# Leinster - Basic Category Theory - Selected problem solutions for Chapter 5

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### 4.1.27

Suppose the right hand square is a pullback. We need to prove the left hand square is a pullback if and only if the full rectangle, which composes both squares, is a pullback.

**Only if** Assume the left hand square and right hand squares are pullbacks. Show full rectangle is a pullback, that is show $g_1 = p_2.p_{12}\overline{f}$, and $f_2 = p_1.\overline{f}$

$$\begin{tikzcd}
A \arrow[drr, bend left, "f_1"] \arrow[dr, dashed, "\overline{f}"'] \arrow[drr, dashed, red, "\overline{g}"] \arrow[ddr, bend right, "f_2"'] \arrow[drrr, dotted, red, bend left=40, "g_1"] \arrow[ddrr, dotted, red, bend right=60, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p_1."] & P_2 \arrow[r, red, "p_2."] \arrow[d, red] & \cdot \arrow[d, red] \\
& \cdot \arrow[r] & \cdot \arrow[r, red] & \cdot
\end{tikzcd}$$

**Left square pullback (black):** For any $f_1$ and $f_2$, there is a unique map $\overline{f}$ such that the left square above commutes.

**Right square pullback (red):** For any $g_1$ and $g_2$, there is a unique $\overline{g}$ such that the red diagram commutes.

Due to the left hand square being a pullback, for each $f_1$, and $f_2$, there is a unique map $\overline{f}$ such that $f_2 = p_1.\overline{f}$ and $f_1 = p_{12}\overline{f}$. Set $f_1 = \overline{g}$. From the right hand side being a pullback, $g_1 = p_2.\overline{g} = p_2.p_{12}\overline{f}$ as required.

**If** Assume the outer rectangle and right hand square are both pullbacks. Show the left hand side square is a pullback, that is $f_1 = p_{12}\overline{h}$, and $h_2 = p_1.\overline{h}$.

$$\begin{tikzcd}
A \arrow[dr, dotted, "\overline{h}"'] \arrow[drr, dotted, red, "\overline{g}"] \arrow[drrr, bend left, "h_1"] \arrow[drrr, bend left=40, dotted, red, "g_1"] \arrow[ddr, bend right, "h_2"'] \arrow[ddrr, bend right=60, dotted, red, "g_2"'] & & & \\
& P_1 \arrow[r, "p_{12}"] \arrow[d, "p1\cdot"] & P_2 \arrow[r, "p_2\cdot"] \arrow[d] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

**Full rectangle pullback (black):** For any $h_1$ and $h_2$, there is a unique $\overline{h}$ such that the black diagram commutes.

Since the right hand square is a pullback, there is a unique $\overline{g}$ such that $g_1 = p_2.\overline{g}$. Since the rectangle is a pullback there exists a unique $\overline{h}$ such that $p_2.p_{12}\overline{h} = h_1$, and $p_1.\overline{h} = h_2$. Set $g_1 = h_1$, then $p_2.p_{12}\overline{h} = p_2.\overline{g}$, so $p_{12}\overline{h} = \overline{g}$. $\overline{g}$ can be regarded as an arbitrary $f_1$, as there is a one to one correspondence with $\overline{g}$ and the arbitrary choice of $g_1$ (or equivalently, $h_1$).