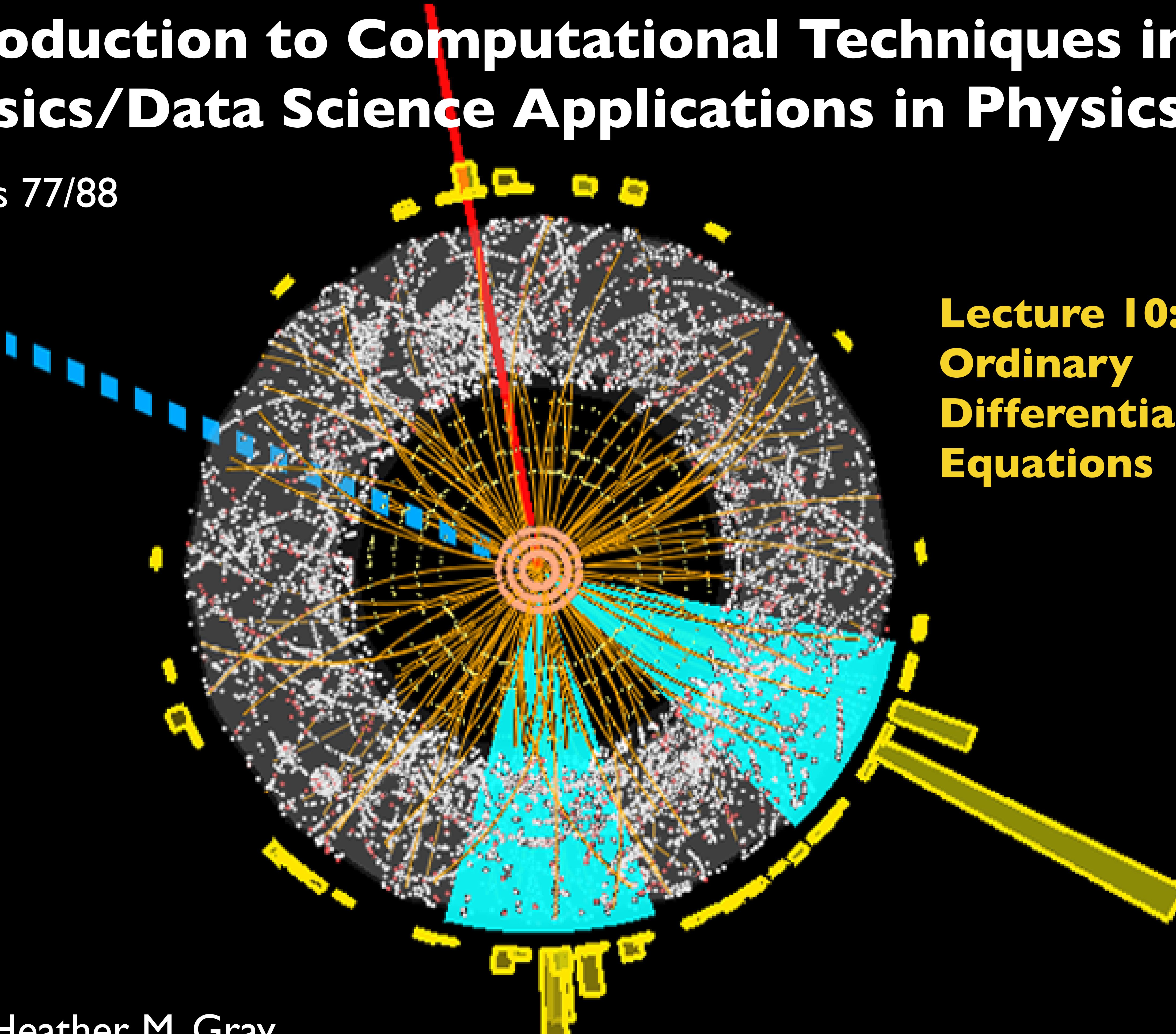
Introduction to Computational Techniques in Physics/Data Science Applications in Physics

Physics 77/88



Lecture 10: Ordinary Differential Equations

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Many thanks to Yury Kolomensky for material used in this course

Differential Equations

- Equations that are composed of
  and their are called differential equations

  - v: variable
  - t: variable

- importance in physics:
  - problems: naturally in terms of
    of some of some
    - Equations of motion: relate to
    - Thermodynamics: gradient to
    - E&M: to

Ordinary Differential Equations (ODEs)

- Most are described by a (set of)
  - e.g. $F(x, t) = m\ddot{x}$
- Solutions are not always
  - Only for the you find in a textbook
    - e.g.
- We will consider the
  - solutions

Classification of ODEs

- ODE can be classified in different ways
    -
        -
        -
        -
    -
        -
        -
    - conditions
        - value problems
        - value problems

Order of ODE

- The of an ordinary differential equation is the order of the
  - Examples
    - $\dfrac{dy}{dx} - y = e^x$: order ODE
    - $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 2y = \cos x$: order ODE
    - $(\dfrac{d^2y}{dx^2})^3 - \dfrac{dy}{dx} + 2y^4 = 1$: order ODE

