1

Nature of the Q.F. is indefinite.

Problem 14. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 + 4x_1x_3 - 8x_2x_3$ to the canonical form by an orthogonal transformation. Find also the rank, index, signature and the nature of the quadratic form.

Solution:

The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The eigen values of this matrix are 0, 3 and 15 and the corresponding eigen vectors are

$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, which are mutually orthogonal.

The normalized modal matrix is $N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

and $N^TAN = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

Now the orthogonal transformation X = NY will reduce the given quadratic form to the canonical form $0y_1^2 + 3y_2^2 + 15y_3^2$.

Also rank = 2, index = 2, signature = 2. The quadratic form is positive semi definite.

Problem 15. Find the orthogonal transformation which reduces the quadratic form $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$ into the canonical form. Determine the rank, index, signature and the nature of the quadratic form.

Solution:

The matrix of the quadratic form is $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

Expanding $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\lambda = 1$ is a root

Dividing $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ by $\lambda - 1$,

$$\begin{array}{|cccc} 1 & -6 & 9 & -4 \\ 0 & 1 & -5 & 4 \\ \hline 1 & -5 & 4 & \underline{0} \end{array}$$

The remaining roots are given by $\lambda^2 - 5\lambda + 4 = 0$

$\lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0$

i.e., $\lambda = 1, 4$

$\therefore$ The eigen values of A are $\lambda = 4, 1, 1$

Case 1 $\lambda = 4$

The eigen vectors are given by $\begin{bmatrix} 2-4 & -1 & 1 \\ -1 & 2-4 & -1 \\ 1 & -1 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_1 - x_2 - 2x_3 = 0$

$-3x_2 - 3x_3 = 0$

A solution is $x_3 = 1$, $x_2 = -1$, $x_1 = 1$.

$\therefore$ The corresponding eigen vector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 2 $\lambda = 1$

The eigen vectors are given by $\begin{bmatrix} 2-1 & -1 & 1 \\ -1 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_1 - x_2 + x_3 = 0$

Put $x_3 = 0$. We get $x_1 = x_2 = 1$. Let $x_1 = x_2 = 1$

$\therefore$ The eigen vector corresponding to $\lambda = 1$ is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

X_1 and X_2 are orthogonal as $X_1^T X_2 = 1 \bullet 0 + (-1) \bullet 1 + 1 \bullet 1 = 0$.

To find another vector $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ corresponding to $\lambda = 1$ such that it is orthogonal to both

X_1 and X_2 and satisfies $x_1 - x_2 + x_3 = 0$

i.e., $X_1.X_3 = 0$, $X_2.X_3 = 0$ and $a - b + c = 0$

i.e., $1.a - 1.b + 1.c = 0$, $1.a + 1.b + 0.c = 0$ and $a - b + c = 0$.

i.e., $a - b + c = 0$ and $a + b = 0$

i.e., $a = -b$ and $c = 2b$

Put $b = 1$, so that $a = -1$, $c = 2$

$$\therefore \qquad X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

The modal matrix is $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Hence the normalized modal matrix is $N = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$

$\therefore$ The required orthogonal transformation is X = NY will reduce the given quadratic form to the canonical form.

C.F= $4y_1^2 + y_2^2 + y_3^2$

Rank of the quadratic form = 3, index = 3, signature = 3. The quadratic form is positive definite.

MA8251 – MATHEMATICS-II

VECTOR CALCULUS

1.Vector differential operator $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$.

2.Gradient of $\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

3. Divergence of $\vec{F} = \nabla \bullet \vec{F}$.

$$4.\, curl\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

5.If $\vec{F}$ *is solenoidal then* $\nabla \bullet \vec{F} = 0$

6. If $\vec{F}$ *is irrotational then* $\nabla \times \vec{F} = 0$

7.Maximum directional derivative = $|\nabla\phi|$

8.Directional derivative of ϕ **in the direction of** $\vec{a}$ **=** $\frac{\nabla\phi \bullet \vec{a}}{|\vec{a}|}$

9.Angle between two normal to the surface is cos θ **=** $\frac{\nabla\phi_1 \bullet \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$ **(given points)**

10.Unit normal vector , $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

11.In the surface integral $ds = \frac{dx\,dy}{|\hat{n}\bullet\vec{k}|}$**,** $ds = \frac{dy\,dz}{|\hat{n}\bullet\vec{i}|}$**,** $ds = \frac{dz\,dx}{|\hat{n}\bullet\vec{j}|}$

12. <u>Green's Theorem</u>

If P(x ,y) and Q(x ,y) are continuous function with continuous partial derivatives in a region R of the xy plane bounded by a simple closed curve C, then $\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

Where C is the curve described in positive direction

13.Gauss Divergence Theorem

The surface integral of the normal component of a vector function F over a closed surface S enclosing a volume V is equal to the volume integral of the divergence of F taken throughout the volume

$\iint\limits_{S} F \bullet \hat{n}\, dS = \iiint\limits_{V} div\vec{F}\, dv = \iiint\limits_{V} \nabla \bullet \vec{F}\, dv$ where , $\vec{n}$ is the unit out ward normal to the surface S .

14. **Stoke's Theorem**

If S is an open surface bounded by a simple closed curve C and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in S and on C, then

$$\iint\limits_{S} curl\vec{F} \cdot \hat{n}\, ds = \int\limits_{C} \vec{F} \cdot d\vec{r}$$

Where $\hat{n}$ is the unit vector normal to the surface.

MA8251 - MATHEMATICS II
UNIT II - VECTOR CALCULUS
CLASS NOTES

1. If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find $\phi(x,y,z)$ given that $\phi(1,-2,2)=4.$

Solution:

$$\nabla\phi = \vec{i}\ \frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z} \quad \to (1)$$

Given $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k} \quad \to (2)$

$\therefore$comparing (1) & (2)

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \quad \to (3)$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \quad \to (4)$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2 \quad \to (5)$$

Integrating (3) $w.r.t\ \ x$ (keeping y and z constant)

$$\phi = x^2yz^3 + f_1(y,z)$$

Integrating (3) $w.r.t\ \ y$ (keeping x and z constant)

$$\phi = x^2yz^3 + f_2(y,z)$$

Integrating (3) $w.r.t\ \ z$ (keeping x and y constant)

$$\phi = x^2yz^3 + f_3(y,z)$$

$\therefore \quad \phi = x^2yz^3 + c$ where c is a constant

Given $\phi(1,-2,2)=4 \qquad \therefore\ -16+c=4 \qquad c=20$

2. Find the values of constants a, b, c so that the maximum value of the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ $at\,(1,2,-1)$ has a magnitude 64 in the direction parallel to z-axis.

Solution:

$$\nabla\phi = \vec{i}\ \frac{\partial}{\partial x}(axy^2 + byz + cz^2x^3) + \vec{j}\ \frac{\partial}{\partial y}(axy^2 + byz + cz^2x^3) + \vec{k}\ \frac{\partial}{\partial z}(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)\vec{i} + (2axy + bz)\vec{j} + (by + 2czx^3)\vec{k}$$

at the point $(1,2,-1)$

$$\nabla\phi = \vec{i}\,(4a+3c) + \vec{j}\,(4a-b) + \vec{k}\,(2b-2c) \quad \to (1)$$

The Directional Derivative is Maximum in the direction of $\nabla\phi$ i.e. in the direction of $\vec{i}\,(4a+3c) + \vec{j}\,(4a-b) + \vec{k}\,(2b-2c)$. But it is given that directional derivative is maximum in the direction of z-axis i.e., in the direction of $0\vec{i} + 0\vec{j} + \vec{k}$. Therefore , $\nabla\phi$ and z-axis are parallel.

$$\frac{4a+3c}{0}=\frac{4a-b}{0}=\frac{2b-2c}{1}=l\ ,\ \text{(say)}$$

$4a+3c=0 \quad \rightarrow (2)$

$4a-b=0 \quad \rightarrow (3)$

substituting in eq.(1),

$\nabla\phi=(2b-2c)\vec{k}$

Maximum value of directional derivative is $|\nabla\phi|$. But it is given as 64.

$|\nabla\phi|=64$

$\left|(2b-2c)\vec{k}\right|=64$

2b-2c=64 , b-c=32

From eq (2) & (3)

4a+3c=0 , 4a-b=0,

Solving, b=-3c

Substituting in b - c = 32 , - 4 c = 32

a = 6 , b = 24 , c = - 8

3. Prove that $Curl\ Curl\ \vec{F}=grad\ div\ \vec{F}-\nabla^2\vec{F}$

Solution:

Let $\vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$

$$Curl\ \vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\vec{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\vec{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$Curl\ (Curl\ \vec{F})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \left(\dfrac{\partial F_3}{\partial y}-\dfrac{\partial F_2}{\partial z}\right)_1 & \left(\dfrac{\partial F_1}{\partial z}-\dfrac{\partial F_3}{\partial x}\right) & \left(\dfrac{\partial F_2}{\partial x}-\dfrac{\partial F_1}{\partial y}\right)\end{vmatrix}$$

$$=\sum\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right]\vec{i}$$

$$=\sum\left[\left(\frac{\partial^2 F_2}{\partial y\,\partial x}+\frac{\partial^2 F_3}{\partial z\,\partial x}\right)-\left(\frac{\partial^2 F_1}{\partial y^2}-\frac{\partial^2 F_1}{\partial z^2}\right)\right]\vec{i}$$

$$=\sum\left[\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_2}{\partial x\,\partial y}+\frac{\partial^2 F_3}{\partial x\,\partial z}\right)-\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right]\vec{i}$$

$$=\sum\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_1\right]\vec{i}$$

$$=\sum\left[\frac{\partial}{\partial x}\left(\nabla.\ \vec{F}\right)-\nabla^2 F_1\right]\vec{i}$$

$$=\left[\vec{i}\,\frac{\partial}{\partial x}\left(\nabla.\ \vec{F}\right)+\vec{j}\,\frac{\partial}{\partial y}\left(\nabla.\ \vec{F}\right)+\vec{k}\,\frac{\partial}{\partial z}\left(\nabla.\ \vec{F}\right)\right]-\nabla^2\left[F_1\vec{i}+F_2\vec{j}+F_3\vec{k}\right]$$

$$curl(curl\vec{F})=\nabla(\nabla.\ \vec{F})-\nabla^2\vec{F}$$

4. ***Prove that $curl\left(grad\varphi\right)=0$, using Stoke's thoerem*** **(APR/MAY 2017)**

Solution : $\iint_s curl\vec{F}.ds=\int_c \vec{F}.dr$

Let , $\vec{F}=grad\phi$

$$\iint_s curl(grad\phi).ds=\int_c grad\phi.dr$$

$$=\int_c(\nabla\phi).d\vec{r}$$

$$=\int_c(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}).(dx\vec{i}+dy\vec{j}+dz\vec{k})$$

$$=\int_c\frac{\partial\phi}{\partial x}.dx+\frac{\partial\phi}{\partial y}.dy+\frac{\partial\phi}{\partial z}.dz$$

$$=\int_c d\phi=0$$

Since , for any open 2 sided surface S , provided it is bounded by the sa,e simple closed curve C.
Hence , R.H.S :

$$\iint curl(grad\phi)=0$$ **(for any S, hence for any $d\vec{s}$) .**

5. Find 'a' and 'b' so that the surfaces $ax^3-by^2z=(a+3)x^2$ and $4x^2y-z^3=11$ cut orthogonally at (2, –1, –3)

Solution:

Let $\phi_1=ax^3-by^2z-(a+3)x^2$, $\qquad\phi_2=4x^2y-z^3-11$

$\nabla\phi_1=[3ax^2-(a+3)2x]\vec{i}-2byz\vec{j}-by^2\vec{k}$

$\nabla\phi_2=8xy\vec{i}-4x^2\vec{j}-3z^2\vec{k}$

At $(2,-1,-3)$ $\nabla\phi_1=(8a-12)\vec{i}-6b\vec{j}-b\vec{k}$

$\nabla\phi_2=16\vec{i}-16\vec{j}-27\vec{k}$

Since the surfaces cut orthogonally at $(2,-1,-3)$, $\nabla\phi_1.\nabla\phi_2=0$

$\Rightarrow -16(8a-12)-16(6b)+27b=0$

$\Rightarrow -128a+192-69b=0$

$\Rightarrow 128a + 69b = 192 \qquad \rightarrow (1)$

Since the points $(2,-1,-3)$ lies on the surface $\phi(x,y,z)=0$, we have

$8a + 3b - 4a = 12$

$\Rightarrow 4a + 3b = 12 \qquad \rightarrow (2)$

Solving (1)&(2) we get $a = -2.333 \quad b = 7.111$

6. Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $r^2 = x^2 + y^2 + z^2$. Find the value of $\nabla^2\left(\frac{1}{r}\right)$.

