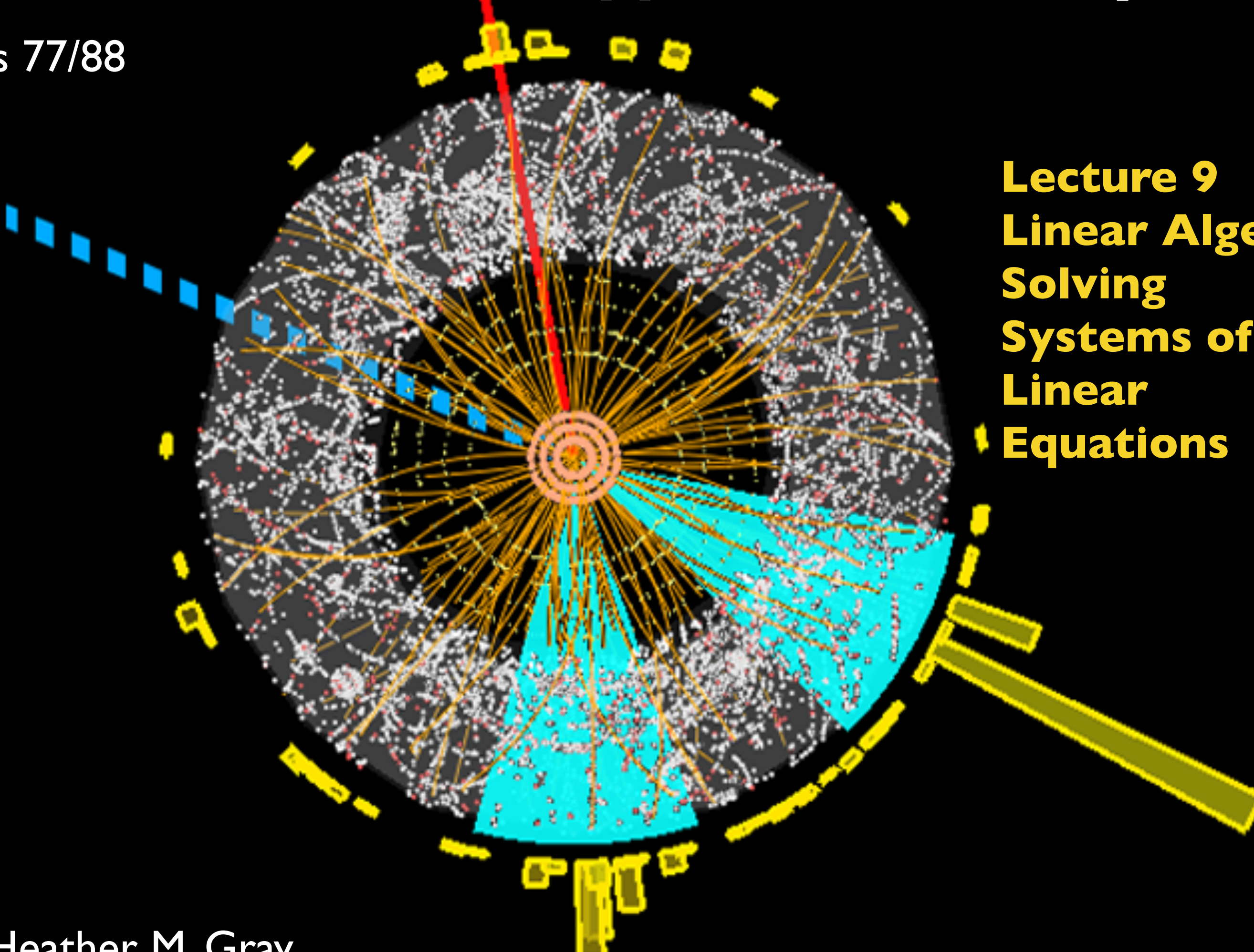
Introduction to Computational Techniques in Physics/Data Science Applications in Physics

Physics 77/88



Lecture 9
Linear Algebra: Solving Systems of Linear Equations

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Many thanks to Amin Jazaeri, Yury Kolomensky for material used in this course

