# Leitner - Basic Category Theory - Problem solutions

## Adam Barber

August 17, 2021

### 0.10

Let S be a set. The indiscrete topological space I(S) is the space whose set of points is S and whose only open subsets are  $\emptyset$  and S. To find a universal property satisfied by the space I(S) proceed as follows. With this topology any map from a topological space to S is continuous.

Parroting the wording of the question, let us rephrase this in universal parlance. Define a function  $i: S \to I(S)$ , by  $i(s) = s, s \in S$ . Then I(S) has the following property.



For all topological spaces X and all functions  $f: X \to S$  there exists a unique continuous map  $\overline{f}: X \to I(S)$ . What it says is all maps into an indiscrete space are continuous. It also says that given S, the universal property determines I(S) and i, up to isomorphism.

#### 0.11

The universal property that is satisfied by the pair  $(ker(\theta), \iota)$  is depicted in the diagram below.

$$ker(\theta) \xrightarrow{\iota} G \xrightarrow{\theta} H$$

$$\exists ! \overline{f} \mid \\ F$$

The statement of the universal property is as follows. For any  $f: F \to G$  such that  $\theta \circ f = \epsilon \circ f$ , there is a unique  $\overline{f}: F \to ker(\theta)$  such that the diagram above commutes. That is  $f = \iota \circ \overline{f}$ .

#### 0.13

(a)

Choose  $\phi(\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n a_i r^i$ . Then  $\phi$  with  $\phi(x) = r$  is a homomorphism that satisfies additive and multiplicative properties. To prove uniqueness assume there is another homomorphism  $\psi$ , with  $\psi(x) = r$ . Then  $\psi(\sum_{i=1}^n a_i x^i) = \sum_{i=1} a_i \psi(x) = \sum_{i=1} a_i r^i$  by properties of a homomorphism. So  $\psi = \phi$ .

(b)

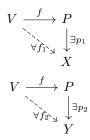
 $\iota \colon \mathbb{Z}[x] \to A$  maps  $\sum_{i=1}^n p_i x^i$  to  $\sum_{i=1}^n p_i a^i$ , using  $\iota(x) = a$ , the multiplicative property of a homomorphism to get  $\iota(x^i) = \iota(x)^i$ , and the additive property to get  $\iota(p_i)\iota(x)^i = p_i\iota(x)^i$  remembering  $p_i$  is in  $\mathbb{Z}$ .

Going in the direction  $A \to \mathbb{Z}[x]$  we know as provided in (b) that, taking  $R = \mathbb{Z}[x]$ , and  $\phi = \iota'$ , there exists a unique ring homomorphism such that  $\iota'(a) = x$ . So  $\iota'$  maps  $\sum_{i=1}^n p_i a^i$  to  $\sum_{i=1}^n p_i x^i$  and  $\iota' \circ \iota = 1_{\mathbb{Z}[x]}$ . Also using definitions of  $\iota$  and  $\iota'$  easily yields  $\iota \circ \iota' = 1_A$ .

#### 0.14

(a)

For the triangles below to commute, we need, as stated in the question  $p_1 \circ f = f_1$  and  $p_2 \circ f = f_2$ .



Choosing  $P = X \times Y$ ,  $p_1$  and  $p_2$  as below makes the triangles commute.

$$p_1: X \times Y \to X$$
  
 $p_2: X \times Y \to Y$ 

(b)

Proving uniqueness involves taking two arbitrary cones with the property stated in (a). Taking  $(P, p_1, p_2)$  and  $(P', p_1', p_2')$  we know from (a) that for all cones  $(V, f_1, f_2)$  there exists a unique linear map  $f: V \to P'$  such that  $p_1' \circ f = f_1$ ,  $p_2' \circ f = f_2$ . In this statement choose V = P', then referring to the triangles in (a), observe there exists a  $f: P \to P'$  such that  $p_1' \circ f = p_1$ ,  $p_2' \circ f = p_2$ .

**Comment** The choice of P and p notation hinted very heavily that this is a projection of a product.

(c)

We need to define the cocone  $(Q,q_1,q_2)$  with the property, for all cocones  $(V,f_1,f_2)$  there exists a unique linear map  $f:Q\to V$  such that  $f\circ q_1=f_1$  and  $f\circ q_2=f_2$ . Choose  $Q=X\times Y,\,q_1:X\to X\oplus Y,\,q_2:Y\to X\oplus Y$ 

Comment This is the dual of the product in (b), the coproduct.