Solution:

$$\nabla\left(r^n\right) = \vec{i}\ \frac{\partial\left(r^n\right)}{\partial x} + \vec{j}\ \frac{\partial\left(r^n\right)}{\partial y} + \vec{k}\ \frac{\partial\left(r^n\right)}{\partial z}$$

$$= \vec{i}\ \ nr^{n-1}\frac{x}{r} + \vec{j}\ \ nr^{n-1}\frac{y}{r} + \vec{k}\ \ nr^{n-1}\frac{z}{r}$$

$$= \vec{i}\, nr^{n-2}x + \vec{j}\, nr^{n-2}y + \vec{k}\, nr^{n-2}z$$

$$= nr^{n-2}(x\vec{i} + y\vec{j} + z\vec{k}) \ (\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k})$$

$$\therefore \nabla\left(r^n\right) = nr^{n-2}\vec{r}.$$

Now

$$\nabla^2\left(r^n\right) = \nabla.\nabla\left(r^n\right) = \nabla.\left(nr^{n-2}\vec{r}\right)$$

$$= n\nabla.\left(r^{n-2}\vec{r}\right)$$

$$= n\left[\nabla\left(r^{n-2}\right).\vec{r} + r^{n-2}\nabla\left(\vec{r}\right)\right] \qquad \because \nabla\left(\vec{r}\right) = 3$$

$$= n\left[(n-2)r^{n-4}\vec{r}.\vec{r} + 3r^{n-2}\right]$$

$$= n\left[(n-2)r^{n-4}r^2 + 3r^{n-2}\right]$$

$$\therefore \nabla^2\left(r^n\right) = n(n+1)r^{n-2}$$

7. **(i)Find the work done in moving a particle in the force field** $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ **along the curve defined by** $x^2 = 4y$ **,** $3x^2 = 8z$ **from** $x = 0$ **to** $x = 2$

Given :

$$x^2 = 4y$$

$$\Rightarrow y = \frac{x^2}{4} \Rightarrow dy = \frac{2x}{4} = \frac{x}{2}$$

$$Also, 3x^2 = 8z$$

$$\Rightarrow z = \frac{3x^2}{8}$$

$$\Rightarrow dz = \frac{6x}{8} = \frac{3x}{4}$$

Work done = $\int_c \vec{F}.d\vec{r}$

$$= \int_c (3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_c 3x^2dx + (2xz - y)dy + zdz$$

$$= \int_{x=0}^{2} 3x^2dx + (2x.\frac{3x^2}{8} - \frac{x^2}{4}).\frac{x}{2}.dx + \frac{3x^2}{8}.\frac{3x}{4}dx$$

$$= \int_{x=0}^{2} (3x^2 + \frac{3x^4}{8} - \frac{x^3}{8} + \frac{9x^3}{32})dx$$

$$= \left[\frac{3x^3}{3} + \frac{3x^5}{40} - \frac{x^4}{32} + \frac{9x^4}{128}\right]_0^2$$

$$= 8 + \frac{12}{5} - \frac{1}{2} + \frac{9}{8}$$

$$= \frac{320 + 96 - 20 + 45}{40} = \frac{441}{40}$$

8. Verify Green's theorem for $\int_c \left[x^2(1+y)dx + (x^3 + y^3)dy\right]$ where C is the boundary of the region defined by the lines x = $\pm 1$and y = ± 1.

Solution:

Given $\int_c x^2(1+y)dx + (y^3 + x^3)dy$

$M = x^2(1+y)$ $\quad\quad$ $N = y^3 + x^3$

$\frac{\partial M}{\partial y} = x^2$ $\quad\quad$ $\frac{\partial N}{\partial y} = 3x^2$

By Green's theorem $\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial y} \right) dxdy$

Consider $\iint_R \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial y} \right) dxdy = \int_{-1}^{1}\int_{-1}^{1} (3x^2 - x^2) dydx = \int_{-1}^{1}\int_{-1}^{1} (2x^2) dydx$

$= \int_{-1}^{1} 2\left[\frac{x^3}{3} \right]_{-1}^{1} dy = \int_{-1}^{1} 2\left[\frac{1}{3} + \frac{1}{3} \right] dy = \int_{-1}^{1} \left[\frac{4}{3} \right] dy = \left[\frac{4}{3} \right] [y]_{-1}^{1} = \frac{8}{3} \qquad \to (1)$

Consider

$\int_c Mdx + Ndy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

Along AB, $y = -1$, $dy = 0$ and x varies from -1 to 1

$\therefore \int_{AB} Mdx + Ndy = \int_{-1}^{1} x^2 (1-1) dx = 0$

Along BC, $x = 1$, $dx = 0$ and y varies from -1 to 1

$\therefore \int_{BC} Mdx + Ndy = \int_{-1}^{1} (y^3 + 1) dy = \left[\frac{y^4}{4} + y \right]_{-1}^{1} = 2$

Along CD, $y = 1$, $dy = 0$ and x varies from 1 to -1

$\therefore \int_{CD} Mdx + Ndy = \int_{1}^{-1} 2x^2 dx = \left[\frac{2x^3}{3} \right]_{1}^{-1} = -\frac{4}{3}$

Along DA, $x = -1$, $dx = 0$ and y varies from 1 to -1

$\therefore \int_{DA} Mdx + Ndy = \int_{1}^{-1} (y^3 - 1) dy = \left[\frac{y^4}{4} - y \right]_{1}^{-1} = \frac{1}{4} + 1 - \frac{1}{4} + 1 = 2$

$\int_C Mdx + Ndy = 0 + 2 - \frac{4}{3} + 2 = 4 - \frac{4}{3} = \frac{8}{3} \qquad \to (2)$

$\therefore (1) = (2)$ Hence the theorem is verified.

9. Verify Gauss divergence theorem for $\vec{F} = \left(x^2 - yz\right)\vec{i} + \left(y^2 - zx\right)\vec{j} + \left(z^2 - xy\right)\vec{k}$ taken over the rectangular parallelopiped bounded by the planes x = 0, x = a, y = 0, y = b, z = 0, and z = c.

Solution:

By Gauss – Divergence theorem $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V div\vec{F} \cdot dV$

Evaluation of LHS:

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_{S_1} \vec{F} \cdot \hat{n}\, ds + \iint_{S_2} \vec{F} \cdot \vec{n} ds + \ldots + \iint_{S_6} \vec{F} \cdot \vec{n} ds$$

Over S_1: x = 0, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (yz)\,dy\,dz = \int_0^c \left[z\left(\frac{y^2}{2}\right)_0^b \right] dz = \frac{b^2}{2}\left(\frac{z^2}{2}\right)_0^c = \frac{b^2c^2}{4}$$

Over S_2: x = a, $\hat{n} = \vec{i}$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (-yz + a^2)\,dy\,dz = \int_0^b \left[-y\left(\frac{z^2}{2}\right)_0^c + a^2\,[z]_0^c \right] dy$$

$$= -\frac{c^2}{2}\left(\frac{y^2}{2}\right)_0^b + ca^2\,[y]_0^b = a^2bc - \frac{b^2c^2}{4}$$

Over S_3: y = 0, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (xz)\,dx\,dz = \int_0^c \left(\frac{x^2}{2}z\right)_0^a dz = \frac{a^2}{2}\left(\frac{c^2}{2}\right) = \frac{a^2c^2}{4}$$

Over S_4: y = b, $\hat{n} = \vec{j}$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (-xz + b^2)\,dx\,dz = \int_0^c \left[-z\left(\frac{a^2}{2}\right) + b^2a \right] dz = ab^2c - \frac{a^2c^2}{4}$$

Over S_5: z = 0, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (xy)\,dx\,dy = \int_0^b \left[y\left(\frac{x^2}{2}\right)_0^a \right] dy = \frac{a^2b^2}{4}$$

Over S_6: z = c, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (-xy + c^2)\,dx\,dy = \int_0^b \left[-y\left(\frac{a^2}{2}\right) + c^2a \right] dy = abc^2 - \frac{a^2b^2}{4}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + a\ b^2c - \frac{a^2c^2}{4} + \frac{a^2b^2}{4} + a\ bc^2 - \frac{a^2b^2}{4}$$

$$= a^2bc + ab^2c + abc^2 = abc\,(a+b+c)$$

Evaluation of RHS:

$$\nabla.\vec{F} = 2(x+y+z)$$

$$\iiint_V \nabla.\vec{F}\,dV = \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2}+xy+xz\right]_0^a dy\,dz$$

$$=2\int_0^c\int_0^b \left[\frac{a^2}{2}+ay+az\right]dy\,dz$$

$$=2\int_0^c \left[\frac{a^2}{2}y+a\frac{y^2}{2}+ayz\right]_0^b dz$$

$$=2\left[\frac{a^2bz}{2}+\frac{ab^2z}{2}+\frac{abz^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]= a^2bc+ab^2c+abc^2 = abc(a+b+c)$$

Hence, Gauss divergence theorem is verified.

10. Verify Stoke's theorem for the vector $\vec{F} = xy\vec{i} - 2yz\,\vec{j} - xz\vec{k}$, where S is the open surface of the rectangular parallelopiped formed by the planes x = 0, y = 0, z=0, x = 1, y = 2 and z = 3 above the XOY plane.

Solution:

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S \nabla\times\vec{F}.\vec{n}\,ds$

Evaluation of $L.H.S$:

$$\int_C \vec{F}.\vec{dr} = \int_{OA} \vec{F}.\vec{dr} + \int_{AB} \vec{F}.\vec{dr} + \int_{BD} \vec{F}.\vec{dr} + \int_{DO} \vec{F}.\vec{dr}$$

Along OA : $y=0,\ z=0,\ dy=0,\ dz=0$

$$\int_{OA} \vec{F}.\vec{dr} = 0$$

Along AB : $x=1,\ z=0,\ dx=0,\ dz=0$

$$\int_{AB} \vec{F}.\vec{dr} = \int_{AE} 0 = 0$$

Along BD : $y=2,\ z=0,\ dy=0, dz=0$

$$\int_{BD} \vec{F}.\vec{dr} = \int_{BD} (2x)\,dx = \int_1^0 2x\ dx = \left[\frac{2x^2}{2}\right]_1^0 = 0-1=-1$$

Along DO: $x=0,\ z=0, dx=0, dz=0$

$$\int_{DO} \vec{F}.\vec{dr} = \int_{DO} 0 = 0$$

$$\therefore \int_{C} \vec{F}.\vec{dr} = 0+0-1+0=-1$$

Evaluation of RHS:

$$\iint_{S} \nabla\times\vec{F}.\hat{n}\ ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

Given, $\vec{F} = (xy)\vec{i} - 2yz\vec{j} - xz\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix} = 2y\vec{i} + (-z)\vec{j} - x\vec{k}$$

Over S_1: $x = 0$, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \nabla\times\vec{F}.\hat{n}\ ds = \int_0^3\int_0^2 \left[2y\vec{i}\right].(-\vec{i})dydz$$

$$= \int_0^3\int_0^2 -2y dy\, dz$$

$$= \int_0^3\int_0^2 -2y\, dy\, dz = \int_0^3 \left[\frac{-2y^2}{2}\right]_0^2 dz$$

$$= -4(z)_0^3 = -12$$

Over S_2: $x = 1$, $\hat{n} = \vec{i}$

$$\iint_{S_2} \nabla\times\vec{F}.\hat{n}\ ds = \int_0^3\int_0^2 \left[2y\vec{i}\right].\left(\vec{i}\right)\ dydz$$

$$= \int_0^3\int_0^2 2y\, dy\, dz = \int_0^3 \left[\frac{2y^2}{2}\right]_0^2 dz = 12$$

Over S_3: $y = 0$, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \nabla\times\vec{F}.\hat{n}\ ds = \int_0^3\int_0^1 \left[-z\vec{j}\right]\left(-\vec{j}\ \right)dxdz$$

$$= \int_0^3\int_0^1 (z)\ dx\, dz$$

$$= \int_0^3 (xz)_0^1 = \int_0^3 (z)dz$$

$$= \left(\frac{z^2}{2}\right)_0^3 = \frac{9}{2}$$

Over S_4: $y = 1$, $\hat{n} = \vec{j}$

$$\iint_{S_4} \nabla\times\vec{F}.\hat{n}\ ds = \int_0^3\int_0^1 -z\,\vec{j}.\vec{j}\,dxdz$$

$$= \int_0^3\int_0^1 (-z)\ dx\,dz = \int_0^3 (-xz)_0^1\ dz$$

$$= \left(\frac{-z^2}{2}\right)_0^3 = -\frac{9}{2}$$

Over S_5 : $z = 1$, $\hat{n} = \vec{k}$

$$\iint_{S_5} \nabla\times\vec{F}.\hat{n}\ ds = \int_0^2\int_0^1 \left(-x\vec{k}\right).\vec{k}dxdy$$

$$= \int_0^2\int_0^1 (-x)dxdy = \int_0^2 \left(-\frac{x^2}{2}\right)_0^1 dy$$

$$= \int_0^2 \left(\frac{-1}{2}\right) dy = \left(\frac{-1}{2}\right)(y)_0^2 = -1$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} \quad = -12 + 12 + \frac{9}{2} - \frac{9}{2} - 1 = -1$$

$\therefore$ L.HS = R.HS.