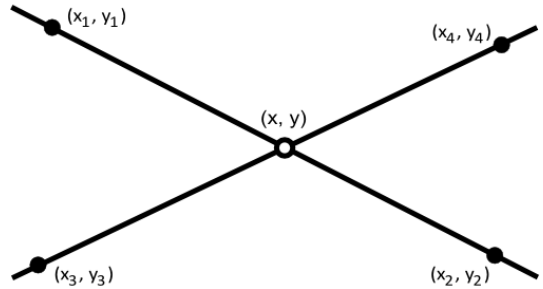
Example: Finding Intersection of Two Lines

- Example 1:
  - $y = 2 - x$
  - $y = x - 1$
- Solution:

  $2 - x = x - 1$

  $3 = 2x$

  - $\Rightarrow x = \frac{3}{2}$
  - $y = \frac{3}{2} - 1 = \frac{1}{2}$
  - $(x, y) = \left(\frac{3}{2}, \frac{1}{2}\right)$



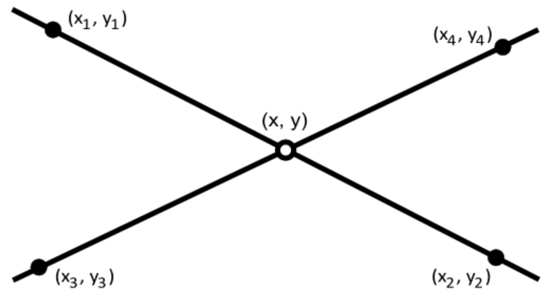
$x + y = 2$

$-x + y = -1$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Example: Finding Intersection of Two Lines

- Example 2 (generic):
  - $a_{11}x + a_{12}y = c_1$
  - $a_{21}x + a_{22}y = c_2$



- Solution:
  - $\frac{a_{11}}{a_{12}}x + y = \frac{c_1}{a_{12}}$; $\frac{a_{21}}{a_{22}}x + y = \frac{c_2}{a_{22}}$

$$\frac{c_1}{a_{12}} - \frac{a_{11}}{a_{12}}\cdot x = \frac{c_2}{a_{22}} - \frac{a_{21}}{a_{22}}\cdot x$$

  - $\frac{c_1}{a_{12}} - \frac{c_2}{a_{22}} = \left(\frac{a_{11}}{a_{12}} - \frac{a_{21}}{a_{22}}\right)x$

$$\Rightarrow x = \frac{\frac{c_1}{a_{12}} - \frac{c_2}{a_{22}}}{\frac{a_{11}}{a_{12}} - \frac{a_{21}}{a_{22}}}$$

Substitute and solve for y

Linear Algebraic Equations

- $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots \underline{a_{1n}x_n} &= c_1 \\ a_{\underline{21}}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n &= c_2 \\ \vdots \qquad\qquad &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots a_{nn}x_n &= c_n \end{aligned}$$

- $$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}}_{\vec{x}} = \underbrace{\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{Bmatrix}}_{\vec{c}}$$

- In matrix format
  - $A\vec{x} = \vec{c}$
- Solution
  - $\underline{A^{-1}A}\,\vec{x} = A^{-1}\vec{c}$

    $\Rightarrow \vec{x} = A^{-1}\vec{c}$

Linear Algebra Primer: Matrices

- Inverse
  - $A \underline{A^{-1}} = I$

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ & & \ddots \; 1 & \\ 0 & \cdots & & 1 \end{pmatrix}$$

- Transpose
  - $A^T = A_{\underline{ji}}$

$$A = \begin{bmatrix} \underline{a} & b \\ c & d \end{bmatrix} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

- Conjugate transpose
  - $A^\dagger$

$$A^\dagger = \begin{bmatrix} a^\dagger & c^\dagger \\ b^\dagger & d^\dagger \end{bmatrix}$$

- Trace
  - $\mathrm{Tr}\, A = \sum_i A_{ii}$

$$\mathrm{Tr}[A] = a + d$$

Linear Algebra Primer: Matrices

- Symmetric
  - $A_{ij} = A_{ji}$ or $A^T = A$ $\qquad A = \begin{bmatrix} d & b \\ b & d \end{bmatrix}$
- Unitary
  - $A^{-1} = A^\dagger$
    $\Rightarrow A^\dagger A = I$
- Normal
  - $A A^\dagger = A^\dagger A$

