Leinster - Basic Category Theory - Selected problem solutions for Chapter 2

Adam Barber

October 29, 2021

2.1.16

(a) Interesting adjoint functors to G-sets.

The trivial group functor I sends a set to a **G**-set with the trivial action $gx = x$. Interesting functors

Orbit functor sends a G-set with underlying set elements a of A to:

$A_G = \{g \cdot a, g \in G\}$

Fixed point functor sends a G-set with underlying set elements a of A to:

$A^G = \{a \text{ such that } g \cdot a = a \text{ for all } g \in G, a \in A\}$

Fixed point functor - right adjoint Morphisms in a G-set are functions on the underlying set, where f commutes with g for every $g \in G$.

There is a bijection for each $A \in \mathbf{Set}$ and $B \in [G, \mathbf{Set}]$ as follows

$$[G, \mathbf{Set}](I(A), B) \to \mathbf{Set}(A, B^G)$$
$$\psi \mapsto \overline{\psi}$$

$\overline{\psi}$ sends each element a of A to $\psi(a)$ if $g \cdot a = a$, otherwise it sends a to $\psi(\emptyset)$.

$$\mathbf{Set}(A, B^G) \to [G, \mathbf{Set}](I(A), B)$$
$$\phi \mapsto \overline{\phi}$$

ϕ sends each $a \in A$ in the underlying set of the G-set to the G-set $(g, \overline{\phi}(a)), g \in G$.

Orbit functor - left adjoint There is a bijection for each $A \in [G, \mathbf{Set}]$ and $B \in \mathbf{Set}$ as follows

$$\mathbf{Set}(A_G, B) \to [G, \mathbf{Set}](A, I(B))$$
$$\psi \mapsto \overline{\psi}$$

So each morphism in **Set** sends the set formed by the orbits of an element a of A, call this a_G, to $\psi(a_G)$, where ψ is a function of sets. Choose a G-set morphism $\overline{\psi} = \psi$, where $\overline{\psi}$ commutes with g for every g in G.

$$[G, \mathbf{Set}](A, I(B)) \to \mathbf{Set}(A_G, B)$$
$$\phi \mapsto \overline{\phi}$$

Choose $\overline{\phi}$ to be a disjoint union of each orbit of a in A, $\overline{\phi}(a) = \amalg\{\phi(g \cdot a), g \in G\}$

2.1.17

Write $\mathcal{O}(X)$ for the poset of open subsets of a topological space X ordered by inclusion.

$\Delta : \mathbf{Set} \to [\mathcal{O}(X)^{op}, \mathbf{Set}]$

Write $\mathcal{P}$ for the presheaf functor category, and $P \in \mathcal{P}$ for the functor which maps $\mathcal{O}(X)^{op}$ to **Set**. Take open sets U, V, such that $U \subseteq V$ in X. A presheaf consists of

- restriction maps, $P(V) \to P(U)$, these are morphisms which enforce some sort of ordering of the mapped sets,
- and the actual mapped sets $P(U), P(V)$ which are called sections.

Since the question specifies a constant presheaf, by definition, the restriction maps of ΔA are identity maps. And the sections are just the A. Specifically $\Delta A(U) = A$ for subsets U of X, and $\Delta A(\to) = 1_A$ for morphisms.

Write $\Gamma P = P(X)$ for the **global** sections functor which takes an element of $\mathcal{P}$ to a **Set**.

Intuition Defining the sections functor to map the entire space in this way means in **Set** we have a map into every section. You can think of the presheaf as loosely a blueprint on how to sort the sections. So if we provide a sorting of some of these sections — a presheaf, and a mapping of these sections, then we can get back a new presheaf. The 'new' presheaf is of course a blueprint on how to sort the mapped sections.

For A in **Set** and B in $\mathcal{P}$

$$\mathbf{Set}(A, \Gamma B) \to \mathcal{P}(\Delta A, B)$$
$$\phi \mapsto \Delta A^{\phi}$$

Where, continuing with our intuition of sorting, we simply 'resort' using composition of ϕ on the functor morphisms and objects as follows.

$$\Delta A^{\phi} = \begin{cases} \phi \cdot U & \text{on objects } U \\ \phi \cdot 1_A & \text{on morphisms} \end{cases}$$