Hence Stoke's theorem is verified.

11. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ and C is the straight line from A(0,0,0) to B(2,1,3)

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}.d\vec{r} = 3x^2dx + (2xz - y)dy + zdz$.

The equation of AB is $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say) $\quad \left(\because \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}\right)$

$\Rightarrow x = 2t \Rightarrow dx = 2dt$

$y = t \Rightarrow dy = dt$

$z = 3t \Rightarrow dz = 3dt$

$$\int_C \vec{F}.\ \vec{dr} = \int_0^1 3x^2dx + (2xz - y)dy + zdz$$

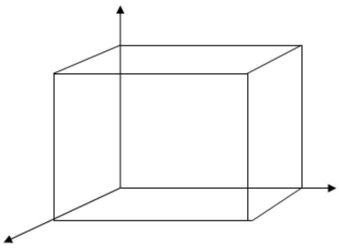
$$= \int_0^1 \left(36t^2 + 8t\right)dt = \left[36\frac{t^3}{3} + 8\left(\frac{t^2}{2}\right)\right]_0^1 = 16$$

12. Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cube bounded by the planes x=0,y=0 , z=0 ,x=1 ,y=1 and z=1.

Solution:Gauss Divergence Theorem is

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V \nabla.\vec{F}dv$$

$$div.\vec{F} = \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)$$



$$= 2x + 2y + 2z$$

$$\iiint_V div\vec{F}.dv = \int_0^1\int_0^1\int_0^1 (2x + 2y + 2z)dxdydz$$

$$= \int_0^1\int_0^1 \left[x^2 + 2xy + 2xz\right]_0^1 dydz$$

$$= \int_0^1 \left[y + y^2 + 2zy\right]_0^1 dz$$

$$= \left[2z + z^2\right]_0^1 = 3 \rightarrow (1)$$

$$= \iint_S \vec{F}.\hat{n}ds = \iint_{s1} + \iint_{s2} + \iint_{s3} + \iint_{s4} + \iint_{s5} + \iint_{s6}$$

Now ; S_1 is the Surface OABC

$$\hat{n} = -k; z = 0; ds = dxdy$$

$$\therefore \iint_{S_1} \vec{F}.\hat{n}ds = \iint_{S_1} \left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)\left(-\vec{k}\right)dxdy$$

$$= \iint_{S_1} z^2 dxdy = 0$$

$$\iint_{S_2} \vec{F}.\hat{n}ds = \iint_{S_2} z^2 dxdy = \int_0^1\int_0^1 dxdy$$

$$= = \int_0^1 [x]_0^1 dy = dy = \int_0^1 dy = [y]_0^1 = 1$$

S_3 **is AOGD ,** $\hat{n} = -\vec{j}$ **;y=0,ds=dzdx**

$$\iint_{S_3} = \iint_{S_3} y^2 dzdx = 0$$

S_4 **is BCFE** $\hat{n} = -\vec{j}$ **;y=1,ds=dxdz**

$$\iint_{S_4} = \int_0^1 \int_0^1 y^2 dxdz = \int_0^1 \int_0^1 dxdz$$

$$= \int_0^1 [x]_0^1 dz = \int_0^1 dz = [z]_0^1 = 1$$

S_5 **is ABED** $\hat{n} = \hat{i}$ **;x=1,ds=dydz**

$$\iint_{S_5} = \int_0^1 \int_0^1 x^2 dydz = \int_0^1 [y]_0^1 dz = \int_0^1 dz$$

$$= [z]_0^1 = 1$$

S_6 **is OCFG** $\hat{n} = -\hat{i}$ **;x=0,ds=dydz**

$$\iint_{S_6} = \iint_{S_6} - x^2 dydz = 0$$

$$\int\int_S \vec{F}.\hat{n}ds = 1+0+1+0+1+0$$

=3 $\rightarrow (2)$

$\therefore (1) = (2)$

Hence:Gauss Divergence Theorem is Verified

UNIT - III
MA8251 - ANALYTIC FUNCTIONS

1. C-R Equations In Cartesian Coordinates : $\boxed{u_x = v_y \text{ and } u_y = -v_x}$

$$f'(z) = u_x + iv_x$$

C-R Equations In Polar Coordinates : $\boxed{u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta}$

$$f'(z) = e^{-i\theta} [u_r + iv_r]$$

2. Milne Thomson Method

If u is given $f(z) = \int [u_x(z,0) - i u_y(z,0)] dz + ic$

If v is given $f(z) = \int [v_y(z,0) + i v_x(z,0)] dz + c$

3. $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$

4. Re[f(z)] = u = $\frac{f(z) + f(\bar{z})}{2}$ **5.** $|f(z)|^2 = f(z) f(\bar{z})$

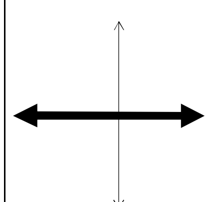
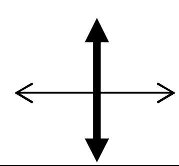
6. Fixed point (Invariant points) are obtained by replacing w = z

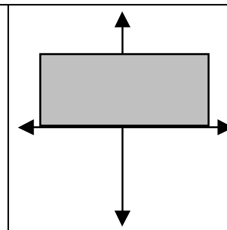
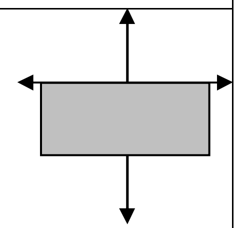
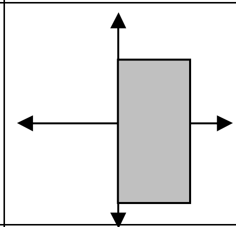
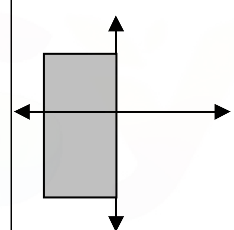
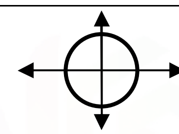
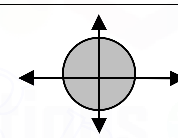
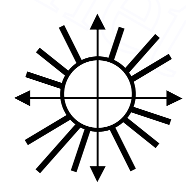
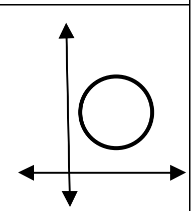
7. Critical points are given by $\frac{dw}{dz} = 0$ *and* $\frac{dz}{dw} = 0$

8. Bilinear transformation which maps the points z_1, z_2 *and* z_3 **of Z – plane onto the points** w_1, w_2 *and* w_3 **of W- plane is given by**

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

9.

S.No.	Description	Z – plane	W - plane	Fig
1	Real axis	y = 0	v = 0	
2	Imaginary axis	x = 0	u = 0	

3	**Upper half plane**	**y > 0**	**v > 0**	
4	**Lower half plane**	**y < 0**	**v < 0**	
5	**Right half plane**	**x** > 0	**u > 0**	
6	**Left half plane**	**x** < 0	**u < 0**	
7	**Unit circle with centre at the origin**	$\lvert z \rvert = 1$	$\lvert w \rvert = 1$	
8	**Interior of Unit circle with centre at the origin**	$\lvert z \rvert < 1$	$\lvert w \rvert < 1$	
9	**Exterior of Unit circle with centre at the origin**	$\lvert z \rvert > 1$	$\lvert w \rvert > 1$	
10	**Circle with centre at a and radius r**	$\lvert z - a \rvert = r$	$\lvert w - a \rvert = r$	

Unit-3

Analytic function

Analytic function:

Let $f(z)$ be a single valued function and it is said to be analytic at a point z_0 if $\lim_{z \to z_0} f(z)$ exists & derivatives exists at all pts. of neighbourhood of z.

Necessary & sufficient condition for a function to be analytic:

A function $f(z) = u + iv$ is said to be analytic in a region R if & only if

(i) du/dx, du/dy, dv/dx, $\frac{dv}{dy}$ are continuous in R

and (ii) C.R. equations are satisfied

ie) $u_x = v_y$ and $u_y = -v_x$.

C-R Equations in Polar form:

Let $z = x+iy = re^{i\theta}$

& $w = f(z) = u(r,\theta) + iv(r,\theta)$ then the

C-R equation is given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Note: (1) Analytic fn or regular fn or holomorphic fn

(2) Analytic $\Rightarrow$ differentiable.

(1) Prove that the function $f(z) = \bar{z}$ is nowhere differentiable.

Sol: Let $z = x+iy$

$$f(z) = \bar{z} = \overline{x+iy} = x-iy$$

ie) $u+iv = x-iy$

$u = x$	$v = -y$
$u_x = 1$	$v_x = 0$
$u_y = 0$	$v_y = -1$

Here $u_x \neq v_y$ & $u_x \neq -v_x$.

CR eqns. are not satisfied

$\Rightarrow f(z)$ is not analytic $\Rightarrow f(z)$ is nowhere differentiable.

(2) Show that $f(z) = |z|^2$ is differentiable at $z=0$ but not analytic at $z=0$.

Sol: Let $z = x+iy$ $\quad \bar{z} = x-iy$

$$|z|^2 = z\bar{z} = x^2+y^2$$

$$\therefore f(z) = |z|^2 = x^2+y^2$$

ie) $u+iv = x^2+y^2$

Here $u = x^2+y^2$	$v = 0$
$u_x = 2x$	$v_x = 0$
$u_y = 2y$	$v_y = 0$.

At the origin $(0,0)$

$$u_x = 0 = v_x \quad \& \quad u_y = 0 = -v_y.$$

$\therefore$ C.R equns are ~~eq~~ satisfied & $\frac{\partial u}{\partial x}$, u_y, v_x, v_y are continuous.

$\therefore f(z) = |z|^2$ is only differentiable at $(0,0)$

But in the neighbourhood of $z=0$, C.R equns are not satisfied. Hence it is not analytic.

(3) Show that the function $w = e^z$ is analytic everywhere in the complex plane.

Proof: Let $z = x+iy$

& $w = e^z = e^{x+iy} = e^x . e^{iy}$ 79, (88), 90, 101

ie) $u+iv = e^x(\cos y + i\sin y)$

$\therefore u = e^x \cos y$ | $v = e^x \sin y$

$u_x = e^x \cos y$ | $v_x = e^x \sin y$

$u_y = -e^x \sin y$ | $v_y = e^x \cos y$

$\therefore u_x = v_x$ & $u_y = -v_x$

CR equs. are satisfied.

$\Rightarrow$ the gn. fn. is analytic.

(4) Test the analyticity of the function $w = \sin z$

$\cos i\theta = \cosh\theta$

$\sin i\theta = i\sinh\theta$

Sol: Let $w = f(z) = \sin z$

$u+iv = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i\cos x \sinh y$

$\therefore u = \sin x \cosh y$ | $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ | $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ | $v_y = \cos x \cosh y$ 191

Here $u_x = v_y$ & $u_y = -v_x$. 57, 53, 65, 69, 89, 103, 109, 94, 87

$\Rightarrow$ CR eqns. are satisfied

$\Rightarrow f(z)$ is analytic.

(5) Find the values of a & b such that the function $f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$ is analytic. Also find $f'(z)$.

Sol: $f(z) = x^2+ay^2-2xy + i(bx^2-y^2+2xy) = u+iv$

Here $u = x^2+ay^2-2xy$ | $v = bx^2-y^2+2xy$

$u_x = 2x-2y$ | $v_x = 2bx+2y$

$u_y = 2ay-2x$ | $v_y = -2y+2x$.

