Leinster - Basic Category Theory - Selected problem solutions for Chapter 2

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2.1.16

(a) Interesting adjoint functors to G-sets.

The trivial group functor I sends a set to a **G**-set with the trivial action $gx = x$. Interesting functors

Orbit functor sends a G-set with underlying set elements a of A to:

$A_G = \{g \cdot a, g \in G\}$

Fixed point functor sends a G-set with underlying set elements a of A to:

$A^G = \{a \text{ such that } g \cdot a = a \text{ for all } g \in G, a \in A\}$

Fixed point functor - right adjoint Morphisms in a G-set are functions on the underlying set, where f commutes with g for every $g \in G$.

There is a bijection for each $A \in \mathbf{Set}$ and $B \in [G, \mathbf{Set}]$ as follows

$$[G, \mathbf{Set}](I(A), B) \to \mathbf{Set}(A, B^G)$$
$$\psi \mapsto \overline{\psi}$$

$\overline{\psi}$ sends each element a of A to $\psi(a)$ if $g \cdot a = a$, otherwise it sends a to $\psi(\emptyset)$.

$$\mathbf{Set}(A, B^G) \to [G, \mathbf{Set}](I(A), B)$$
$$\phi \mapsto \overline{\phi}$$

ϕ sends each $a \in A$ in the underlying set of the G-set to the G-set $(g, \overline{\phi}(a)), g \in G$.

Orbit functor - left adjoint There is a bijection for each $A \in [G, \mathbf{Set}]$ and $B \in \mathbf{Set}$ as follows

$$\mathbf{Set}(A_G, B) \to [G, \mathbf{Set}](A, I(B))$$
$$\psi \mapsto \overline{\psi}$$

So each morphism in **Set** sends the set formed by the orbits of an element a of A, call this a_G, to $\psi(a_G)$, where ψ is a function of sets. Choose a G-set morphism $\overline{\psi} = \psi$, where $\overline{\psi}$ commutes with g for every g in G.

$$[G, \mathbf{Set}](A, I(B)) \to \mathbf{Set}(A_G, B)$$
$$\phi \mapsto \overline{\phi}$$

Choose $\overline{\phi}$ to be a disjoint union of each orbit of a in A, $\overline{\phi}(a) = \amalg\{\phi(g \cdot a), g \in G\}$

2.1.17

Write $\mathcal{O}(X)$ for the poset of open subsets of a topological space X ordered by inclusion.

$\Delta : \mathbf{Set} \to [\mathcal{O}(X)^{op}, \mathbf{Set}]$

Write $\mathcal{P}$ for the presheaf functor category, and $P \in \mathcal{P}$ for the functor which maps $\mathcal{O}(X)^{op}$ to **Set**. Take open sets U, V, such that $U \subseteq V$ in X. A presheaf consists of

- restriction maps, $P(V) \to P(U)$, these are morphisms which enforce some sort of ordering of the mapped sets,
- and the actual mapped sets $P(U), P(V)$ which are called sections.

Since the question specifies a constant presheaf, by definition, the restriction maps of ΔA are identity maps. And the sections are just the A. Specifically $\Delta A(U) = A$ for subsets U of X, and $\Delta A(\to) = 1_A$ for morphisms.

Write $\Gamma P = P(X)$ for the **global** sections functor which takes an element of $\mathcal{P}$ to a **Set**.

We are required to show a bijection:

For A in **Set** and B in $\mathcal{P}$

$$\mathbf{Set}(A, \Gamma B) \to \mathcal{P}(\Delta A, B)$$

and

$$\mathcal{P}(\Delta A, B) \to \mathbf{Set}(A, \Gamma B)$$

The maps between the presheaf functors in $\mathcal{P}$ are natural transformations. Natural transformations are a collection of maps α_A: $\{\Delta A(A) \to B(A)\}_{A \in \mathcal{A}}$. For $U \subseteq V \subseteq X$ we have the commuting square:

$$\begin{tikzcd} \Delta A(X) \arrow[r, "1_A"] \arrow[d, "\alpha_X"] & \Delta A(V) \arrow[r, "1_A"] \arrow[d, "\alpha_V"] & \Delta A(U) \arrow[d, "\alpha_U"] \\ B(X) \arrow[r, "B(f)"] & B(V) \arrow[r, "B(f)"] & B(U) \end{tikzcd} \tag{1}$$

Recall $\Delta A(\cdot) = A$. Then the morphism in **Set** is represented by α_X above. As visible from the figure above this corresponds one to one with each α_A in $\mathcal{A}$, so the bijection holds. Dually using the exact same reasoning Π, the left adjoint of Δ is the presheaf evaluation at the empty set, $\Pi(P) = P(\emptyset)$.

