

Leinster - Basic Category Theory - Selected problem solutions for Chapter 5

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5.1.34

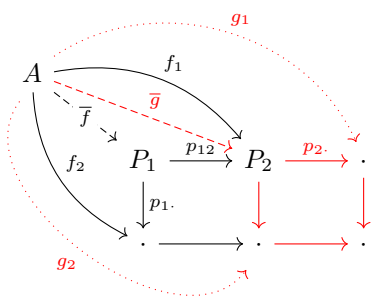
The equaliser square is not necessarily a pullback. There is no reason why any function into the X would commute with a unique function into E , composed with i .

The converse is true though, a pullback implies an equaliser, when the square is set up as in the question.

5.1.35

Suppose the right hand square is a pullback. We need to prove the left hand square is a pullback if and only if the full rectangle, which composes both squares, is a pullback.

Only if Assume the left hand square and right hand squares are pullbacks. Show full rectangle is a pullback, that is show $g_1 = p_2.p_{12}\bar{f}$, and $f_2 = p_1.\bar{f}$

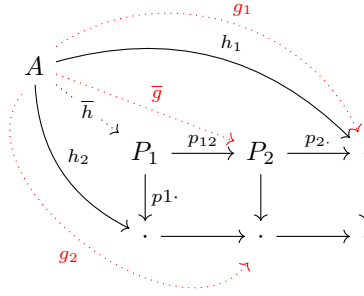


Left square pullback (black): For any f_1 and f_2 , there is a unique map \bar{f} such that the left square above commutes.

Right square pullback (red): For any g_1 and g_2 , there is a unique \bar{g} such that the red diagram commutes.

Due to the left hand square being a pullback, for each f_1 , and f_2 , there is a unique map \bar{f} such that $f_2 = p_1.\bar{f}$ and $f_1 = p_{12}\bar{f}$. Set $f_1 = \bar{g}$. From the right hand side being a pullback, $g_1 = p_2.\bar{g} = p_2.p_{12}\bar{f}$ as required.

If Assume the outer rectangle and right hand square are both pullbacks. Show the left hand side square is a pullback, that is $f_1 = p_{12}\bar{h}$, and $h_2 = p_1.\bar{h}$, for any f_1, h_2 .



Full rectangle pullback (black): For any h_1 and h_2 , there is a unique \bar{h} such that the black diagram commutes.

Since the right hand square is a pullback, for any g_1 , there is a unique \bar{g} such that $g_1 = p_2.\bar{g}$. Since the rectangle is a pullback, for any h_1 , there exists a unique \bar{h} such that $p_2.p_{12}\bar{h} = h_1$, and $p_1.\bar{h} = h_2$. Set $g_1 = h_1$, then $p_2.p_{12}\bar{h} = p_2.\bar{g}$, so $p_{12}\bar{h} = \bar{g}$. \bar{g} can be regarded as an arbitrary f_1 , as there is a one to one correspondence with \bar{g} and the arbitrary choice of g_1 , or equivalently, h_1 .

5.1.36

(a) If $(L \xrightarrow{p_I} D(I))_{i \in I}$ is a limit cone, there exists a unique h such that $p_I \circ h = f_I$. However we are given that $p_I \circ h = p_I \circ h' = f_I$, so h must equal h' .

(b) When I is the two object discrete category, say $X \times Y$, $\mathcal{A} = \mathbf{Set}$, and $A = 1$, the statement in (a) says if $x = x', y = y'$, then $(x, y) = (x', y')$.

5.1.37

For any $A \in \mathcal{A}$, and all maps $I \xrightarrow{u} J$, a cone on D is

$$\begin{array}{ccc}
A & \xrightarrow{f_I} & D(I) \\
& \searrow f_J & \downarrow Du \\
& & D(J)
\end{array} \tag{1}$$

A limit of D is a cone $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$, such that for any cone on D with vertex A (1), there exists a unique map $\bar{f}: A \rightarrow L$ such that $p_J \circ \bar{f} = f_J$, for all $J \in \mathbf{I}$.