Since $f(z)$ is analytic, CR eqns are satisfied

ie) $u_x = v_y$ | $u_y = -v_x$

ie) $2x-2y = 2x-2y$ | $2ay-2x = -2bx-2y$.

comparing the coeffts of x & y, we get

$2a = -2$ | $-2 = -2b$

$\therefore a = -1$ | $b = 1$

$\therefore f(z) = x^2+ay^2-2xy + i(bx^2-y^2+2xy)$

$= x^2-y^2-2xy + i(x^2-y^2+2xy)$

$f'(z) = u_x + iv_x$

$= 2x-2y+i(2x+2y)$

$= 2[x-y+i(x+y)]$

$= 2[x+i^2y+ix+iy]$

$= 2[(x+iy)+i(x+iy)]$

$= 2[(x+iy)(1+i)] = 2(1+i)z$ //.

(6) P.T $f(z) = \sinh z$ is analytic.

$i\sinh x = \sin ix$

$\therefore \sinh x = -i\sin ix$

Sol: $\sinh z = \sinh(x+iy)$

$= -i\sin i(x+iy)$

$= -i\sin(ix-y)$

$= -i[\sin ix\cos y - \cos ix\sin y]$

$= -i(i\sinh x\cos y - \cosh x\sin y]$

$u+iv = \sinh x\cos y + i\cosh x\sin y$

7. Prove that $f(z) = \log z$ is analytic.

Sol: Let $f(z) = \log z = \log(x+iy)$.

$$u+iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}(b/a)$

$\therefore u = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2}(\cancel{2}x)$$

$$= x/x^2+y^2$$

$$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2}(\cancel{2}y)$$

$$= \frac{y}{x^2+y^2}$$

$v = \tan^{-1}(y/x)$

$$\frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2}\cdot\frac{x(0)-y(1)}{x^2}$$

$$= \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{-y}{\cancel{x^2}}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{x(1)-y(0)}{x^2}$$

$$= \frac{\cancel{x^2}}{x^2+y^2}\cdot\frac{x}{\cancel{x^2}} = \frac{x}{x^2+y^2}$$

Here $u_x = v_y$ & $u_y = -v_x$.

$\Rightarrow$ CR eqns are satisfied

$\Rightarrow$ $f(z)$ is analytic.

Properties of Analytic function:

Results:

1. $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ is known as Laplace equation in 2-dimension.

2. $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \rightarrow$ Laplace eqn. in 3 dimension.

3. $\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \rightarrow$ Laplace eqn in Polar co-ordinates.

Property 1:

The real and imaginary parts of an analytic function $w = u+iv$ satisfy the Laplace equation in 2-D cartesian co-ordinates.

Proof:

Given $w = u+iv$ is analytic function

To prove u & v satisfy Laplace eqn.

(i.e) To prove $\nabla^2 u = 0$ & $\nabla^2 v = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Big| \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Proof: Given $w = u+iv$ is analytic

$\Rightarrow$ C.R eqns. are satisfied.

$\Rightarrow u_x = v_y$ & $u_y = -v_x$

(i.e) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ — (1) & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ — (2)

Diff. (1) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{— (3)}$$

Diff (2) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{— (4)}$$

(3) + (4) gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$\Rightarrow \nabla^2 u = 0 \Rightarrow u$ satisfies Laplace equation.

Diff. (1) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \text{---} (5)$$

Diff. (1) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow -\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x^2} \quad \text{---} (6)$$

(5) + (6) gives $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} = 0$

$\Rightarrow v$ satisfies the Laplace equation.

Property 2:

An analytic function with constant real part is constant.

Proof:

Given $f(z) = u + iv$ is analytic.

and real part of $f(z) = $ const

i.e. $u = $ const.

To prove: $f(z)$ is const i.e. $f'(z) = 0$.

Given $f(z)$ is analytic

$\Rightarrow$ C-R eqns are satisfied

$\Rightarrow u_x = v_y$ & $u_y = -v_x$ --- (1)

Gn. $u = $ const $\Rightarrow u_x = 0, u_y = 0$

$\Rightarrow v_y = 0, -v_x = 0$ (by (1)).

$f'(z) = u_x + iv_x = 0 + i0 = 0.$

$\Rightarrow f(z) = $ const.

HW

1. An analytic fn. with constant imaginary part is constant.
2. The real & imaginary parts of an analytic function $w = u(r,\theta) + iv(r,\theta)$ satisfy the Laplace equation in Polar co-ordinates.

Property 3:

An analytic function with constant modulus is constant.

Proof:

Let $f(z) = u + iv$ be an analytic function.

Given $|f(z)|$ is const.

ie) $|f(z)| = \sqrt{u^2+v^2} = c$ (const)

$\Rightarrow u^2+v^2 = c^2$ —— (1)

Given $f(z)$ is analytic

$\Rightarrow$ CR eqns. are satisfied

$\Rightarrow u_x = v_y$ —— (2) & $u_y = -v_x$ —— (3)

To prove: $f(z) = $ const $\Rightarrow f'(z) = 0$.

Diff. (1) w.r.t. x, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{—— (4)}$$

Diff. (1) w.r.t. y, we get

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

ie) $-u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial x} = 0$ —— (5) (by CR eqns (2) & (3))

(4) $\times u \Rightarrow u^2\dfrac{\partial u}{\partial x} + uv\dfrac{\partial v}{\partial x} = 0$

(5) $\times v \Rightarrow v^2\dfrac{\partial u}{\partial x} - uv\dfrac{\partial v}{\partial x} = 0$

adding, $(u^2+v^2)\partial u/\partial x = 0$

ie) $(u^2+v^2)\frac{\partial u}{\partial x} = 0$

Since $u^2+v^2 = c^2 \neq 0$,

$$\frac{\partial u}{\partial x} = 0$$

$\Rightarrow u$ is a const.

(4) $\times v$ $-$ (5) $\times u$ $\Rightarrow$

$$uv\cancel{\frac{\partial u}{\partial x}} + v^2\frac{\partial v}{\partial x} - uv\cancel{\frac{\partial u}{\partial x}} + u^2\frac{\partial v}{\partial x} = 0$$

$$(u^2+v^2)\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \frac{\partial v}{\partial x} = 0 \quad (\because u^2+v^2 = c^2 \neq 0)$$

$$\therefore f'(z) = u_x + iv_x = 0 + i0 = 0$$

$\Rightarrow f'(z) = 0$

$\Rightarrow f'(z) =$ const.

Note: The family of two curves intersects orthogonally if the product of their slopes is -1 ie) $m_1 m_2 = -1$.

Property 4:

If $f(z) = u+iv$ is analytic then the family of curves $u(x,y) = c_1$ & $v(x,y) = c_2$ intersects orthogonally where c_1 & c_2 are consts.

Proof: Given $f(z) = u+iv$ is analytic

$\Rightarrow$ CR eqns. are satisfied

$\Rightarrow u_x = v_y$ & $u_y = -v_x$

consider $u(x,y) = c_1$ then by total derivative,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$0 = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow \frac{\partial u}{\partial y}dy = -\frac{\partial u}{\partial x}dx$$

Proof:

Let $f(z) = u+iv$

Given $f(z)$ is an analytic function

$u_x = v_y$ and $u_y = -v_x$

20

66, 68, 70, 72, 75, 79, 82, 81, 85, 90, 94, 101 (12)

$f(\bar{z}) = u - iv$

$f(z)\cdot f(\bar{z}) = (u+iv)(u-iv) = u^2+v^2$

$\therefore |f(z)|^2 = u^2+v^2$

Consider L.H.S $= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2+v^2) + \partial^2/\partial y^2 (u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) + \frac{\partial^2}{\partial x^2}(v^2) + \frac{\partial^2}{\partial y^2}(v^2) \longrightarrow (1)$$

Now $\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$

$$\therefore \frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x}\right]$$

$$= 2\left(u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right)$$

$||^{ly}$ $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$$\therefore \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u(0)\right]$$

($\because u$ is harmonic)

$$= 2\left[u_x^2 + u_y^2\right] = 2|f'(z)|^2$$

Also $\frac{\partial^2}{\partial x^2}(v^2) + \frac{\partial^2}{\partial y^2}(v^2) = 2|f'(z)|^2$

$$\therefore (1) \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2 \quad //$$

i.e) $m_1 = \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y}$.

Also $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$$\therefore 0 = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\Rightarrow \frac{\partial v}{\partial y}dy = -\frac{\partial v}{\partial x}dx$$

$$\therefore m_2 = \frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = \frac{+\partial u/\partial y}{\partial u/\partial x}$$

(by C.R. eqns $u_x = v_y$ & $u_y = -v_x$)

To prove: $m_1 m_2 = -1$

$$\therefore m_1 m_2 = \frac{-\partial u/\partial x}{\partial u/\partial y} \times \frac{\partial u/\partial y}{\partial u/\partial x} = -1.$$

$\therefore$ The family of two curves intersects orthogonally.

Harmonic function:

Any function which has continuous 2nd order partial derivatives & which satisfies Laplace eqn. is called Harmonic fn.

Conjugate Harmonic function:

Two Harmonic fns. u & v such that $f(z) = u+iv$ is an analytic fn. then each is called the conjugate harmonic fn. of the other

i.e) u is harmonic conjugate to v

|||ly v is harmonic conjugate to u.

Problems:

1. If $f(z) = u+iv$ is a regular function of z in a domain D then $\nabla^2\left[|f(z)|^2\right] = 4\,|f'(z)|^2$.

(2) Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Proof: Let $z = x+iy$

$\bar{z} = x - iy$.

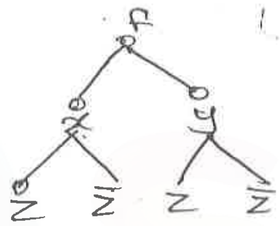
$z+\bar{z} = 2x$ & $z - \bar{z} = 2iy$.

$\therefore x = \frac{z+\bar{z}}{2}$ | $y = \frac{z-\bar{z}}{2i}$

$\frac{\partial x}{\partial z} = 1/2$ | $\frac{\partial y}{\partial z} = \frac{1}{2i}$

$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$ | $\frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$

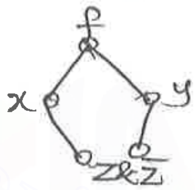
Consider a function $f = f(x,y)$ then by total derivative,



$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{-1}{2i}\right)$$

$$= \frac{1}{2}\left[\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right]$$



$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial}{\partial z}\left[\frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{1}{i}\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial z} - \frac{1}{i}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial z}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{2}\right) + \frac{\partial^2 f}{\partial y \partial x}\left(\frac{+1}{2i}\right) - \frac{1}{i}\left(\frac{\partial^2 f}{\partial x \partial y}\left(\frac{1}{2}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{+1}{2i}\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \cancel{\frac{1}{2i}\frac{\partial^2 f}{\partial y \partial x}} - \cancel{\frac{1}{2i}\frac{\partial^2 f}{\partial x \partial y}} + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\right]$$

$$= \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

ie) $4\frac{\partial^2}{\partial z \partial \bar{z}}(f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$

$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. Hence proved.

(3) If $f(z)$ is an analytic function then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0.$$

(or) $\log|f(z)|$ is harmonic. $|z|^2 = z\bar{z}$

Proof:

consider $\log|f(z)| = \log\left(f(z)\,\overline{f(z)}\right)^{1/2}$

$$= \log\left(f(z)\,f(\bar{z})\right)^{1/2}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 4\frac{\partial^2}{\partial z\partial\bar{z}}\log\left(f(z)f(\bar{z})\right)^{1/2}$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\frac{1}{2}\log\left(f(z)\,f(\bar{z})\right)$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\log f(z)+\log f(\bar{z})\right]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\bar{z}}\left(\log f(z)+\log f(\bar{z})\right)\right]$$

$$= 2\frac{\partial}{\partial z}\left[0+\frac{1}{f(\bar{z})}\cdot f'(\bar{z})\right]$$

$$= 2\frac{\partial}{\partial z}\left[\frac{f'(\bar{z})}{f(\bar{z})}\right] = 2\cdot(0) = 0 \,//.$$

Hence proved.

