Leinster - Basic Category Theory - Problem solutions Adam Barber
0.10:

Let S be a set. The indiscrete topological space $I(S)$ is the space whose set of points is S and whose only open subsets are $\emptyset$ and S. To find a universal property satisfied by the space $I(S)$ proceed as follows. With this topology any map from a topological space to S is continuous.

Parroting the wording of the question, let us rephrase this in universal parlance. Define a function $i : S \to I(S)$, by $i(s) = s, s \in S$. Then $I(S)$ has the following property.

$$S \xrightarrow{\quad i \quad} I(S)$$
$$\nwarrow \qquad \uparrow \overline{f}$$
$$\forall f \qquad$$
$$X$$

For all topological spaces X and all functions $f : X \to S$ there exists a unique continuous map $\overline{f} : X \to I(S)$. What it says is all maps into an indiscrete space are continuous. It also says that given S, the universal property determines $I(S)$ and i, up to isomorphism.

0.11

The universal property that is satisfied by the pair $(ker(\theta), \iota)$ is depicted in the diagram below.

$$ker(\theta) \xrightarrow{\quad \iota \quad} G \xrightarrow[\theta]{\quad} H$$
$$\uparrow \exists! \overline{f} \qquad \nearrow \forall f$$
$$F$$

0.13:
(a)
Choose $\phi(\sum_{i=1}^{n} a_i x^i) = \sum_{i=1}^{n} a_i r^i$. Then ϕ with $\phi(x) = r$ is a homomorphism that satisfies additive and multiplicative properties. To prove uniqueness assume there is another homomorphism ψ, with $\psi(x) = r$. Then $\psi(\sum_{i=1}^{n} a_i x^i) = \sum_{i=1} a_i \psi(x) = \sum_{i=1} a_i r^i$ by properties of a homomorphism. So $\psi = \phi$.
(b)
$\iota \colon \mathbb{Z}[x] \to A$ maps $\sum_{i=1}^{n} p_i x^i$ to $\sum_{i=1}^{n} p_i a^i$, using $\iota(x) = a$, the multiplicative property of a homomorphism to get $\iota(x^i) = \iota(x)^i$, and the additive property to get $\iota(p_i)\iota(x)^i = p_i \iota(x)^i$ remembering p_i is in $\mathbb{Z}$.

Going in the direction $A \to \mathbb{Z}[x]$ we know as provided in (b) that, taking $R = \mathbb{Z}[x]$, and $\phi = \iota'$, there exists a unique ring homomorphism such that $\iota'(a) = x$. So ι' maps $\sum_{i=1}^{n} p_i a^i$ to $\sum_{i=1}^{n} p_i x^i$ and $\iota' \circ \iota = 1_{\mathbb{Z}[x]}$. Also using definitions of ι and ι' easily yields $\iota \circ \iota' = 1_A$.