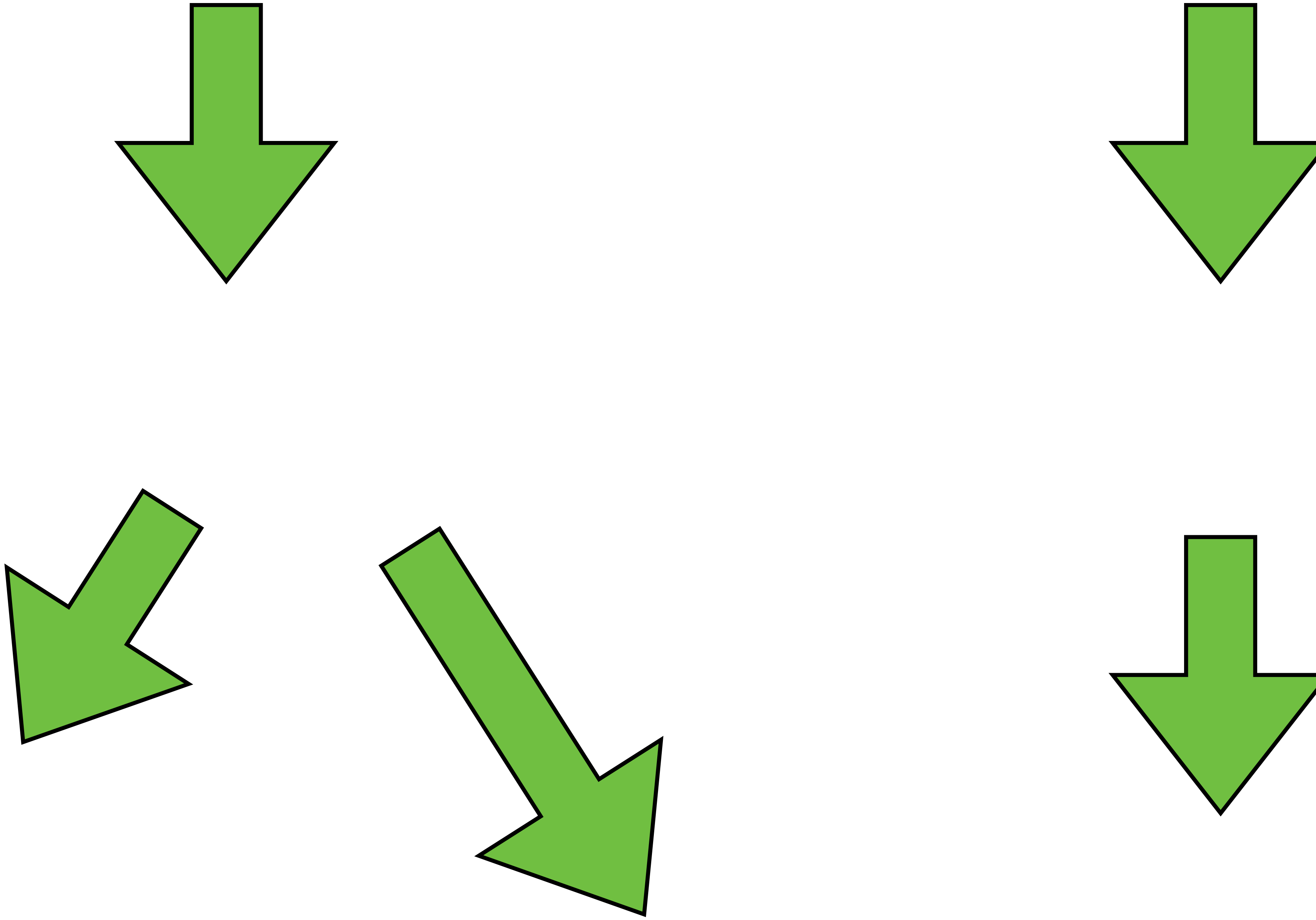
Linear vs Nonlinear ODEs

- An ODE is if the and its appear to power
  - And there is of the unknown function and/ or its derivatives
  - Examples
    - $\frac{dy}{dx} - y = e^x$ ODE
    - $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2x^2y = \cos(x)$ ODE
    - $(\frac{d^2y}{dx^2})^3 - \frac{dy}{dx} + \sqrt{y} = 1$ ODE

Initial Conditions

- problems
  - The are at of the
    - e.g. $y'' + 2y' + y = e^{-2x}$
- problems
  - The are at one point of the independent variable
    - e.g. $y'' + 2y' + y = e^{-2x}$