(4). If $f(z)$ is an analytic function then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f(z)|^{p-2}\,|f'(z)|^2.$$

$$\text{LHS} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}|f(z)|^p$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\left[\left(f(z)\,\overline{f(z)}\right)^{1/2}\right]^p.$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\left[f(z)\cdot f(\bar{z})\right]^{p/2}.$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\bar{z}}\left(f(z)^{p/2}\,f(\bar{z})^{p/2}\right)\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z)^{p/2} \cdot p/2\, f(\bar{z})^{p/2-1} \cdot f'(\bar{z}) + f(\bar{z})\cdot(0)\right]$$

$$= \frac{4p}{2}\frac{\partial}{\partial z}\left[f(z)^{p/2} f(\bar{z})^{p/2-1} f'(\bar{z})\right]$$

$$= 2p\, f(\bar{z})^{p/2-1} f'(\bar{z})\, \frac{\partial}{\partial z}\left[f(z)^{p/2}\right]$$

$$= \cancel{2}p\, f(\bar{z})^{p/2-1} f'(\bar{z}) \cdot \frac{p}{\cancel{2}} f(z)^{p/2-1} \cdot f'(z)$$

$$= p^2\left[f(z)^{p/2-1} f(\bar{z})^{p/2-1}\right]\left[f'(z) f'(\bar{z})\right]$$

$$= p^2\left[f(z) f(\bar{z})\right]^{\frac{p-2}{2}}\left[f'(z) f'(\bar{z})\right]$$

$$= p^2 |f(z)|^{p-2} |f'(z)|^2 = \text{R.H.S.}$$

Hence proved.

(5) H.W 1. Prove that $\nabla^2 |\mathrm{Re}\, f(z)|^2 = 2|f'(z)|^2$

2. P.T $\nabla^2 |\mathrm{Im}\, f(z)|^2 = 2 |f'(z)|^2$.

(2) $$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right] f$$

$$\therefore 2\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \longrightarrow (1)$$

$$\text{IIIly } 2\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \longrightarrow (2)$$

$$(1)\times(2) \Rightarrow 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} - \frac{1}{i^2}\frac{\partial^2}{\partial y^2}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \;//$$

(6) P.T. $v = 3x^2y - y^3$ is harmonic

P.T. $v = x^2 - y^2 + 2xy - 3x - 2y$ is harmonic

P.T $u = e^x(\cos y - \sin y)$ is harmonic

CONSTRUCTION OF AN ANALYTIC FUNCTION:

Milne's Thomson method:

I. When real part $u(x,y)$ is given:

Step 1: Find $\frac{du}{dx}$ which is equal to $\phi_1(x,y)$

Step 2: Find $\frac{du}{dy}$ which is equal to $\phi_2(x,y)$

Step 3: Find $\phi_1(z,0)$ & $\phi_2(z,0)$ by replacing x by z & y by 0 resp.

Step 4: $f(z)$ is obtained by the formula

$$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]dz.$$

(1) If $u = e^x \sin y$ find $f(z) = u+iv$.

Sol: Given $u = e^x \sin y$

$\phi_1(x,y) = u_x = e^x \sin y$ | $\phi_2(x,y) = u_y = e^x \cos y$.

$\therefore \phi_1(z,0) = 0$ & $\phi_2(z,0) = e^z$

$$\therefore f(z) = \int(\phi_1(z,0) - i\phi_2(z,0))dz$$

$$= \int(0 - ie^z)dz = -i\int e^z dz$$

$$= -ie^z + c.$$

(2) Determine the analytic function where the real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Sol: Given: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\phi_1(x,y) = u_x = 3x^2 - 3y^2 + 6x$$

$$\phi_2(x,y) = u_y = -6xy - 6y$$

$\therefore \phi_1(z,0) = 3z^2 + 6z$ & $\phi_2(z,0) = 0$

$$\therefore f(z) = \int(\phi_1(z,0) - i\phi_2(z,0))dz$$

$$= \int(3z^2 + 6z)dz$$

$$= \cancel{3}\left[\frac{z^3}{\cancel{3}}\right] + 6\left(\frac{z^2}{2}\right) + c$$

$$= z^3 + 3z^2 + c.$$

(3) Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic & determine its conjugate also find $f(z)$

Sol:

Given $u = \frac{1}{2}\log(x^2+y^2)$

$\cancel{\phi_1(x,y)} = u_x = \frac{1}{2} \cdot \frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$

$\cancel{\phi_2(x,y)} = \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$

$= \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+y^2}(\cancel{2}y)$

$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2} = 0$

$\Rightarrow u$ is harmonic.

To find $f(z)$

$u = \frac{1}{2}\log(x^2+y^2)$

$\phi_1(x,y) = u_x = \frac{x}{x^2+y^2}$ & $\phi_2(x,y) = u_y = \frac{y}{x^2+y^2}$

$\therefore \phi_1(z,0) = \frac{z}{z^2} = \frac{1}{z}$ $\quad \phi_2(z,0) = 0$.

$\Rightarrow f(z) = \int (\phi_1(z,0) - i\,\phi_2(z,0))\,dz$

$= \int \frac{1}{z}\,dz = \log z + c$.

$re^{i\theta} = x+iy$

$r(\cos\theta + i\sin\theta)$

$r = \sqrt{x^2+y^2}$

To find v $\quad f(z) = \log z + c$

$= \log(x+iy) + c$

$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x) + c$

$\log(re^{i\theta})$

$= \log r + i\theta$

$= \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$

$\Rightarrow v = \tan^{-1}(y/x)$ //

(A) $u = e^{2x}\sin 2y$

$\phi_1(x,y) = u_x = 2e^{2x}\sin 2y$

$\phi_2(x,y) = u_y = 2e^{2x}\cos 2y$

$\therefore \phi_1(z,0) = 0$ & $\phi_2(z,0) = 2e^{2z}$

By Milne's Thomson method,

$\therefore f(z) = \int (\phi_1(z,0) - i\phi_2(z,0))dz$

$= \int (0 - i2e^{2z})dz$

$= -2i\left(\frac{e^{2z}}{2}\right) + c = -ie^{2z} + c$

7) Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

Sol: Let $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\phi_1(x,y) = u_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(0 + 2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\therefore \phi_1(z,0) = \frac{(1 - \cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\phi_2(x,y) = u_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$\therefore \phi_2(z,0) = 0$

By Milne's Thomson method,

$\therefore f(z) = \int [\phi_1(z,0)dz - i\phi_2(z,0)]dz$

$= -\int \operatorname{cosec}^2 z\, dz = \cot z + c$

II. When Imaginary part v is given.

Step 1: Find $\frac{dv}{dy}$ which is equal to $\phi_1(x,y)$

Step 2: Find $\frac{dv}{dx}$ which is equal to $\phi_2(x,y)$

Step 3: Find $\phi_1(z,0)$ & $\phi_2(z,0)$ by replacing x by z & y by 0 respectively

Step 5: $f(z)$ can be obtained by the formula

$$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz.$$

① Find the regular fn whose imaginary part is $e^{-x}(x\cos y + y\sin y)$.

Sol: $v = e^{-x}(x\cos y + y\sin y]$

$$\phi_1(x,y) = v_y = e^{-x}[-x\sin y + y\cos y + \sin y]$$

$$\therefore \phi_1(z,0) = e^{-z}[0+0+0] = 0.$$

$$\phi_2(x,y) = v_x = e^{-x}[\cos y] + (x\cos y + y\sin y)(-e^{-x})$$

$$\phi_2(z,0) = e^{-z} + (z+0)(e^{-z})$$

$$= e^{-z}[1-z] = e^{-z} - ze^{-z}$$

$\therefore$ By Milne's Thomson method,

$$f(z) = \int(\phi_1(z,0) + i\phi_2(z,0))\,dz$$

$$= \int(0 + i(e^{-z} - ze^{-z}))\,dz$$

$$= i\left[\frac{\cancel{e^{-z}}}{-1} + ze^{-z} + \cancel{e^{-z}}\right] + c$$

$$= ie^{z}(z) + c$$

$u'' = 0$
$u = z$ $du = 1$
$dv = e^{-z}$ u'
$v_1 = -e^{-z}$
$v_2 = -e^{-z}$ $v_2 =$

$uv_1 - u'v_2 + u''v_3$

$-[-ze^{-z} -$

$u = z$ $dv = e^{-z}dz$
$u' = 1$ $v = -e^{-z}$
$v_1 = e^{-z}$
$v_2 = -e^{-z}$

$ze^{-z} - e^{-z}$

(2) Find the analytic function $w = u + iv$ given $v = e^{-2xy}\sin(x^2-y^2)$

Sol: Given $v = e^{-2xy}\sin(x^2-y^2)$

$$\phi_1(x,y) = v_y = e^{-2xy}\left[\cos(x^2-y^2)(-2y)\right] + \sin(x^2-y^2)(-2xe^{-2xy})$$

$$\therefore \phi_1(z,0) = e^0[0] + \sin(z^2-0^2)(-2ze^0)$$

$$= \sin z^2(-2z) = -2z\sin z^2$$

$$\phi_2(x,y) = v_x = e^{-2xy}\left((2x)\cos(x^2-y^2)\right) + \sin(x^2-y^2)\cdot e^{-2xy}(-2y)$$

$$\therefore \phi_2(z,0) = e^0[2z\cos z^2] + 0 = 2z\cos z^2$$

By Milne's Thomson method,

$$f(z) = \int(\phi_1(z,0) + i\phi_2(z,0))dz$$

$$= \int(-2z(\sin z^2) + i2z\cos z^2)dz$$

$$= -2\int z\sin z^2 dz + i2\int z\cos z^2 dz \qquad \text{Put } t = z^2$$

$$dt = 2z\,dz$$

$$= -\frac{2}{2}\int \sin t\,dt + \frac{2i}{2}\int \cos t\,dt \qquad \Rightarrow \frac{dt}{2} = z\,dz$$

$$= -[-\cos t] + i\sin t] + c = \cos t + i\sin t + c$$

$$\therefore f(z) = \cos z^2 + i\sin z^2 + c = e^{iz^2} + c.$$

26 79, 90, 110, 111, 112, 109

(3) Verify if the fn $e^{-2x}\cos 2y$ can be real / imaginary part of an analytic function.

Sol:

22 | 60, 63, 65, 72, 74, 79, 87, 90, 98, 105, 108, 113

(4) If $v = x^2 - y^2 + \frac{x}{x^2+y^2}$ find $f(z)$ & u.

Sol:

$$\phi_1(x,y) = \frac{\partial v}{\partial y} = -2y + \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2}$$

$$\therefore \phi_1(z,0) = 0 + 0 = 0$$

$$\phi_2(x,y) = \frac{\partial v}{\partial x} = 2x + \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\therefore \phi_2(z,0) = 2z + \left(\frac{-z^2}{(z^2)^2}\right) = 2z - \frac{1}{z^2}$$

By Milne's thomson method,

$$f(z) = \int (\phi_1(z,0) + i\phi_2(z,0))\,dz$$

$$= \int \left(0 + i\left(2z - \frac{1}{z^2}\right)\right) dz$$

$$= 2i\int z\,dz - i\int z^{-2}\,dz$$

$$= \cancel{2}i\left(\frac{z^2}{\cancel{2}}\right) - i\left(\frac{z^{-1}}{-1}\right) + c$$

$$= i\left(z^2 + \frac{1}{z}\right) + c.$$

To find u:

$$f(z) = i\left[(x+iy)^2 + \frac{1}{x+iy}\right] + c$$

$$= i\left[x^2 + (iy)^2 + 2ixy + \frac{x-iy}{x^2+y^2}\right] + c$$

$$= i\left[x^2 - y^2 + 2ixy + \frac{x-iy}{x^2+y^2}\right] + c$$

$$= i\left[x^2 - y^2 + \frac{x}{x^2+y^2} + i\left(2xy - \frac{y}{x^2+y^2}\right)\right] + c$$

$$= i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) - \left(2xy - \frac{y}{x^2+y^2}\right) + c$$

a+iv

$$f(z) = \frac{y}{x^2+y^2} - 2xy + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right)$$

$$\therefore u = \frac{y}{x^2+y^2} - 2xy$$

60, 63, 65, 68, 72

24, 79, 82, 90

8, 15, 17

65, 60

S1: Multiply $f(z)$ by coeff. of u + coeff. of $v(-i)$
S2: It converts the fn into $F(z)=U+iV$

(9) IV: $u+v$ & $u-v$ are given

(1) If $f(z)=u+iv$ is an analytic function and $u-v=e^x(\cos y-\sin y)$. Find $f(z)$ interms of z.

79, 88, 90, 101

Sol: Given $f(z)=u+iv$

$$(1+i)f(z)=(1+i)(u+iv)$$
$$=u+iv+iu-v$$
$$=(u-v)+i(u+v).$$

Here $F(z)=U+iV$

where $F(z)=(1+i)f(z)$

$U=u-v$ & $V=u+v$.

