

Leinster - Basic Category Theory - Selected problem solutions for Chapter 2

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November 24, 2021

2.1.16

(a) Interesting adjoint functors to G -sets.

The trivial group functor I sends a set to a \mathbf{G} -set with the trivial action $gx = x$.
Interesting functors

Orbit functor sends a G -set with underlying set elements a of A to:

$$A_G = \{g \cdot a, g \in G\}$$

Fixed point functor sends a G -set with underlying set elements a of A to:

$$A^G = \{a \text{ such that } g \cdot a = a \text{ for all } g \in G, a \in A\}$$

Fixed point functor - right adjoint Morphisms in a G -set are functions on the underlying set, where f commutes with g for every $g \in G$.

There is a bijection for each $A \in \mathbf{Set}$ and $B \in [G, \mathbf{Set}]$ as follows

$$\begin{aligned} [G, \mathbf{Set}](I(A), B) &\rightarrow \mathbf{Set}(A, B^G) \\ \psi &\mapsto \bar{\psi} \end{aligned}$$

$\bar{\psi}$ sends each element a of A to $\psi(a)$ if $g \cdot a = a$, otherwise it sends a to $\psi(\emptyset)$.

$$\begin{aligned} \mathbf{Set}(A, B^G) &\rightarrow [G, \mathbf{Set}](I(A), B) \\ \phi &\mapsto \bar{\phi} \end{aligned}$$

ϕ sends each $a \in A$ in the underlying set of the G -set to the G -set $(g, \bar{\phi}(a)), g \in G$.

Orbit functor - left adjoint There is a bijection for each $A \in [G, \mathbf{Set}]$ and $B \in \mathbf{Set}$ as follows

$$\begin{aligned} \mathbf{Set}(A_G, B) &\rightarrow [G, \mathbf{Set}](A, I(B)) \\ \psi &\mapsto \bar{\psi} \end{aligned}$$

So each morphism in \mathbf{Set} sends the set formed by the orbits of an element a of A , call this a_G , to $\psi(a_G)$, where ψ is a function of sets. Choose a G -set morphism $\bar{\psi} = \psi$, where $\bar{\psi}$ commutes with g for every g in G .

$$\begin{aligned} [G, \mathbf{Set}](A, I(B)) &\rightarrow \mathbf{Set}(A_G, B) \\ \phi &\mapsto \bar{\phi} \end{aligned}$$

Choose $\bar{\phi}$ to be a disjoint union of each orbit of a in A , $\bar{\phi}(a) = \coprod \{\phi(g \cdot a), g \in G\}$

2.1.17

Write $\mathcal{O}(X)$ for the poset of open subsets of a topological space X ordered by inclusion.

$$\Delta : \mathbf{Set} \rightarrow [\mathcal{O}(X)^{op}, \mathbf{Set}]$$

Write \mathcal{P} for the presheaf functor category, and $P \in \mathcal{P}$ for the functor which maps $\mathcal{O}(X)^{op}$ to \mathbf{Set} . Take open sets U, V , such that $U \subseteq V$ in X . A presheaf consists of

- restriction maps, $P(V) \rightarrow P(U)$, these are morphisms which enforce some sort of ordering of the mapped sets,
- and the actual mapped sets $P(U), P(V)$ which are called sections.

Since the question specifies a constant presheaf, by definition, the restriction maps of ΔA are identity maps. And the sections are just the A . Specifically $\Delta A(U) = A$ for subsets U of X , and $\Delta A(\rightarrow) = 1_A$ for morphisms.

Write $\Gamma P = P(X)$ for the **global** sections functor which takes an element of \mathcal{P} to a \mathbf{Set} .

We are required to show a bijection:

For A in \mathbf{Set} and B in \mathcal{P}

$$\mathbf{Set}(A, \Gamma B) \rightarrow \mathcal{P}(\Delta A, B)$$

and

$$\mathcal{P}(\Delta A, B) \rightarrow \mathbf{Set}(A, \Gamma B)$$

The maps between the presheaf functors in \mathcal{P} are natural transformations. Natural transformations are a collection of maps $\alpha_A: \{\Delta A(A) \rightarrow B(A)\}_{A \in \mathcal{A}}$. For $U \subseteq V \subseteq X$ we have the commuting square:

$$\begin{array}{ccccc} \Delta A(X) & \xrightarrow{1_A} & \Delta A(V) & \xrightarrow{1_A} & \Delta A(U) \\ \downarrow \alpha_X & & \downarrow \alpha_V & & \downarrow \alpha_U \\ B(X) & \xrightarrow{B(f)} & B(V) & \xrightarrow{B(f)} & B(U) \end{array} \quad (1)$$

Recall $\Delta A(\cdot) = A$. Then the morphism in \mathbf{Set} is represented by α_X above. As visible from the figure above this corresponds one to one with each α_A in \mathcal{A} , so the bijection holds. Dually using the exact same reasoning Π , the left adjoint of Δ is the presheaf evaluation at the empty set, $\Pi(P) = P(\emptyset)$.

