Leinster - Basic Category Theory - Selected problem solutions for Chapter 3

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3.1.1

There are bijections

$$\begin{aligned} (A+B, C) &\leftrightarrow ((A,B), \Delta C) \\ f &\leftrightarrow \overline{f} \end{aligned}$$

where $\overline{f} = (f, f)$

$$\begin{aligned} (\Delta A, (B, C)) &\leftrightarrow (A, B \times C) \\ g = (p, q) &\leftrightarrow \overline{g} \end{aligned}$$

where $\overline{g}(x) = (p(x), q(x))$

So the sum is left adjoint to Δ, and the product is its right adjoint.

3.1.2

We are given the definition of a sequence, where there is a unique function x such that the square below commutes.

We have $x_0 = a$, and $x_{n+1} = r(x_n)$.

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} \\ X & \xrightarrow{r} & X \end{array}$$

This is precisely the definition of the comma category $(\mathbb{N} \Rightarrow X)$, where objects are $(n \in \mathbb{N}, x, t \in X)$.

3.2.12

(a)

$$\theta(S) = \bigcup \theta(R) \supseteq \bigcup R = S$$

But $\theta^2(S) = \theta(S)$, so $\theta(S) \subseteq S$.

Taken together, the above implies $\theta(S) = S$.

(b)

$$\begin{aligned} & A \subseteq B \\ \implies & f[A] \subseteq f[B] \\ \implies & gf[A] \subseteq gf[B] \end{aligned}$$

g and f are taken to be injections here. We need to prove there is a bijection between A and B. **Note:** this does not follow immediately from g and f being injections.

Take $\theta(S) = A - g[B \setminus f[S]]$. Then $S_1 \subseteq S_2 \implies \theta(S_1) \subseteq \theta(S_2)$. Since f, g and hence θ is order preserving, we may apply the result in (a). Specifically, there exists S such that $S = A - g[B \setminus f[S]] \implies g[B \setminus f[S]] = A \setminus S$.

(c) We need to prove a bijection between A and B to deduce the theorem. Consider $h\colon A \to B$

$$h(x) = \begin{cases} f(x), & x \in S, \\ g^{-1}(x), & x \in A \setminus S \end{cases}$$

f has a codomain of $f[S]$, so every element of the codomain has a preimage in S. We are given that f is injective.

g is injective and hence invertible. Using the result in (b) we have a direct expression for g^{-1}. Hence we have $gh = 1_A$, and $hg = 1_B$, for x in $A \setminus S$.

An alternative proof, has a similar basic idea, of partitioning the domain of the bijection around the fixed point. **Sketch proof** Set $A_0 = A$. $A_{i+1} = gfA_i$. Define $k(x) = gf(x)$ if $x \in A_i$ for some i, otherwise $k(x) = x$. To prove k is surjective comes down to two cases. Suppose $y \in A_n$, for some n then A_{n-1} is the x-value such that $k(x) = y$. If y is not in A_n for any arbitrarily large n, then we must have $k(x) = x$.

3.2.14

Need to prove that for any family $(A_i)_{i'\in I}$ of objects of $\mathcal{A}$, there is some object of $\mathcal{A}$ not isomorphic to A_i for $i \in I$. It suffices to prove for A in $F(S)$, $F : \mathbf{Set} \to \mathcal{A}$, then we know the condition holds for $\mathcal{A}$. Now UF is injective by Exercise 2.3.11, so U is injective on objects A of $F(S)$. So if UA_i is not isomorphic to UA_j, this would imply A_i is not isomorphic to A_j. So we need to prove for a given i, $|UA_i| < |\mathcal{P}(UA)|$:

$$|UA_i| \leq |\Sigma UA_i| < |\mathcal{P}(UA)|$$

The strict equality due to Theorem 3.2.2.

3.2.15

The key point here is that *Set* is not small. I think of *Set* as a power set of an arbitrary family of sets, as in the proof for Proposition 3.2.4. Set is locally small however, as for any two objects A and B, the functions between A and B form a set. This question is a little too wooly for me, I struggled, without the necessary background, to reason my way though so many ambiguities that presented themselves. Here is a shot.

(a) **Mon** is equivalent to a single object category, which is small. So **Mon** is essentially small.

(b) $\mathbb{Z}$, the group of integers viewed as a one object category, is locally small. Groups are just an 'enriched' set.

(c) The ordered set of integers still has a large class of isomorphism classes (?) My guess here is it locally small, as there is one map between each two objects.

(d) Using the existence of a left adjoint proved in 3.2.16, and the result of 3.2.14, tells us the class of isomorphism classes of **Cat** is large. So **Cat** is not essentially small. For locally small we would require the set of natural transformations between **Cat** and **Set** be a set. There is one component for each object in **Cat** which is small, hence the morphisms form a single element set. *Cat* is locally small. (?)

(e) **Guess**. Same reasoning as (a), locally small.