58, 62, 68, 66, 69, 78, 71, 73, 75, 90, 83, 84, 85, 85, 87, 94, 97, 99, 101, 109

Given: $U=u-v=e^x(\cos y-\sin y)$, a real part

$$\therefore \phi_1(x,y)=\frac{dU}{dx}=e^x(\cos y-\sin y)$$
$$\phi_1(z,0)=e^z$$
$$\phi_2(x,y)=\frac{dU}{dy}=e^x(-\sin y-\cos y)$$
$$\therefore \phi_2(z,0)=e^z(0-1)=-e^z$$

By Milne's Thomson method,

$$F(z)=\int(\phi_1(x,y)-i\phi_2(x,y))dz$$
$$=\int(e^z+ie^z)dz$$
$$F(z)=e^z+ie^z+c=(1+i)e^z+c$$

i.e) $(1+i)f(z)=(1+i)e^z+c$

$$\Rightarrow \boxed{f(z)=e^z+c_1}$$

(2) Find $f(z)$ given $u+v=\frac{x}{x^2+y^2}$ & $f(1)=1$

Sol: Let $f(z)=u+iv$

$$(1-i)f(z)=(1-i)(u+iv)$$
$$=u+iv-iu+v=(u+v)+i(v-u).$$

Let $F(z) = U + iV$

where $F(z) = (1-i)f(z)$

$U = u+v$ & $V = v-u$.

Given: $U = u+v = \dfrac{x}{x^2+y^2}$

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = \frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \phi_1(z,0) = \frac{-z^2}{z^4} = \frac{-1}{z^2}.$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = \frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore \phi_2(z,0) = 0.$$

By Milne's Thomson method,

$$F(z) = \int (\phi_1(z,0) + i\phi_2(z,0))\,dz$$

$$= \int \frac{-1}{z^2}\,dz = \frac{-z^{-1}}{(-1)} + C = \frac{1}{(z)} + C.$$

ie) $(1-i)f(z) = \dfrac{1}{z} + C$.

$$f(z) = \frac{1}{z} \times \frac{1}{1-i} + C_1$$

$$\boxed{f(z) = \frac{1+i}{2z} + C_1}$$

$f(1) = 1$

$\Rightarrow \dfrac{1+i}{2} + C_1 = 1$

$\Rightarrow C_1 = 1 - \dfrac{1+i}{2}$

$= \dfrac{2-1-i}{2}$

$= \dfrac{1-i}{2}$

(8) If $2u+v = e^x(\cos y - \sin y)$ find $f(z)$

Sol: Let $f(z) = u+iv$.

$$\therefore (2-i)f(z) = (2-i)(u+iv)$$

$$= 2u + 2iv - iu - i^2 v$$

$$= (2u+v) + i(2v-u).$$

Let $F(z) = U+iV$ where $F(z) = (2-i)f(z)$

$\therefore U = 2u+v,\ V = 2v-u$

Here $U = 2u+v = e^{x}(\cos y - \sin y) \rightarrow$ real part

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^{x}(\cos y - \sin y)$$

$$\therefore \phi_1(z,0) = e^{z}$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^{x}(-\sin y - \cos y)$$

$$\therefore \phi_2(z,0) = e^{z}(-0-1) = -e^{z}$$

By Milne's Thomson method,

$$F(z) = \int (\phi_1(z,0) - i\,\phi_2(z,0))\,dz$$

$$= \int (e^{z} - i(-e^{z}))\,dz$$

$$= \int (e^{z} + ie^{z})\,dz = e^{z} + ie^{z} + c.$$

$$F(z) = e^{z}(1+i) + c$$

i.e.) $(2-i)f(z) = e^{z}(1+i) + c$

$$\therefore f(z) = \frac{e^{z}(1+i)}{(2-i)} + c_1$$

$$= \frac{e^{z}(1+i)(2+i)}{4+1} + c_1$$

$$= \frac{e^{z}}{5}(2+i+2i-1) + c_1$$

$$f(z) = \frac{e^{z}}{5}(1+3i) + c_1.$$

HW Given $u - 2v = e^{x}(\cos y - \sin y)$. Find $F(z)$.

Q.3 Find if $f(z) = u+iv$ is an A.F of z & if $u = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, find v $\left[v = -\dfrac{2\sinh 2y}{e^{2y} + e^{-2y} - 2\cos 2x} + c \right]$

Q.4 Find the A.F for c $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ is the real part. Hence determine the A.F $u+iv$ for c $u+v$ is the above function. $\dfrac{1+i}{2}\cot z + c$

Bilinear transformation:

The transformation $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation. It is also known as Mobius transformation or linear fractional transformation. ($\because \frac{az+b}{cz+d}$ is a fraction formed by the linear fns)

Fixed points (or) Invariant points:

(In the transformation $w = f(z)$, if $w = z$ ~~then these set of points is called~~ then it is called as fixed pts. or invariant pts.) X

Fixed pts (or) invariant points of a transformation $w = f(z)$ are the points that are mapped onto itself & they are kept fixed under the transformation. Thus they are obtained from $w = f(z) = z$.

Note:

Fixed points (or) Invariant points of the transformation $w = \frac{az+b}{cz+d}$ is obtained from

$$z = \frac{az+b}{cz+d}.$$

Formula:

The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Cross Ratio:

Given four points z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

(1) Obtain the invariant points of the transformation

(i) $w = \frac{z-1}{z+1}$ (ii) $w = 2 - \frac{2}{z}$

Sol:

(i) The invariant pts are given by.

$z = \frac{z-1}{z+1}$

$\Rightarrow z^2 + \cancel{z} = \cancel{z} - 1$

$z^2 - 1 = 0$

$z = \pm i.$

(ii) $z = 2 - \frac{2}{z}$

$z^2 = 2z - 2$

$z^2 - 2z + 2 = 0$

$z = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$

(2) Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, i, -i$ resp.

Sol: Given $z_1 = -2,\ z_2 = 0,\ z_3 = 2$

$w_1 = 0,\ w_2 = i,\ w_3 = -i$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i+i)}{(w+i)(i-0)} = \frac{(z+2)(0-2)}{(z-2)(0+2)}$$

$$\frac{2w\cancel{i}}{\cancel{i}(w+i)} = \frac{-\cancel{2}(z+2)}{\cancel{2}(z-2)}$$

$2w(z-2) = -(z+2)(w+i)$

$2wz - 4w = -wz - 2w - iz - 2i$

$2wz - 4w + wz + 2w = -i(z+2)$

$3wz - 2w = -i(z+2)$

$w(3z-2) = -i(z+2)$

$\therefore w = \frac{-i(z+2)}{3z-2}$ //

2/4/ 61, 77, 79, 83, 84

3) Find the Bilinear transformation which maps $z = 1, i, -1$ onto $w = i, 0, -i$. Hence Find the fixed points.

Sol: Given: $z_1 = 1,\ z_2 = i,\ z_3 = -1$

$w_1 = i;\ w_2 = 0,\ w_3 = -i$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{\cancel{i}(w-i)}{-\cancel{i}(w+i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \times \frac{i+1}{i+1} \qquad \frac{w-i}{w+i} = \frac{zi+z-i-1}{zi-z+i-1}$$

$w(zi - z + i - 1) - zi^2 + \cancel{zi} - i^2 + i$

$$-\left(\frac{w-i}{w+i}\right) = \frac{(z-1)(i^2+1+2i)}{(z+1)(-1-1)} \qquad = w(zi+z-i-1) + zi^2 \cancel{+zi} - i^2 i$$

$w[\cancel{z} - z + i \cancel{-1} - \cancel{zi} - \cancel{z} + i + 1)$

$$\cancel{-}\left(\frac{w-i}{w+i}\right) = \frac{(z-1)(2i)}{(z+1)(-2)} = \cancel{-}\frac{(z-1)\,i}{(z+1)} \qquad = -2zi - 2i$$

$w(-2z+2i)$

$$(w-i)(z+1) = (zi-i)(w+i)$$

$$wz + w - iz - i = iwz - z - iw + 1 \quad \text{—— (1)}$$

$$wz + w - iwz + iw = 1 - z + iz + i$$

$$w(z + 1 - zi + i] = 1 - z + i(z+1)$$

$$w = \frac{(1-z) + i(1+z)}{(1+z) + i(1-z)} \quad //$$

Fixed points are given by

$$z = \frac{(1-z) + i(1+z)}{(1+z) + i(1-z)}$$

$$(1) \Rightarrow z^2 + z - i\cancel{z} - i = iz^2 - z - i\cancel{z} + 1$$

$$z^2 + z - i - iz^2 + z - 1 = 0.$$

$$z^2(1-i) + 2z - (i+i) = 0.$$

$$\therefore z = \frac{-2 \pm \sqrt{4 + 4(1-i)(1+i)}}{2(1-i)}$$

ie) $z = \dfrac{-2 \pm \sqrt{4+4(2)}}{2(1+i)}$

$= \dfrac{-2 \pm \sqrt{12}}{2(1+i)} = \dfrac{-2 \pm 2\sqrt{3}}{2(1+i)} = \dfrac{-1 \pm \sqrt{3}}{(1+i)}$

$\therefore z = \dfrac{-1+\sqrt{3}}{(1+i)}$ (or) $\dfrac{-1-\sqrt{3}}{1+i} = z$

$z = \dfrac{(-1+\sqrt{3})(1-i)}{2}$ | $z = \dfrac{(-1-\sqrt{3})(1-i)}{2}$ //.

The fixed points are $\dfrac{-1+\sqrt{3}}{2}(1-i)$ & $\dfrac{-1-\sqrt{3}}{2}(1-i)$.

(3) Find the bilinear transformation that maps the points $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ onto the points $w_1 = 0$, $w_2 = i$, $w_3 = \infty$. Find the fixed points

Sol: The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{\cancel{w_3}\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\cancel{z_1}\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\cancel{z_1}\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)} = \frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{-w}{-i} = \frac{-i}{-z}$$

$$wz = i^2 = -1$$

$$\boxed{\therefore w = \frac{-1}{z}}$$

Fixed points are given by

$$z = \frac{-1}{z}$$

$$z^2 + 1 = 0.$$

$$z^2 = -1 \Rightarrow z = \pm i.$$

H/W

(1) $z = 0, 1, \infty$ & $w = i, 1, -1$

$w = \dfrac{z+i}{1+iz}$

(2) $z = -1, 0, 1$ & $w = 0, i, 3i$

$w = -3i\left(\dfrac{1+z}{z-3}\right)$

(5) Find the bilinear transformation which maps the point $1, i, -1$ onto the points $0, 1, \infty$. Show that the transformation maps the interior of the unit circle of the z-plane onto the upper half of the w-plane. 5

Sol: Given: $z_1 = 1, \; z_2 = i, \; z_3 = -1$

$w_1 = 0, \; w_2 = 1, \; w_3 = \infty$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{\cancel{w_3}\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w = \frac{(z-1)}{z+1} \times \frac{1+i}{-1+i} \times \frac{-1-i}{-1-i}$$

$$= \frac{(z-1)}{(z+1)} \times \frac{-(1+i)^2}{1^2-i^2}$$

$$= \frac{(z-1)}{(z+1)} \times \frac{-(\cancel{1}+\cancel{i^2}+2i)}{1+1} = \frac{-(z-1)(\cancel{2}i)}{(z+1)(\cancel{2})} = -i\left(\frac{z-1}{z+1}\right)$$

$$\boxed{w = \frac{-iz+i}{z+1}}$$

$wz + w = -iz + i$

$\Rightarrow wz + w + iz - i = 0$.

