Leinster - Basic Category Theory - Selected problem solutions for Chapter 5

Adam Barber

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5.1.34

The equaliser square is not necessarily a pullback. There is no reason why any function into the X would commute with a unique function into E, composed with i.

The converse is true though, a pullback implies an equaliser, when the square is set up as in the question.

5.1.35

Suppose the right hand square is a pullback. We need to prove the left hand square is a pullback if and only if the full rectangle, which composes both squares, is a pullback.

Only if Assume the left hand square and right hand squares are pullbacks. Show full rectangle is a pullback, that is show $g_1 = p_{2\cdot}.p_{12}\overline{f}$, and $f_2 = p_{1\cdot}.\overline{f}$

$$
\begin{tikzcd}
A \arrow[dr, dashed, "\overline{f}"] \arrow[drr, bend left, "f_1"] \arrow[drr, dashed, "\overline{g}"] \arrow[ddr, bend right, "f_2"'] \arrow[drrr, dotted, bend left=50, "g_1"] \arrow[ddrr, dotted, bend right=60, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p_{1\cdot}"] & P_2 \arrow[r, "p_{2\cdot}"] \arrow[d] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

Left square pullback (black): For any f_1 and f_2, there is a unique map $\overline{f}$ such that the left square above commutes.

Right square pullback (red): For any g_1 and g_2, there is a unique $\overline{g}$ such that the red diagram commutes.

Due to the left hand square being a pullback, for each f_1, and f_2, there is a unique map $\overline{f}$ such that $f_2 = p_1.\overline{f}$ and $f_1 = p_{12}\overline{f}$. Set $f_1 = \overline{g}$. From the right hand side being a pullback, $g_1 = p_2.\overline{g} = p_2.p_{12}\overline{f}$ as required.

If Assume the outer rectangle and right hand square are both pullbacks. Show the left hand side square is a pullback, that is $f_1 = p_{12}\overline{h}$, and $h_2 = p_1.\overline{h}$, for any f_1, h_2.

$$
\begin{tikzcd}
A \arrow[dr, dotted, "\overline{h}"'] \arrow[drr, dotted, red, "\overline{g}"] \arrow[drrr, bend left, "h_1"] \arrow[drrr, bend left=50, dotted, red, "g_1"] \arrow[ddr, bend right, "h_2"'] \arrow[ddrr, bend right=50, dotted, red, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p1\cdot"] & P_2 \arrow[r, "p_2\cdot"] \arrow[d] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

Full rectangle pullback (black): For any h_1 and h_2, there is a unique $\overline{h}$ such that the black diagram commutes.

Since the right hand square is a pullback, for any g_1, there is a unique $\overline{g}$ such that $g_1 = p_2.\overline{g}$. Since the rectangle is a pullback, for any h_1, there exists a unique $\overline{h}$ such that $p_2.p_{12}\overline{h} = h_1$, and $p_1.\overline{h} = h_2$. Set $g_1 = h_1$, then $p_2.p_{12}\overline{h} = p_2.\overline{g}$, so $p_{12}\overline{h} = \overline{g}$. $\overline{g}$ can be regarded as an arbitrary f_1, as there is a one to one correspondence with $\overline{g}$ and the arbitrary choice of g_1, or equivalently, h_1.

5.1.36

(a) If $(L \xrightarrow{p_I} D(I))_{i\in I}$ is a limit cone, there exists a unique h such that $p_I \circ h = f_I$. However we are given that $p_I \circ h = p_I \circ h' = f_I$, so h must equal h'.

(b) When I is the two object discrete category, say $X \times Y$, $\mathcal{A} = \mathbf{Set}$, and $A = 1$, the statement in (a) says if $x = x', y = y'$, then $(x, y) = (x', y')$.

5.1.37

For any $A \in \mathcal{A}$, and all maps $I \xrightarrow{u} J$, a cone on D is

$$\begin{tikzcd} A \arrow[r, "f_I"] \arrow[dr, "f_J"'] & D(I) \arrow[d, "Du"] \\ & D(J) \end{tikzcd} \tag{1}$$

A limit of D is a cone $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$, such that for any cone on D with vertex A (1), there exists a unique map $\overline{f} \colon A \to L$ such that $p_J \circ \overline{f} = f_J$, for all $J \in \mathbf{I}$.

