Leinster - Basic Category Theory - Selected problem solutions for Chapter 5

Adam Barber

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5.1.34

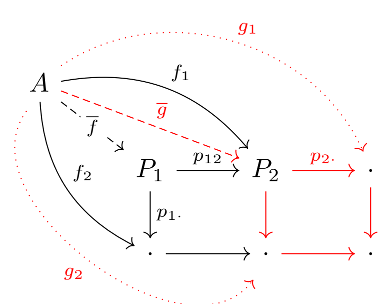
The equaliser square is not necessarily a pullback. There is no reason why any function into the X would commute with a unique function into E, composed with i.

The converse is true though, a pullback implies an equaliser, when the square is set up as in the question.

5.1.35

Suppose the right hand square is a pullback. We need to prove the left hand square is a pullback if and only if the full rectangle, which composes both squares, is a pullback.

Only if Assume the left hand square and right hand squares are pullbacks. Show full rectangle is a pullback, that is show $g_1 = p_2.p_{12}\overline{f}$, and $f_2 = p_1.\overline{f}$



Left square pullback (black): For any f_1 and f_2, there is a unique map $\overline{f}$ such that the left square above commutes.

Right square pullback (red): For any g_1 and g_2, there is a unique $\overline{g}$ such that the red diagram commutes.

Due to the left hand square being a pullback, for each f_1, and f_2, there is a unique map $\overline{f}$ such that $f_2 = p_1.\overline{f}$ and $f_1 = p_{12}\overline{f}$. Set $f_1 = \overline{g}$. From the right hand side being a pullback, $g_1 = p_2.\overline{g} = p_2.p_{12}\overline{f}$ as required.

If Assume the outer rectangle and right hand square are both pullbacks. Show the left hand side square is a pullback, that is $f_1 = p_{12}\overline{h}$, and $h_2 = p_1.\overline{h}$, for any f_1, h_2.

$$\begin{tikzcd}
A \arrow[drrr, bend left, "h_1"] \arrow[drrr, bend left=50, dotted, red, "g_1"] \arrow[dr, dotted, "\overline{h}"'] \arrow[drr, dotted, red, "\overline{g}"] \arrow[ddr, bend right, "h_2"'] \arrow[ddrr, bend right=60, dotted, red, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p1\cdot"] & P_2 \arrow[r, "p_2\cdot"] \arrow[d] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Full rectangle pullback (black): For any h_1 and h_2, there is a unique $\overline{h}$ such that the black diagram commutes.

Since the right hand square is a pullback, for any g_1, there is a unique $\overline{g}$ such that $g_1 = p_2.\overline{g}$. Since the rectangle is a pullback, for any h_1, there exists a unique $\overline{h}$ such that $p_2.p_{12}\overline{h} = h_1$, and $p_1.\overline{h} = h_2$. Set $g_1 = h_1$, then $p_2.p_{12}\overline{h} = p_2.\overline{g}$, so $p_{12}\overline{h} = \overline{g}$. $\overline{g}$ can be regarded as an arbitrary f_1, as there is a one to one correspondence with $\overline{g}$ and the arbitrary choice of g_1, or equivalently, h_1.

5.1.36

(a) If $(L \xrightarrow{p_I} D(I))_{i \in I}$ is a limit cone, there exists a unique h such that $p_I \circ h = f_I$. However we are given that $p_I \circ h = p_I \circ h' = f_I$, so h must equal h'.

(b) When I is the two object discrete category, say $X \times Y$, $\mathcal{A} = \mathbf{Set}$, and $A = 1$, the statement in (a) says if $x = x', y = y'$, then $(x, y) = (x', y')$.

5.1.37

For any $A \in \mathcal{A}$, and all maps $I \xrightarrow{u} J$, a cone on D is

$$\begin{tikzcd} A \arrow[r, "f_I"] \arrow[dr, "f_J"'] & D(I) \arrow[d, "Du"] \\ & D(J) \end{tikzcd} \tag{1}$$

A limit of D is a cone $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$, such that for any cone on D with vertex A (1), there exists a unique map $\overline{f}\colon A \to L$ such that $p_J \circ \overline{f} = f_J$, for all $J \in \mathbf{I}$.

We have the set $\{(x_I)_{I \in \mathbf{I}} | x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and } (Du)(x_I) = x_J, \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\}$.

