Leinster - Basic Category Theory - Selected problem solutions for Chapter 2

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2.1.16

(a) Interesting adjoint functors to G-sets.

The trivial group functor I sends a set to a **G**-set with the trivial action $gx = x$. Interesting functors

Orbit functor sends a G-set with underlying set elements a of A to:

$A_G = \{g \cdot a, g \in G\}$

Fixed point functor sends a G-set with underlying set elements a of A to:

$A^G = \{a \text{ such that } g \cdot a = a \text{ for all } g \in G, a \in A\}$

Fixed point functor - right adjoint Morphisms in a G-set are functions on the underlying set, where f commutes with g for every $g \in G$.

There is a bijection for each $A \in \mathbf{Set}$ and $B \in [G, \mathbf{Set}]$ as follows

$$[G, \mathbf{Set}](I(A), B) \to \mathbf{Set}(A, B^G)$$
$$\psi \mapsto \overline{\psi}$$

$\overline{\psi}$ sends each element a of A to $\psi(a)$ if $g \cdot a = a$, otherwise it sends a to $\psi(\emptyset)$.

$$\mathbf{Set}(A, B^G) \to [G, \mathbf{Set}](I(A), B)$$
$$\phi \mapsto \overline{\phi}$$

ϕ sends each $a \in A$ in the underlying set of the G-set to the G-set $(g, \overline{\phi}(a)), g \in G$.

Orbit functor - left adjoint There is a bijection for each $A \in [G, \mathbf{Set}]$ and $B \in \mathbf{Set}$ as follows

$$\mathbf{Set}(A_G, B) \to [G, \mathbf{Set}](A, I(B))$$
$$\psi \mapsto \overline{\psi}$$

So each morphism in **Set** sends the set formed by the orbits of an element a of A, call this a_G, to $\psi(a_G)$, where ψ is a function of sets. Choose a G-set morphism $\overline{\psi} = \psi$, where $\overline{\psi}$ commutes with g for every g in G.

$$[G, \mathbf{Set}](A, I(B)) \to \mathbf{Set}(A_G, B)$$
$$\phi \mapsto \overline{\phi}$$

Choose $\overline{\phi}$ to be a disjoint union of each orbit of a in A, $\overline{\phi}(a) = \amalg\{\phi(g \cdot a), g \in G\}$

2.1.17

Write $\mathcal{O}(X)$ for the poset of open subsets of a topological space X ordered by inclusion.

$\Delta : \mathbf{Set} \to [\mathcal{O}(X)^{op}, \mathbf{Set}]$

Write $\mathcal{P}$ for the presheaf functor category, and $P \in \mathcal{P}$ for the functor which maps $\mathcal{O}(X)^{op}$ to **Set**. Take open sets U, V, such that $U \subseteq V$ in X. A presheaf consists of

- restriction maps, $P(V) \to P(U)$, these are morphisms which enforce some sort of ordering of the mapped sets,
- and the actual mapped sets $P(U), P(V)$ which are called sections.

Since the question specifies a constant presheaf, by definition, the restriction maps of ΔA are identity maps. And the sections are just the A. Specifically $\Delta A(U) = A$ for subsets U of X, and $\Delta A(\to) = 1_A$ for morphisms.

Write $\Gamma P = P(X)$ for the **global** sections functor which takes an element of $\mathcal{P}$ to a **Set**.

We are required to show a bijection:

For A in **Set** and B in $\mathcal{P}$

$$\mathbf{Set}(A, \Gamma B) \to \mathcal{P}(\Delta A, B)$$

and

$$\mathcal{P}(\Delta A, B) \to \mathbf{Set}(A, \Gamma B)$$

The maps between the presheaf functors in $\mathcal{P}$ are natural transformations. Natural transformations are a collection of maps α_A: $\{\Delta A(A) \to B(A)\}_{A \in \mathcal{A}}$. For $U \subseteq V \subseteq X$ we have the commuting square:

$$\begin{tikzcd}
\Delta A(X) \arrow[r, "1_A"] \arrow[d, "\alpha_X"] & \Delta A(V) \arrow[r, "1_A"] \arrow[d, "\alpha_V"] & \Delta A(U) \arrow[d, "\alpha_U"] \\
B(X) \arrow[r, "B(f)"] & B(V) \arrow[r, "B(f)"] & B(U)
\end{tikzcd} \tag{1}$$

Recall $\Delta A(\cdot) = A$. Then the morphism in **Set** is represented by α_X above. As visible from the figure above this corresponds one to one with each α_A in $\mathcal{A}$, so the bijection holds. Dually using the exact same reasoning Π, the left adjoint of Δ is the presheaf evaluation at the empty set, $\Pi(P) = P(\emptyset)$.

