

Leinster - Basic Category Theory - Selected problem solutions for Chapter 5

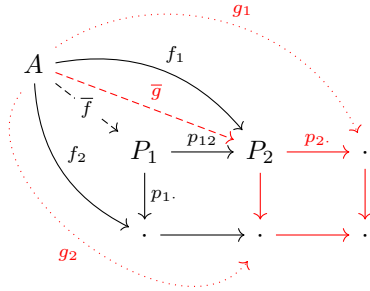
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4.1.27

Suppose the right hand square is a pullback. We need to prove the left hand square is a pullback if and only if the full rectangle, which composes both squares, is a pullback.

Only if Assume the left hand square and right hand squares are pullbacks. Show full rectangle is a pullback, that is show $g_1 = p_2.p_{12}\bar{f}$, and $f_2 = p_1.\bar{f}$

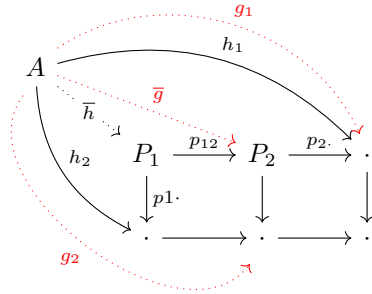


Left square pullback (black): For any f_1 and f_2 , there is a unique map \bar{f} such that the left square above commutes.

Right square pullback (red): For any g_1 and g_2 , there is a unique \bar{g} such that the red diagram commutes.

Due to the left hand square being a pullback, for each f_1 , and f_2 , there is a unique map \bar{f} such that $f_2 = p_1.\bar{f}$ and $f_1 = p_{12}\bar{f}$. Set $f_1 = \bar{g}$. From the right hand side being a pullback, $g_1 = p_2.\bar{g} = p_2.p_{12}\bar{f}$ as required.

If Assume the outer rectangle and right hand square are both pullbacks. Show the left hand side square is a pullback, that is $f_1 = p_{12}\bar{h}$, and $h_2 = p_1.\bar{h}$.



Full rectangle pullback (black): For any h_1 and h_2 , there is a unique \bar{h} such that the black diagram commutes.

Since the right hand square is a pullback, there is a unique \bar{g} such that $g_1 = p_2.\bar{g}$. Since the rectangle is a pullback there exists a unique \bar{h} such that $p_2.p_{12}\bar{h} = h_1$, and $p_1.\bar{h} = h_2$. Set $g_1 = h_1$, then $p_2.p_{12}\bar{h} = p_2.\bar{g}$, so $p_{12}\bar{h} = \bar{g}$. \bar{g} can be regarded as an arbitrary f_1 , as there is a one to one correspondence with \bar{g} and the arbitrary choice of g_1 (or equivalently, h_1).