Determinant

- If the determinant of a matrix is non-zero then solutions exist (and vice versa)
- Our Generic Example 2:
  - $a_{11}x + a_{12}y = c_1$
  - $a_{21}x + a_{22}y = c_2$
- $$x = \frac{\frac{c_1}{a_{12}} - \frac{c_2}{a_{22}}}{\left[\frac{a_{11}}{a_{12}} - \frac{a_{21}}{a_{22}}\right]}, y = \frac{\frac{c_2}{a_{21}} - \frac{c_1}{a_{11}}}{\left[\frac{a_{11}}{a_{12}} - \frac{a_{21}}{a_{22}}\right]}$$
- If $\det A = \frac{a_{11}}{a_{12}} - \frac{a_{21}}{a_{2}} \neq 0$; x and y are finite

Determinant of a Matrix

- The determinant is a single parameter that can be used to characterize the matrix

- $$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- $$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- $$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Aside: Levi-Civita Symbol

- Define $\epsilon_{ij} = \begin{cases} +1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{if } i=j \end{cases}$

- e.g. $\begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- For matrix A $\Rightarrow \det A = \sum_{i,j=1}^{2} \varepsilon_{ij} a_{1i} a_{2j}$

Aside: Levi-Civita Symbol

- Extending to 3 dimensions
- $$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (3,2,1),\ (1,3,2) \text{ or } (2,1,3) \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i \end{cases}$$
- A famous application
- $$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} e_i a^j b^k$$
- $$(a \times b)^i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} a^j b^k = \varepsilon_{ijk} a^j b^k$$

Einstein summation: sum over repeated indices is implied

Aside: Levi-Civita Symbol

- We can generalize to matrices of arbitrary size
  - $\epsilon_{i_1 i_2 \cdots i_n} = \begin{cases} (-1)^p \epsilon_{12\ldots n} = (-1)^p & \text{if } i_1 \neq i_2 \neq \cdots \neq i_n \\ 0 & \text{otherwise} \end{cases}$
    - if p is the parity of the permutation from $i_1 \ldots i_n$ to $1 \ldots n$
      - the number of pairwise interchanges to get from order $i_1 \ldots i_n$ to $1 \ldots n$
- Then in n dimensions $A = [a_{ij}]$ and
  - $\det(A) = \epsilon_{i_1 i_2 \ldots i_n} a_{1 i_1} a_{2 i_2} a_{n a_n}$

Cramer's Rule

- M_{ij} is the determinant of the matrix M with the ith row and jth column removed
- $(-1)^{ij}$ is called the cofactor of element a_{ij}
- $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \; \forall j$

Cramer's Rule

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ c_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ c_3 & & & & \\ \vdots & & & & \\ c_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & & & & \\ a_{n1} & a_{n2} & & \cdots & a_{nn} \end{vmatrix}}$$

$$x_j = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & c_1 & a_{ij+1} & \cdots & a_{in} \\ & & & c_2 & & & \\ & & & \vdots & & & \\ a_{n1} & \cdots & a_{nj-1} & c_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}$$

Practical Details: Cramer's Rule

- $3n^2$ operations for the determinant
- $3n^2$ operations for every unknown
- Unstable for larges matrix
- Large error propagation
- Good for small matrices ($n < 20$)

Gaussian Elimination

- Divide each row by the leading element
- Subtract row 1 from all the other rows
- Move to the second row and continue

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

Gaussian Elimination

- Divide each row by the leading element

$$
\begin{bmatrix}
\frac{a_{11}}{a_{11}} & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{11}}{a_{11}} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{a_{31}}{a_{31}} & \frac{a_{32}}{a_{31}} & \cdots & \frac{a_{3n}}{a_{31}} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{a_{n1}}{a_{n1}} & \frac{a_{n2}}{a_{n1}} & \cdots & \frac{a_{nn}}{a_{n1}}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ \vdots \\ \vdots \\ x_3 \\ \vdots \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
\frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_3}{a_{31}} \\ \vdots \\ \frac{c_n}{a_{n1}}
\end{bmatrix}
$$

Gaussian Elimination

- Subtract row 1 from all the other rows

$$
\begin{bmatrix}
1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\
\vdots & \vdots & \vdots & \vdots & \\
0 & \frac{a_{32}}{a_{31}} - \frac{a_{12}}{a_{11}} & \frac{a_{33}}{a_{31}} - \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{3n}}{a_{31}} - \frac{a_{1n}}{a_{11}} \\
\vdots & \vdots & \vdots & \vdots & \\
0 & \frac{a_{n2}}{a_{n1}} - \frac{a_{12}}{a_{11}} & \frac{a_{n3}}{a_{n1}} - \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{nn}}{a_{n1}} - \frac{a_{12}}{a_{11}}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ \vdots \\ \vdots \\ x_2 \\ \vdots \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
\frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_3}{a_{31}} - \frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_n}{a_{n1}} - \frac{c_1}{a_{11}}
\end{bmatrix}
$$

