# Leinster - Basic Category Theory - Selected problem solutions for Chapter 5

Adam Barber

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### 5.1.34

The equaliser square is not necessarily a pullback. There is no reason why any function into the $X$ would commute with a unique function into $E$, composed with $i$.

The converse is true though, a pullback implies an equaliser, when the square is set up as in the question.

### 5.1.35

Suppose the right hand square is a pullback. We need to prove the left hand square is a pullback if and only if the full rectangle, which composes both squares, is a pullback.

**Only if** Assume the left hand square and right hand squares are pullbacks. Show full rectangle is a pullback, that is show $g_1 = p_2.p_{12}\overline{f}$, and $f_2 = p_1.\overline{f}$

$$
\begin{tikzcd}
A \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "f_2"'] \arrow[dr, dashed, "\overline{f}"] \arrow[drr, dashed, "\overline{g}"'] \arrow[drrr, dotted, bend left=40, "g_1"] \arrow[ddrr, dotted, bend right=60, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p_1\cdot"] & P_2 \arrow[r, "p_2\cdot"] \arrow[d] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

**Left square pullback (black):** For any $f_1$ and $f_2$, there is a unique map $\overline{f}$ such that the left square above commutes.

**Right square pullback (red):** For any $g_1$ and $g_2$, there is a unique $\overline{g}$ such that the red diagram commutes.

Due to the left hand square being a pullback, for each $f_1$, and $f_2$, there is a unique map $\overline{f}$ such that $f_2 = p_1.\overline{f}$ and $f_1 = p_{12}\overline{f}$. Set $f_1 = \overline{g}$. From the right hand side being a pullback, $g_1 = p_2.\overline{g} = p_2.p_{12}\overline{f}$ as required.

**If** Assume the outer rectangle and right hand square are both pullbacks. Show the left hand side square is a pullback, that is $f_1 = p_{12}\overline{h}$, and $h_2 = p_1.\overline{h}$, for any $f_1, h_2$.

$$\begin{tikzcd}
A \arrow[rrrd, bend left, "h_1"] \arrow[rrrd, dotted, red, bend left=60, "g_1"] \arrow[dr, dashed, "\overline{h}"'] \arrow[rrd, dotted, red, "\overline{g}"] \arrow[rdd, bend right, "h_2"'] \arrow[rrdd, dotted, red, bend right=60, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p1."] & P_2 \arrow[r, "p_2."] \arrow[d] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

**Full rectangle pullback (black):** For any $h_1$ and $h_2$, there is a unique $\overline{h}$ such that the black diagram commutes.

Since the right hand square is a pullback, for any $g_1$, there is a unique $\overline{g}$ such that $g_1 = p_2.\overline{g}$. Since the rectangle is a pullback, for any $h_1$, there exists a unique $\overline{h}$ such that $p_2.p_{12}\overline{h} = h_1$, and $p_1.\overline{h} = h_2$. Set $g_1 = h_1$, then $p_2.p_{12}\overline{h} = p_2.\overline{g}$, so $p_{12}\overline{h} = \overline{g}$. $\overline{g}$ can be regarded as an arbitrary $f_1$, as there is a one to one correspondence with $\overline{g}$ and the arbitrary choice of $g_1$, or equivalently, $h_1$.

### 5.1.36

**(a)** If $(L \xrightarrow{p_I} D(I))_{i \in I}$ is a limit cone, there exists a unique $h$ such that $p_I \circ h = f_I$. However we are given that $p_I \circ h = p_I \circ h' = f_I$, so $h$ must equal $h'$.

**(b)** When $I$ is the two object discrete category, say $X \times Y$, $\mathcal{A} = \mathbf{Set}$, and $A = 1$, the statement in (a) says if $x = x', y = y'$, then $(x, y) = (x', y')$.

### 5.1.37

For any $A \in \mathcal{A}$, and all maps $I \xrightarrow{u} J$, a cone on D is

$$
\begin{tikzcd}
A \arrow[r, "f_I"] \arrow[dr, "f_J"'] & D(I) \arrow[d, "Du"] \\
& D(J)
\end{tikzcd}
\tag{1}
$$

A limit of $D$ is a cone $(L \xrightarrow{p_I} D(I))_{I\in\mathbf{I}}$, such that for any cone on $D$ with vertex A (1), there exists a unique map $\overline{f}\colon A \to L$ such that $p_J \circ \overline{f} = f_J$, for all $J \in \mathbf{I}$.

