Leitner - Basic Category Theory - Problem solutions

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0.10

Let S be a set. The indiscrete topological space $I(S)$ is the space whose set of points is S and whose only open subsets are $\emptyset$ and S. To find a universal property satisfied by the space $I(S)$ proceed as follows. With this topology any map from a topological space to S is continuous.

Parroting the wording of the question, let us rephrase this in universal parlance. Define a function $i\colon S \to I(S)$, by $i(s) = s, s \in S$. Then $I(S)$ has the following property.

$$\begin{tikzcd}
S \arrow[r, "i"] & I(S) \\
& X \arrow[ul, dashed, "\forall f"] \arrow[u, "\overline{f}"]
\end{tikzcd}$$

For all topological spaces X and all functions $f\colon X \to S$ there exists a unique continuous map $\overline{f}\colon X \to I(S)$. What it says is all maps into an indiscrete space are continuous. It also says that given S, the universal property determines $I(S)$ and i, up to isomorphism.

0.11

The universal property that is satisfied by the pair $(ker(\theta), \iota)$ is depicted in the diagram below.

$$\begin{tikzcd}
ker(\theta) \arrow[r, "\iota"] & G \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\epsilon"'] & H \\
F \arrow[u, dashed, "\exists!\overline{f}"] \arrow[ur, "\forall f"'] & &
\end{tikzcd}$$

The statement of the universal property is as follows. For any $f\colon F \to G$ such that $\theta \circ f = \epsilon \circ f$, there is a unique $\overline{f}\colon F \to ker(\theta)$ such that the diagram above commutes. That is $f = \iota \circ \overline{f}$.

0.13

(a)

Choose $\phi(\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n a_i r^i$. Then ϕ with $\phi(x) = r$ is a homomorphism that satisfies additive and multiplicative properties. To prove uniqueness assume there is another homomorphism ψ, with $\psi(x) = r$. Then $\psi(\sum_{i=1}^n a_i x^i) = \sum_{i=1} a_i \psi(x) = \sum_{i=1} a_i r^i$ by properties of a homomorphism. So $\psi = \phi$.

(b)

$\iota\colon \mathbb{Z}[x] \to A$ maps $\sum_{i=1}^n p_i x^i$ to $\sum_{i=1}^n p_i a^i$, using $\iota(x) = a$, the multiplicative property of a homomorphism to get $\iota(x^i) = \iota(x)^i$, and the additive property to get $\iota(p_i)\iota(x)^i = p_i \iota(x)^i$ remembering p_i is in $\mathbb{Z}$.

Going in the direction $A \to \mathbb{Z}[x]$ we know as provided in (b) that, taking $R = \mathbb{Z}[x]$, and $\phi = \iota'$, there exists a unique ring homomorphism such that $\iota'(a) = x$. So ι' maps $\sum_{i=1}^n p_i a^i$ to $\sum_{i=1}^n p_i x^i$ and $\iota' \circ \iota = 1_{\mathbb{Z}[x]}$. Also using definitions of ι and ι' easily yields $\iota \circ \iota' = 1_A$.

0.14

(a)

For the triangles below to commute, we need, as stated in the question $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, dashed, "\forall f_1"'] & P \arrow[d, "\exists p_1"] \\ & X \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, dashed, "\forall f_2"'] & P \arrow[d, "\exists p_2"] \\ & Y \end{tikzcd}$$

Choosing $P = X \times Y$, p_1 and p_2 as below makes the triangles commute.

$$p_1\colon X \times Y \to X$$
$$p_2\colon X \times Y \to Y$$

(b)

Proving uniqueness involves taking two arbitrary cones with the property stated in (a). Taking (P, p_1, p_2) and (P', p_1', p_2') we know from (a) that for all cones (V, f_1, f_2) there exists a unique linear map $f\colon V \to P'$ such that $p_1' \circ f = f_1$, $p_2' \circ f = f_2$. In this statement choose $V = P'$, then referring to the triangles in (a), observe there exists a $f\colon P \to P'$ such that $p_1' \circ f = p_1$, $p_2' \circ f = p_2$.

Comment The choice of P and p notation hinted very heavily that this is a projection of a product.

(c)

We need to define the cocone (Q, q_1, q_2) with the property, for all cocones (V, f_1, f_2) there exists a unique linear map $f \colon Q \to V$ such that $f \circ q_1 = f_1$ and $f \circ q_2 = f_2$. Choose $Q = X \times Y$, $q_1 \colon X \to X \oplus Y$, $q_2 \colon Y \to X \oplus Y$

Comment This is the dual of the product in (b), the coproduct. Set equivalent is the disjoint union.