

# Leinster - Basic Category Theory - Selected problem solutions for Chapter 4

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## 4.1.27

$H_A$  is naturally isomorphic if and only if  $\alpha_A : H_A(B) \rightarrow H_{A'}(B)$  is isomorphic for all  $B$  in  $\mathcal{A}$ .

The naturality square of  $H_A$  is constructed below. For every map  $g : B' \rightarrow B$ ,  $B, B'$  in  $\mathcal{A}$ , the following square commutes

$$\begin{array}{ccc} H_A(B) & \xrightarrow{H_A(g) = - \circ g} & H_A(B') \\ \downarrow \alpha_A & & \downarrow \alpha_{A'} \\ H_{A'}(B) & \xrightarrow{H_{A'}(g) = - \circ g} & H_{A'}(B') \end{array}$$

Moreover, since  $H_A \cong H_{A'}$ ,  $\alpha_A$  is an isomorphism for every  $B$  in  $\mathcal{A}$ .

Now consider the square below. For an arbitrary  $B$  in  $\mathcal{A}$  we need to show there is a bijection between the component  $\alpha_A$  and a morphism  $\bar{f}$  in  $\mathcal{A}$ .

$$\begin{array}{ccc} \mathcal{A}(B, A) & \xrightarrow{\alpha_A = f \circ -} & \mathcal{A}(B, A') \\ (-)^B \uparrow & & \downarrow (-)(B) \\ A & \xrightarrow{\bar{f}} & A' \end{array}$$

But  $(-)^B$  and  $(-)(B)$  are unique and inverses so take  $f, \bar{f} : A \rightarrow A'$  as  $\bar{f}(A) = f(A^B)(B) = A'$  and we have our required bijection, hence  $A$  and  $A'$  are isomorphic. Note that the isomorphism of  $\alpha_A$  must hold for all  $B$  in  $\mathcal{A}$ . To see why, suppose there exists a  $B$ , such that  $\alpha_A$  is a bijective morphism but not an isomorphism, then  $\bar{f}$  is not an isomorphism, and we have a contradiction. Suppose, alternatively, there exists a  $B$ , such that  $\alpha_A$  is not bijective, then our expression for  $\bar{f}$  implies  $\bar{f}$  is not bijective, and again we have a contradiction.

#### 4.1.27 - another less convoluted attempt

With  $f: A \rightarrow B$ ,  $A, B \in \mathcal{A}$  consider the following diagram

$$\begin{array}{ccc} H_A(B) & \xrightarrow{-\circ f} & H_A(A) \\ \downarrow \alpha_B & & \downarrow \alpha_A \\ H_B(B) & \xrightarrow{-\circ f} & H_B(A) \end{array}$$

We require  $g \circ f = 1_A$  and  $f \circ g = 1_B$ .

Set  $g: B \rightarrow A$ . By naturality above square commutes, so

$$\begin{aligned} \alpha_A(g \circ f) &= (\alpha_B \circ g)(f), \\ \alpha_A(g \circ f) &= 1_B(f), \\ g \circ f &= \alpha_A^{-1} 1_B f = 1_{H_A(A)} = 1_A, \end{aligned}$$

the last equality just being a matter of notation. The other direction proceeds analogously.

#### 4.1.27 - even shorter version

$H_A \cong H_{A'}$ . Both sides are functors from  $\mathcal{A}^{op}$  to **Set**. Functors preserve identity, so  $H_A(A) = 1_{H_A(A)} = 1_A$ , and similarly  $H_{A'}(A') = 1_{H_{A'}(A')} = 1_{A'}$ . So  $1_A \cong 1_{A'}$ .

#### 4.1.28

Here we construct a bijection between the set  $U_p(G)$  and a group homomorphism  $\phi$ .

$$\begin{array}{ccc} U_p(G) & \xrightarrow{h} & U_p(H) \\ \downarrow \alpha_G & & \downarrow \alpha_H \\ \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G) & \xrightarrow{h} & \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H) \end{array}$$

$U_p(G)$  is the set of  $\{g \in G: g^p = 1\}$ .