We have the set $\{(x_I)_{I \in \mathbf{I}} | x_I \in D(I)$ for all $I \in \mathbf{I}$ and $(Du)(x_I) = x_J,$ for all $I \xrightarrow{u} J$ in $\mathbf{I}\}$. The product limit formed is easier seen graphically. There is a family of maps for each $I \in \mathbf{I}$, each with

$$\begin{tikzcd} 1 \arrow[r, "f_I"] \arrow[dr, "f_J"'] & x_I \in D(I) \arrow[d, "Du"] \\ & x_J \in D(J) \end{tikzcd}$$

Then fix $p_J = Du$, $\overline{f} = f_I$, and we have from the definition of a cone and (1) above $p_J \circ \overline{f} = f_J$, for all $J \in \mathbf{I}$. $\overline{f}$ is also unique. To see this assume there are two maps $\overline{f}$ and $\overline{f}'$, that make the above triangle commute. Then $Du \circ \overline{f} = Du \circ \overline{f}'$, for all maps $I \to J$. Set $I = J$ to retrieve $\overline{f} = \overline{f}'$. This family of maps we have described is precisely the definition of a product given in 5.1.7. So the set of x_I can be written $\prod_{I \in \mathbf{I}} D(I)$.

So if any cone exists in **Set**, then a limit exists. Does a cone always exist in **Set**?

5.1.38

(a) We are given maps s and t,

$$\prod_{I \in \mathbf{I}} D(I) \overset{s}{\underset{t}{\rightrightarrows}} \prod_{J \xrightarrow{u} K \text{ in } \mathbf{I}} D(K)$$

The u-component of s is the composite

$$\prod_{I \in \mathbf{I}} D(I) \xrightarrow{pr_J} D(J) \xrightarrow{Du} D(K)$$

The u-component of t is pr_K.

The fork property of the equalizer, says that the below diagram commutes for all maps u, $J \xrightarrow{u} K$ in $\mathbf{I}$, essentially that maps $(A \to D(J))_{J \in \mathbf{I}}$ are a cone on D.

$$
\begin{tikzcd}
A \arrow[d] & \\
\prod_{I\in\mathbf{I}} D(I) \arrow[r, "pr_J"] \arrow[dr, "pr_K"'] & D(J) \arrow[d, "Du"] \\
& D(K)
\end{tikzcd}
\tag{2}
$$

The other important property of the equalizer is that for any fork, or as above, cone, there exists a unique map $\overline{f}\colon A \to L$ such that

$$
\begin{tikzcd}
A \arrow[d, "\overline{f}"] \arrow[dr, "f"] & \\
L \arrow[r, "i"] & \prod_{I\in\mathbf{I}} D(I)
\end{tikzcd}
\tag{3}
$$

commutes.

Now $(L \xrightarrow{pr_J \circ i} D(J))_{J\in\mathbf{J}}$ is a cone, as it factors through $\prod_{I\in\mathbf{I}} D(I)$, as A does in (2). (3) also implies $pr_J \circ i \circ \overline{f} = f_J$ for all J, where $f_J\colon A \to D(J) = pr_J \circ f$.

(b) The definition of a finite limit is a limit of shape $\mathbf{I}$ for some finite category I. So to show a limit is finite, we must show the diagram the limit maps into is indexed by a finite category. Finite categories have only finitely many maps. So binary products, terminal objects, equalizers and pullbacks are all finite limits. From part (a) we know if $\mathcal{A}$ has all products and equalizers then $\mathcal{A}$ has all limits. If we however restrict the products to binary products, then by definition limits of $\mathcal{A}$ will be finite.

5.1.39

A pullback (5.7) from page 114, with Z as the terminal object, collapses to a binary product. The key point here is that the limit is unique up to isomorphism, so limits in a category with pullbacks and a terminal object are binary products, and hence finite.

5.1.40

We are given $X \xrightarrow{m} A$, and $X' \xrightarrow{m'} A$ are monics in **Set**. **Monic**(A) is the full subcategory of the slice category $\mathcal{A}/A$, whose objects have as their maps the monics. Recall in $\mathcal{A}/A$, objects are tuples (X, m) such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "m"'] & & X' \arrow[dl, "m'"] \\ & A & \end{tikzcd}$$

Isomorphic implies equal images: Note that if m and m' are isomorphic, then f must be a bijection. The bijection can then be written $m = m' \circ f$, and $m' = m \circ f^{-1}$. Note also the mm and $m\prime$, by virtue of them as monics, are injective. **Intuition:** We can essentially roundtrip on the triangle above, starting from an element in the image of m (or conversely m'), and map it to an element in the image of m' (respectively m). Explicitly we can write $\{m(x), x \in X\} = \{m' \circ f(x), x \in X\} = \{m'(x), x \in X'\}$.

Equal images implies isomorphic: If images of m and m' are equal,

$|{m'}^{-1}(A)| = |m^{-1}(A)|$, which implies a bijection between X and X', and hence a bijection between maps m and m' as in the previous paragraph.