We have the set $\{(x_I)_{I \in \mathbf{I}} | x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and } (Du)(x_I) = x_J, \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\}$. The product limit formed is easier seen graphically. There is a family of maps for each $I \in \mathbf{I}$, each with

$$\begin{array}{ccc}
1 & \xrightarrow{f_I} & x_I \in D(I) \\
& \searrow f_J & \downarrow Du \\
& & x_J \in D(J)
\end{array}$$

Then fix $p_J = Du$, $\bar{f} = f_I$, and we have from the definition of a cone and (1) above $p_J \circ \bar{f} = f_J$, for all $J \in \mathbf{I}$. \bar{f} is also unique. To see this assume there are two maps \bar{f} and \bar{f}' , that make the above triangle commute. Then $Du \circ \bar{f} = Du \circ \bar{f}'$, for all maps $I \rightarrow J$. Set $I = J$ to retrieve $\bar{f} = \bar{f}'$. This family of maps we have described is precisely the definition of a product given in 5.1.7. So the set of x_I can be written $\prod_{I \in \mathbf{I}} D(I)$.

So if any cone exists in **Set**, then a limit exists. Does a cone always exist in **Set**?

5.1.38

(a) We are given maps s and t ,

$$\prod_{I \in \mathbf{I}} D(I) \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} \prod_{J \xrightarrow{u} K \text{ in } \mathbf{I}} D(K)$$

The u -component of s is the composite

$$\prod_{I \in \mathbf{I}} D(I) \xrightarrow{pr_J} D(J) \xrightarrow{Du} D(K)$$

The u -component of t is pr_K .

The fork property of the equalizer, says that the below diagram commutes for all maps $u, J \xrightarrow{u} K$ in \mathbf{I} , essentially that maps $(A \rightarrow D(J))_{J \in \mathbf{I}}$ are a cone on D .

$$\begin{array}{ccc}
A & & \\
\downarrow & & \\
\prod_{I \in \mathbf{I}} D(I) & \xrightarrow{pr_J} & D(J) \\
& \searrow pr_K & \downarrow Du \\
& & D(K)
\end{array} \tag{2}$$

The other important property of the equalizer is that for any fork, or as above, cone, there exists a unique map $\bar{f}: A \rightarrow L$ such that

$$\begin{array}{ccc}
A & & \\
\downarrow \bar{f} & \searrow f & \\
L & \xrightarrow{i} & \prod_{I \in \mathbf{I}} D(I)
\end{array} \tag{3}$$

commutes.

Now $(L \xrightarrow{pr_J \circ i} D(J))_{J \in \mathbf{J}}$ is a cone, as it factors through $\prod_{I \in \mathbf{I}} D(I)$, as A does in (2). (3) also implies $pr_J \circ i \circ \bar{f} = f_J$ for all J , where $f_J: A \rightarrow D(J) = pr_J \circ f$.

(b) The definition of a finite limit is a limit of shape \mathbf{I} for some finite category \mathbf{I} . So to show a limit is finite, we must show the diagram the limit maps into is indexed by a finite category. Finite categories have only finitely many maps. So binary products, terminal objects, equalizers and pullbacks are all finite limits. From part (a) we know if \mathcal{A} has all products and equalizers then \mathcal{A} has all limits. If we however restrict the products to binary products, then by definition limits of \mathcal{A} will be finite.

5.1.39

A pullback (5.7) from page 114, with Z as the terminal object, collapses to a binary product. The key point here is that the limit is unique up to isomorphism, so limits in a category with pullbacks and a terminal object are binary products, and hence finite.

5.1.40

We are given $X \xrightarrow{m} A$, and $X' \xrightarrow{m'} A$ are monics in **Set**. **Monic**(A) is the full subcategory of the slice category \mathcal{A}/A , whose objects have as their maps the monics. Recall in \mathcal{A}/A , objects are tuples (X, m) such that the following

diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & \searrow m & \swarrow m' \\ & A & \end{array}$$

Isomorphic implies equal images: Note that if m and m' are isomorphic, then f must be a bijection. The bijection can then be written $m = m' \circ f$, and $m' = m \circ f^{-1}$. Note also the m and m' , by virtue of them as monics, are injective. **Intuition:** We can essentially roundtrip on the triangle above, starting from an element in the image of m (or conversely m'), and map it to an element in the image of m' (respectively m). Explicitly we can write $\{m(x), x \in X\} = \{m' \circ f(x), x \in X\} = \{m'(x), x \in X'\}$.

