

# Leitner - Basic Category Theory - Problem solutions

Adam Barber

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## 0.10

Let  $S$  be a set. The indiscrete topological space  $I(S)$  is the space whose set of points is  $S$  and whose only open subsets are  $\emptyset$  and  $S$ . To find a universal property satisfied by the space  $I(S)$  proceed as follows. With this topology any map from a topological space to  $S$  is continuous.

Parroting the wording of the question, let us rephrase this in universal parlance. Define a function  $i : S \rightarrow I(S)$ , by  $i(s) = s, s \in S$ . Then  $I(S)$  has the following property.

$$\begin{array}{ccc} S & \xrightarrow{i} & I(S) \\ & \nwarrow \forall f & \uparrow \bar{f} \\ & & X \end{array}$$

For all topological spaces  $X$  and all functions  $f : X \rightarrow S$  there exists a unique continuous map  $\bar{f} : X \rightarrow I(S)$ . What it says is all maps into an indiscrete space are continuous. It also says that given  $S$ , the universal property determines  $I(S)$  and  $i$ , up to isomorphism.

## 0.11

The universal property that is satisfied by the pair  $(\ker(\theta), \iota)$  is depicted in the diagram below.

$$\begin{array}{ccccc} \ker(\theta) & \xrightarrow{\iota} & G & \xrightarrow[\epsilon]{\theta} & H \\ \uparrow \exists! \bar{f} & & \nearrow \forall f & & \\ F & & & & \end{array}$$

The statement of the universal property is as follows. For any  $f : F \rightarrow G$  such that  $\theta \circ f = \epsilon \circ f$ , there is a unique  $\bar{f} : F \rightarrow \ker(\theta)$  such that the diagram above commutes. That is  $f = \iota \circ \bar{f}$ .

**0.13****(a)**

Choose  $\phi(\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n a_i r^i$ . Then  $\phi$  with  $\phi(x) = r$  is a homomorphism that satisfies additive and multiplicative properties. To prove uniqueness assume there is another homomorphism  $\psi$ , with  $\psi(x) = r$ . Then  $\psi(\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n a_i \psi(x) = \sum_{i=1}^n a_i r^i$  by properties of a homomorphism. So  $\psi = \phi$ .

**(b)**

$\iota: \mathbb{Z}[x] \rightarrow A$  maps  $\sum_{i=1}^n p_i x^i$  to  $\sum_{i=1}^n p_i a^i$ , using  $\iota(x) = a$ , the multiplicative property of a homomorphism to get  $\iota(x^i) = \iota(x)^i$ , and the additive property to get  $\iota(p_i)\iota(x)^i = p_i \iota(x)^i$  remembering  $p_i$  is in  $\mathbb{Z}$ .

Going in the direction  $A \rightarrow \mathbb{Z}[x]$  we know as provided in (b) that, taking  $R = \mathbb{Z}[x]$ , and  $\phi = \iota'$ , there exists a unique ring homomorphism such that  $\iota'(a) = x$ . So  $\iota'$  maps  $\sum_{i=1}^n p_i a^i$  to  $\sum_{i=1}^n p_i x^i$  and  $\iota' \circ \iota = 1_{\mathbb{Z}[x]}$ . Also using definitions of  $\iota$  and  $\iota'$  easily yields  $\iota \circ \iota' = 1_A$ .