We have the set $\{(x_I)_{I\in\mathbf{I}} | x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and } (Du)(x_I) = x_J, \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\}$. The product limit formed is easier seen graphically. There is a family of maps for each $I \in \mathbf{I}$, each with

$$
\begin{tikzcd}
1 \arrow[r, "f_I"] \arrow[dr, "f_J"'] & x_I \in D(I) \arrow[d, "Du"] \\
& x_J \in D(J)
\end{tikzcd}
$$

Then fix $p_J = Du$, $\overline{f} = f_I$, and we have from the definition of a cone and (1) above $p_J \circ \overline{f} = f_J$, for all $J \in \mathbf{I}$. $\overline{f}$ is also unique. To see this assume there are two maps $\overline{f}$ and $\overline{f}'$, that make the above triangle commute. Then $Du \circ \overline{f} = Du \circ \overline{f}'$, for all maps $I \to J$. Set $I = J$ to retrieve $\overline{f} = \overline{f}'$. This family of maps we have described is precisely the definition of a product given in 5.1.7. So the set of $x_I$ can be written $\prod_{I\in\mathbf{I}} D(I)$.

So if any cone exists in **Set**, then a limit exists. Does a cone always exist in **Set**?

### 5.1.38

**(a)** We are given maps $s$ and $t$,

$$\prod_{I\in\mathbf{I}} D(I) \overset{s}{\underset{t}{\rightrightarrows}} \prod_{J \xrightarrow{u} K \text{ in } \mathbf{I}} D(K)$$

The $u$-component of $s$ is the composite

$$\prod_{I\in\mathbf{I}} D(I) \xrightarrow{pr_J} D(J) \xrightarrow{Du} D(K)$$

The $u$-component of $t$ is $pr_K$.

The fork property of the equalizer, says that the below diagram commutes for all maps $u$, $J \xrightarrow{u} K$ in $\mathbf{I}$, essentially that maps $(A \to D(J))_{J\in\mathbf{I}}$ are a cone on $D$.

$$\begin{tikzcd}
A \arrow[d] & \\
\prod_{I\in\mathbf{I}} D(I) \arrow[r, "pr_J"] \arrow[dr, "pr_K"'] & D(J) \arrow[d, "Du"] \\
& D(K)
\end{tikzcd} \tag{2}$$

The other important property of the equalizer is that for any fork, or as above, cone, there exists a unique map $\overline{f}\colon A \to L$ such that

$$\begin{tikzcd}
A \arrow[d, "\overline{f}"] \arrow[dr, "f"] & \\
L \arrow[r, "i"] & \prod_{I\in\mathbf{I}} D(I)
\end{tikzcd} \tag{3}$$

commutes.

Now $(L \xrightarrow{pr_J \circ i} D(J))_{J\in\mathbf{J}}$ is a cone, as it factors through $\prod_{I\in\mathbf{I}} D(I)$, as $A$ does in (2). (3) also implies $pr_J \circ i \circ \overline{f} = f_J$ for all $J$, where $f_J\colon A \to D(J) = pr_J \circ f$.

**(b)** The definition of a finite limit is a limit of shape $\mathbf{I}$ for some finite category $I$. So to show a limit is finite, we must show the diagram the limit maps into is indexed by a finite category. Finite categories have only finitely many maps. So binary products, terminal objects, equalizers and pullbacks are all finite limits. From part (a) we know if $\mathcal{A}$ has all products and equalizers then $\mathcal{A}$ has all limits. If we however restrict the products to binary products, then by definition limits of $\mathcal{A}$ will be finite.

### 5.1.39

A pullback (5.7) from page 114, with $Z$ as the terminal object, collapses to a binary product. The key point here is that the limit is unique up to isomorphism, so limits in a category with pullbacks and a terminal object are binary products, and hence finite.

### 5.1.40

We are given $X \xrightarrow{m} A$, and $X' \xrightarrow{m'} A$ are monics in **Set**. **Monic**$(A)$ is the full subcategory of the slice category $\mathcal{A}/A$, whose objects have as their maps the monics. Recall in $\mathcal{A}/A$, objects are tuples $(X, m)$ such that the following diagram commutes

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "m"'] & & X' \arrow[dl, "m'"] \\
& A &
\end{tikzcd}
$$

**Isomorphic implies equal images:** Note that if $m$ and $m'$ are isomorphic, then $f$ must be a bijection. The bijection can then be written $m = m' \circ f$, and $m' = m \circ f^{-1}$. Note also the m$m$ and $m\prime$, by virtue of them as monics, are injective. **Intuition:** We can essentially roundtrip on the triangle above, starting from an element in the image of $m$ (or conversely $m'$), and map it to an element in the image of $m'$ (respectively $m$). Explicitly we can write $\{m(x), x \in X\} = \{m' \circ f(x), x \in X\} = \{m'(x), x \in X'\}$.

**Equal images implies isomorphic:** If images of $m$ and $m'$ are equal,

$|{m'}^{-1}(A)| = |m^{-1}(A)|$, which implies a bijection between $X$ and $X'$, and hence a bijection between maps $m$ and $m'$ as in the previous paragraph.

**5.1.41**

$$
\begin{tikzcd}
Y \arrow[dr, dashed, "\overline{f}"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& X \arrow[r, "1"] \arrow[d, "1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}
$$

From the pullback diagram, for all commuting maps, that is for all $p, q, f \circ p = f \circ q \implies p = q$, if and only if the diagram above is a pullback.

