Leinster - Basic Category Theory - Selected problem solutions for Chapter 6

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6.2.20

(a)

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{\alpha} & Y \end{array} \tag{1}$$

where X, Y are functors in the category $[A, \mathcal{S}]$

By Lemma 5.1.32 α is monic in $[A, \mathcal{S}]$ if and only if the above square is a pullback. Using Theorem 6.2.5 there is a pullback whose image under the evaluation functor ev_A is a pullback for each $A \in \mathbf{A}$ in $\mathcal{S}$. So by the lemma α_A is monic for all $A \in \mathcal{A}$. The other direction holds by virtue of the same theorem, that there is only one way to extend the pullback on $\mathcal{S}$ for each A, to a pullback on α_A to a pullback on $[A, \mathcal{S}]$

(b) Monics in $[\mathbf{A}^{\mathbf{op}}, \mathbf{Set}]$ are epics in $[\mathbf{A}, \mathbf{Set}]$ and vice versa.

6.2.21

(a) Use a cardinality argument as follows. There is only one identity map represented by the left hand side of the following expresssion. $H_A(A) \cong X(A) + Y(A)$, for all A in $\mathcal{A}$. Which means that either $X(A)$ or $Y(A)$ must be the empty set for all A in $\mathcal{A}$.

6.2.22

The category of elements can be represented by $(1 \to X)$, where 1 is a single element set. The comma category commuting diagram becomes

$$
\begin{tikzcd}
1 \arrow[r] \arrow[dr] & x \arrow[d, "Xf"] \\
& x'
\end{tikzcd}
\tag{2}
$$

where $x \in X(A)$, and $x' \in X(A')$, and $f : A' \to A$. The above diagram shows under our choice of comma category that $Xf(x) = x'$ as required.

6.2.23

A category of elememts with a terminal object by definition is equivalent to the definition of a representation as a universal element in (4.6).

6.2.24

Let E be a functor in the functor category $[\mathbf{A}^{op}, \mathbf{Set}]$ and $E \to X$ be an object of the slice category, where X is a presheaf on $\mathbf{A}$. We need an equivalance functor to map $E \to X$ to some $[\mathbf{B}^{op}, \mathbf{Set}]$. For a given A, and consider $\alpha_A : E(A) \to X(A)$. For a $x \in X(A)$ back out the definition of E with

$$\beta_A(x) = \{e : \alpha_A(e) = x\} \tag{3}$$

where $e \in E(A), x \in X(A)$.

So now we have (A, x) pairs as in the definition of the category of elements in Definition 6.2.16, and can construct a functor using them informally as $(A, x) \to \beta_A(x)$. However we do need to show that $\beta_A(x)$ and $\beta_{A'}(x)$ induce an f such that $(Xf)(x') = x$. Because the morphism of E to X is a natural transformation we know that with $f \colon A \to A'$, that $\alpha_A(e) = (Xf)(\alpha_{A'}e')$ taken with (3) means $(Xf)(x') = x$ as required.

In the other direction, if E is a presheaf on the category of elements,

$$E(a) = \coprod_{x \in X(a)} E(a, x) \tag{4}$$

and for $e \in E(a), x \in X(a)$ define $f(e) = x$.

Source of ideas for this proof.[1]

6.2.25

(a) i. Functoriality of $\mathrm{Lan}_F X$

[1]https://math.stackexchange.com/questions/3633646/every-slice-of-a-presheaf-category-is-again-a-presheaf-category

Let the diagram given for our colimit be $D_B := X$, with $(A, FA \to B)$ in $(F \Rightarrow B)$. To prove $\mathrm{Lan}_F X$ is a functor we need to consider $\mathrm{Lan}_F X$ on morphisms $f \colon B \to B'$, and $f' \colon B' \to B''$. f and f' induce the maps presented below:

$$\begin{tikzcd} D_B \arrow[r, "p_I"] \arrow[d, "D(f)"] & Lan_F(B) \arrow[d, "L(f)"] \\ D_{B'} \arrow[r, "p_{I'}"] \arrow[d, "D(f')"] & Lan_F(B') \arrow[d, "L(f')"] \\ D_{B''} \arrow[r, "p_{I''}"] & Lan_F(B'') \end{tikzcd} \tag{5}$$

The map from $\mathrm{Lan}_F B \to \mathrm{Lan}_F B''$ is a unique map by the colimit property of $\mathrm{Lan}_F B$ and hence $L(f')L(f) = L(f'f)$ as required.

(a) ii. Bijection between $\mathrm{Lan}_F X \to Y$ and $X \to YF$.

Consider the cocone $\left(X(A) \xrightarrow{p_I} \mathrm{Lan}_F X(B)\right)_{\{FA \to B\}}$, for all $(A, FA \to B)$ in $(F \Rightarrow B)$. To form the bijection required, make the canonical choice of $1_F(A)$ in $(F \Rightarrow F(A))$ and reevaluate p_I above which now becomes $X(A) \to \mathrm{Lan}_F XF(A)$. With this choice there is a single base of the cocone $X(A)$ for every $\mathrm{Lan}_F XF(A)$ so we can form the required bijection between $\mathrm{Lan}_F X \to Y$ and $X \to YF$. The task remaining is to prove naturality between $\mathrm{Lan}_F X$ and X.