Analytical vs Numerical Solutions

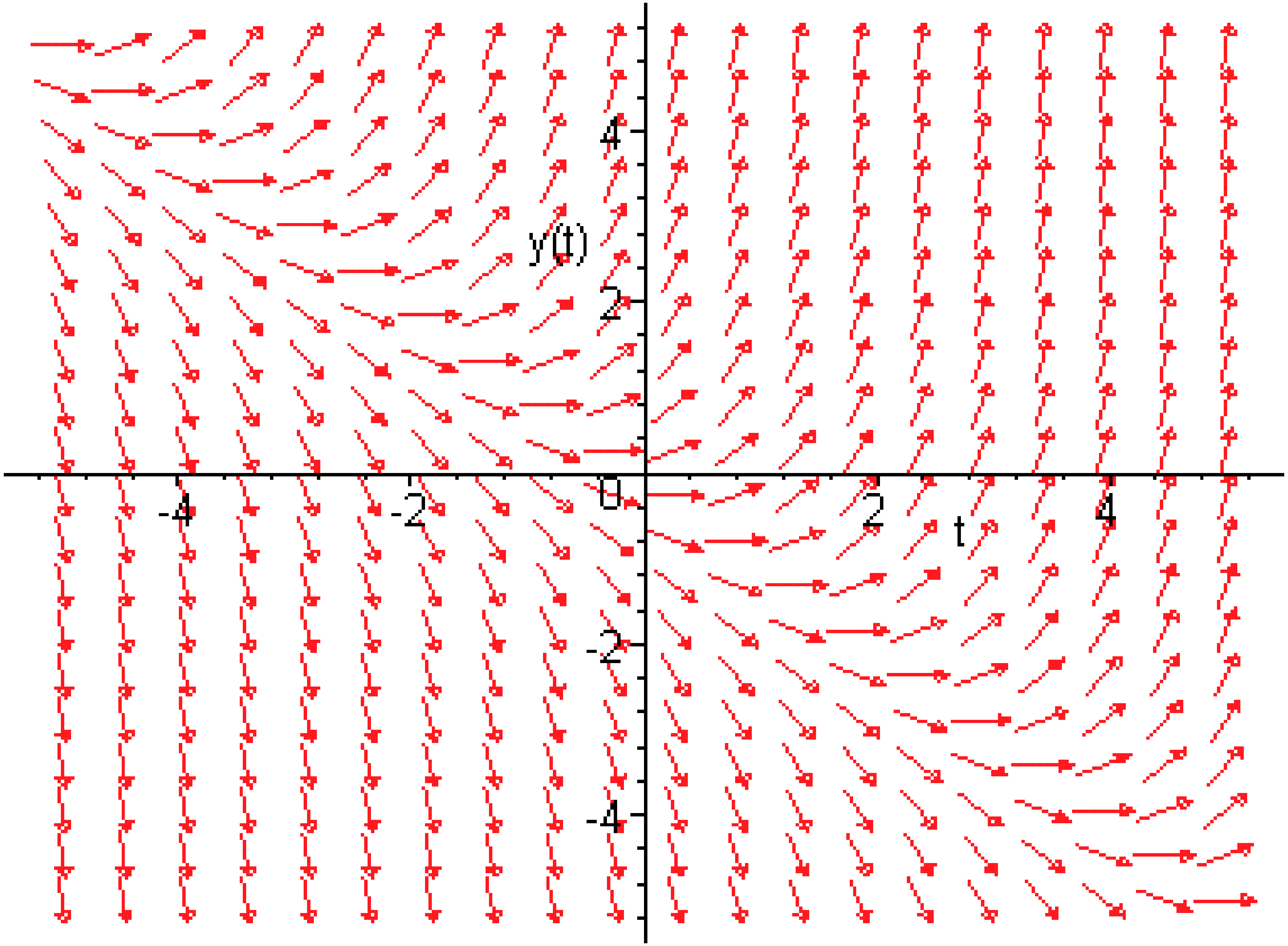


Visualization: Direction Fields

- Consider the differential equation
- Equation specifies at each in the
- Gives the that a to the must have at
- A plot of drawn at in the $x - y$ plane showing the of the s
  - field
- field gives the