$z(w+i) + (w-i) = 0$

$z(w+i) = -(w-i)$

$\therefore z = -\left(\frac{w-i}{w+i}\right)$

Thus the region $|z| < 1$ gives the region

$$\left|-\frac{(w-i)}{w+i}\right| < 1$$

Let $w = u+iv$, we get

$$\left|\frac{-(u+iv-i)}{u+iv+i}\right| < 1$$

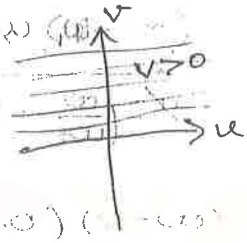
$$\left|\frac{-u-iv+i}{u+iv+i}\right| < 1$$

$$\left|\frac{-u+i(1-v)}{u+i(1+v)}\right| < 1$$

$$\Rightarrow |-u+i(1-v)| < |u+i(1+v)|$$

ie) $\sqrt{u^2+(1-v)^2} < \sqrt{u^2+(1+v)^2}$

Squaring on both sides, we get



$$u^2+(1-v)^2 < u^2+(1+v)^2$$

$$\cancel{u^2}+\cancel{1}+\cancel{v^2}-2v < \cancel{u^2}+\cancel{1}+\cancel{v^2}+2v$$

$$-4v < 0$$

$$-v < 0$$

$$\Rightarrow v > 0$$

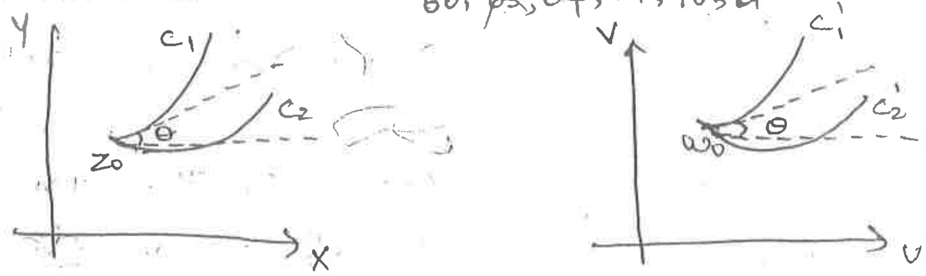
Hence the interior of the unit circle is mapped onto the upper half of the w-plane.

(6) Find the bilinear transformation that maps into the points $1+i, -i, 2-i$ of the z-plane into the points $0, 1, i$ of the w-plane

11/4

22, 23, 27, 28, 30, 32, 36, 37, 39, 40, 41, 42, 45, 49, 50, 53, 54, 57, ~~58~~, 59, 60, ~~63~~, 64, 67, 70, ~~71~~

Conformal mapping:



z-plane   w-plane

Consider the transformation $w = f(z)$. Under this transformation, a point z_0 and any two curves c_1 & c_2 passing through z_0 in the z-plane will be mapped onto a point w_0 & two curves c_1' & c_2' in the w-plane.

If the angle btw. c_1 & c_2 at z_0 is the same as the angle between c_1' & c_2' at w_0 both in magnitude & sense, then the transformation $w = f(z)$ is said to be conformal at the point z_0.

Defn:

(1) A transformation that preserves angle between every pair of curves through a point both in magnitude & sense is called conformal at that point

(2) Isogonal mapping

A transformation that preserves angle between every pair of curves through a point in magnitude but altered in direction is called isogonal at that point.

Note:

A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if $f'(z_0) \neq 0$.

Critical Points:

The point at which the mapping $w=f(z)$ is not conformal i.e) $f'(z)=0$ is called critical point of mapping.

The critical points of the transformation $w=f(z)$ are given by $\frac{dw}{dz}=0$ & $\frac{dz}{dw}=0$.

① Find the critical points of the transformation $w = z+\frac{1}{z}$.

Sol:

$$w = z+\frac{1}{z}$$

$$\frac{dw}{dz} = 1-\frac{1}{z^2} = \frac{z^2-1}{z^2}$$

Critical points occur at $\frac{dw}{dz}=0$ & $\frac{dz}{dw}=0$

i.e) $\frac{z^2-1}{z^2}=0$	$\frac{z^2}{z^2-1}=0$.
$\Rightarrow z=\pm 1$	$z=0$.

$\therefore$ critical points are $z=\pm1, z=0$.

② Find the points such that $w=f(z)=\sin z$ is not conformal.

$$w=\sin z$$

$$\frac{dw}{dz}=\cos z \quad \therefore \frac{dz}{dw}=\frac{1}{\cos z}$$

Critical pts. occur at $\frac{dw}{dz}=0, \frac{dz}{dw}=0$

$\cos z=0$	$\frac{1}{\cos z}=0$.
$z=\pm(2n-1)\frac{\pi}{2}$	$1=0$ which is impossible.
$n=1,2,\ldots$	

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$\therefore$ critical pts. are $z=(2n-1)\frac{\pi}{2}, n\in \mathbb{Z}$.

③ $w^2=(z-\alpha)(z-\beta)$

$2w\frac{dw}{dz}=(z-\alpha)+(z-\beta)$	$\frac{dz}{dw} \Rightarrow \frac{2w}{(z-\alpha)+(z-\beta)}=0$
$\frac{dw}{dz}=0 \Rightarrow z-\alpha+z-\beta=0$	$2w=0$
$z=(\alpha+\beta)/2$	$w=0$.
	$z=\alpha, z=\beta$

Some standard Transformation:

I. Translation:

The transformation $w = c + z$, $c \to$ complex constant

Let $z = x+iy$, $w = u+iv$, $c = a+ib$

$\therefore w = z + c$

$\Rightarrow u+iv = x+iy+a+ib$

$u+iv = (x+a)+i(y+b)$

$u = x+a$ and $v = y+b$.

This transformation translates any region in z-plane onto a region with same shape & size in same orientation in w-plane.

① Find the image of $2x+y-3=0$ under the transformation $w = z+2i$.

Sol: $w = z+2i$

$u+iv = x+iy+2i$

$= x + i(y+2)$

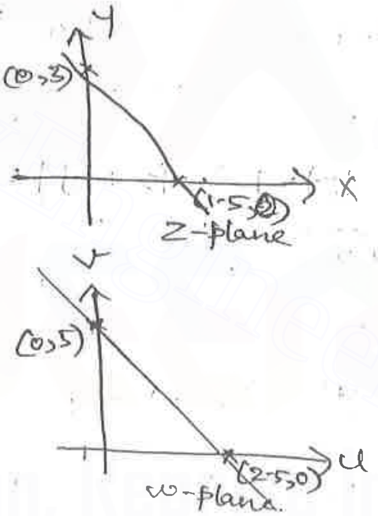
$\therefore u = x$ & $v = y+2$

$x = u$ & $y = v-2$

Given: $2x+y-3=0$

$2u + v-2-3=0$

$2u+v-5=0$



Hence the image of $2x+y-3=0$ is $2u+v-5=0$, a st line in w-plane.

② Draw the image of the square whose vertices are at (0,0), (1,0), (1,1) & (0,1) in the z-plane under the transformation $w = (1+i)z$. ~~What~~

Sol:

Let $w = (1+i)z$

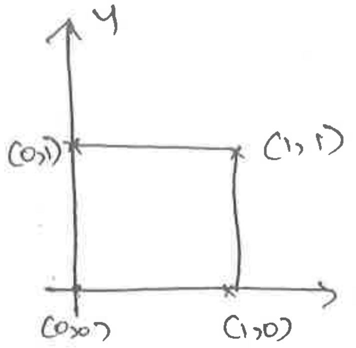
$u+iv = (1+i)(x+iy)$

$u+iv = x+iy+ix-y$

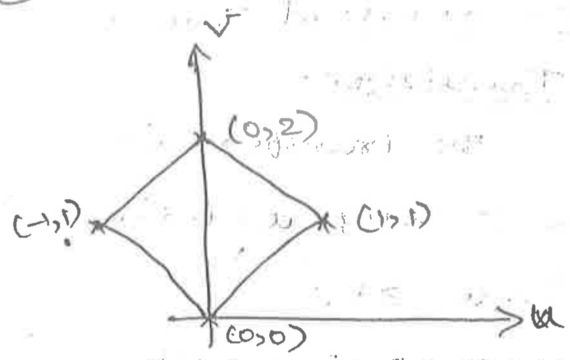
$= x-y+i(x+y)$

$u = x-y$ & $v = x+y$

pts z-plane (x, y)	pts w-plane ($u=x-y$, $v=x+y$)
(0, 0)	(0, 0)
(1, 0)	(1, 1)
(1, 1)	(0, 2)
(0, 1)	(−1, 1)



Z-plane



ω-plane

(3) Find the image of $|z|=1$ by the transformation $\omega = z+2+4i$.

Sol: $\omega = z+2+4i$

$u+iv = x+iy+2+4i$

$= (x+2)+i(y+4)$

$\therefore u = x+2$ & $v = y+4$

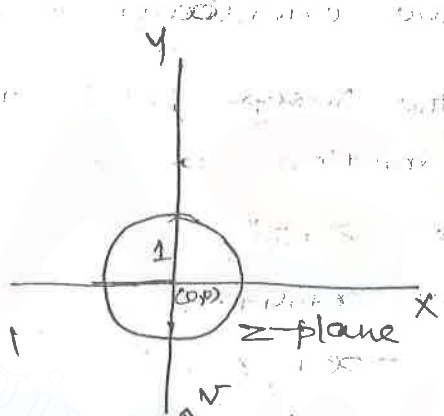
$\Rightarrow x = u-2 \mid y = v-4$

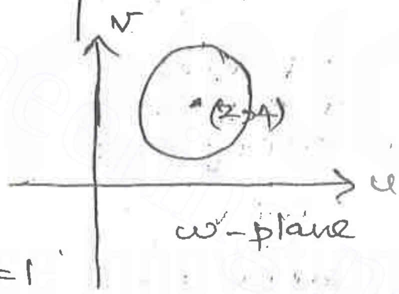
Given: $|z|=1 \Rightarrow |x+iy| = 1$

ie) $\sqrt{x^2+y^2} = 1$

$x^2+y^2 = 1$

$(u-2)^2+(v-4)^2 = 1$





Hence the circle $x^2+y^2=1$ is mapped into $(u-2)^2+(v-4)^2=1$ in ω-plane which is also a circle with centre (2,4) & radius 1.

Q. Find the image of $|z|=a$ under the transf $W=2z$ / Ans $|\omega|=2a$

HW What is the region of the ω-plane into which the rectangular region in the z-plane bounded by the lines $x=0, y=0, x=1, y=2$ is mapped under the transformation $\omega = z+2-i$

Sol: $\omega = z+(2-i)$

$u = x+2 \mid v = y-1$

$x = u-2 \quad y = v-1$

$x=0 \Rightarrow u=2 \quad \mid \quad y=0 \Rightarrow v=-1$

$x=1 \Rightarrow u=3 \quad \mid \quad y=2 \Rightarrow v=1$.

Magnification : $w = cz$, $c \to$ a real constant

$u + iv = c(x+iy) = cx + icy$

$\therefore u = cx$ & $v = cy$.

(1) Find the image of the circle $|z| = 2$ under the transformation $w = 5z$.

$\to$ $w = 5z$

$u + iv = 5x + i5y$

$u = 5x$ & $v = 5y$

$\therefore x = \frac{u}{5}$ | $y = \frac{v}{5}$

Given: $|z| = 2$

$\sqrt{x^2+y^2} = 2$

$x^2 + y^2 = 4$

$\left(\frac{u}{5}\right)^2 + \left(\frac{v}{5}\right)^2 = 4$

$u^2 + v^2 = 100$

ie) $|w|^2 = 10^2$ which is a circle in w-plane

$\therefore |w| = 10$

$\therefore$ The image of the circle $|z| = 2$ is $|w| = 10$ under $w = 5z$.

84, 90, 101, 108

(2) Find the image of the circle $|z| = \lambda$ under the transformation. $w = 5z$

$\to$ $w = 5z$

$u + iv = 5x + 5iy$

$u = 5x$ & $v = 5y$

$\therefore x = u/5$ & $y = v/5$

Given: $|z| = \lambda$

$|z|^2 = \lambda^2$

$x^2 + y^2 = \lambda^2$

$\frac{u^2}{25} + \frac{v^2}{25} = \lambda^2$

$u^2 + v^2 = (5\lambda)^2$

$|w|^2 = (5\lambda)^2$

The image of $|z| = \lambda$ in the z-plane is transformed into $|w| = 5\lambda$ in the w-plane under the transformation $w = 5z$.

Inversion & reflection: [The transf. $w = 1/z$ represents inversion w.r.t the unit $\odot$ $|z|=1$]

S.T. the transformation $w = \frac{1}{z}$ transforms all circles and st-lines in the z-plane into circles or st-lines in w-plane.

