Leinster - Basic Category Theory - Selected problem solutions for Chapter 4

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4.1.27

H_A is naturally isomorphic if and only if $\alpha_A : H_A(B) \to H_{A'}(B)$ is isomorphic for all B in $\mathcal{A}$.

The naturality square of H_A is constructed below. For every map $g : B' \to B$, B, B' in $\mathcal{A}$, the following square commutes

$$
\begin{tikzcd}
H_A(B) \arrow[r, "H_A(g) = - \circ g"] \arrow[d, "\alpha_A"] & H_A(B') \arrow[d, "\alpha_{A'}"] \\
H_{A'}(B) \arrow[r, "H_{A'}(g) = - \circ g"] & H_{A'}(B')
\end{tikzcd}
$$

Moreover, since $H_A \cong H_{A'}$, α_A is an isomorphism for every B in $\mathcal{A}$.

Now consider the square below. For an arbitrary B in $\mathcal{A}$ we need to show there is a bijection between the component α_A and a morphism $\overline{f}$ in $\mathcal{A}$.

$$
\begin{tikzcd}
\mathcal{A}(B, A) \arrow[r, "\alpha_A = f \circ -"] & \mathcal{A}(B, A') \arrow[d, "(-)(B)"] \\
A \arrow[u, "(-)^B"] \arrow[r, "\overline{f}"] & A'
\end{tikzcd}
$$

But $(-)^B$ and $(-)(B)$ are unique and inverses so take $f, \overline{f} \colon A \to A'$ as $\overline{f}(A) = f(A^B)(B) = A'$ and we have our required bijection, hence A and A' are isomorphic. Note that the isomorphism of α_A must hold for all B in $\mathcal{A}$. To see why, suppose there exists a B, such that α_A is a bijective morphism but not an isomorphism, then $\overline{f}$ is not an isomorphism, and we have a contradiction. Suppose, alternatively, there exists a B, such that α_A is not bijective, then our expression for $\overline{f}$ implies $\overline{f}$ is not bijective, and again we have a contradiction.

4.1.27 - another less convoluted attempt

With $f\colon A \to B$, $A, B \in \mathcal{A}$ consider the following diagram

$$\begin{tikzcd}
H_A(B) \arrow[r, "-\circ f"] \arrow[d, "\alpha_B"] & H_A(A) \arrow[d, "\alpha_A"] \\
H_B(B) \arrow[r, "-\circ f"] & H_B(A)
\end{tikzcd}$$

We require $g \circ f = 1_A$ and $f \circ g = 1_B$.

Set $g\colon B \to A$. By naturality above square commutes, so

$$\begin{aligned}
\alpha_A(g \circ f) &= (\alpha_B \circ g)(f), \\
\alpha_A(g \circ f) &= 1_B(f), \\
g \circ f &= \alpha_A^{-1} 1_B f = 1_{H_A(A)} = 1_A,
\end{aligned}$$

the last equality just being a matter of notation.

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Here we construct a bijection between the set $U_p(G)$ and a group homomorphism ϕ.

$$\begin{tikzcd}
U_p(G) \arrow[r, "h"] \arrow[d, "\alpha_G"] & U_p(H) \arrow[d, "\alpha_H"] \\
\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G) \arrow[r, "h"] & \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H)
\end{tikzcd}$$

$U_p(G)$ is the set of $\{g \in G\colon g^p = 1\}$.

For the present question, take an arbitrary g in G. Set $\phi(1) = g$. By the properties of a homormorphism we shall see this maps the additive group $\mathbb{Z}/p\mathbb{Z}$ to $U_p(G)$. ϕ preserves the identity so $\phi(0) = 1$. Since $\phi(1+1) = g^2$, generally $\phi(n) = g^n$. So $\phi(p) = g^p = \phi(0) = 1$. So ϕ maps to a group with order p, or simply order 1 if g is the element of the trivial group. So $\mathbb{Z}/p\mathbb{Z}$ sees groups of order p or 1. This result means we have the required bijection, α and α^{-1} in the diagram above. Observing the diagram we just need to specify how morphisms in $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ work. They are simply group homomorphisms h, that take $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G)$ to $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H)$. So referring to the diagram, naturality holds, and we can conclude $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ and U_p are naturally isomorphic. [1]

[1] A well known result is as follows. For a group homomorphism $\psi : G_1 \to G_2$, let $g \in G_1$ be of finite order n. Then $\psi(g)$ divides the order of g. Because $g^n = e_1$ implies $\psi(g)^n = \psi(g^n) = \psi(e_1) = e_2$. So if p is prime then the resulting homomorphism maps to a group of order p or 1.