# Leitner - Basic Category Theory - Problem solutions

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### 0.10

Let $S$ be a set. The indiscrete topological space $I(S)$ is the space whose set of points is S and whose only open subsets are $\emptyset$ and $S$. To find a universal property satisfied by the space $I(S)$ proceed as follows. With this topology any map from a topological space to S is continuous.

Parroting the wording of the question, let us rephrase this in universal parlance. Define a function $i : S \to I(S)$, by $i(s) = s, s \in S$. Then $I(S)$ has the following property.

$$\begin{tikzcd}
S \arrow[r, "i"] & I(S) \\
& X \arrow[u, "\overline{f}"] \arrow[ul, dashed, "\forall f"]
\end{tikzcd}$$

For all topological spaces $X$ and all functions $f : X \to S$ there exists a unique continuous map $\overline{f} : X \to I(S)$. What it says is all maps into an indiscrete space are continuous. It also says that given $S$, the universal property determines $I(S)$ and $i$, up to isomorphism.

### 0.11

The universal property that is satisfied by the pair $(ker(\theta), \iota)$ is depicted in the diagram below.

$$\begin{tikzcd}
ker(\theta) \arrow[r, "\iota"] & G \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\epsilon"'] & H \\
F \arrow[u, dashed, "\exists! \overline{f}"] \arrow[ur, "\forall f"'] & &
\end{tikzcd}$$

The statement of the universal property is as follows. For any $f : F \to G$ such that $\theta \circ f = \epsilon \circ f$, there is a unique $\overline{f} : F \to ker(\theta)$ such that the diagram above commutes. That is $f = \iota \circ \overline{f}$.

**0.13**

**(a)**

Choose $\phi(\sum_{i=1}^{n} a_i x^i) = \sum_{i=1}^{n} a_i r^i$. Then $\phi$ with $\phi(x) = r$ is a homomorphism that satisfies additive and multiplicative properties. To prove uniqueness assume there is another homomorphism $\psi$, with $\psi(x) = r$. Then $\psi(\sum_{i=1}^{n} a_i x^i) = \sum_{i=1} a_i \psi(x) = \sum_{i=1} a_i r^i$ by properties of a homomorphism. So $\psi = \phi$.

**(b)**

$\iota \colon \mathbb{Z}[x] \to A$ maps $\sum_{i=1}^{n} p_i x^i$ to $\sum_{i=1}^{n} p_i a^i$, using $\iota(x) = a$, the multiplicative property of a homomorphism to get $\iota(x^i) = \iota(x)^i$, and the additive property to get $\iota(p_i)\iota(x)^i = p_i \iota(x)^i$ remembering $p_i$ is in $\mathbb{Z}$.

Going in the direction $A \to \mathbb{Z}[x]$ we know as provided in (b) that, taking $R = \mathbb{Z}[x]$, and $\phi = \iota'$, there exists a unique ring homomorphism such that $\iota'(a) = x$. So $\iota'$ maps $\sum_{i=1}^{n} p_i a^i$ to $\sum_{i=1}^{n} p_i x^i$ and $\iota' \circ \iota = 1_{\mathbb{Z}[x]}$. Also using definitions of $\iota$ and $\iota'$ easily yields $\iota \circ \iota' = 1_A$.

## (0.14)

**(a)**

For the triangles below to commute, we need, as stated in the question $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, dashed, "\forall f_1"'] & P \arrow[d, "\exists p_1"] \\ & X \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, dashed, "\forall f_2"'] & P \arrow[d, "\exists p_2"] \\ & Y \end{tikzcd}$$

Choosing $P = X \times Y$, $p_1$ and $p_2$ as below makes the triangles commute.

$$p_1 : P \to X$$
$$p_2 : P \to Y$$

## (0.14)

**(b)**

Proving uniqueness involves taking two arbitrary cones with the property stated in (a). Taking $(P, p_1, p_2)$ and $(P', p_1', p_2')$ we know from (a) that for all cones $(V, f_1, f_2)$ there exists a unique linear map $f : V \to P'$ such that $p_1' \circ f = f_1$, $p_2' \circ f = f_2$. In this statement choose $V = P'$, then referring to the triangles in (a), observe there exists a $f : P \to P'$ such that $p_1' \circ f = p_1$, $p_2' \circ f = p_2$.