

Leinster - Basic Category Theory - Selected problem solutions for Chapter 6

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6.2.20

(a)

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ \downarrow 1 & & \downarrow \alpha \\ X & \xrightarrow{\alpha} & Y \end{array} \quad (1)$$

where X, Y are functors in the category $[A, \mathcal{S}]$

By Lemma 5.1.32 α is monic in $[A, \mathcal{S}]$ if and only if the above square is a pullback. Using Theorem 6.2.5 there is a pullback whose image under the evaluation functor ev_A is a pullback for each $A \in \mathbf{A}$ in \mathcal{S} . So by the lemma α_A is monic for all $A \in \mathcal{A}$. The other direction holds by virtue of the same theorem, that there is only one way to extend the pullback on \mathcal{S} for each A , to a pullback on α_A to a pullback on $[A, \mathcal{S}]$

(b) Monics in $[\mathbf{A}^{\text{op}}, \mathbf{Set}]$ are epics in $[\mathbf{A}, \mathbf{Set}]$ and vice versa.

6.2.21

(a) Use a cardinality argument as follows. There is only one identity map represented by the left hand side of the following expression. $H_A(A) \cong X(A) + Y(A)$, for all A in \mathcal{A} . Which means that either $X(A)$ or $Y(A)$ must be the empty set for all A in \mathcal{A} .

6.2.22

The category of elements can be represented by $(1 \rightarrow X)$, where 1 is a single element set. The comma category commuting diagram becomes

$$\begin{array}{ccc}
1 & \longrightarrow & x \\
& \searrow & \downarrow Xf \\
& & x'
\end{array} \tag{2}$$

where $x \in X(A)$, and $x' \in X(A')$, and $f : A' \rightarrow A$. The above diagram shows under our choice of comma category that $Xf(x) = x'$ as required.

6.2.23

A category of elements with a terminal object by definition is equivalent to the definition of a representation as a universal element in (4.6).

6.2.24

Let E be a functor in the functor category $[\mathbf{A}^{op}, \mathbf{Set}]$ and $E \rightarrow X$ be an object of the slice category, where X is a presheaf on \mathbf{A} . We need an equivalence functor to map $E \rightarrow X$ to some $[\mathbf{B}^{op}, \mathbf{Set}]$. For a given A , and consider $\alpha_A : E(A) \rightarrow X(A)$. For a $x \in X(A)$ back out the definition of E with

$$\beta_A(x) = \{e : \alpha_A(e) = x\} \tag{3}$$

where $e \in E(A), x \in X(A)$.

So now we have (A, x) pairs as in the definition of the category of elements in Definition 6.2.16, and can construct a functor using them informally as $(A, x) \rightarrow \beta_A(x)$. However we do need to show that $\beta_A(x)$ and $\beta_{A'}(x)$ induce an f such that $(Xf)(x') = x$. Because the morphism of E to X is a natural transformation we know that with $f : A \rightarrow A'$, that $\alpha_A(e) = (Xf)(\alpha_{A'}(e'))$ taken with (3) means $(Xf)(x') = x$ as required.

In the other direction, if E is a presheaf on the category of elements,

$$E(a) = \bigsqcup_{x \in X(a)} E(a, x) \tag{4}$$

and for $e \in E(a), x \in X(a)$ define $f(e) = x$.

Source of ideas for this proof.¹

6.2.25

(a) i. Functoriality of $\text{Lan}_F X$

¹<https://math.stackexchange.com/questions/3633646/every-slice-of-a-presheaf-category-is-again-a-presheaf-category>

Let the diagram given for our colimit be $D_B := X$, with $(A, FA \rightarrow B)$ in $(F \Rightarrow B)$. To prove $\text{Lan}_F X$ is a functor we need to consider $\text{Lan}_F X$ on morphisms $f: B \rightarrow B'$, and $f': B' \rightarrow B''$. f and f' induce the maps presented below:

$$\begin{array}{ccc}
D_B & \xrightarrow{p_I} & \text{Lan}_F(B) \\
\downarrow D(f) & & \downarrow L(f) \\
D_{B'} & \xrightarrow{p_{I'}} & \text{Lan}_F(B') \\
\downarrow D(f') & & \downarrow L(f') \\
D_{B''} & \xrightarrow{p_{I''}} & \text{Lan}_F(B'')
\end{array} \tag{5}$$

The map from $\text{Lan}_F B \rightarrow \text{Lan}_F B''$ is a unique map by the colimit property of $\text{Lan}_F B$ and hence $L(f')(f) = L(f'f)$ as required.

(a) ii. Bijection between $\text{Lan}_F X \rightarrow Y$ and $X \rightarrow YF$.

Consider the cocone $(X(A) \xrightarrow{p_I} \text{Lan}_F X(B))_{\{FA \rightarrow B\}}$, for all $(A, FA \rightarrow B)$ in $(F \Rightarrow B)$. To form the bijection required, make the canonical choice of $1_F(A)$ in $(F \Rightarrow F(A))$ and reevaluate p_I above which now becomes $X(A) \rightarrow \text{Lan}_F X F(A)$. With this choice there is a single base of the cocone $X(A)$ for every $\text{Lan}_F X F(A)$ so we can form the required bijection between $\text{Lan}_F X \rightarrow Y$ and $X \rightarrow YF$. The task remaining is to prove naturality between $\text{Lan}_F X$ and X .