Gaussian Elimination

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & 1 & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & 1 & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ \vdots \\ c'_n \end{bmatrix}$$

- Back substitution
  - $x_n = c'_n$
  - $x_i = c'_n - \sum_{j=i+1}^{n} a'_{ij} x_j$

Gaussian Elimination: Practical Issues

- Division by zero
  - May occur in forward elimination steps
- Rounding off error
  - Prone to rounding off errors

Gaussian Elimination: Example

- Let's look at the following system of equations
- We're going to use five significant figures with chopping

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

- After forward elimination

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & 1 & -588.23524 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 \\ -588.33323 \\ 0.9992235 \end{bmatrix}$$

Gaussian Elimination: Example

- Next, we apply back substitution

$$\begin{bmatrix} 1 & -.7 & 0 \\ 0 & 1 & -588.23524 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ -588.3323 \\ 0.9992235 \end{bmatrix}$$

- $x_3 =$ 0.99993
- $x_2 =$ -1.5
- $x_1 =$ -0.35000

Gaussian Elimination: Example

- Compare the calculated values with the exact solution

- $$[X]_{\text{exact}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- $$[X]_{\text{calculated}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

Now let's see how this looks in python

Gaussian Elimination: Improvements

- We can increase the number of significant digits
  - Decreases the rounding off error
  - Does not avoid division
- Gaussian elimination with partial pivoting
  - Avoids division by zero
  - Reduces rounding off error

Partial Pivoting

- Gaussian elimination with partial pivoting applies row switching to normal Gaussian elimination
- At the beginning of the kth step of forward elimination, find the maximum of

  $|a_{kk}|, |a_{k+1,k}|, \ldots |a_{nk}|$

- If the maximum of the values is $|a_{pk}|$ in the pth row $k \leq p \leq n$
  - Switch rows p and k
- Gaussian elimination with partial pivoting ensures that each step of forward elimination is performed with the pivoting element a_{kk} having the largest absolute value

Example

- Consider the same system of equations
  - $10x_1 - 7x_2 = 7$
  - $-3x_1 + 2.099x_2 + 3x_3 = 3.901$
  - $5x_1 - x_2 + 5x_3 = 6$
- In matrix form
- $$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$
- Let's solve them using Gaussian elimination with partial pivoting with five significant digits with chopping

Partial Pivoting: Example

- Forward elimination: step 1
  - Examine the values of the first colum
  - |10| , |-3| , |5| or 10 , 3 and 5
- The largest absolute value is 10, which means, following the rules of partial pivoting, we don't switch anything

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Forward elimination

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & -0.0034 & 2 \\ 0 & -0.9 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.00033 \\ -0.5 \end{bmatrix}$$

Partial Pivoting: Example

- Forward elimination, step 2
  - Examine the values of the second column
    - |-0.0034| and |-0.9| or 0.0034 and 0.9
  - The largest absolute value is 0.9 so switch row 2 and 3

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & -0.0034 & 2 \\ 0 & -0.9 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.000333 \\ -0.5 \end{bmatrix}$$

Row swap

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & -\underline{0.9} & -1 \\ 0 & -0.0034 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -0.5 \\ 2.00033 \end{bmatrix}$$

Partial Pivoting: Example

- Perform forward elimination

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & -0.9 & -1 \\ 0 & -0.0034 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -0.5 \\ 2.000333 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & 1 & 1.1111 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.55555 \\ 0.99922 \end{bmatrix}$$

Partial Pivoting Example

- Solve the equations through back substitution

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & 1 & 1.1111 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.55555 \\ 0.999223 \end{bmatrix}$$

- $x_3 =$ 0.99922
- $x_2 =$ -1
- $x_1 =$ 0

Partial Pivoting: Example

- Compare the calculated and exact solution
- That they are precisely equal is a coincidence, but it illustrates the advantage of partial pivoting
- $[X]_{\text{exact}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
- $[X]_{\text{calculated}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0.99923 \end{bmatrix}$

Matrix Factorization

- Assume that our matrix **A** can be written as $\mathbf{A} = \mathbf{VU}$ where **V** and **U** are triangular matrices

