Leinster - Basic Category Theory - Selected problem solutions for Chapter 3

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4.1.27

Take the naturality square of H_A, and construct with A, A' in the following way. For every map $f : A \to A'$, the following square commutes.

$$\begin{tikzcd} H_A(A) \arrow[r, "H_A(f)=-\circ f"] \arrow[d, "\alpha_A=f\circ-"] & H_A(A') \arrow[d, "\alpha_{A'}=f\circ-"] \\ H_{A'}(A) \arrow[r, "H_{A'}(f)=-\circ f"] & H_{A'}(A') \end{tikzcd}$$

Moreover, since $H_A \cong H_{A'}$, α_A is an isomorphism for every A in the codomain of f. Consider the left hand side component α_A, which represents a map of $\mathbf{1} \to H_{A'}(A)$. Now a map of $\mathbf{1}$ into $\mathcal{A}$ is the same thing as an object of $\mathcal{A}$, so we take $\alpha_A = \mathcal{A}(A, A')$, that is α_A represents an isomorphism in $\mathcal{A}$, and $A \cong A'$.

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Here we construct a bijection between the set $U_p(G)$ and a group homomorphism ϕ.

$$\begin{tikzcd} U_p(G) \arrow[r, "h"] \arrow[d, "\alpha_G"] & U_p(H) \arrow[d, "\alpha_H"] \\ \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G) \arrow[r, "h"] & \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H) \end{tikzcd}$$

$U_p(G)$ is the set of $\{g \in G \colon g^p = 1\}$.

For the present question, take an arbitrary g in G. Set $\phi(1) = g$. By the properties of a homormorphism we shall see this maps the additive group $\mathbb{Z}/p\mathbb{Z}$ to $U_p(G)$. ϕ preserves the identity so $\phi(0) = 1$. Since $\phi(1+1) = g^2$, generally $\phi(n) = g^n$. So $\phi(p) = g^p = \phi(0) = 1$. So ϕ maps to a group with order p, or simply order 1 if g is the element of the trivial group. So $\mathbb{Z}/p\mathbb{Z}$ sees groups of order p or 1. This result means we have the required bijection, α and α^{-1} in the diagram above. Observing the diagram we just need to specify how morphisms in $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ work. They are simply group homomorphisms h, that take $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G)$ to $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H)$. So referring to the diagram, naturality holds, and we can conclude $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ and U_p are naturally isomorphic. [1]

[1] A well known result is as follows. For a group homomorphism $\psi : G_1 \to G_2$, let $g \in G_1$ be of finite order n. Then $\psi(g)$ divides the order of g. Because $g^n = e_1$ implies $\psi(g)^n = \psi(g^n) = \psi(e_1) = e_2$. So if p is prime then the resulting homomorphism maps to a group of order p or 1.