We are already given that the projections $p_J((x_I)_{I \in \mathbf{I}}) = x_J$. Now I visualise p_J in the product sense, something which just slices up the $(x_I)_{I \in \mathbf{I}}$ into each dimension. There is only one way, up to isomorphism, to do this slicing up, so p_J is unique. So for an $f_J\colon A \to D(J)$ we now have to show there exists a unique map $\overline{f}$ such that $p_J \circ \overline{f} = f_J$. For existence, choose $\overline{f} = f(x_I))_{I \in \mathbf{I}}$. Uniqueness of $\overline{f}$ follows from that it commutes with p_J which is unique. 1

5.1.38

(a) We are given maps s and t,

$$\prod_{I \in \mathbf{I}} D(I) \overset{s}{\underset{t}{\rightrightarrows}} \prod_{J \xrightarrow{u} K \text{ in } \mathbf{I}} D(K)$$

The u-component of s is the composite

$$\prod_{I \in \mathbf{I}} D(I) \xrightarrow{pr_J} D(J) \xrightarrow{Du} D(K)$$

The u-component of t is pr_K.

The fork property of the equalizer, says that the below diagram commutes for all maps u, $J \xrightarrow{u} K$ in $\mathbf{I}$, essentially that maps $(A \to D(J))_{J \in \mathbf{I}}$ are a cone on D.

$$\begin{tikzcd} A \arrow[d] & \\ \prod_{I \in \mathbf{I}} D(I) \arrow[r, "pr_J"] \arrow[dr, "pr_K"'] & D(J) \arrow[d, "Du"] \\ & D(K) \end{tikzcd} \tag{2}$$

The other important property of the equalizer is that for any fork, or as above, cone, there exists a unique map $\overline{f}\colon A \to L$ such that

$$\begin{tikzcd} A \arrow[d, "\overline{f}"] \arrow[dr, "f"] & \\ L \arrow[r, "i"] & \prod_{I \in \mathbf{I}} D(I) \end{tikzcd} \tag{3}$$

commutes.

Now $(L \xrightarrow{pr_J \circ i} D(J))_{J \in \mathbf{J}}$ is a cone, as it factors through $\prod_{I \in \mathbf{I}} D(I)$, as A does in (2). (3) also implies $pr_J \circ i \circ \overline{f} = f_J$ for all J, where $f_J \colon A \to D(J) = pr_J \circ f$.

(b) The definition of a finite limit is a limit of shape $\mathbf{I}$ for some finite category I. So to show a limit is finite, we must show the diagram the limit maps into is indexed by a finite category. Finite categories have only finitely many maps. So binary products, terminal objects, equalizers and pullbacks are all finite limits. From part (a) we know if $\mathcal{A}$ has all products and equalizers then $\mathcal{A}$ has all limits. If we however restrict the products to binary products, then by definition limits of $\mathcal{A}$ will be finite.

5.1.39

A pullback (5.7) from page 114, with Z as the terminal object, collapses to a binary product. The key point here is that the limit is unique up to isomorphism, so limits in a category with pullbacks and a terminal object are binary products, and hence finite.

5.1.40

We are given $X \xrightarrow{m} A$, and $X' \xrightarrow{m'} A$ are monics in **Set**. **Monic**(A) is the full subcategory of the slice category $\mathcal{A}/A$, whose objects have as their maps the monics. Recall in $\mathcal{A}/A$, objects are tuples (X, m) such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "m"'] & & X' \arrow[dl, "m'"] \\ & A & \end{tikzcd}$$

Isomorphic implies equal images: Note that if m and m' are isomorphic, then f must be a bijection. The bijection can then be written $m = m' \circ f$, and $m' = m \circ f^{-1}$. Note also the mm and $m\prime$, by virtue of them as monics, are injective. **Intuition:** We can essentially roundtrip on the triangle above, starting from an element in the image of m (or conversely m'), and map it to an element in the image of m' (respectively m). Explicitly we can write $\{m(x), x \in X\} = \{m' \circ f(x), x \in X\} = \{m'(x), x \in X'\}$.

Equal images implies isomorphic: If images of m and m' are equal, $|{m'}^{-1}(A)| = |m^{-1}(A)|$, which implies a bijection between X and X', and hence a bijection between maps m and m' as in the previous paragraph.

5.1.41

$$
\begin{tikzcd}
Y \arrow[dr, dashed, "\overline{f}"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& X \arrow[r, "1"] \arrow[d, "1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}
$$

From the pullback diagram, for all commuting maps, that is for all $p, q, f \circ p = f \circ q \implies p = q$, if and only if the diagram above is a pullback.