For the left adjoint to Π, Λ, and for A in **Set** and B in $\mathcal{P}$, we need to show a bijection between:

$$\mathcal{P}(\Lambda A, B) \leftrightarrow \mathbf{Set}(A, \Pi(B))$$

To try and cobble together a definition of the presheaf functor Λ, start with the naturality diagram representing morphisms in $\mathcal{P}$:

$$\begin{tikzcd}
\Lambda(U) \arrow[r, "A(f)"] \arrow[d, "\alpha_U"] & \Lambda(\emptyset) \arrow[d, "\alpha_\emptyset"] \\
B(U) \arrow[r, "B(f)"] & B(\emptyset)
\end{tikzcd}$$

Note that $\Pi(B) = B(\emptyset)$. Start by choosing $\Lambda(\emptyset) = A$, so the morphism in **Set** is $\alpha_\emptyset$. Our choice of Λ needs to make this diagram commute for all U in $\mathcal{O}(X)^{op}$. For $U \neq \emptyset$ we could try $\Lambda(U) = A$, however to force the square above to commute with this choice, will impose some structure on the presheaf B. Rather, try setting $\Lambda(U) = \emptyset$ for $U \neq \emptyset$. Choosing the initial object $\emptyset$ of $\mathcal{O}(X)^{op}$, means there is one map out of the top LHS of the square in the above diagram, and the square commutes as required.

We also have

$$\mathcal{P}(A, \nabla B) \leftrightarrow \mathbf{Set}(\Gamma A, B)$$

∇, the right adjoint to Γ can be obtained dually, by swapping **Set** with $\mathbf{Set}^{op}$ and $\mathcal{O}(X)^{op}$ with $\mathcal{O}(X)$. This is simply a relabelling which has the effect of reversing the chain of adjoint functors stated in the question. We then apply analogous reasoning, take $\nabla(U) = \{*\}$, for $U \neq X$, and $\nabla(X) = B$.

2.2.11

The full subcategory where η_a is an isomorphism

2.2.12

(a) **Heuristic sort of proof** if the counit, $FG(f) \to f, B \in \mathcal{B}$ is isomorphic then a mapping back exists $f \to FG(f)$ such that their composition is the identity. So for a given B and B' and $FG(f)$ and f are one to one. Which necessarily means f and Gf are one to one, so G is full and faithful.

Algebraic proof From (2.2) the naturality axiom states:

$$\overline{(FG(B) \xrightarrow{\epsilon} B \xrightarrow{q} B'} = G(B) \xrightarrow{1_{G(B)}} G(B) \xrightarrow{G(q)} G(B')$$

ϵ injective implies faithful: $G(q_1) = G(q_2) \implies \epsilon q_1 = \epsilon q_2 \implies q_1 = q_2$

faithful implies ϵ injective: $\epsilon q_1 = \epsilon q_2 \implies G(q_1) = G(q_2) \implies q_1 = q2$

ϵ is injective implies full: For a given $h = G(q)$, need to find $q : \mathcal{B} \to \mathcal{B}$ inducing h. We know from naturality equation above that $G(q) = \overline{q\epsilon}$. ϵ needs to be invertible to retrieve q and hence satisfy fullness requirement.

full implies ϵ is injective: Put $B' = FG(B)$ in the naturality condition above to give:

$$\overline{(FG(B) \xrightarrow{\epsilon} B \xrightarrow{\lambda} FG(B)} = G(B) \xrightarrow{1_{G(B)}} G(B) \xrightarrow{G(\lambda)} GFG(B)$$

Using fullness choose λ such that $G\lambda = \eta$. Then

$$\begin{aligned} \overline{1_{FG}}(B) &= \eta_G(B), \text{ therefore} \\ 1_{FG}(B) &= \overline{\eta_G}(B) = \lambda\epsilon_G(B). \end{aligned}$$

So ϵ has an inverse and is therefore injective.

2.2.13

(a) We have sets S, T, a function $f\colon S \to T$. $P(S)$ denotes the set of all subsets of S. The functor f^* takes elements of T to their inverse under f. Looking for left and right adjoints of f^*. We can immediately see the left adjoint of f^* is f from below.

$$P(S)(A, f^{-1}(B)) \cong P(T)(f(A), B) \tag{2}$$

Now to find the right adjoint of f^*, G below:

$$P(T)(A, G(B)) \cong P(S)(f^{-1}(A), B)$$

Dualising

$$P(T)^{op}(G(B), A) \cong P(S)^{op}(B, f^{-1}(A)) \tag{3}$$

Equation (3) is (2) up to an isomorphism. So we choose $G = f$ here. In fact the power set P is self adjoint. Loosely, this means we have an isomorphism between the opposite category and the original category. We use this isomorphism to get a representation of G in $P(T)$, with the right adjoint sending B to $\overline{f(\overline{B})}$.

