Leinster - Basic Category Theory - Selected problem solutions for Chapter 3

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3.1.1

There are bijections

$$\begin{aligned}(A+B,C) &\leftrightarrow ((A,B),\Delta C)\\ f &\leftrightarrow \overline{f}\end{aligned}$$

where $\overline{f} = (f,f)$

$$\begin{aligned}(\Delta A,(B,C)) &\leftrightarrow (A,B\times C)\\ g=(p,q) &\leftrightarrow \overline{g}\end{aligned}$$

where $\overline{g}(x) = (p(x),q(x))$

So the sum is left adjoint to Δ, and the product is its right adjoint.

3.1.2

We are given the definition of a sequence, where there is a unique function x such that the square below commutes.

We have $x_0 = a$, and $x_{n+1} = r(x_n)$.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "s"] \arrow[d, "x"] & \mathbb{N} \arrow[d, "x"] \\ X \arrow[r, "r"] & X \end{tikzcd}$$

This is precisely the definition of the comma category $(\mathbb{N} \Rightarrow X)$, where objects are $(n \in \mathbb{N}, x, t \in X)$.

3.2.12

(a)

$$\theta(S) = \bigcup \theta(R) \supseteq \bigcup R = S$$

But $\theta^2(S) = \theta(S)$, so $\theta(S) \subseteq S$.

Taken together, the above implies $\theta(S) = S$.

(b)

$$\begin{aligned} & A \subseteq B \\ \implies & fA \subseteq fB \\ \implies & gfA \subseteq gfB \end{aligned}$$

g and f are taken to be injections here. We need to prove there is a bijection between A and B. **Note:** this does not follow immediately from g and f being injections.

Take $\theta(S) = A - g(B \setminus fS)$. Then $S_1 \subseteq S_2 \implies \theta(S_1) \subseteq \theta(S_2)$. Since f, g and hence θ is order preserving, we may apply the result in (a). Specifically, there exists S such that $S = A - g(B \setminus fS) \implies g(B \setminus fS) = A \setminus S$.

(c) We need to prove a bijection between A and B to deduce the theorem. Consider $h \colon A \to B$

$$h(x) = \begin{cases} f(x), & x \in S, \\ g^{-1}(x), & x \in A \setminus S \end{cases}$$

f has a codomain of fS, so every element of the codomain has a preimage in S. We are given that f is injective.

g is injective and hence invertible. Using the result in (b) we have a direct expression for g^{-1}. Hence we have $gh = 1_A$, and $hg = 1_B$, for x in $A \setminus S$.

3.2.14

Need to prove that for any family $(A_i)_{i`\in I}$ of objects of $\mathcal{A}$, there is some object of $\mathcal{A}$ not isomorphic to A_i for $i \in I$. It suffices to prove for A in $F(S)$, $F : \mathbf{Set} \to \mathcal{A}$, then we know the condition holds for $\mathcal{A}$. Now UF is injective by Exercise 2.3.11, so U is injective on objects A of $F(S)$. So if UA_i is not isomorphic to UA_j, this would imply A_i is not isomorphic to A_j. So we need to prove for a given i, $|UA_i| < |\mathcal{P}(UA)|$:

$$|UA_i| \leq |\Sigma UA_i| < |\mathcal{P}(UA)|$$

The strict equality due to Theorem 3.2.2.