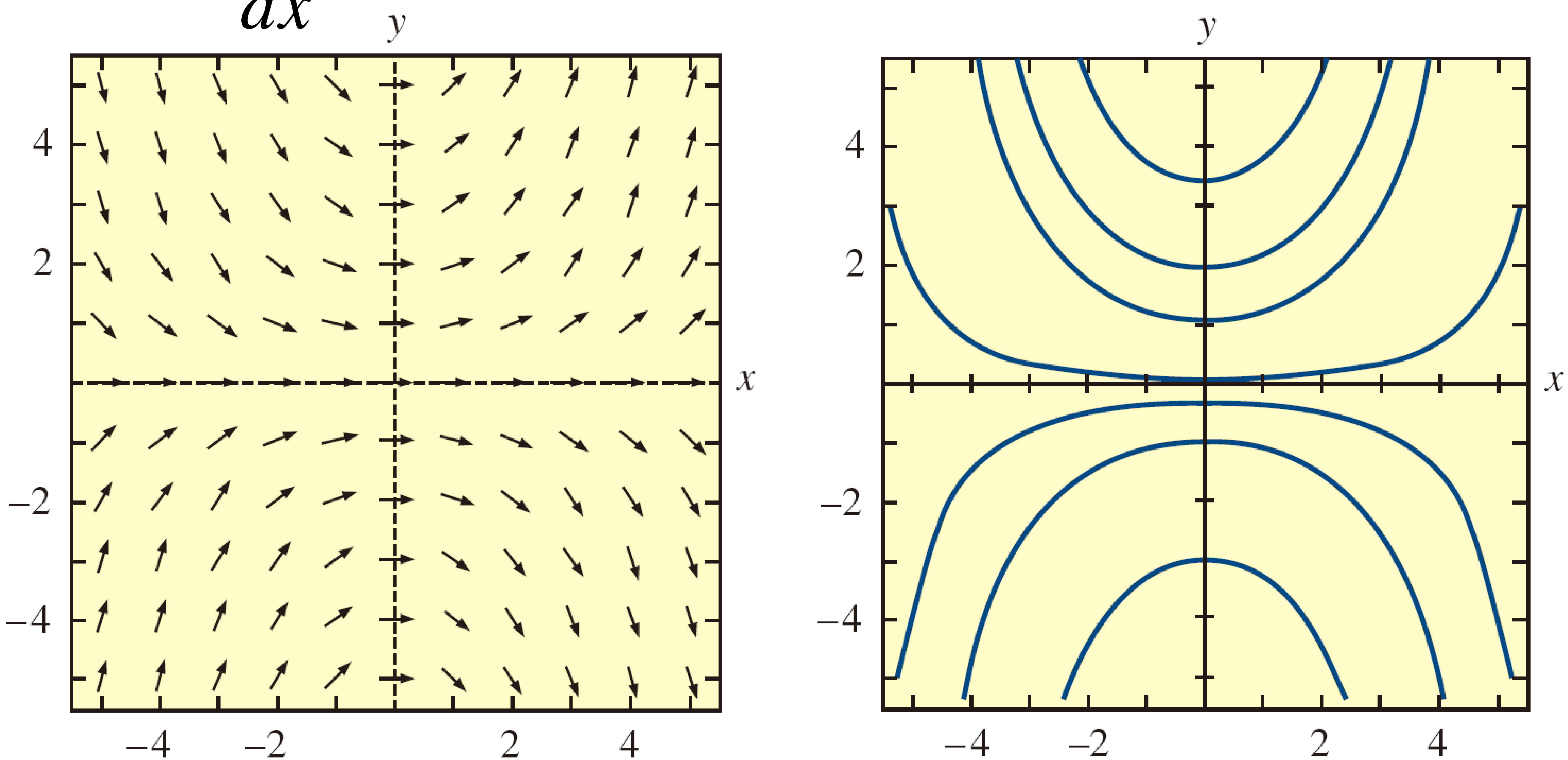
Example

$$f(t, y) =$$



Example

$$\frac{dy}{dx} = \qquad\qquad y =$$



Direction fields can help in selecting a class of

Methods to solve Differential Equations

- Try a , see if it
  - reduce the set of possible solutions
-
  - Reduce the equation to a

Example: Analytical Solution

- An to a differential equation is a that satisfies the equation
- Example
  - $\dfrac{dx}{dt} + x(t) = 0$
- Solution
  - $x(t) =$
- Proof
  - $\dfrac{dx(t)}{dt} =$
  - $\Rightarrow$

Stability and Chaos

- Solution of an ODE can be
    - Solutions resulting from of the remain to the original solution
-
    - Solutions resulting from the perturbations to the original solution
    - Solutions resulting from perturbations the original solutions

