# Leinster - Basic Category Theory - Selected problem solutions for Chapter 4

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**4.1.27**

$H_A$ is naturally isomorphic if and only if $\alpha_A : H_A(B) \to H_{A'}(B)$ is isomorphic for all $B$ in $\mathcal{A}$.

The naturality square of $H_A$ is constructed below. For every map $g : B' \to B$, $B, B'$ in $\mathcal{A}$, the following square commutes

$$\begin{tikzcd} H_A(B) \arrow[r, "H_A(g)=-\circ g"] \arrow[d, "\alpha_A"] & H_A(B') \arrow[d, "\alpha_{A'}"] \\ H_{A'}(B) \arrow[r, "H_{A'}(g)=-\circ g"] & H_{A'}(B') \end{tikzcd}$$

Moreover, since $H_A \cong H_{A'}$, $\alpha_A$ is an isomorphism for every $B$ in $\mathcal{A}$.

Now consider the square below. For an arbitrary $B$ in $\mathcal{A}$ we need to show there is a bijection between the component $\alpha_A$ and a morphism $\overline{f}$ in $\mathcal{A}$.

$$\begin{tikzcd} \mathcal{A}(B, A) \arrow[r, "\alpha_A=f\circ -"] & \mathcal{A}(B, A') \arrow[d, "(-)(B)"] \\ A \arrow[u, "(-)^B"] \arrow[r, "\overline{f}"] & A' \end{tikzcd}$$

But $(-)^B$ and $(-)(B)$ are unique and inverses so take $f, \overline{f} \colon A \to A'$ as $\overline{f}(A) = f(A^B)(B) = A'$ and we have our required bijection, hence $A$ and $A'$ are isomorphic. Note that the isomorphism of $\alpha_A$ must hold for all $B$ in $\mathcal{A}$. To see why, suppose there exists a $B$, such that $\alpha_A$ is a bijective morphism but not an isomorphism, then $\overline{f}$ is not an isomorphism, and we have a contradiction. Suppose, alternatively, there exists a $B$, such that $\alpha_A$ is not bijective, then our expression for $\overline{f}$ implies $\overline{f}$ is not bijective, and again we have a contradiction.

**4.1.27 - another less convoluted attempt**

With $f\colon A \to B$, $A, B \in \mathcal{A}$ consider the following diagram

$$\begin{tikzcd} H_A(B) \arrow[r, "-\circ f"] \arrow[d, "\alpha_B"] & H_A(A) \arrow[d, "\alpha_A"] \\ H_B(B) \arrow[r, "-\circ f"] & H_B(A) \end{tikzcd}$$

We require $g \circ f = 1_A$ and $f \circ g = 1_B$.

Set $g\colon B \to A$. By naturality above square commutes, so

$$\begin{aligned} \alpha_A(g \circ f) &= (\alpha_B \circ g)(f), \\ \alpha_A(g \circ f) &= 1_B(f), \\ g \circ f &= \alpha_A^{-1} 1_B f = 1_{H_A(A)} = 1_A, \end{aligned}$$

the last equality just being a matter of notation. The other direction proceeds analogously.

**4.1.27 - even shorter version**

$H_A \cong H_{A'}$. Both sides are functors from $\mathcal{A}^{op}$ to **Set**. Functors preserve identity, so $H_A(A) = 1_{H_A(A)} = 1_A$, and similarly $H_{A'}(A') = 1_{H_{A'}(A')} = 1_{A'}$. So $1_A \cong 1_{A'}$.

**4.1.28**

Here we construct a bijection between the set $U_p(G)$ and a group homomorphism $\phi$.

$$\begin{tikzcd} U_p(G) \arrow[r, "h"] \arrow[d, "\alpha_G"] & U_p(H) \arrow[d, "\alpha_H"] \\ \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G) \arrow[r, "h"] & \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H) \end{tikzcd}$$

$U_p(G)$ is the set of $\{g \in G\colon g^p = 1\}$.

For the present question, take an arbitrary $g$ in $G$. Set $\phi(1) = g$. By the properties of a homormorphism we shall see this maps the additive group $\mathbb{Z}/p\mathbb{Z}$ to $U_p(G)$. $\phi$ preserves the identity so $\phi(0) = 1$. Since $\phi(1+1) = g^2$, generally $\phi(n) = g^n$. So $\phi(p) = g^p = \phi(0) = 1$. So $\phi$ maps to a group with order $p$, or simply order 1 if $g$ is the element of the trivial group. So $\mathbb{Z}/p\mathbb{Z}$ sees groups of order $p$ or 1. This result means we have the required bijection, $\alpha$ and $\alpha^{-1}$ in the diagram above. Observing the diagram we just need to specify how morphisms in $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ work. They are simply group homomorphisms $h$, that take $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G)$ to $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H)$. So referring to the diagram, naturality holds, and we can conclude $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ and $U_p$ are naturally isomorphic. [1]

**4.1.29**

Here we show a natural isomorphism between $\mathbf{CRing} \to \mathbf{Set}$ and $\mathbf{CRing}(\mathbb{Z}[x], -)$. Let $R$ be a ring, and $h\colon R \to S$ be a ring homomorphism.

$$\begin{tikzcd} U(R) \arrow[r, "h"] \arrow[d, "\alpha_R"] & U(S) \arrow[d, "\alpha_S"] \\ \mathbf{CRing}(\mathbb{Z}[x], R) \arrow[r, "h"] & \mathbf{CRing}(\mathbb{Z}[x], S) \end{tikzcd}$$

For $\alpha_R$, we only require the **elements** of $r \in U(R)$ to construct the ring homomorphism from $\mathbb{Z}[x]$ to $R$.

For $\alpha_R^{-1}$, we simply forget the ring structure by applying $U$.

From (0.13) we have there exists a unique ring homomorphism $\phi\colon \mathbb{Z}[x] \to R$ such that $\phi(x) = r$. One can observe this ring homomorphism is analogous to the group homormorphism in (4.1.28), it is completely determined by the choice of $\phi(x)$. So the maps $\mathbf{CRing}(\mathbb{Z}[x], R)$ are essentially the same as elements of $R$, and the description of the bijection is complete. The only thing remaining is to verify naturality, which here boils down to the morphisms $h$ being the same, whether acting on sets or rings.

[1] A well known result is as follows. For a group homomorphism $\psi : G_1 \to G_2$, let $g \in G_1$ be of finite order $n$. Then $\psi(g)$ divides the order of $g$. Because $g^n = e_1$ implies $\psi(g)^n = \psi(g^n) = \psi(e_1) = e_2$. So if $p$ is prime then the resulting homomorphism maps to a group of order $p$ or 1.