Proof: consider the transformation $w = \frac{1}{z}$

$$\text{ie) } u+iv = \frac{1}{x+iy}$$

$$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right).$$

$\therefore u = \frac{x}{x^2+y^2}$ & $v = \frac{-y}{x^2+y^2}$ and $u^2+v^2 = \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$

Consider the general equation of a circle

$$a(x^2+y^2) + 2gx + 2fy + c = 0. \quad \text{---} (1)$$

$\div$ by x^2+y^2,

$$a + 2g\frac{x}{x^2+y^2} + 2f\cdot\frac{y}{x^2+y^2} + c\cdot\frac{1}{x^2+y^2} = 0.$$

$$a + 2gu - 2fv + c(u^2+v^2) = 0. \quad \text{---} (2)$$

The transformed equation is

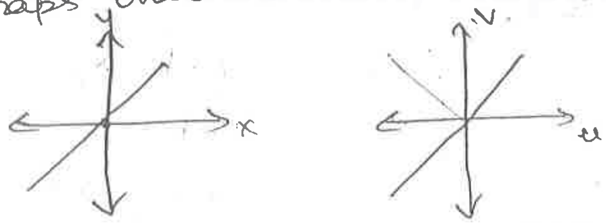
$$c(u^2+v^2) + 2gu - 2fv + a = 0.$$

Case(i): $a=0,\ c=0$

$(1) \Rightarrow 2gx + 2fy = 0$

$(2) \Rightarrow 2gu - 2fv = 0.$

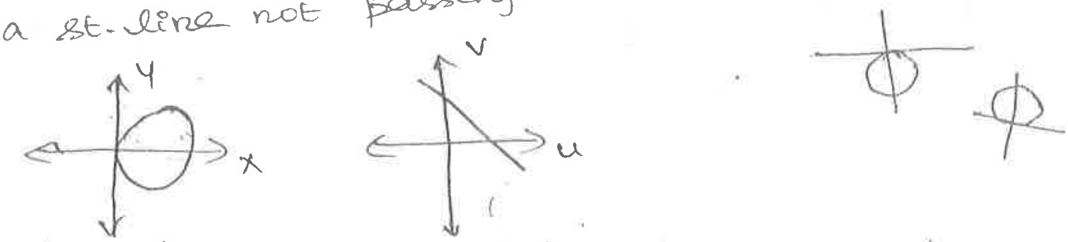
Straight line passes through the origin of z-plane maps onto st. line passing through the origin in w-plane



Case(ii): $a \neq 0,\ c=0$. $(1) \Rightarrow a(x^2+y^2) + 2gx + 2fy = 0$

$(2) \Rightarrow a + 2gu - 2fv = 0$.

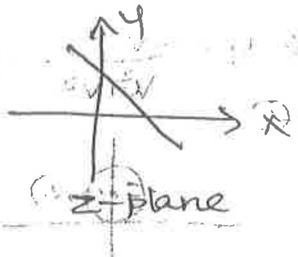
A circle through the origin in z-plane maps onto a st. line not passing through the origin in w-plane

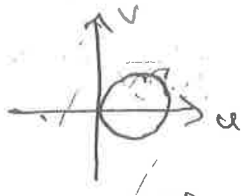


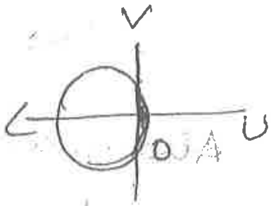
Case (iii): $a=0, c\neq 0$. ① $\Rightarrow 2gx+2fy+c=0$

② $\Rightarrow 2gu-2fv+c(u^2+v^2)=0$

The st. lines not passing through the origin in z-plane maps into circles through the origin in w-plane.



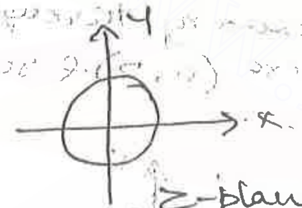


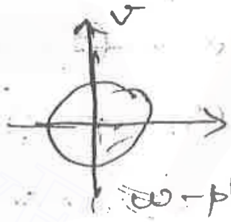


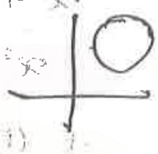
Case (iv): $a\neq 0, c\neq 0$ ① $\Rightarrow a(x^2+y^2)+2gx+2fy+c=0$

② $\Rightarrow c(u^2+v^2)+2gu-2fv+a=0$

Circles not passing through the origin in z-plane maps into circles not passing through the origin.

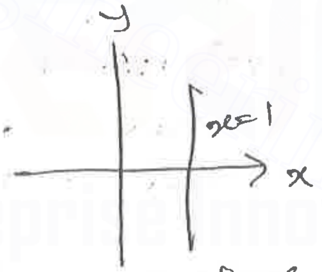






① Find the image of $x=1$ under the transformation $w=\frac{1}{z}$

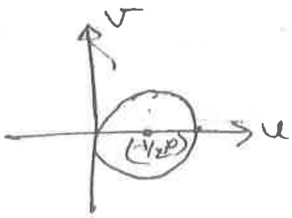
Sol: Let $w=1/z$, $z=1/w$



$$\Rightarrow x+iy=\frac{1}{u+iv}$$

$$=\frac{u-iv}{u^2+v^2}$$

$$x=\frac{u}{u^2+v^2} \quad \& \quad y=\frac{-v}{u^2+v^2}$$



Given: $x=1 \Rightarrow \frac{u}{u^2+v^2}=1$

$$\Rightarrow u^2+v^2=u$$

$u^2+v^2-u=0. \rightarrow$ circle with centre $(-g,-f)=(1/2,0)$

$\rightarrow$ radius $\sqrt{g^2+f^2-c}=1/2$

$\therefore$ The image of $x=1$ is the circle with centre $(1/2,0)$ & radius $1/2$

(2) Find the image of $|z-2i|=2$ under the transformation $w=1/z$.

Sol: Let $w=1/z \Rightarrow z=\frac{1}{w}$

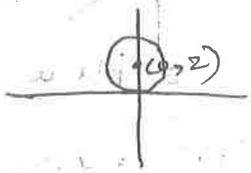
i.e) $x+iy=\frac{1}{u+iv} \Rightarrow x=\frac{u}{u^2+v^2}$ — (1) & $y=\frac{-v}{u^2+v^2}$ — (2)

Also given

$|z-2i|=2$

$|x+iy-2i|=2$

$|x+(y-2)i|=2 \Rightarrow \sqrt{x^2+(y-2)^2}=2$



$\Rightarrow x^2+(y-2)^2=4$

$x^2+y^2+4-4y=4$

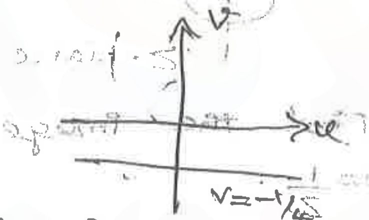
$2g=0 \Rightarrow g=0$

$2f=-4 \Rightarrow f=-2$

$x^2+y^2-4y=0$ — (3) a circle passing through $(0,0)$ with centre $(0,2)$ & radius 2.

Sub. (1) & (2) in (3), we get

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\frac{4v}{u^2+v^2}=0$$



$u^2+v^2+4v(u^2+v^2)=0$

$(u^2+v^2)(1+4v)=0$ $(\because u^2+v^2\neq 0)$ w-plane

$1+4v=0 \Rightarrow v=-1/4$

which is a st. line in w-plane.

(3) Find the image of $|z-1|=1$ in the complex plane under the mapping $w=\frac{1}{z}$

$\rightarrow$ $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$$x+iy=\frac{1}{u+iv}\times\frac{u-iv}{u-iv}$$

$$=\frac{u-iv}{u^2+v^2}=\frac{u}{u^2+v^2}+i\left(\frac{-v}{u^2+v^2}\right)$$

$\therefore x=\frac{u}{u^2+v^2}$ — (1) & $y=\frac{-v}{u^2+v^2}$ — (2)

Given: $|z-1|=1$

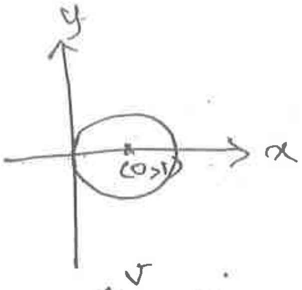
$|x+iy-1|=1 \Rightarrow |(x-1)+iy|=1$

$(x-1)^2+y^2=1$

$x^2-2x+1+y^2=1$

$x^2-2x+y^2=1$ — (3)

Sub. (1) & (2) in (3), we get



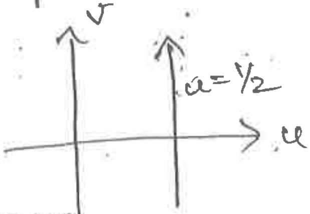
$$\frac{u^2}{(u^2+v^2)^2} - \frac{2u}{u^2+v^2} + \frac{v^2}{(u^2+v^2)^2} = 0$$

$$u^2 - 2u(u^2+v^2) + v^2 = 0$$

$$(u^2+v^2)(1-2u) = 0$$

$$1-2u = 0$$

$$u = \frac{1}{2}$$



which is a st. line in the w-plane.

(3) What will be the image of a circle with centre origin in the XY plane under the trans. $w = 1/z$.

Sol: $w = \frac{1}{z} \rightarrow z = \frac{1}{w}$

$$x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

$$\therefore x = \frac{u}{u^2+v^2} \quad \text{---(1)} \quad \& \quad y = \frac{-v}{u^2+v^2}$$

Also given $x^2+y^2 = r^2$

$$\text{i.e.) } \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = r^2$$

$$\frac{\cancel{u^2+v^2}}{(u^2+v^2)^{\cancel{2}}} = r^2$$

$$u^2+v^2 = \frac{1}{r^2}$$

which is a circle with centre $(0,0)$ & radius $1/r$ in w-plane.

(4) Determine the image of $1 < x < 2$ under the mapping $w = \frac{1}{z}$

Sol: $z = \frac{1}{w} \Rightarrow x = \frac{u}{u^2+v^2}$ ---(1) & $y = \frac{-v}{u^2+v^2}$

Given $x > 1$

$$\Rightarrow \frac{u}{u^2+v^2} > 1$$

$$\Rightarrow u > u^2+v^2$$

$$\therefore u^2+v^2-u < 0$$

$$(u^2-u)+v^2 < 0$$

$$\left(u^2-u+\tfrac{1}{4}\right)-\tfrac{1}{4}+v^2 < 0$$

$$\left(u-\tfrac{1}{2}\right)^2+(v-0)^2 < \left(\tfrac{1}{2}\right)^2$$

$2g = -1$, $g = -1/2$, $\therefore -g = 1/2$ | $2f = 0$, $-f = 0$

centre $(1/2, 0)$ & radius $1/2$

This is the region lies inside ~~which is a the circle~~ with centre $(1/2, 0)$ & radius $1/2$

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$x < 2$

$u < 2u^2+2v^2$

$u^2+v^2 > \frac{u}{2}$

$u^2+v^2-\frac{u}{2} > 0$

$(u-\frac{1}{4})^2+v^2 > (\frac{1}{4})^2$

$2g = -\frac{1}{2}$ | $2f=0$

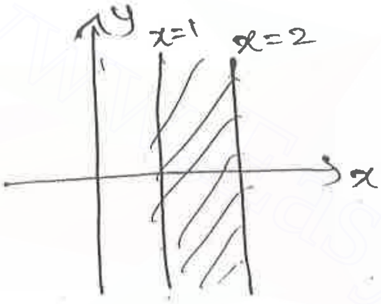
$g = -\frac{1}{4}$ | $f=0$

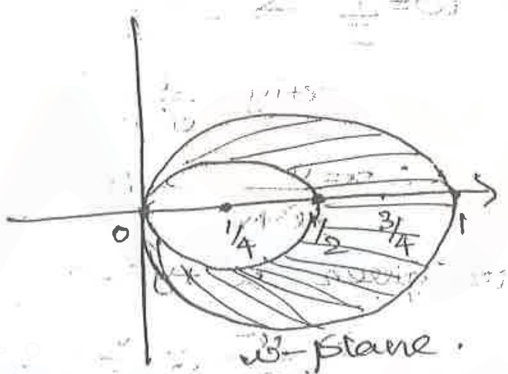
$\therefore$ centre $(-g,-f) = (\frac{1}{4}, 0)$

radius $= \frac{1}{4}$.

This is the region lies outside the circle with centre $(\frac{1}{4}, 0)$ & radius $\frac{1}{4}$

$\therefore$ The image of $1 < x < 2$ is the region lies btw. the circles $(u-\frac{1}{4})^2+v^2 > (\frac{1}{4})^2$ & $(u-\frac{1}{2})^2+v^2 < (\frac{1}{2})^2$





<u>HW</u> Find the image of the infinite strips. (i) $\frac{1}{4} < y < \frac{1}{2}$ (ii) $0 < y < \frac{1}{2}$ under the trans. $w = \frac{1}{z}$