Numerical Solutions

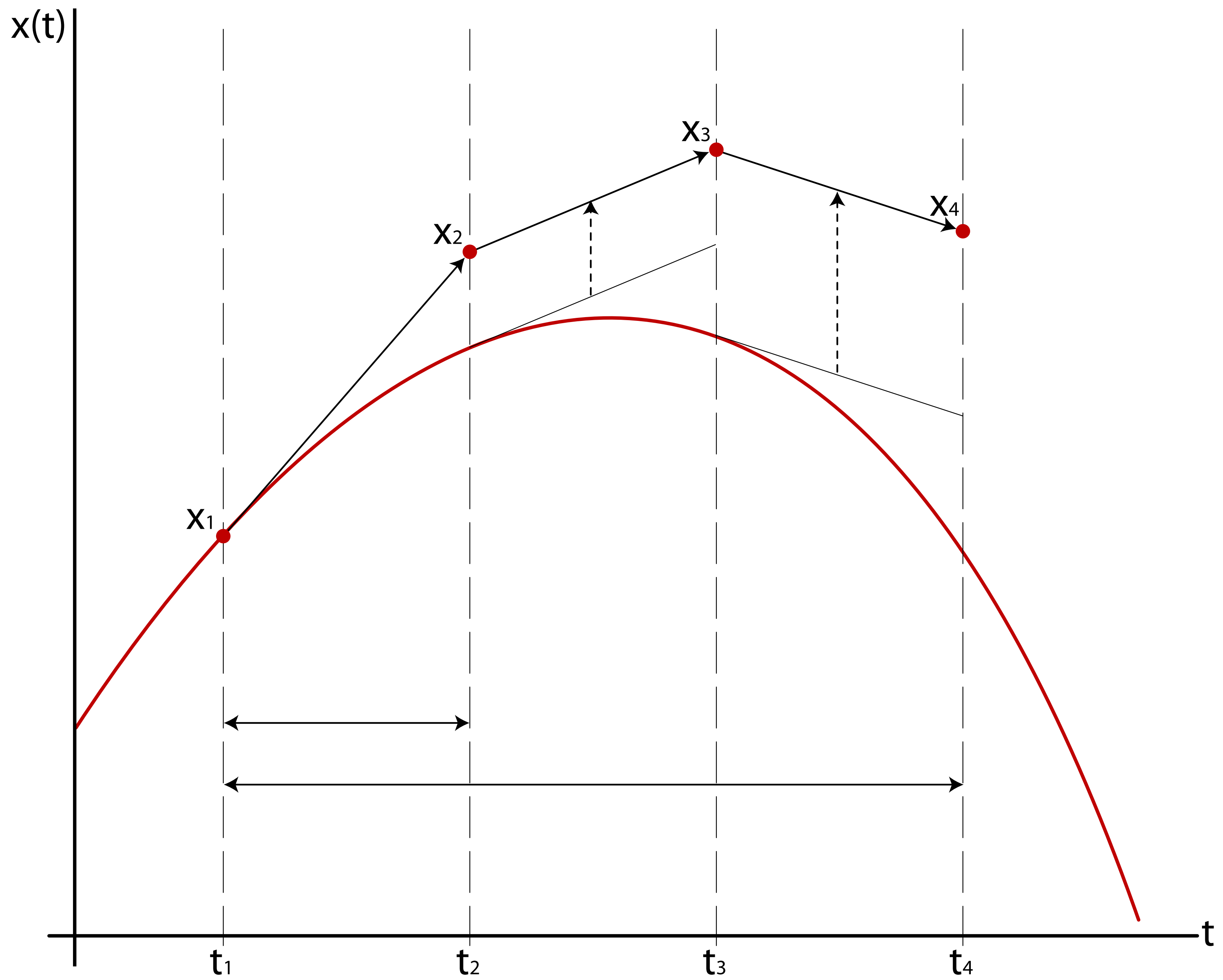
- solution values are in moving across the where the is sought
  - i.e. need to solve differential equations in a
- In stepping from to the , incur some
  - Next approximate lie on a from the one we started from
- or of solutions determines, in part, whether such are or with time

Euler Method

- Example: find the of a
  - $x(x, t) =$
- Rewrite with partial differences
  - $dx =$
  - $\Rightarrow \Delta x =$
- Implementation
  -
  -

This is known as the Euler method

Euler Method



A. Ayers, Computational Physics with Python

Euler Method

- Precision limited by
    - Δt to reduce
- Calculation time scales with the
    - $N_{\text{steps}} \sim$ where τ is the to be integrated
- So far limited to

Euler Method for 2^{nd} Order ODE

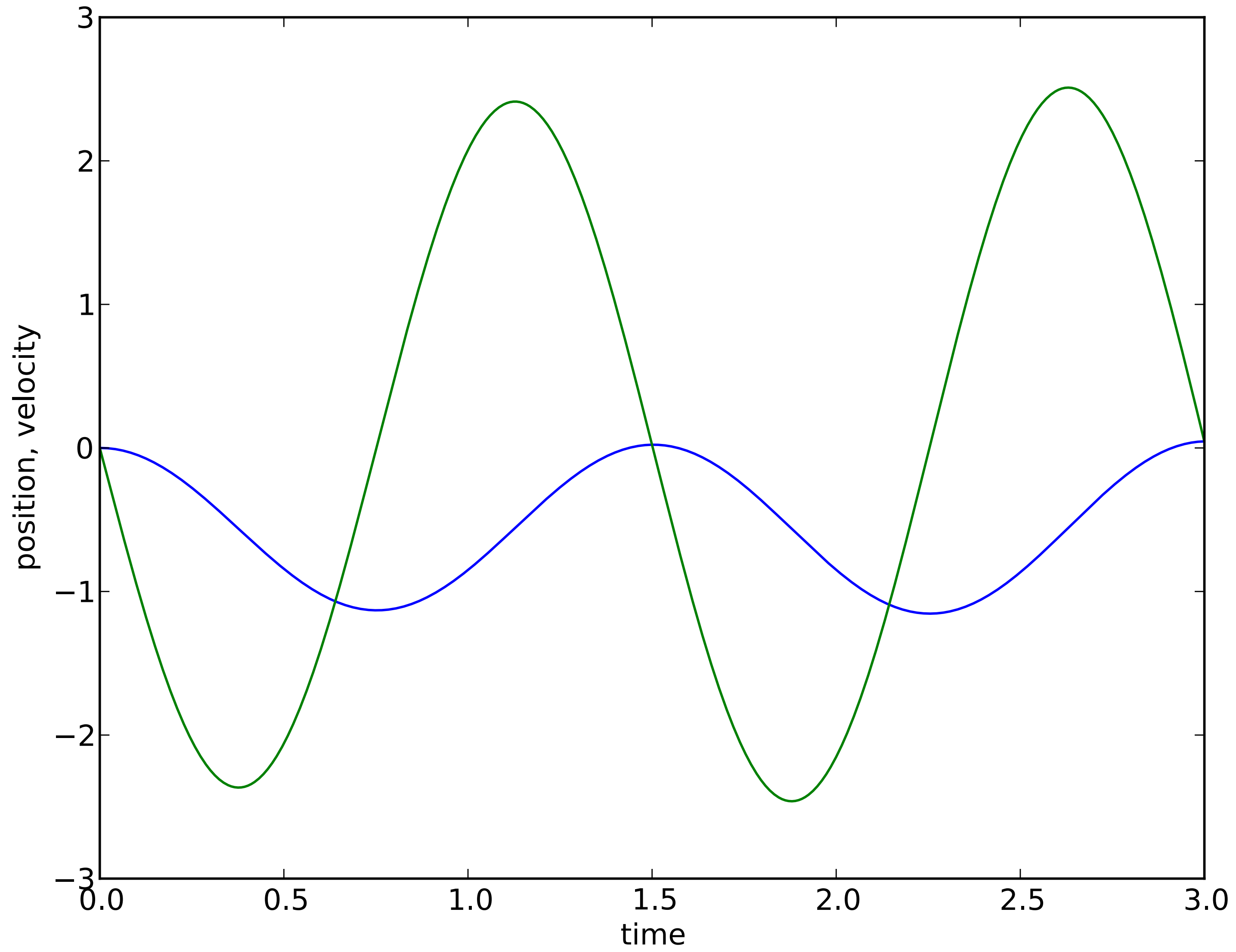
- Standard trick: convert a order ODE into a of order ODEs
  - Example:
    -
    - Define
      -
      -
  - Solutions can be obtained by applying the Euler method