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} =$$

- $$\overset{V}{\begin{bmatrix} v_{11} & 0 & 0 & \cdots & 0 \\ v_{21} & v_{22} & 0 & \cdots & 0 \\ v_{31} & v_{32} & v_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nn} \end{bmatrix}} \overset{U}{\begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{32} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}}$$

Matrix Factorization

- We generally look to solve $[A][x] = [C]$
- Now we can decompose $[A]$ into $[V]$ and $[U]$, so

  $[v][u][x] = [C]$
- Then we solve $[v][z] = [C]$ for $[z]$
- And then solve $[u][x] = [z]$ for $[x]$

Matrix Factorization

- Method: Decompose $[A]$ to $[V]$ and $[U]$
- $$[A] = [V]\,.\,[U] = \begin{bmatrix} \underline{1} & 0 & 0 \\ v_{21} & \underline{1} & 0 \\ v_{31} & v_{32} & \underline{1} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$
- $[U]$ is the same as the coefficient matrix at the end of the elimination step
- $[V]$ is obtained using the multipliers that were used in the

LU Decomposition: Example

- Let's start by finding $[U]$ matrix using the forward elimination procedure of Gaussian elimination
- $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$
- $\text{Row2} - \left[\frac{\text{Row1}}{25}\right] \times 64 = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$
- $\text{Row3} - \left[\frac{\text{Row1}}{25}\right] \times 144 = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Lower-Upper Decomposition: Example

- Finding the $[U]$ matrix using the forward elimination process of Gaussian elimination
- $$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$
- $$\text{Row3} - \left[\frac{\text{Row2}}{-4.8}\right] \times -16.8 = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$
- $$[U] - \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

LU Decomposition: Example

- Finding the $[V]$ matrix using the multipliers used during the forward elimination process
- From the first step of forward elimination
  - $\begin{bmatrix} 1 & 0 & 0 \\ v_{21} & 1 & 0 \\ v_{31} & v_{32} & 1 \end{bmatrix}$
  - $\Rightarrow v_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$
  - $\Rightarrow v_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$

LU Decomposition: Example

- From the second step of forward elimination
- $$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$
- $\Rightarrow v_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$

LU Decomposition: Example

- $$[V] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

- Cross-check: does $[V][U] = [A]$?
-
- $$[V][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} =$$

25   5   1   25   5   1

2.56×25   2.56×5 − 4.8   2.56 − 1.56   = 64   8   1

5.76×25   5.76×5 − 3.5×5   5.76 + 3.5×(−1.56) + 0.7   144   12   1

LU Decomposition: Example

- Now let's use VU factorization to solve the following set of linear equations
- $$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$
- Using the procedure for finding the $[V]$ and $[U]$ matrices
- $$[A] = [V][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

LU Decomposition: Example

- Complete the forward substitution to solve for $[Z] : [L][Z] = [X]$
- $z_1 = 106.8$

  $\Rightarrow z_2 = 177.2 - 2.56 z_1$
- $= 177.2 - 2.56(106.8)$

  $= -96.2$

  $z_3 = 278.2 - 5.76 z_1 - 3.5 z_2$
- $= 279.2 - 5.76(106.8) - 3.5(-96.21)$

  $= 0.735$
- $$\Rightarrow [Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

LU Decomposition: Example

- Now set $[U][X] = [Z]$
- $$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -.48 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$
- Solve for $[X]$
- The three equations become
  - $25a_1 + 5a_2 + a_3 = 106.8$
  - $-48a_2 - 1.56a_3 = 96.21$
  - $0.7a_3 = 0.735$

LU Decomposition: Example

- From the 3rd equation
  - $0.7a_3 = 0.735$
  - $\Rightarrow a_3 = \dfrac{0.735}{0.7}$

    $= 1.050$
- Substitute a_3 into the second equation
  - $-4.8a_2 - 1.56a_3 = -96.21$
  - $\Rightarrow a_2 = \dfrac{-96.21 + 1.56a_3}{-4.8}$

    $= \dfrac{-96.21 + 1.56(1.050)}{-4.8}$

    $= 19.70$

LU Decomposition: Example

- Substituting a_3 and a_2 using the first equation
  - $25a_1 + 5a_2 + a_3 = 106.8$

$$\Rightarrow a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

- $$= \frac{106.8 - 5(19.70) - 1.050}{25}$$

$$= 0.2900$$

- Hence the final solution is

- $$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$