For the left adjoint to Π, Λ, and for A in **Set** and B in $\mathcal{P}$, we need to show a bijection between:

$$\mathcal{P}(\Lambda A, B) \leftrightarrow \mathbf{Set}(A, \Pi(B))$$

To try and cobble together a definition of the presheaf functor Λ, start with the naturality diagram representing morphisms in $\mathcal{P}$:

$$\begin{tikzcd} \Lambda(U) \arrow[r, "A(f)"] \arrow[d, "\alpha_U"] & \Lambda(\emptyset) \arrow[d, "\alpha_\emptyset"] \\ B(U) \arrow[r, "B(f)"] & B(\emptyset) \end{tikzcd}$$

Note that $\Pi(B) = B(\emptyset)$. Start by choosing $\Lambda(\emptyset) = A$, so the morphism in **Set** is $\alpha_\emptyset$. Our choice of Λ needs to make this diagram commute for all U in $\mathcal{O}(X)^{op}$. For $U \neq \emptyset$ we could try $\Lambda(U) = A$, however to force the square above to commute with this choice, will impose some structure on the presheaf B. Rather, try setting $\Lambda(U) = \emptyset$ for $U \neq \emptyset$. Choosing the initial object $\emptyset$ of $\mathcal{O}(X)^{op}$, means there is one map out of the top LHS of the square in the above diagram, and the square commutes as required.

We also have

$$\mathcal{P}(A, \nabla B) \leftrightarrow \mathbf{Set}(\Gamma A, B)$$

∇, the right adjoint to Γ can be obtained dually, by swapping **Set** with $\mathbf{Set}^{op}$ and $\mathcal{O}(X)^{op}$ with $\mathcal{O}(X)$. This is simply a relabelling which has the effect of reversing the chain of adjoint functors stated in the question. We then apply analogous reasoning, take $\nabla(U) = \{*\}$, for $U \neq X$, and $\nabla(X) = B$.

2.2.11

The full subcategory where η_a is an isomorphism

2.2.13

(a) We have sets S, T, a function $f\colon S \to T$. $P(S)$ denotes the set of all subsets of S. The functor f^* takes elements of T to their inverse under f. Looking for left and right adjoints of f^*. We can immediately see the left adjoint of f^* is f from below.

$$P(S)(A, f^{-1}(B)) \cong P(T)(f(A), B) \tag{2}$$

Now to find the right adjoint of f^*, G below:

$$P(T)(A, G(B)) \cong P(S)(f^{-1}(A), B)$$

Dualising

$$P(T)^{op}(G(B), A) \cong P(S)^{op}(B, f^{-1}(A)) \tag{3}$$

Equation (3) is (2) up to an isomorphism. So we choose $G = f$ here. In fact the power set P is self adjoint. Loosely, this means we have an isomorphism between the opposite category and the original category. We use this isomorphism to get a representation of G in $P(T)$, with the right adjoint sending B to $\overline{f(\overline{B})}$.

Because f is a bijection, elements in T that are not in $f(\overline{B})$ must have elements in B as their preimage. So $\overline{f(\overline{B})}$ consists of all sets of T where $f^{-1}(T) \subseteq B$. In summary the left adjoint of f^* is

$$F(S) = \{t \in T,\ \exists s \in S:\ s \in f^{-1}(t)\}$$

F represents choosing elements of T such that some element of S is in the inverse image of f.

and the right adjoint

$$G(S) = \{t \in T,\ \forall s \in S:\ s \in f^{-1}(t)\}$$

G represents choosing elements of T such that every element of S is in the inverse image of f.