Generalize

- Rewrite in vector form
    - $y =$
- Vector of derivatives is
    - $\dot{y} =$
- Euler solution
    - $y_{i+1} =$
- See notebok for implementation

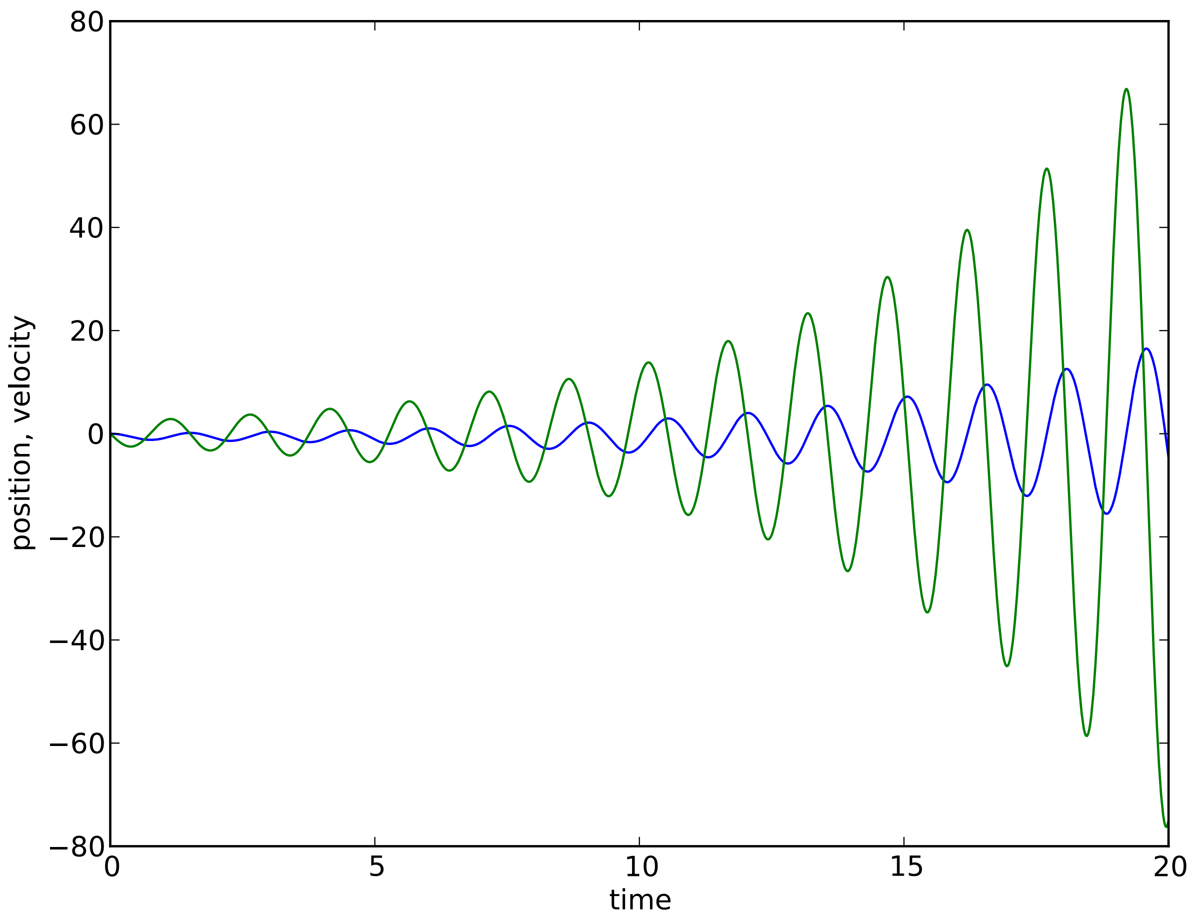
Solutions

- Example:
  - Mass on a vertical spring
    - $F =$



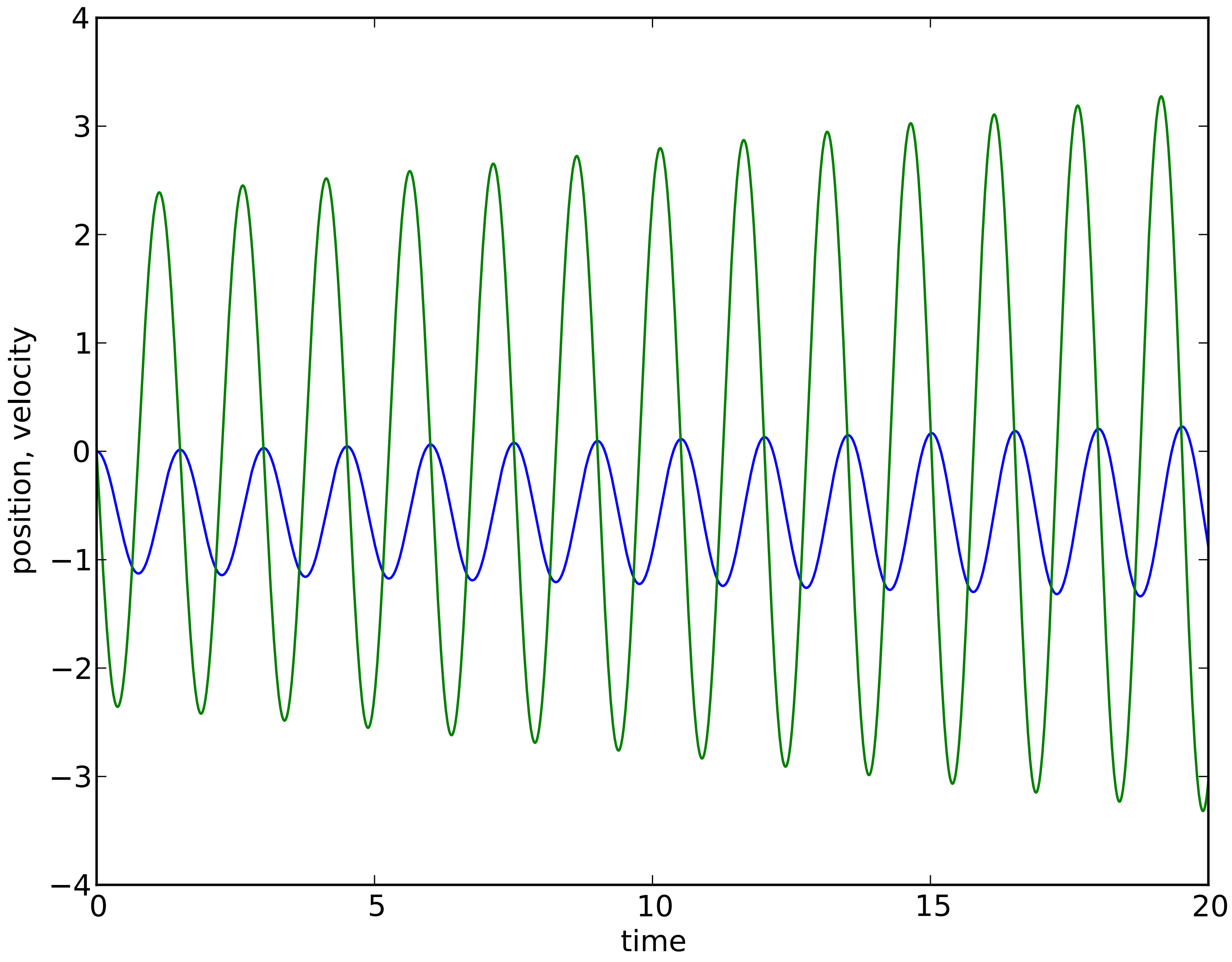
Problems

- Euler method
  - Energy is



Example with a total time of 20 seconds and N =1000

Problems



Example with a total time of 20 seconds and N = 10000

Better, but the is still

Euler-Cromer Method

- Trick that works for (SHO)
  - Replace derivative with derivative
    - $y_{i+1} = y_i + \dot{y}_{i+1} \Delta t$
- Not a , so we need to do better

Runge-Kutta Methods

- General case:
  - Find a function $y(t)$ with its time derivative
  - Apply the chain rule

$$\ddot{y} =$$

$$=$$

$$g_a \equiv$$

$$g_{ab} \equiv$$

$$\bullet \quad =$$

$$=$$

Runge-Kutta Methods

- Similarly,
  - $\dddot{y} =$
  - Reminder

    $$g_a \equiv \frac{\partial y}{\partial a}$$
  - $$g_{ab} \equiv \frac{\partial^2 y}{\partial a \partial b}$$