5.1.42

The given square is a pullback, which means for a fixed f, m, f', m', any other commuting square factors through it as follows.

$$
\begin{tikzcd}
Y \arrow[dr, dashed, "\overline{f}"] \arrow[drr, bend left, "f\overline{f}"] \arrow[ddr, bend right, "m'\overline{f}"'] & & \\
& X' \arrow[r, "f"] \arrow[d, "m'"] & X \arrow[d, "m"] \\
& A' \arrow[r, "f'"] & A
\end{tikzcd}
$$

We know from the properties of a pullback that $\overline{f} : Y \to X'$ is unique for each distinct pair of maps, $Y \to X$, and $Y \to A'$, such that the diagram above commutes.

In the following we use the contrapositive form of monic, so for maps x, x', f is monic if $x \neq x' \implies f \circ x \neq f \circ x'$.

Now we know m is monic, so consider two distinct $\overline{f}_1$ and $\overline{f}_2$ in respect of two commuting diagrams as above. There must indeed be two distinct $mf\overline{f}_1 \neq mf\overline{f}_2$, such that each respective diagram commutes. Since the outer arrows commute, $mf\overline{f}_1 = f'm'\overline{f}_1$, and $mf\overline{f}_2 = f'm'\overline{f}_2$. So $f'm'\overline{f}_1 \neq f'm'\overline{f}_2 \implies m'\overline{f}_1 \neq m'\overline{f}_2$, and m' is monic.

5.2.21

The equaliser is a map f below such that $si = ti$, together with a universal property. The coequaliser is a map p satisfying $ps = pt$, and universal with this property.

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \arrow[r, "p"] & C \end{tikzcd}$$

If f is isomorphic then there is a $\overline{f}$ such that $f\overline{f} = 1_E, \overline{f}f = 1_X$, so $sf\overline{f} = s = tf\overline{f} = t$.

In the opposite direction, we need to show if $s = t$, then the equaliser exists and is isomorphic. To do this we will use the universal property of the equaliser. Specifically, any f that is a fork factors through i as below

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \\ A \arrow[u, "\overline{f}"] \arrow[ur, "f"'] & & \end{tikzcd}$$

Since $s = t$ we can choose any function f and it will be a fork, and hence an equaliser exists. Immediately we can see that if we choose $f = 1_X$ then we have $i \circ \overline{i} = 1_X$, where $\overline{i}$ is the unique morphism depicted by $\overline{f}$ in the diagram below. Now we need to show $\overline{i} \circ i = 1_E$. Put $f = i\overline{i}i$ below, then there is a unique h such that

$$i\overline{i}i = ih. \tag{4}$$

This implies $h = \overline{i}i$. Substituting $i\overline{i} = 1_X$ into (4) yields $h = 1_E$. So $\overline{i}i = 1_E$.

Proof for the coequaliser works the same, but dualised.

5.2.22

(a) The coequaliser of

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "1"'] & X \end{tikzcd} \tag{5}$$

in **Set** is described as follows. Let $\sim$ be the equivalence relation between domain and codomain of f, $x \sim fx$, for all $x \in X$. The coequaliser () is then the quotient map $p \colon X \to X/\sim$.

(b) The quotient needs to be something quite degenerate in that no subsets of it can be open, in order to form an indiscrete topology as the question requires. So this rules out any type of g-orbit quotient which would take the points on the circle to g-orbits on the plane, e.g $e^{i2\pi z} \mapsto z \mod 1$. Any such mapping would map an open arc in S^1 to a family of open sets in $\mathbb{R}$, and hence not have the indiscrete topology.

So here is a solution. Let X be the circle $S^1 = \{e^{i2\pi x}, x \in [0,1)\}$. Define an equivalence relation $x \sim y$ if $y - x \in \mathbb{Q}$. A set $V = S^1/\mathbb{Q}$ is open if $\{x \in [0,1)\colon [x] \in V\}$ is open in $[0,1)$. Here $[x]$ is the equivalence class of x, i.e

$$\{x \in [0,1)\colon y - x \in \mathbb{Q}\}$$

If $V \subset S^1/\mathbb{Q}$ is open and nonempty then $\{x \in [0,1) : [x] \in V\}$ should contain an interval $(a,b), a \geq 0, a < 1$. So $(a,b) \subset \{x \in [0,1) : [x] \in V\}$, which means that

$$x \in (a,b) \implies [x] \in V$$

But consider $z \in [0,1)$. Then there exists a $z' \in (a,b)$, such that $z - z' \in \mathbb{Q}$.

