Leitner - Basic Category Theory - Problem solutions

Adam Barber

August 17, 2021

0.10

Let S be a set. The indiscrete topological space I(S) is the space whose set of points is S and whose only open subsets are \emptyset and S. To find a universal property satisfied by the space I(S) proceed as follows. With this topology any map from a topological space to S is continuous.

Parroting the wording of the question, let us rephrase this in universal parlance. Define a function $i: S \to I(S)$, by $i(s) = s, s \in S$. Then I(S) has the following property.



For all topological spaces X and all functions $f\colon X\to S$ there exists a unique continuous map $\overline{f}\colon X\to I(S)$. What it says is all maps into an indiscrete space are continuous. It also says that given S, the universal property determines I(S) and i, up to isomorphism.

0.11

The universal property that is satisfied by the pair $(ker(\theta), \iota)$ is depicted in the diagram below.

$$ker(\theta) \xrightarrow{\iota} G \xrightarrow{\theta} H$$

$$\exists ! \overline{f} \mid \\ F$$

The statement of the universal property is as follows. For any $f \colon F \to G$ such that $\theta \circ f = \epsilon \circ f$, there is a unique $\overline{f} \colon F \to ker(\theta)$ such that the diagram above commutes. That is $f = \iota \circ \overline{f}$.

0.13

(a)

Choose $\phi(\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n a_i r^i$. Then ϕ with $\phi(x) = r$ is a homomorphism that satisfies additive and multiplicative properties. To prove uniqueness assume there is another homomorphism ψ , with $\psi(x) = r$. Then $\psi(\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n a_i \psi(x) = \sum_{i=1}^n a_i r^i$ by properties of a homomorphism. So $\psi = \phi$.

(b)

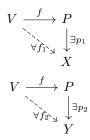
 $\iota \colon \mathbb{Z}[x] \to A$ maps $\sum_{i=1}^n p_i x^i$ to $\sum_{i=1}^n p_i a^i$, using $\iota(x) = a$, the multiplicative property of a homomorphism to get $\iota(x^i) = \iota(x)^i$, and the additive property to get $\iota(p_i)\iota(x)^i = p_i\iota(x)^i$ remembering p_i is in \mathbb{Z} .

Going in the direction $A \to \mathbb{Z}[x]$ we know as provided in (b) that, taking $R = \mathbb{Z}[x]$, and $\phi = \iota'$, there exists a unique ring homomorphism such that $\iota'(a) = x$. So ι' maps $\sum_{i=1}^n p_i a^i$ to $\sum_{i=1}^n p_i x^i$ and $\iota' \circ \iota = 1_{\mathbb{Z}[x]}$. Also using definitions of ι and ι' easily yields $\iota \circ \iota' = 1_A$.

0.14

(a)

For the triangles below to commute, we need, as stated in the question $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$.



Choosing $P = X \times Y$, p_1 and p_2 as below makes the triangles commute.

$$p_1: X \times Y \to X$$

 $p_2: X \times Y \to Y$

(b)

Proving uniqueness involves taking two arbitrary cones with the property stated in (a). Taking (P, p_1, p_2) and (P', p'_1, p'_2) we know from (a) that for all cones (V, f_1, f_2) there exists a unique linear map $f \colon V \to P'$ such that $p'_1 \circ f = f_1$, $p'_2 \circ f = f_2$. In this statement choose V = P', then referring to the triangles in (a), observe there exists a $f \colon P \to P'$ such that $p'_1 \circ f = p_1$, $p'_2 \circ f = p_2$.

Comment The choice of P and p notation hinted very heavily that this is a projection of a product.

(c)

We need to define the cocone (Q,q_1,q_2) with the property, for all cocones (V,f_1,f_2) there exists a unique linear map $f\colon Q\to V$ such that $f\circ q_1=f_1$ and $f\circ q_2=f_2$. Choose $Q=X\times Y,\,q_1\colon X\to X\oplus Y,\,q_2\colon Y\to X\oplus Y$

Comment This is the dual of the product in (b), the coproduct. Set equivalent is the disjoint union.