Leinster - Basic Category Theory - Selected problem solutions for Chapter 3

Adam Barber

December 22, 2021

3.1.1

There are bijections

$$\begin{aligned} (A+B, C) &\leftrightarrow ((A,B), \Delta C) \\ f &\leftrightarrow \overline{f} \end{aligned}$$

where $\overline{f} = (f, f)$

$$\begin{aligned} (\Delta A, (B,C)) &\leftrightarrow (A, B \times C) \\ g = (p,q) &\leftrightarrow \overline{g} \end{aligned}$$

where $\overline{g}(x) = (p(x), q(x))$

So the sum is left adjoint to Δ, and the product is its right adjoint.

3.1.2

We are given the definition of a sequence, where there is a unique function x such that the square below commutes.

We have $x_0 = a$, and $x_{n+1} = r(x_n)$.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "s"] \arrow[d, "x"] & \mathbb{N} \arrow[d, "x"] \\ X \arrow[r, "r"] & X \end{tikzcd}$$

This is precisely the definition of the comma category $(\mathbb{N} \Rightarrow X)$, where objects are $(n \in \mathbb{N}, x, t \in X)$.

3.2.12

(a)

$$\theta(S) = \bigcup \theta(R) \supseteq \bigcup R = S$$

But $\theta^2(S) = \theta(S)$, so $\theta(S) \subseteq S$.

Taken together, the above implies $\theta(S) = S$.

(b)

$$\begin{aligned} & A \subseteq B \\ \implies & fA \subseteq fB \\ \implies & gfA \subseteq gfB \end{aligned}$$

g and f are taken to be injections here.

So gf is order preserving. Which means by (a) there is some $gf(S) = S$, $S \in A$. This implies $gf(A \setminus S) = A \setminus S$. Since g is an injection $f(A \setminus S) = B \setminus fS$, giving $g(B \setminus fS) = A \setminus S$. $(*)$.

(c) So by the injectivity of f and g we can vary A and B, leading to the isomorphism below

$$\begin{aligned} (g \circ f)(A \setminus S) &= A \setminus S, \text{ and} \\ (f \circ g)(B \setminus fS) &= B \setminus fS \end{aligned}$$

3.2.14

Need to prove that for any family $(A_i)_{i`\in I}$ of objects of $\mathcal{A}$, there is some object of $\mathcal{A}$ not isomorphic to A_i for $i \in I$. It suffices to prove for A in $F(S)$, $F : \mathbf{Set} \to \mathcal{A}$, then we know the condition holds for $\mathcal{A}$. Now UF is injective by Exercise 2.3.11, so U is injective on objects A of $F(S)$. So if UA_i is not isomorphic to UA_j, this would imply A_i is not isomorphic to A_j. So we need to prove for a given i, $|UA_i| < |\mathcal{P}(UA)|$:

$$|UA_i| \leq |\Sigma UA_i| < |\mathcal{P}(UA)|$$

The strict equality due to Theorem 3.2.2.