**5.1.42**

The given square is a pullback, which means for a fixed $f, m, f', m'$, any other commuting square factors through it as follows.

$$
\begin{tikzcd}
Y \arrow[dr, dashed, "\overline{f}"] \arrow[drr, bend left, "f\overline{f}"] \arrow[ddr, bend right, "m'\overline{f}"'] & & \\
& X' \arrow[r, "f"] \arrow[d, "m'"] & X \arrow[d, "m"] \\
& A' \arrow[r, "f'"] & A
\end{tikzcd}
$$

We know from the properties of a pullback that $\overline{f} : Y \to X'$ is unique for each distinct pair of maps, $Y \to X$, and $Y \to A'$, such that the diagram above commutes.

In the following we use the contrapositive form of monic, so for maps $x, x'$, $f$ is monic if $x \neq x' \implies f \circ x \neq f \circ x'$.

Now we know $m$ is monic, so consider two distinct $\overline{f}_1$ and $\overline{f}_2$ in respect of two commuting diagrams as above. There must indeed be two distinct $mf\overline{f}_1 \neq mf\overline{f}_2$, such that each respective diagram commutes. Since the outer arrows commute, $mf\overline{f}_1 = f'm'\overline{f}_1$, and $mf\overline{f}_2 = f'm'\overline{f}_2$. So $f'm'\overline{f}_1 \neq f'm'\overline{f}_2 \implies m'\overline{f}_1 \neq m'\overline{f}_2$, and $m'$ is monic.

**5.2.21**

The equaliser is a map $f$ below such that $si = ti$, together with a universal property. The coequaliser is a map $p$ satisfying $ps = pt$, and universal with this property.

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \arrow[r, "p"] & C \end{tikzcd}$$

If $f$ is isomorphic then there is a $\overline{f}$ such that $f\overline{f} = 1_E, \overline{f}f = 1_X$, so $sf\overline{f} = s = tf\overline{f} = t$.

In the opposite direction, we need to show if $s = t$, then the equaliser exists and is isomorphic. To do this we will use the universal property of the equaliser. Specifically, any $f$ that is a fork factors through $i$ as below

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \\ A \arrow[u, "\overline{f}"] \arrow[ur, "f"'] & & \end{tikzcd}$$

Since $s = t$ we can choose any function $f$ and it will be a fork, and hence an equaliser exists. Immediately we can see that if we choose $f = 1_X$ then we have $i \circ \overline{i} = 1_X$, where $\overline{i}$ is the unique morphism depicted by $\overline{f}$ in the diagram below. Now we need to show $\overline{i} \circ i = 1_E$. Put $f = i\overline{i}i$ below, then there is a unique $h$ such that

$$i\overline{i}i = ih. \tag{4}$$

This implies $h = \overline{i}i$. Substituting $i\overline{i} = 1_X$ into (4) yields $h = 1_E$. So $\overline{i}i = 1_E$.

Proof for the coequaliser works the same, but dualised.

**5.2.22**

**(a)** The coequaliser of

$$X \underset{1}{\overset{f}{\rightrightarrows}} X \tag{5}$$

in **Set** is described as follows. Let $\sim$ be the equivalence relation between domain and codomain of $f$, $x \sim fx$, for all $x \in X$. The coequaliser (5) is then the quotient map $p\colon X \to X/\sim$.

**(b)** WIP

**5.2.23**

**(a)** We have the inclusion $f : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$. If $g \circ f = g' \circ f$, we need to show $g = g'$. $g \circ f$ is essentially a group homomorphism restricted to a domain of $\mathbb{N}$, whereas $g$ is the respective homomorphism on the expanded domain of $\mathbb{Z}$. The idea here is to stitch together a group homomorphism on the expanded domain using the group homomorphism on $\mathbb{N}$.

Define, for $x \in \mathbb{N}$

$$h(x) = \begin{cases} g \circ f(x), & x >= 0, \\ g \circ f(-x), & x < 0 \end{cases}$$

and define $h'$ analogously.

Since $g \circ f(-x) = -g \circ f(x) = -g' \circ f(x) = g' \circ f(-x)$, then $h = h'$ on $\{\forall z \in \mathbb{Z}\}$.

**(b)** We have $i : \mathbb{Z} \to \mathbb{Q}$, and $hi = h'i$. We need to show $h = h'$, on the full domain of $\mathbb{Q}$. A rational number is defined as $p/q, p, q \in \mathbb{Z}, q \neq 0$. The answer is a similar concept to (a), stitch together a ring homomorphism from the $hi$ on $\mathbb{Z}$, and show equality holds on the full domain of $\mathbb{Q}$.

Define $h(x) = h(p)h(1/q) = \frac{hi(p)}{hi(q)}$, and it follows that $h = h'$ as required.