Hence $[z] = [z'] \in V$. Hence $V = S^1/\mathbb{Q}$. So our quotient has the indiscrete topology. [1]

Cardinality Informally the elements of the quotient are of the form $\{a+q, q \in \mathbb{Q}\}$, where $a \in [0,1)$, so uncountable as is $[0,1)$.

5.2.23

(a) We have the inclusion $f : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$. If $g \circ f = g' \circ f$, we need to show $g = g'$. $g \circ f$ is essentially a group homomorphism restricted to a domain of $\mathbb{N}$, whereas g is the respective homomorphism on the expanded domain of $\mathbb{Z}$. The idea here is to stitch together a group homomorphism on the expanded domain using the group homomorphism on $\mathbb{N}$.

Define, for $x \in \mathbb{N}$

$$h(x) = \begin{cases} g \circ f(x), & x >= 0, \\ g \circ f(-x), & x < 0 \end{cases}$$

and define h' analogously.

Since $g \circ f(-x) = -g \circ f(x) = -g' \circ f(x) = g' \circ f(-x)$, then $h = h'$ on $\{\forall z \in \mathbb{Z}\}$.

(b) We have $i : \mathbb{Z} \to \mathbb{Q}$, and $hi = h'i$. We need to show $h = h'$, on the full domain of $\mathbb{Q}$. A rational number is defined as $p/q, p, q \in \mathbb{Z}, q \neq 0$. The answer is a similar concept to (a), stitch together a ring homomorphism from the hi on $\mathbb{Z}$, and show equality holds on the full domain of $\mathbb{Q}$.

Define $h(x) = h(p)h(1/q) = \frac{hi(p)}{hi(q)}$, and it follows that $h = h'$ as required.

[1] https://math.stackexchange.com/questions/4452608/quotient-map-on-s1-such-that-that-the-quotient-is-an-uncountable-space-with-the

5.2.24

(a)

$$\begin{tikzcd}
X \arrow[rr, "f"] & & X' \\
& A \arrow[ul, "e"'] \arrow[ur, "e'"] &
\end{tikzcd}$$

e, e' in $\mathbf{Epic}(A)$. The equivalence relation on A is written

$$a \sim a' \implies f(a) = f(a')$$

Same equivalence relation implies isomorphism Consider the following two equivalence relations

$$a \sim a' \implies e(a) = e(a'), \tag{6}$$

$$a \approx a' \implies e'(a) = e'(a') \tag{7}$$

From Chapter 3 p70, we know that a map $p : A \to A/\sim$ sending an element to its equivalence class, has a universal property. Any function $f : A \to B$ such that

$$\forall a, a' \in A, a \sim a' \implies f(a) = f(a')$$

factorises uniquely through p, as in the diagram

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[dr, "f"] & A/\sim \arrow[d, "\bar{f}"] \\
& C
\end{tikzcd}$$

Now if indeed both e and e' induce the **same** equivalance relation, then $a \sim a'$ also implies $e'(a) = e'(a')$. We can then use the universal property of the relation $\sim$ as follows

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "e'"] & X/\sim \arrow[d, "\bar{e}"] \\
& A'
\end{tikzcd}$$

And similarly, with the relation $\approx$ (7) we have

$$\begin{tikzcd}
X \arrow[r, "e'"] \arrow[dr, "e"] & X/\approx \arrow[d, "\overline{e'}"] \\
& A
\end{tikzcd}$$

Equating from the diagrams, $A = X/\sim$, and $A' = X/\approx$. So the quotient sets are isomorphic, and isomorphisms of e and e' follow.

Note The surjective property of e and e' is implicit in them being quotient maps. Speicifcally each equivalence class has at least one member.

Aside More generally isomorphisms are coequalizers of identity morphisms:

$$X \underset{1}{\overset{1}{\rightrightarrows}} X$$

Isomorphism implies same equivalence relation If e and e' are isomorphic then there is an injective f such that

$$\begin{aligned} e' &= fe, \text{ and} \\ e &= f^{-1}e' \end{aligned}$$

So any relation $R = \{(a,a') \colon e(a) = e(a')\} = \{(a,a') \colon e'(a) = e'(a')\}$ by injectivity of f.

