Leinster - Basic Category Theory - Selected problem solutions for Chapter 3

Adam Barber

December 22, 2021

3.1.1

There are bijections

$$
\begin{aligned}
(A+B, C) &\leftrightarrow ((A,B), \Delta C) \\
f &\leftrightarrow \overline{f}
\end{aligned}
$$

where $\overline{f} = (f, f)$

$$
\begin{aligned}
(\Delta A, (B, C)) &\leftrightarrow (A, B \times C) \\
g = (p, q) &\leftrightarrow \overline{g}
\end{aligned}
$$

where $\overline{g}(x) = (p(x), q(x))$

So the sum is left adjoint to Δ, and the product is its right adjoint.

3.1.2

We are given the definition of a sequence, where there is a unique function x such that the square below commutes.

We have $x_0 = a$, and $x_{n+1} = r(x_n)$.

$$
\begin{tikzcd}
\mathbb{N} \arrow[r, "s"] \arrow[d, "x"] & \mathbb{N} \arrow[d, "x"] \\
X \arrow[r, "r"] & X
\end{tikzcd}
$$

This is precisely the definition of the comma category $(\mathbb{N} \Rightarrow X)$, where objects are $(n \in \mathbb{N}, x, t \in X)$.

3.2.12

(a)

$$\theta(S) = \bigcup \theta(R) \supseteq \bigcup R = S$$

But $\theta^2(S) = \theta(S)$, so $\theta(S) \subseteq S$.

Taken together, the above implies $\theta(S) = S$.

(b)

$$\begin{aligned} & A \subseteq B \\ \implies & fA \subseteq fB \\ \implies & gfA \subseteq gfB \end{aligned}$$

g and f are taken to be injections here. We need to prove there is a bijection between A and B. **Note:** this does not follow immediately from g and f being injections.

So gf is order preserving. Which means by (a) there is some $gf(S) = S$, $S \in A$. So we know how the bijection behaves on S, but we need to specify its behaviour for $A \setminus S$. Set $g(B) = A$. Then $g(B) - S = g(B) - gf(S) = g(B \setminus fS) = A \setminus S$. (*).

(c) We need to prove surjectivity of g to deduce the theorem. So for all $a \in A$ we need some b such that $g(b) = a$. For $a \in S$ this holds immediately since $gf(S) = S$. For $a \in A \setminus S$, we have the equation for g derived in (b).

3.2.14

Need to prove that for any family $(A_i)_{i`\in I}$ of objects of $\mathcal{A}$, there is some object of $\mathcal{A}$ not isomorphic to A_i for $i \in I$. It suffices to prove for A in $F(S)$, $F : \mathbf{Set} \to \mathcal{A}$, then we know the condition holds for $\mathcal{A}$. Now UF is injective by Exercise 2.3.11, so U is injective on objects A of $F(S)$. So if UA_i is not isomorphic to UA_j, this would imply A_i is not isomorphic to A_j. So we need to prove for a given i, $|UA_i| < |\mathcal{P}(UA)|$:

$$|UA_i| \leq |\Sigma UA_i| < |\mathcal{P}(UA)|$$

The strict equality due to Theorem 3.2.2.