For the left adjoint to Π , Λ , and for A in \mathbf{Set} and B in \mathcal{P} , we need to show a bijection between:

$$\mathcal{P}(\Lambda A, B) \leftrightarrow \mathbf{Set}(A, \Pi(B))$$

To try and cobble together a definition of the presheaf functor Λ , start with the naturality diagram representing morphisms in \mathcal{P} :

$$\begin{array}{ccc} \Lambda(U) & \xrightarrow{A(f)} & \Lambda(\emptyset) \\ \downarrow \alpha_U & & \downarrow \alpha_{\emptyset} \\ B(U) & \xrightarrow{B(f)} & B(\emptyset) \end{array}$$

Note that $\Pi(B) = B(\emptyset)$. Start by choosing $\Lambda(\emptyset) = A$, so the morphism in \mathbf{Set} is α_{\emptyset} . Our choice of Λ needs to make this diagram commute for all U in $\mathcal{O}(X)^{op}$. For $U \neq \emptyset$ we could try $\Lambda(U) = A$, however to force the square above to commute with this choice, will impose some structure on the presheaf B . Rather, try setting $\Lambda(U) = \emptyset$ for $U \neq \emptyset$. Choosing the initial object \emptyset of $\mathcal{O}(X)^{op}$, means there is one map out of the top LHS of the square in the above diagram, and the square commutes as required.

We also have

$$\mathcal{P}(A, \nabla B) \leftrightarrow \mathbf{Set}(\Gamma A, B)$$

∇ , the right adjoint to Γ can be obtained dually, by swapping \mathbf{Set} with \mathbf{Set}^{op} and $\mathcal{O}(X)^{op}$ with $\mathcal{O}(X)$. This is simply a relabelling which has the effect of reversing the chain of adjoint functors stated in the question. We then apply analogous reasoning, take $\nabla(U) = \{*\}$, for $U \neq X$, and $\nabla(X) = B$.

2.2.11

The full subcategory where η_a is an isomorphism

2.2.12

(a) **Heuristic sort of proof** if the counit, $FG(f) \rightarrow f, B \in \mathcal{B}$ is isomorphic then a mapping back exists $f \rightarrow FG(f)$ such that their composition is the identity. So for a given B and B' and $FG(f)$ and f are one to one. Which necessarily means f and Gf are one to one, so G is full and faithful.

Algebraic proof From (2.2) the naturality axiom states:

$$\overline{(FG(B) \xrightarrow{\epsilon} B \xrightarrow{q} B')} = G(B) \xrightarrow{1_{G(B)}} G(B) \xrightarrow{G(q)} G(B')$$

ϵ injective implies faithful: $G(q_1) = G(q_2) \implies \epsilon q_1 = \epsilon q_2 \implies q_1 = q_2$

faithful implies ϵ injective: $\epsilon q_1 = \epsilon q_2 \implies G(q_1) = G(q_2) \implies q_1 = q_2$

ϵ is injective implies full: For a given $h = G(q)$, need to find $q : \mathcal{B} \rightarrow \mathcal{B}$ inducing h . We know from naturality equation above that $G(q) = \overline{q\epsilon}$. ϵ needs to be invertible to retrieve q and hence satisfy fullness requirement.

full implies ϵ is injective: Put $B' = FG(B)$ in the naturality condition above to give:

$$\overline{(FG(B) \xrightarrow{\epsilon} B \xrightarrow{\lambda} FG(B))} = G(B) \xrightarrow{1_{G(B)}} G(B) \xrightarrow{G(\lambda)} GFG(B)$$

Using fullness choose λ such that $G\lambda = \eta$. Then

$$\begin{aligned} \overline{1_{FG}(B)} &= \eta_G(B), \text{ therefore} \\ 1_{FG}(B) &= \overline{\eta_G}(B) = \lambda \epsilon_G(B) \end{aligned}$$

So ϵ has an inverse and is therefore injective.

2.2.13

(a) We have sets S, T , a function $f : S \rightarrow T$. $P(S)$ denotes the set of all subsets of S . The functor f^* takes elements of T to their inverse under f . Looking for left and right adjoints of f^* . We can immediately see the left adjoint of f^* is f from below.

$$P(S)(A, f^{-1}(B)) \cong P(T)(f(A), B) \quad (2)$$

Now to find the right adjoint of f^* , G below:

$$P(T)(A, G(B)) \cong P(S)(f^{-1}(A), B)$$

Dualising

$$P(T)^{op}(G(B), A) \cong P(S)^{op}(B, f^{-1}(A)) \quad (3)$$

Equation (3) is (2) up to an isomorphism. So we choose $G = f$ here. In fact the power set P is self adjoint. Loosely, this means we have an isomorphism between the opposite category and the original category. We use this isomorphism to get a representation of G in $P(T)$, with the right adjoint sending B to $f(\overline{B})$.

