# Leinster - Basic Category Theory - Selected problem solutions for Chapter 3

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### 3.1.1

There are bijections

$$\begin{gathered}(A+B, C) \leftrightarrow ((A,B), \Delta C) \\ f \leftrightarrow \overline{f}\end{gathered}$$

where $\overline{f} = (f, f)$

$$\begin{gathered}(\Delta A, (B,C)) \leftrightarrow (A, B \times C) \\ g = (p,q) \leftrightarrow \overline{g}\end{gathered}$$

where $\overline{g}(x) = (p(x), q(x))$

So the sum is left adjoint to $\Delta$, and the product is its right adjoint.

### 3.1.2

We are given the definition of a sequence, where there is a unique function $x$ such that the square below commutes.

We have $x_0 = a$, and $x_{n+1} = r(x_n)$.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "s"] \arrow[d, "x"] & \mathbb{N} \arrow[d, "x"] \\ X \arrow[r, "r"] & X \end{tikzcd}$$

This is precisely the definition of the comma category $(\mathbb{N} \Rightarrow X)$, where objects are $(n \in \mathbb{N}, x, t \in X)$.

**3.2.12**

**(a)**

$$\theta(S) = \bigcup \theta(R) \supseteq \bigcup R = S$$

But $\theta^2(S) = \theta(S)$, so $\theta(S) \subseteq S$.

Taken together, the above implies $\theta(S) = S$.

**(b)**

$$\begin{aligned} & A \subseteq B \\ \implies & fA \subseteq fB \\ \implies & gfA \subseteq gfB \end{aligned}$$

$g$ and $f$ are taken to be injections here. We need to prove there is a bijection between $A$ and $B$. **Note:** this does not follow immediately from $g$ and $f$ being injections.

Take $\theta(S) = A - g(B \setminus fS)$. Then $S_1 \subseteq S_2 \implies \theta(S_1) \subseteq \theta(S_2)$. Since $f, g$ and hence $\theta$ is order preserving, we may apply the result in (a). Specifically, there exists $S$ such that $S = A - g(B \setminus fS) \implies g(B \setminus fS) = A \setminus S$.

**(c)** We need to prove a bijection between $A$ and $B$ to deduce the theorem. Consider $h \colon A \to B$

$$h(x) = \begin{cases} f(x), & x \in S, \\ g^{-1}(x), & x \in A \setminus S \end{cases}$$

$f$ has a codomain of $fS$, so every element of the codomain has a preimage in $S$. We are given that $f$ is injective.

$g$ is injective and hence invertible. Using the result in (b) we have a direct expression for $g^{-1}$. Hence we have $gh = 1_A$, and $hg = 1_B$, for $x$ in $A \setminus S$.

An alternative proof, has a similar basic idea, of partitioning the domain of the bijection around the fixed point. **Sketch proof** Set $A_0 = A$. $A_{i+1} = gfA_i$. Define $k(x) = gf(x)$ if $x \in A_i$ for some $i$, otherwise $k(x) = x$. To prove $k$ is surjective comes down to two cases. Suppose $y \in A_n$, for some $n$ then $A_{n-1}$ is the $x$-value such that $k(x) = y$. If $y$ is not in $A_n$ for any arbitrarily large $n$, then we must have $k(x) = x$.

**3.2.14**

Need to prove that for any family $(A_i)_{i`\in I}$ of objects of $\mathcal{A}$, there is some object of $\mathcal{A}$ not isomorphic to $A_i$ for $i \in I$. It suffices to prove for $A$ in $F(S)$, $F : \mathbf{Set} \to \mathcal{A}$, then we know the condition holds for $\mathcal{A}$. Now $UF$ is injective by Exercise 2.3.11, so $U$ is injective on objects $A$ of $F(S)$. So if $UA_i$ is not isomorphic to $UA_j$, this would imply $A_i$ is not isomorphic to $A_j$. So we need to prove for a given $i$, $|UA_i| < |\mathcal{P}(UA)|$:

$$|UA_i| \leq |\Sigma UA_i| < |\mathcal{P}(UA)|$$

The strict equality due to Theorem 3.2.2.

**3.2.15**

The key point here is that *Set* is not small. I think of *Set* as a power set of an arbitrary family of sets, as in the proof for Proposition 3.2.4. Set is locally small however, as for any two objects $A$ and $B$, the functions between $A$ and $B$ form a set. This question is a little too wooly for me, I struggled, without the necessary background, to reason my way though so many ambiguities that presented themselves. Here is a shot.

**(a)** **Mon** is equivalent to a single object category, which is small. So **Mon** is essentially small.

**(b)** $\mathbb{Z}$, the group of integers viewed as a one object category, is locally small. Groups are just an 'enriched' set.

**(c)** The ordered set of integers still has a large class of isomorphism classes (?) My guess here is it locally small, as there is one map between each two objects.

**(d)** Using the existence of a left adjoint proved in 3.2.16, and the result of 3.2.14, tells us the class of isomorphism classes of **Cat** is large. So **Cat** is not essentially small. For locally small we would require the set of natural transformations between **Cat** and **Set** be a set. There is one component for each object in **Cat** which is small, hence the morphisms form a single element set. *Cat* is locally small. (?)

**(e)** **Guess**. Same reasoning as (a), locally small.