5.2.25

(a) An **equaliser** of s and t, is a map i such that s is a fork, that is $si = ti$. Furthermore any other fork f factors through i as below

$$\begin{tikzcd} A \arrow[d, "\bar{e}"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"] & X \end{tikzcd}$$

Split monic $\implies$ regular monic If we have $em = 1_A$, then m is an equaliser of

$$X \underset{1}{\overset{1}{\rightrightarrows}} X$$

and the universal map is

$$\begin{tikzcd} A \arrow[d, "fe"'] \arrow[dr, "f"] & \\ E \arrow[r, "m"] & B \end{tikzcd}$$

Regular monic $\implies$ monic m has to be monic for the universal property of m as an equaliser to hold. The universal property states that for all forks f factor through m. To see this, consider two such forks $f_1 \neq f_2$, with their corresponding unique maps $\overline{f}_1 \neq \overline{f}_2$. To get a contradiction assume m is not monic and $m\overline{f}_1 = m\overline{f}_2$. Then the triangle does not commute and the universal property does not hold.

Alternative more elegant proof[2]

This is a much simpler proof than my attempt above, that an equaliser is monic.

Assume $i\colon E \to A$ equalises $f\colon A \to B, g\colon A \to B$. For maps $j, l : D \to E$ such that $i \circ j = i \circ l$, we know $f(i \circ j) = (f \circ i) \circ j = (g \circ i) \circ j = g(i \circ j)$. Reminder: maps in a category are associative. So $i \circ j$ is a fork. By the universal property there exists a *unique* k such that $i \circ k = i \circ j$. So $j = k = l$, and $i \circ j = i \circ l \implies j = l$ as required.

(b) Need to find two maps that a monic $m\colon A \to B$ in **Ab** equalises. They are $g : B \to B/\mathrm{im}(M)$, and the zero map $0 : B \to 0$. The zero object in **Ab** is by definition the initial and terminal object, which is the trivial group. So m equalises f and g. You can see g composing m means the codomain will be $\mathrm{im}(m)/\mathrm{im}(m)$ which is isomorphic to 0.

For the second part need to find a monic m which we can't invert. $f : \mathbb{Z} \to \mathbb{Q}$ works.

(c)

$$A \xrightarrow{m} B \underset{g}{\overset{f}{\rightrightarrows}} C$$

The question is to find a monic in **Top** which is not regular.

The monics $m : A \to B$ in **Top** are injections. So if we consider the restriction of B to the image of m, and equip it with the subspace topology, then we have a homeomorphism from A onto $m[A]$, or an embedding of A. Now consider two maps $f, g : B \to \{0, 1\}$, where 0,1 has the indiscrete topology, f maps everything to 1 and g maps the elements of $m[A]$ to 1. Then the equalizer $fm = gm$ represents the subspace embeddings. Specifically, the equalizer is a set consisting of all embeddings of X for all $X \subseteq A$. The **regular** monics are those where an equalizer exists, and where an equalizer exists in **Top**, it is the subspace embeddings as described. This is because equalizers are a limit, and limits are unique up to isomorphism.

Now we need a monic that is **not** regular. This is a monic that does not admit an equalizer of some f, g, as above. We know all regular monics in **Top** are the inclusion of a subspace. We also know that all injections are monics. So we simply need to find an injection between spaces which is not the inclusion of a

[2] https://math.stackexchange.com/questions/81296/every-equalizer-is-monic

subspace. So equip A with a finer topology than that induced by the subspace topology of B. Under a finer topology for A there is at least one open set in A mapping to a closed set in $m(A)$. However this means we have a $g^{-1}(b), b \in 0, 1$ that is closed, hence g is not continuous at this point, so no such pair of mappings in **Top** exist.

5.2.26

(a)

Regular epic and monic $\implies$ isomorphic Regular epic means we have maps f, g

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

of which e is a coequalizer. We are given e is also monic, which taken with the coequalizer property means $f = g$. Since $f = g$, 1 is factorised by e, due to the universal property of the coequalizer. This is presented below

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "f"'] & B \arrow[r, "e"] \arrow[dr, "1"'] & Y \arrow[d, dotted, "l"] \\ & & Z \end{tikzcd}$$

So we have $le = 1$, for some $l \colon Y \to Z$. Applying e to both sides gives $ele = e \implies el = 1$, since e is epic.

Isomorphic $\implies$ monic Suppose we have an isomorphic $f : X \to Z$ where $f \circ e_1 = f \circ e_2$

Since f is an isomorphism there is a $\overline{f} : Z \to X$ such that $\overline{f}f = 1_X$. So applying $\overline{f}$ to both sides yields $e_1 = e_2$.