Because f is a bijection, elements in T that are not in $f(\overline{B})$ must have elements in B as their preimage. So $\overline{f(\overline{B})}$ consists of all sets of T where $f^{-1}(T) \subseteq B$. In summary the left adjoint of f^* is

$$F(S) = \{t \in T, \exists s \in S : s \in f^{-1}(t)\}$$

F represents choosing elements of T such that some element of S is in the inverse image of f .

and the right adjoint

$$G(S) = \{t \in T, \forall s \in S : s \in f^{-1}(t)\}$$

G represents choosing elements of T such that every element of S is in the inverse image of f .

(b) We are asked to interpret, in light of the results in (a.), the unit $\eta: 1_T \rightarrow G \circ F$, and counit $\epsilon: F \circ G \rightarrow 1_S$, for all adjunctions.

Consider the $R(x, y)$ as a set in $X \times Y$ and S as a set in X . In all of the below I interpret **set inclusion as logical implication**.

Description of the functors used follows.

- \forall_y takes a set $R(x, y)$ and returns $S(x)$ with preimage in $R(x, y)$ for all y . So each element in $R(x, y)$ inducing $S(x)$ is fully contained in $R(x, y)$.
- p^* the inverse image functor takes a set $S(x)$ and returns its preimage $R(x, y)$. This inclusion of X into $X \times Y$ adds a variable in Y . To use parlance of first order logic the statement is free in y .
- \exists_Y takes a set $R(x, y)$ and returns $S(x)$ with preimage in $R(x, y)$ for at least one y .

We know from (a) that $\exists_Y \dashv p^* \dashv \forall_Y$

$p^* \dashv \forall_Y$

- $\eta: 1_X \rightarrow \forall_Y \circ p^*$ Plug in as argument to both sides of the implication the set $S(x)$. Evaluating the RHS, applying p^* results in the product of $S(x)$ with Y , the set $R(x, y)$. So η can be interpreted as $S(x) \implies R(x, y) \forall y$.

- $\epsilon : p^* \circ \forall_Y \rightarrow 1_{X \times Y}$. (??) The universal functor is projection of a subset of x -values for the set $R(x, y)$ passed to the functor. Applying p^* to yield say $R^Y(x, y)$ makes the statement on the LHS free in y , so it requires assignment for the statement to be meaningful. If we were to assign y on the LHS for corresponding to the y value for each RHS predicate, elementwise, then we essentially just have the statement that $R^Y(x, y) \implies R(x, y)$.

$\exists_Y \dashv p^*$

- $\eta : 1_{X \times Y} \rightarrow p^* \circ \exists_Y$. (??) The existence functor is just projection of all the x values for a given set of (x, y) . The resulting statement on the right hand side is free in Y after applying p^* so requires assignment to be meaningful. If we were to assign y on the RHS for corresponding to the y value for each LHS predicate, elementwise, then we essentially just have an identity.
- $\epsilon : \exists_Y \circ p^* \rightarrow 1_X$ Plug in as argument to both sides of the implication the set $S(x)$. Applying p^* returns the set $R(x, y)$. So ϵ can be interpreted as $\exists y : R(x, y) \implies S(x)$.

2.2.14 Natural transformations for $[\mathcal{A}, \mathcal{B}]$.

$$\begin{array}{ccc} FA & \xrightarrow{F(f)} & FA' \\ \downarrow \alpha_A & & \downarrow \alpha_{A'} \\ GA & \xrightarrow{G(f)} & GA' \end{array}$$

$Y \in [\mathcal{B}, \mathcal{I}], F^*(Y) = Y \circ F$. Natural transformations for $[[\mathcal{A}, \mathcal{I}], [\mathcal{B}, \mathcal{I}]]$.

$$\begin{array}{ccc} F^*Y = Y \circ F & \xrightarrow{F^*(f)} & F^*Y' = Y' \circ F \\ \downarrow \alpha_Y^* & & \downarrow \alpha_{Y'}^* \\ G^*Y = Y \circ G & \xrightarrow{G^*(f)} & G^*Y' = Y' \circ G \end{array}$$

So it is evident from comparing the above natural transformation diagrams that we have the relationship $\alpha_Y^* = Y \circ \alpha$

Using first triangle inequality starting at point $F^*(Y)$ we have

$$\begin{aligned} \epsilon_{F^*(Y)}^* F^*(\eta_Y^*) F^*(Y) &= F^*(Y)(\epsilon F) F^*(\eta_Y^*) F^*(Y) \\ &= Y F(\epsilon F)(\eta F) \\ &= Y F 1_F \text{ since } \epsilon F \circ F \eta = 1_F \\ &= F^*(Y) \end{aligned}$$

The other triangle follows similarly. So by Theorem 2.2.5 we have an adjunction between F^* and G^* .