Runge-Kutta Methods

- Taylor expansion:

$$y(t + \Delta t) = y(t)$$

$$+$$

$$+$$

- $$+$$

$$+$$

- Compare to an alternative polynomial expansion:
  - $y(t + \Delta t) =$

Runge-Kutta Methods

- Polynomials:
  - $k_1 =$
  - $k_2 =$
  - $k_3 =$
  - $\vdots$
  - $k_n =$
- Coefficients and are determined by against Taylor expansion
  - Determined by
  - i.e. order RK, order RK, etc

2nd order RK

- $y(t + \Delta t) =$
- $k_2 =$

  $=$
- Follows:
  - $y(t + \Delta t) =$
    - $\alpha_1 + \alpha_2 =$
    - $\alpha_1 v_{21} =$

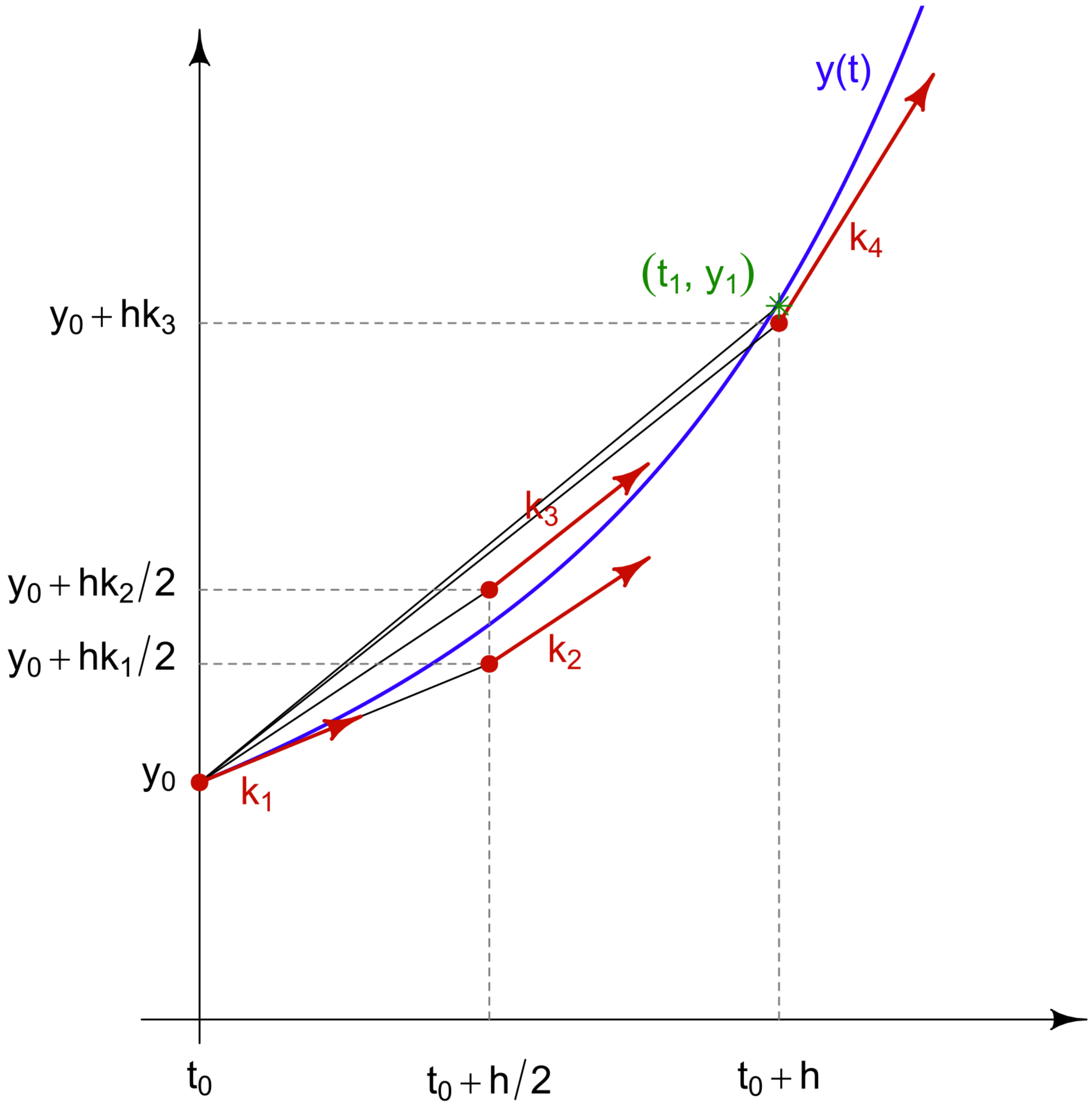
2nd order RK

- Standard solution:
  - $v_{21} =$
  - $\alpha_1 = \alpha_2 =$
    - $y(t + \Delta t) =$
    - $k_1 =$
    - $k_2 =$

4th Order RK

- $y(t + \Delta t) =$ $+ O(\Delta t^4)$
    - $k_1 =$
    - $k_2 =$
    - $k_3 =$
    - $k_4 =$
- 4th order RK is a
    - good trade off between and

Runge-Kutta Illustration



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