Isomorphic $\implies$ regular epic The coequalizer of

$$\begin{tikzcd} A \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & B \end{tikzcd}$$

is precisely an ismorphic e. The universal property can be checked, loosely it proceeds along the same lines as in **Regular epic and monic $\implies$ isomorphic**.

(b)

Epic $\implies$ regular epic Each epic map in **Set** can be regarded as a quotient map $p\colon A \to A/\sim$ of an equivalence class $\sim$. This quotient map has the property that $p(a) = p(a') \iff a \sim a'$, for $a, a' \in A$. So here we are using the given epic map to define the equivalence relation $\sim$ on A. Along the same lines as Example 5.2.9, p is the coequalizer of maps s, t in A.

Split epic $\implies$ regular epic

$$\begin{tikzcd} Y \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & Z \end{tikzcd}$$

$$g \circ f = g' \circ f \implies g = g'$$

The axiom of choice in **Set** says that, for an epic c, there is a s such that $cs = 1_X$. So

$$\begin{tikzcd} X \arrow[r, shift left, "s \circ c"] \arrow[r, shift right, "1"'] & X \arrow[r, "c"] \arrow[dr, "g"'] & Y \arrow[d, dotted, "\overline{g}"] \\ & & Z \end{tikzcd}$$

So in **Set**, c is a coequalizer of the maps $s \circ c$ and 1. c is also universal as indicated above, this can be confirmed by checking for any g that factors through c that the diagram commutes. The only thing remaning to prove is uniqueness of c, but this follows directly from the epic property, since c is epic.

Regular epic $\implies$ split epic The map 1, is factorised by the coequalizer e, and from this fact we can derive the existence of a right inverse from the universal property. Similar reasoning to 5.2.26 (a).

Regular epic $\implies$ epic e is a regular epic means that for any h such that $hf = hg$, there is a unique $\overline{h}$ such that $h = \overline{h}e$, and $ef = eg$. So suppose we have two maps gg', such that $g \circ e = g' \circ e$. But this factorisation is unique according to the stated universal property of e, so $g = \overline{h} = g'$, and e is epic.

Spit epic $\implies$ epic Split epic means there exists a c such that $e \circ c = 1$. Suppose there is a split epic e such that $g \circ e = g'e$. Apply both sides to c to get $g = g'$.

(c) Take $\phi\colon \mathbb{Z}_4 \to \mathbb{Z}_2$. Rewrite as abstract cyclic groups $\{1, h\}$, and $\{1, g, g^2, g^3\}$. So $\phi(g) = h$ and ϕ is surjective and it follows in *Grp* that it is epic. However let's try and construct a right inverse say $\psi : \mathbb{Z} \to \mathbb{Z}$. We have $\psi(1) = 1$, $\psi(h^2) = \psi(1) = 1$, now $\psi(h) = g$ will mean $\psi(h^2) = g^2$, so the only choice to map h then is $\psi(h) = g^2$. However $\phi \circ \psi(h) = 1$, and so ϕ is not split.

For **Top** we need to come up with a map that has no right inverse. Informally, a surjective map f which takes at least one open set to a closed set works. Then a right inverse of f will not be continuous, and hence not exist in **Top**, as all maps there are continuous by definition.

5.2.27

We already have from monics are closed under pullback, from **5.1.42**. For a split epic $e\colon X \to A$ consider the pullback diagram

$$\begin{tikzcd}
A' \arrow[dr, dashed, "\overline{f}"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "sf"'] & & \\
& X' \arrow[r, "e'"] \arrow[d, "f'"] & A' \arrow[d, "f"] \\
& X \arrow[r, "e"] & A
\end{tikzcd}$$

where s is the right inverse of e. So e split epic implies e' has a right inverse, and is a split epic also as required. Only monics and epics closed under pullback.

Monics, split monics, epics and split epics are closed under composition. This can be checked easily against the definitions provided in the text.