Because f is a bijection, elements in T that are not in $f(\overline{B})$ must have elements in B as their preimage. So $\overline{f(\overline{B})}$ consists of all sets of T where $f^{-1}(T) \subseteq B$. In summary the left adjoint of f^* is

$$F(S) = \{t \in T,\ \exists s \in S:\ s \in f^{-1}(t)\}$$

F represents choosing elements of T such that some element of S is in the inverse image of f.

and the right adjoint

$$G(S) = \{t \in T,\ \forall s \in S:\ s \in f^{-1}(t)\}$$

G represents choosing elements of T such that every element of S is in the inverse image of f.

(b) We are asked to interpret, in light of the results in (a.), the unit $\eta\colon 1_T \to G \circ F$, and counit $\epsilon\colon F \circ G \to 1_S$, for all adjunctions.

Consider the $R(x,y)$ as a set in $X \times Y$ and S as as a set in X. In all of the below I interpret **set inclusion as logical implication**.

Description of the functors used follows.

- $\forall_y$ takes a set $R(x,y)$ and returns $S(x)$ with preimage in $R(x,y)$ for all y. So each element in $R(x,y)$ inducing $S(x)$ is fully contained in $R(x,y)$.
- p^* the inverse image functor takes a set $S(x)$ and returns its preimage $R(x,y)$. This inclusion of X into $X \times Y$ adds a variable in Y. To use parlance of first order logic the statement is free in y.
- $\exists_Y$ takes a set $R(x,y)$ and returns $S(x)$ with preimage in $R(x,y)$ for at least one y.

We know from (a) that $\exists_Y \dashv p^* \dashv \forall_Y$

$p^* \dashv \forall_Y$

- $\eta : 1_X \to \forall_Y \circ p^*$ Plug in as argument to both sides of the implication the set $S(x)$. Evaluating the RHS, applying p^* results in the product of $S(x)$ with Y, the set $R(x,y)$. So η can be interpreted as $S(x) \implies R(x,y)\ \forall y$.

- $\epsilon : p^* \circ \forall_Y \to 1_{X \times Y}$. (??) The universal functor is projection of a subset of x-values for the set $R(x,y)$ passed to the functor. Applying p^* to yield say $R^Y(x,y)$ makes the statement on the LHS free in y, so it requires assignment for the statement to be meaningful. If we were to assign y on the LHS for corresponding to the y value for each RHS predicate, elementwise, then we essentially just have the statement that $R^Y(x,y) \implies R(x,y)$.

$\exists_Y \dashv p^*$

- $\eta : 1_{X \times Y} \to p^* \circ \exists_Y$. (??) The existence functor is just projection of all the x values for a given set of (x,y). The resulting statement on the right hand side is free in Y after applying p^* so requires assignment to be meaningful. If we were to assign y on the RHS for corresponding to the y value for each LHS predicate, elementwise, then we essentially just have an identity.
- $\epsilon : \exists_Y \circ p^* \to 1_X$ Plug in as argument to both sides of the implication the set $S(x)$. Applying p^* returns the set $R(x,y)$. So ϵ can be interpreted as $\exists y : R(x,y) \implies S(x)$.

2.2.14 Natural transformations for $[\mathcal{A}, \mathcal{B}]$.

$$\begin{tikzcd} FA \arrow[r, "F(f)"] \arrow[d, "\alpha_A"] & FA' \arrow[d, "\alpha_{A'}"] \\ GA \arrow[r, "G(f)"] & GA' \end{tikzcd}$$

$Y \in [\mathcal{B}, \mathcal{I}], F^*(Y) = Y \circ F$. Natural transformations for $[[\mathcal{A}, \mathcal{I}], [\mathcal{B}, \mathcal{I}]]$.

$$\begin{tikzcd} F^*Y = Y \circ F \arrow[r, "F^*(f)"] \arrow[d, "\alpha^*_Y"] & F^*Y' = Y' \circ F \arrow[d, "\alpha^*_{Y'}"] \\ G^*Y = Y \circ G \arrow[r, "G^*(f)"] & G^*Y' = Y' \circ G \end{tikzcd}$$

So it is evident from comparing the above natural transformation diagrams that we have the relationship $\alpha^*_Y = Y \circ \alpha$

Using first triangle inequality starting at point $F^*(Y)$ we have

$$\begin{aligned} \epsilon^*_{F^*(Y)} F^*(\eta^*_Y) F^*(Y) =& F^*(Y)(\epsilon F) F^*(\eta^*_Y) F^*(Y) \\ =& YF(\epsilon F)(\eta F) \\ =& YF 1_F \text{ since } \epsilon F \circ F\eta = 1_F \\ =& F^*(Y) \end{aligned}$$

The other triangle follows similarly. So by Theorem 2.2.5 we have an adjunction between F^* and G^*.