For the present question, take an arbitrary  $g$  in  $G$ . Set  $\phi(1) = g$ . By the properties of a homomorphism we shall see this maps the additive group  $\mathbb{Z}/p\mathbb{Z}$  to  $U_p(G)$ .  $\phi$  preserves the identity so  $\phi(0) = 1$ . Since  $\phi(1+1) = g^2$ , generally  $\phi(n) = g^n$ . So  $\phi(p) = g^p = \phi(0) = 1$ . So  $\phi$  maps to a group with order  $p$ , or simply order 1 if  $g$  is the element of the trivial group. So  $\mathbb{Z}/p\mathbb{Z}$  sees groups of order  $p$  or 1. This result means we have the required bijection,  $\alpha$  and  $\alpha^{-1}$  in the diagram above. Observing the diagram we just need to specify how morphisms

in  $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$  work. They are simply group homomorphisms  $h$ , that take  $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G)$  to  $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H)$ . So referring to the diagram, naturality holds, and we can conclude  $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$  and  $U_p$  are naturally isomorphic.<sup>1</sup>

#### 4.1.29

Here we show a natural isomorphism between  $\mathbf{CRing} \rightarrow \mathbf{Set}$  and  $\mathbf{CRing}(\mathbb{Z}[x], -)$ . Let  $R$  be a ring, and  $h: R \rightarrow S$  be a ring homomorphism.

$$\begin{array}{ccc} U(R) & \xrightarrow{h} & U(S) \\ \downarrow \alpha_R & & \downarrow \alpha_S \\ \mathbf{CRing}(\mathbb{Z}[x], R) & \xrightarrow{h} & \mathbf{CRing}(\mathbb{Z}[x], S) \end{array}$$

For  $\alpha_R$ , we only require the **elements** of  $r \in U(R)$  to construct the ring homomorphism from  $\mathbb{Z}[x]$  to  $R$ .

For  $\alpha_R^{-1}$ , we simply forget the ring structure by applying  $U$ .

From (0.13) we have there exists a unique ring homomorphism  $\phi: \mathbb{Z}[x] \rightarrow R$  such that  $\phi(x) = r$ . One can observe this ring homomorphism is analogous to the group homomorphism in (4.1.28), it is completely determined by the choice of  $\phi(x)$ . So the maps  $\mathbf{CRing}(\mathbb{Z}[x], R)$  are essentially the same as elements of  $R$ , and the description of the bijection is complete. The only thing remaining is to verify naturality, which here boils down to the morphisms  $h$  being the same, whether acting on sets or rings.

#### 4.1.30

Let  $X$  be a topological space, and  $S$  be the Sierpinski space.

$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{\mathcal{O}(g)} & \mathcal{O}(Y) \\ \downarrow \alpha_X & & \downarrow \alpha_Y \\ (f: X \rightarrow S) & \xrightarrow{- \circ g} & (fg: Y \rightarrow S) \end{array}$$

**Note:** A function  $f$  from one topological space  $X$  into another topological space  $S$  is continuous if and only if for every open set  $V$  in  $S$ ,  $f^{-1}(V)$  is open in  $X$ .

Let  $f$  be the continuous map  $f: X \rightarrow S$ , let the singleton open set in the Sierpinski space  $S$  be  $V$ . Since  $f$  is continuous, we retrieve the open sets of  $X$  by setting  $\alpha_X^{-1} = f^{-1}(V)$ .

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<sup>1</sup>A well known result is as follows. For a group homomorphism  $\psi: G_1 \rightarrow G_2$ , let  $g \in G_1$  be of finite order  $n$ . Then  $\psi(g)$  divides the order of  $g$ . Because  $g^n = e_1$  implies  $\psi(g)^n = \psi(g^n) = \psi(e_1) = e_2$ . So if  $p$  is prime then the resulting homomorphism maps to a group of order  $p$  or 1.

In the  $\alpha_X$  direction take the open sets of  $X$ ,  $\overline{X} = \mathcal{O}(X)$  and map them to  $V$ . So  $\alpha_X = \overline{X} \mapsto (\overline{X} \rightarrow V)$ .

To prove  $\mathcal{O} \cong H_S$ , we need to show the isomorphism is natural, specifically the square above commutes. But in the category *Top* the morphisms are defined as the continuous functions, so there is a one to one correspondence between  $\mathcal{O}(g)$  and  $- \circ g$ , and the square indeed commutes.