

Leinster - Basic Category Theory - Selected problem solutions for Chapter 4

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4.1.27

H_A is naturally isomorphic if and only if $\alpha_A : H_A(B) \rightarrow H_{A'}(B)$ is isomorphic for all B in \mathcal{A} .

The naturality square of H_A is constructed below. For every map $g : B' \rightarrow B$, B, B' in \mathcal{A} , the following square commutes

$$\begin{array}{ccc} H_A(B) & \xrightarrow{H_A(g) = - \circ g} & H_A(B') \\ \downarrow \alpha_A & & \downarrow \alpha_{A'} \\ H_{A'}(B) & \xrightarrow{H_{A'}(g) = - \circ g} & H_{A'}(B') \end{array}$$

Moreover, since $H_A \cong H_{A'}$, α_A is an isomorphism for every B in \mathcal{A} .

Now consider the square below. For an arbitrary B in \mathcal{A} we need to show there is a bijection between the component α_A and a morphism \bar{f} in \mathcal{A} .

$$\begin{array}{ccc} \mathcal{A}(B, A) & \xrightarrow{\alpha_A = f \circ -} & \mathcal{A}(B, A') \\ (-)^B \uparrow & & \downarrow (-)(B) \\ A & \xrightarrow{\bar{f}} & A' \end{array}$$

But $(-)^B$ and $(-)(B)$ are unique and inverses so take $f, \bar{f} : A \rightarrow A'$ as $\bar{f}(A) = f(A^B)(B) = A'$ and we have our required bijection, hence A and A' are isomorphic. Note that the isomorphism of α_A must hold for all B in \mathcal{A} . To see why, suppose there exists a B , such that α_A is a bijective morphism but not an isomorphism, then \bar{f} is not an isomorphism, and we have a contradiction. Suppose, alternatively, there exists a B , such that α_A is not bijective, then our expression for \bar{f} implies \bar{f} is not bijective, and again we have a contradiction.

4.1.27 - another less convoluted attempt

With $f: A \rightarrow B$, $A, B \in \mathcal{A}$ consider the following diagram

$$\begin{array}{ccc} H_A(B) & \xrightarrow{-\circ f} & H_A(A) \\ \downarrow \alpha_B & & \downarrow \alpha_A \\ H_B(B) & \xrightarrow{-\circ f} & H_B(A) \end{array}$$

We require $g \circ f = 1_A$ and $f \circ g = 1_B$.

Set $g: B \rightarrow A$. By naturality above square commutes, so

$$\begin{aligned} \alpha_A(g \circ f) &= (\alpha_B \circ g)(f), \\ \alpha_A(g \circ f) &= 1_B(f), \\ g \circ f &= \alpha_A^{-1} 1_B f = 1_{H_A(A)} = 1_A, \end{aligned}$$

the last equality just being a matter of notation. The other direction proceeds analogously.

4.1.27 - even shorter version

$H_A \cong H_{A'}$. Both sides are functors from \mathcal{A}^{op} to **Set**. Functors preserve identity, so $H_A(A) = 1_{H_A(A)} = 1_A$, and similarly $H_{A'}(A') = 1_{H_{A'}(A')} = 1_{A'}$. So $1_A \cong 1_{A'}$.

4.1.28

Here we construct a bijection between the set $U_p(G)$ and a group homomorphism ϕ .

$$\begin{array}{ccc} U_p(G) & \xrightarrow{h} & U_p(H) \\ \downarrow \alpha_G & & \downarrow \alpha_H \\ \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G) & \xrightarrow{h} & \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H) \end{array}$$

$U_p(G)$ is the set of $\{g \in G: g^p = 1\}$.

For the present question, take an arbitrary g in G . Set $\phi(1) = g$. By the properties of a homomorphism we shall see this maps the additive group $\mathbb{Z}/p\mathbb{Z}$ to $U_p(G)$. ϕ preserves the identity so $\phi(0) = 1$. Since $\phi(1+1) = g^2$, generally $\phi(n) = g^n$. So $\phi(p) = g^p = \phi(0) = 1$. So ϕ maps to a group with order p , or simply order 1 if g is the element of the trivial group. So $\mathbb{Z}/p\mathbb{Z}$ sees groups of order p or 1. This result means we have the required bijection, α and α^{-1} in the diagram above. Observing the diagram we just need to specify how morphisms

in $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ work. They are simply group homomorphisms h , that take $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G)$ to $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H)$. So referring to the diagram, naturality holds, and we can conclude $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ and U_p are naturally isomorphic.¹

4.1.29

Here we show a natural isomorphism between $\mathbf{CRing} \rightarrow \mathbf{Set}$ and $\mathbf{CRing}(\mathbb{Z}[x], -)$. Let R be a ring, and $h: R \rightarrow S$ be a ring homomorphism.

$$\begin{array}{ccc} U(R) & \xrightarrow{h} & U(S) \\ \downarrow \alpha_R & & \downarrow \alpha_S \\ \mathbf{CRing}(\mathbb{Z}[x], R) & \xrightarrow{h} & \mathbf{CRing}(\mathbb{Z}[x], S) \end{array}$$

For α_R , we only require the **elements** of $r \in U(R)$ to construct the ring homomorphism from $\mathbb{Z}[x]$ to R .