2.3.10

Equivalence between (F, G, η, ϵ) means for each A the square below commutes.

$$\begin{tikzcd}
1_A \arrow[r, "f"] \arrow[d, "\alpha_A"] & 1_{A'} \arrow[d, "\alpha_{A'}"] \\
GF(A) \arrow[r, "GF(f)"] & GF(A')
\end{tikzcd}$$

as well as in the other direction, for all $A \in mathcalA$, the below square commutes

$$\begin{tikzcd}
GF(A) \arrow[r, "GF(f)"] \arrow[d, "\alpha_A^{-1}"] & GF(A') \arrow[d, "\alpha_{A'}^{-1}"] \\
A \arrow[r, "f"] & A'
\end{tikzcd}$$

For F left adjoint to G using the definition based upon initial objects, we require

A map $(F(A), \eta_A) \to (B, f)$ in $(A \Rightarrow B)$ is a map $q\colon F(A) \to B$ in $\mathcal{B}$

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, dashed, "f"'] & GF(A) \arrow[d, "GF(q)"] \\
& G(B)
\end{tikzcd}$$

So we choose $GF(q) = \alpha_A^{-1} f$

2.3.11

A map $(F(S), \eta_S) \to (A, f)$ in $(S \Rightarrow U)$ is a map $q\colon F(S) \to A$ in $\mathcal{A}$

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[dr, dashed, "f"'] & UF(S) \arrow[d, "U(q)"] \\
& U(A)
\end{tikzcd}$$

$U(q) \circ \eta_S$ and f commute. So in the case f is injective, then $U(q) \circ \eta_S$ must be injective. This implies in turn that η_S is injective.

2.3.12

Par

- Objects: sets X
- Morphisms: Partial functions, written (f, D), where $f\colon X \to Y,\ X \subseteq D$, morphisms are only defined when $X \subseteq D$.

Set$_*$

- Objects: sets $X \cup \{*\}$
- Morphisms: $f^*(X) = Y, X \subseteq D,$ o.w $\{*\}$

$F\colon \mathbf{Par} \to \mathbf{Set}_*$

$$F(f, D) = x \mapsto \begin{cases} f(X), & \text{if } X \subseteq D. \\ *, & \text{otherwise.} \end{cases}$$
$$F(X) = X \cup \{*\} \text{ on objects}$$

$G\colon \mathbf{Set}_* \to \mathbf{Par}$

$$G(f^*) = (f^*, X \setminus \{*\}),$$
$$G(X) = X \setminus \{*\} \text{ on objects}$$

So we are mapping the undefined value of $\{*\}$ to the empty set. Which means $GF(\{*\}) = \emptyset$. So F and G are not isomorphic. However its seems we can construct a natural isomorphism α_X between $1_{\mathbf{Par}}$ and GF. An easier way to prove equivalence though is to show F is full, faithful and essentially surjective on objects.

F is faithful as for a morphism in $\mathbf{Par}$, $(X \to Y, D)$ there is at most one corresponding morphism in $\mathbf{Set}_*$, described in the definition of F. Alternatively, the domain of f can be recovered from Ff. It is those points which get mapped to something $\neq \{*\}$, since morphisms preserve distinguished elements. But since we have the domain then f can be recovered from Ff, since f is the restriction of Ff to the domain of Ff. [1]

F is full as for a morphism in $\mathbf{Set}_*$, $Ff\colon X \to Y$ there is at least one morphism inducing it in $\mathbf{Par}$, define $L = \{x\colon f(x) = \{*\}\}$, then the preimage partial function is $(f, X \setminus L)$, again by definition of F.

Finally F is essentially surjective on objects, because for all objects $B \in \mathbf{Set}_*$ there exists A in $\mathbf{Par}$ such that $F(A) \cong B$. Specifically $A = B \setminus \{*\}$.

So $\mathbf{Par}$ and $\mathbf{Set}_*$ are equivalent.

[1] https://math.stackexchange.com/questions/884451/why-are-the-category-of-pointed-sets-and-the-category-of-sets-and-partial-functi