(b) We are asked to interpret, in light of the results in (a.), the unit $\eta\colon 1_T \to G \circ F$, and counit $\epsilon\colon F \circ G \to 1_S$, for all adjunctions.

Consider the $R(x, y)$ as a set in $X \times Y$ and S as as a set in X. In all of the below I interpret **set inclusion as logical implication**.

Description of the functors used follows.

- $\forall_y$ takes a set $R(x,y)$ and returns $S(x)$ with preimage in $R(x,y)$ for all y. So each element in $R(x,y)$ inducing $S(x)$ is fully contained in $R(x,y)$.
- p^* the inverse image functor takes a set $S(x)$ and returns its preimage $R(x,y)$. This inclusion of X into $X \times Y$ adds a variable in Y. To use parlance of first order logic the statement is free in y.
- $\exists_Y$ takes a set $R(x,y)$ and returns $S(x)$ with preimage in $R(x,y)$ for at least one y.

We know from (a) that $\exists_Y \dashv p^* \dashv \forall_Y$

$p^* \dashv \forall_Y$

- $\eta : 1_X \to \forall_Y \circ p^*$ Plug in as argument to both sides of the implication the set $S(x)$. Evaluating the RHS, applying p^* results in the product of $S(x)$ with Y, the set $R(x,y)$. So η can be interpreted as $S(x) \implies R(x,y)\,\forall y$.
- $\epsilon : p^* \circ \forall_Y \to 1_{X\times Y}$. (??) The universal functor is projection of a subset of x-values for the set $R(x,y)$ passed to the functor. Applying p^* to yield say $R^Y(x,y)$ makes the statement on the LHS free in y, so it requires assignment for the statement to be meaningful. If we were to assign y on the LHS for corresponding to the y value for each RHS predicate, elementwise, then we essentially just have the statement that $R^Y(x,y) \implies R(x,y)$.

$\exists_Y \dashv p^*$

- $\eta : 1_{X\times Y} \to p^* \circ \exists_Y$. (??) The existence functor is just projection of all the x values for a given set of (x,y). The resulting statement on the right hand side is free in Y after applying p^* so requires assignment to be meaningful. If we were to assign y on the RHS for corresponding to the y value for each LHS predicate, elementwise, then we essentially just have an identity.
- $\epsilon : \exists_Y \circ p^* \to 1_X$ Plug in as argument to both sides of the implication the set $S(x)$. Applying p^* returns the set $R(x,y)$. So ϵ can be interpreted as $\exists y : R(x,y) \implies S(x)$.

2.2.14 Natural transformations for $[\mathcal{A},\mathcal{B}]$.

$$\begin{tikzcd} FA \arrow[r, "F(f)"] \arrow[d, "\alpha_A"] & FA' \arrow[d, "\alpha_{A'}"] \\ GA \arrow[r, "G(f)"] & GA' \end{tikzcd}$$

$Y \in [\mathcal{B},\mathcal{I}], F^*(Y) = Y \circ F$. Natural transformations for $[[\mathcal{A},\mathcal{I}],[\mathcal{B},\mathcal{I}]]$.

$$\begin{array}{ccc} F^*Y = Y \circ F & \xrightarrow{F^*(f)} & F^*Y' = Y' \circ F \\ \downarrow{\scriptstyle \alpha^*_Y} & & \downarrow{\scriptstyle \alpha^*_{Y'}} \\ G^*Y = Y \circ G & \xrightarrow{G^*(f)} & G^*Y' = Y' \circ G \end{array}$$

So it is evident from comparing the above natural transformation diagrams that we have the relationship $\alpha^*_Y = Y \circ \alpha$

Using first triangle inequality starting at point $F^*(Y)$ we have

$$\begin{aligned} \epsilon^*_{F^*(Y)} F^*(\eta^*_Y) F^*(Y) =& F^*(Y)(\epsilon F) F^*(\eta^*_Y) F^*(Y) \\ =& YF(\epsilon F)(\eta F) \\ =& YF1_F \text{ since } \epsilon F \circ F\eta = 1_F \\ =& F^*(Y) \end{aligned}$$

The other triangle follows similarly. So by Theorem 2.2.5 we have an adjunction between F^* and G^*.