Leinster - Basic Category Theory - Selected problem solutions for Chapter 5

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5.1.34

The equaliser square is not necessarily a pullback. There is no reason why any function into the X would commute with a unique function into E, composed with i.

The converse is true though, a pullback implies an equaliser, when the square is set up as in the question.

5.1.35

Suppose the right hand square is a pullback. We need to prove the left hand square is a pullback if and only if the full rectangle, which composes both squares, is a pullback.

Only if Assume the left hand square and right hand squares are pullbacks. Show full rectangle is a pullback, that is show $g_1 = p_{2\cdot}.p_{12}\overline{f}$, and $f_2 = p_{1\cdot}.\overline{f}$

$$
\begin{tikzcd}
A \arrow[drr, bend left, "f_1"] \arrow[dr, dashed, "\overline{f}"'] \arrow[drr, dashed, "\overline{g}"] \arrow[ddr, bend right, "f_2"'] \arrow[drrr, bend left=60, dotted, "g_1"] \arrow[ddrr, bend right=60, dotted, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p_{1\cdot}"] & P_2 \arrow[r, "p_{2\cdot}"] \arrow[d] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

Left square pullback (black): For any f_1 and f_2, there is a unique map $\overline{f}$ such that the left square above commutes.

Right square pullback (red): For any g_1 and g_2, there is a unique $\overline{g}$ such that the red diagram commutes.

Due to the left hand square being a pullback, for each f_1, and f_2, there is a unique map $\overline{f}$ such that $f_2 = p_1.\overline{f}$ and $f_1 = p_{12}\overline{f}$. Set $f_1 = \overline{g}$. From the right hand side being a pullback, $g_1 = p_2.\overline{g} = p_2.p_{12}\overline{f}$ as required.

If Assume the outer rectangle and right hand square are both pullbacks. Show the left hand side square is a pullback, that is $f_1 = p_{12}\overline{h}$, and $h_2 = p_1.\overline{h}$, for any f_1, h_2.

$$
\begin{tikzcd}
A \arrow[dr, dotted, "\overline{h}"'] \arrow[drr, dotted, red, "\overline{g}"] \arrow[drrr, bend left, "h_1"] \arrow[drrr, bend left=50, dotted, red, "g_1"] \arrow[ddr, bend right, "h_2"'] \arrow[ddrr, bend right=60, dotted, red, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p1."] & P_2 \arrow[r, "p_2."] \arrow[d] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

Full rectangle pullback (black): For any h_1 and h_2, there is a unique $\overline{h}$ such that the black diagram commutes.

Since the right hand square is a pullback, for any g_1, there is a unique $\overline{g}$ such that $g_1 = p_2.\overline{g}$. Since the rectangle is a pullback, for any h_1, there exists a unique $\overline{h}$ such that $p_2.p_{12}\overline{h} = h_1$, and $p_1.\overline{h} = h_2$. Set $g_1 = h_1$, then $p_2.p_{12}\overline{h} = p_2.\overline{g}$, so $p_{12}\overline{h} = \overline{g}$. $\overline{g}$ can be regarded as an arbitrary f_1, as there is a one to one correspondence with $\overline{g}$ and the arbitrary choice of g_1, or equivalently, h_1.