Equal images implies isomorphic: If images of m and m' are equal,

$|m'^{-1}(A)| = |m^{-1}(A)|$, which implies a bijection between X and X' , and hence a bijection between maps m and m' as in the previous paragraph.

5.1.41

$$\begin{array}{ccccc} Y & & \xrightarrow{p} & & X \\ & \searrow \bar{f} & & \searrow & \\ & & X & \xrightarrow{1} & X \\ & \searrow q & \downarrow 1 & & \downarrow f \\ & & X & \xrightarrow{f} & Y \end{array}$$

From the pullback diagram, for all commuting maps, that is for all $p, q, f \circ p = f \circ q \implies p = q$, if and only if the diagram above is a pullback.

5.1.42

The given square is a pullback, which means for a fixed f, m, f', m' , any other commuting square factors through it as follows.

$$\begin{array}{ccccc} Y & & \xrightarrow{f\bar{f}} & & X \\ & \searrow \bar{f} & & \searrow & \\ & & X' & \xrightarrow{f} & X \\ & \searrow m'\bar{f} & \downarrow m' & & \downarrow m \\ & & A' & \xrightarrow{f'} & A \end{array}$$

We know from the properties of a pullback that $\bar{f} : Y \rightarrow X'$ is unique for each distinct pair of maps, $Y \rightarrow X$, and $Y \rightarrow A'$, such that the diagram above commutes.

In the following we use the contrapositive form of monic, so for maps x, x', f is monic if $x \neq x' \implies f \circ x \neq f \circ x'$.

Now we know m is monic, so consider two distinct \bar{f}_1 and \bar{f}_2 in respect of two commuting diagrams as above. There must indeed be two distinct $m\bar{f}_1 \neq m\bar{f}_2$, such that each respective diagram commutes. Since the outer arrows commute, $m\bar{f}_1 = f'm'\bar{f}_1$, and $m\bar{f}_2 = f'm'\bar{f}_2$. So $f'm'\bar{f}_1 \neq f'm'\bar{f}_2 \implies m'\bar{f}_1 \neq m'\bar{f}_2$, and m' is monic.

5.1.42

The equaliser is a map f below such that $si = ti$, together with a universal property. The coequaliser is a map p satisfying $ps = pt$, and universal with this property.

$$E \xrightarrow{i} X \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} Y \xrightarrow{p} C$$

If f is isomorphic then there is a \bar{f} such that $f\bar{f} = 1_E$, $\bar{f}f = 1_X$, so $sf\bar{f} = s = tf\bar{f} = t$.

In the opposite direction, we need to show if $s = t$, then the equaliser exists and is isomorphic. To do this we will use the universal property of the equaliser. Specifically, any f that is a fork factors through i as below

$$\begin{array}{ccccc} E & \xrightarrow{i} & X & \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} & Y \\ \bar{f} \uparrow & \nearrow f & & & \\ A & & & & \end{array}$$

Since $s = t$ we can choose any function f and it will be a fork, and hence an equaliser exists. Immediately we can see that if we choose $f = 1_X$ then we have $i \circ \bar{i} = 1_X$, where \bar{i} is the unique morphism depicted by \bar{f} in the diagram below. Now we need to show $\bar{i} \circ i = 1_E$. Put $f = i\bar{i}i$ below, then there is a unique h such that

$$i\bar{i}i = ih. \tag{4}$$

This implies $h = \bar{i}i$. Substituting $i\bar{i} = 1_X$ into (4) yields $h = 1_E$. So $\bar{i}i = 1_E$.

Proof works the same dualised for the coequaliser.