For α_R^{-1} , we simply forget the ring structure by applying U .

From (0.13) we have there exists a unique ring homomorphism $\phi: \mathbb{Z}[x] \rightarrow R$ such that $\phi(x) = r$. One can observe this ring homomorphism is analogous to the group homomorphism in (4.1.28), it is completely determined by the choice of $\phi(x)$. So the maps $\mathbf{CRing}(\mathbb{Z}[x], R)$ are essentially the same as elements of R , and the description of the bijection is complete. The only thing remaining is to verify naturality, which here boils down to the morphisms h being the same, whether acting on sets or rings.

4.1.30

Let X be a topological space, and S be the Sierpinski space.

$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{\mathcal{O}(g)} & \mathcal{O}(Y) \\ \downarrow \alpha_X & & \downarrow \alpha_Y \\ (f: X \rightarrow S) & \xrightarrow{- \circ g} & (fg: Y \rightarrow S) \end{array}$$

Note: A function f from one topological space X into another topological space S is continuous if and only if for every open set V in S , $f^{-1}(V)$ is open in X .

Let f be the continuous map $f: X \rightarrow S$, let the singleton open set in the Sierpinski space S be V . Since f is continuous, we retrieve the open sets of X by setting $\alpha_X^{-1} = f^{-1}(V)$.

¹A well known result is as follows. For a group homomorphism $\psi: G_1 \rightarrow G_2$, let $g \in G_1$ be of finite order n . Then $\psi(g)$ divides the order of g . Because $g^n = e_1$ implies $\psi(g)^n = \psi(g^n) = \psi(e_1) = e_2$. So if p is prime then the resulting homomorphism maps to a group of order p or 1.

In the α_X direction take the open sets of X , $\overline{X} = \mathcal{O}(X)$ and map them to V . So $\alpha_X = \overline{X} \mapsto (\overline{X} \rightarrow V)$.

To prove $\mathcal{O} \cong H_S$, we need to show the isomorphism is natural, specifically the square above commutes. But in the category *Top* the morphisms are defined as the continuous functions, so there is a one to one correspondence between $\mathcal{O}(g)$ and $- \circ g$, and the square indeed commutes.

4.1.32

The naturality square of the composite functors, with $f : A' \rightarrow A$, and $g : B \rightarrow B'$, $p : A \rightarrow B$

$$\begin{array}{ccc} \mathcal{B}(F(A), B) & \xrightarrow{g \circ - \circ f} & \mathcal{B}(F(A'), B') \\ \downarrow \alpha_{A, B} & & \downarrow \alpha_{A', B'} \\ \mathcal{A}(A, G(B)) & \xrightarrow{g \circ - \circ f} & \mathcal{A}(A', G(B')) \end{array}$$

Maps in $\mathcal{B}(F(-), -)$, represented by the top row, are

$$F(A) \xrightarrow{Ff} F(A') \xrightarrow{q} B \xrightarrow{g} B'. \quad (1)$$

Maps in $\mathcal{A}(-, G(-))$, represented by the bottom row, are

$$A \xrightarrow{f} A' \xrightarrow{p} G(B) \xrightarrow{Gg} G(B'). \quad (2)$$

If the above square is a natural isomorphism, then the α are invertible for all f, g , and p . Equivalently the maps (1) and (2) are one to one.

In the other direction, if F and G are adjoint, then the α are one to one, for all f, g , by definition of adjointness (2.1). Also (1) and (2) are one to one by the result of Exercise 2.1.14. Hence natural isomorphism between $\mathcal{B}(F(-), -)$ and $\mathcal{A}(-, G(-))$ follows.

4.2.3

Definition of functor $M^{op} \rightarrow \mathbf{Set}$. There is a single object in M , the underlying set of the group. So the functor is determined by its effect on morphisms.

Note: Morphisms in the group category are simply elements of the group. So the functor is described as $F(m)(x) = x \cdot m$. The contravariant nature of the functor is convention, because we are right applying the group action, rather than left.

(a) Since M has only one object, then every other functor out of M , is isomorphic to $F(m)$ above. So the $F(m)$ is the unique representable functor, up to isomorphism.

(b) Set $\alpha: \underline{M} \rightarrow X$ to be $\alpha(1) = x \cdot m$ for all $m \in M$. Morphisms of G -sets preserve the group action, but have the equivariant property. Specifically, for α , we require $\alpha(x) \cdot g = \alpha(x \cdot g)$, for all x, g . We have

$$\begin{aligned}\alpha(m) &= \alpha(1) \cdot m \\ \alpha(m^2) &= \alpha(1) \cdot m^2 \\ \alpha(m^3) &= \alpha(1) \cdot m^3 \\ &\vdots\end{aligned}\tag{3}$$

So fixing $\alpha(1) = x$ fully determines the M -set map, which is unique. To see it is unique assume there is another map $\beta(1) = x$, then apply (3) analogously to recover $\beta = \alpha$.