5.3.9

Need to prove directly that

$$\mathcal{A}(A, X \times Y) \cong \mathcal{A}(A, X) \times \mathcal{A}(A, Y)$$

naturally in $A, X, Y \in \mathcal{A}$. Begin with naturality in X. With $f'\colon X \to X_1$

$$\begin{tikzcd}
\begin{matrix}\mathcal{A}(A, X \times Y) \\ = h\end{matrix} \arrow[r, "\phi"] \arrow[d, "\alpha_{A,X,Y}"] & \mathcal{A}(A, X_1 \times Y) \arrow[d, "\alpha_{A,X_1,Y}"] \\
\begin{matrix}\mathcal{A}(A, X) \times \mathcal{A}(A, Y) \\ = p_1 h \times p_2 h\end{matrix} \arrow[r, "\psi"] & \begin{matrix}\mathcal{A}(A, X_1) \times \mathcal{A}(A, Y) \\ = f' p_1 h \times p_2 h\end{matrix}
\end{tikzcd} \tag{8}$$

Now the binary product $P = X \times Y$ is such that for all object and maps f_1, f_2 in $\mathcal{A}$, there exists a unique map $\overline{f}\colon A \to P$ such that the below diagram commutes.

$$
\begin{tikzcd}
 & A \arrow[d, dotted, "\bar{f}"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\
 & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X & & Y
\end{tikzcd}
\tag{9}
$$

So by defintion of the product there is a bijection represented by α in the diagram (8). For α, in the downwards direction we have $h \mapsto (p_1 \circ h) \times (p_2 \circ h)$. In the upwards direction we can establish existence and uniqueness of an inducing map $\overline{h}$ from the commuting diagram (9). Specifically $f \times g \mapsto \overline{h}$. The mapping in this direction presupposes the existence of a product for each X and Y, otherwise there would be no commuting diagram with which to identify our inducing map $\overline{h}$. We were of course given in the question $\mathcal{A}$ is a category with binary products.

So we have the required bijection. To prove naturality in X, we must also show the diagram (8) commutes. The following key question arises in evaluating the maps in each direction presented in the diagram. Suppose we have a product of X and Y, $P = X \times Y$. If we have a map $f' : X \mapsto X_1,\ X, X_1 \in \mathcal{A}$, how do we write down the definition of $X_1 \times Y$, using our knowledge of P? Observe that for a map from A to X_1, $f' \circ f_1$, there is a map from P to X_1, $f' \circ p_1$, such that the product diagram commutes for all maps $A \to X_1$. This is by nature of $\mathcal{A}$ as a category, we can construct all maps A to X_1, since we have maps $f\colon A \to X$, and $f'\colon X \to X_1$ at our disposal. Written in equation form, $f' \circ p_1 \circ \overline{f} = f' \circ f_1$ for all maps f', and f_1. And since P and p_1, p_2 are unique up to isomorphism, we have just characterised the product $X_1 \times Y$. Written as a triplet it is $(P, f'p_1, p_2)$. So ϕ maps p_1 to $f'p_1$, and is the identity on P and p_2. And $\psi\colon \mathcal{A}(A, X) \times \mathcal{A}(A, Y) \to \mathcal{A}(A, X_1) \times \mathcal{A}(A, Y) = f \times g \mapsto f'f \times g$. Elementary checking with these maps shows (8) commutes. The proof for naturality in Y follows analogously.

For naturality in A consider, for $h^* = A \mapsto B$

$$
\begin{tikzcd}
\begin{array}{c}\mathcal{A}(B, X \times Y)\\ h\end{array} \arrow[r, "- \circ h^*"] \arrow[d, "\alpha_{B,X,Y}"] & \begin{array}{c}\mathcal{A}(A, X \times Y)\\ hh^*\end{array} \arrow[d, "\alpha_{A,X,Y}"] \\
\begin{array}{c}\mathcal{A}(B, X) \times \mathcal{A}(B, Y)\\ = p_1 h \times p_2 h\end{array} \arrow[r, "- \circ h^* \times - \circ h^*"] & \begin{array}{c}\mathcal{A}(A, X) \times \mathcal{A}(A, Y)\\ = p_1 hh^* \times p_2 hh^*\end{array}
\end{tikzcd}
\tag{10}
$$

Observe that if we 'reseat' our map into P from B to $A \in \mathcal{A}$, P, p_1, p_2 do not change, it is the unique map into P that changes, by precomposing with h^*.

Elementary checks using our α show the above diagram commutes, and hence we have naturality in A.

5.3.11

From 5.1.22 limits L in **Set** were defined as

$$\lim_{\leftarrow \mathbf{I}} D \cong \{(x_I)_{I\in\mathbf{I}} \mid x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and } (Du)(x_I) = x_J \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\} \tag{11}$$

We also have the projections

$$p_J((x_I)_{I\in\mathbf{I}}) = x_J \tag{12}$$

Now, along the lines of Example 5.34, we wish to prove there is only one group structure on $(x_I)_{I\in\mathbf{I}}$ with the property that the p_J are homomorphisms. To do this we need to show, from the properties of a homomorphism, we can identify a unique group structure on $(x_I)_{I\in\mathbf{I}}$, that is multiplication, inverse, and identity map for the group. Here and in the following I am writing $(x_I)_{I\in\mathbf{I}}$, informally to refer to the subset specified in the equation (11) defining $\lim_{\leftarrow \mathbf{I}} D$. Parroting the logic in 5.3.4, let $(x_I)_{I\in\mathbf{I}} \cdot (y_I)_{I\in\mathbf{I}} = (z_I)_{I\in\mathbf{I}}$. Then $z_J = p_J((z_I)_{I\in\mathbf{I}}) = p_J((x_I)_{I\in\mathbf{I}} \cdot p_J((y_I)_{I\in\mathbf{I}}) = x_J y_J$. So it follows directly from the properties of the homomorphism p_J that multiplication on the limit is defined pointwise as follows

$$(x_I)_{I\in\mathbf{I}} \cdot (y_I)_{I\in\mathbf{I}} = (x_I y_I)_{I\in\mathbf{I}}$$

For the inverse suppose $(x_I)_{I\in\mathbf{I}} \cdot (x'_I)_{I\in\mathbf{I}} = 1$. Then, using pointwise group multiplication as above $(x_I)_{I\in\mathbf{I}} \cdot (x'_I)_{I\in\mathbf{I}} = (x_I x'_I)_{I\in\mathbf{I}} = 1 \implies x'_I = x_I^{-1}$.

5.3.13

(a) With $P = F(A)$, we have the adjunction given

$$\begin{tikzcd} \mathbf{Set} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left] \end{tikzcd}$$

and we need to prove $F(S)$ is projective for all sets S. The adjunction means maps $\mathcal{B}(F(A), B)$ are one to one with maps $\mathbf{Set}(A, G(B))$. We need to prove that f epic $\implies$ $\mathcal{B}(F(A), f)$ is epic.

With $f\colon B \to B'$, $B, B' \in \mathcal{B}$ the $\mathcal{B}(F(S), f)$ is shorthand for a map, say g such that $(F(S) \to B) \to (F(S) \to fB))$. Now we know from the properties of an adjunction that g is one to one with some say $\hat{g}$ in **Set** such that $\hat{g}\colon (A \to G(B)) \to (A \to Gf(B))$. The adjunction given is useful because as we are in **Set**, we know that in **Set** epic maps are precisely the surjections. We are given in **Set** that Gf is epic, hence surjective, which implies the map $\hat{g}$ is epic. Specifically, given a map $A \to Gf(B)$, we can identify its inducing map by virtue of the fact that Gf has a right inverse. Since $\hat{g}$ is one to one with g, conclude that its adjoint g is epic.

(b) The key in this question is realising that **maps** of Abelian groups form Abelian groups themselves. Then we can just use the property given in Example 5.2.19, that surjective maps are epic in $\mathbf{Ab}$. We require an object P such that $\mathbf{Ab}(P, f)$ is not epic for an epic f. Suppose $P = \mathbb{Z}_m$. We know by standard results there is an isomorphism of $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{gcd(m,n)}$. So we just need to craftily pick m, p and q such that $f\colon \mathbb{Z}_p \to \mathbb{Z}_q$ is epic, but $\mathbb{Z}_m \to \mathbb{Z}_p$ is not surjective. $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) \to \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$ is not surjective, so that choice works.

(c) Here we need to show a precomposing epic $f\colon V' \to V$ results in an epic map $\mathbf{Vect}_{\mathbf{k}}(f, I)$. Similar to (b) we utilise that the set of linear maps from V to W forms a vector space itself $\mathbf{Hom}(V, W)$, and epic maps in vector spaces are the surjections. So we need to show a map in $\mathbf{Hom}(V, W) \to \mathbf{Hom}(V', W)$ is a surjection. With linear maps $g\colon V \to W$, we have the precomposition $g \mapsto g \circ f$. Now for each element in $\mathbf{Hom}(V', W)$, we need to find an inducing map in $\mathbf{Hom}(V, W)$. We already know f is a surjection in $\mathbf{Vect}_{\mathbf{k}}$, and hence it has a right inverse such that $fe = 1$. So by precomposing a map in $\mathbf{Hom}(V', W)$ with e we can recover g. This completes the proof. For a non injective object in $\mathbf{Ab}$, similarly to (b) choose $f\colon \mathbb{Z}_3 \to \mathbb{Z}_2$, and $I = \mathbb{Z}_2$.