Leitner - Basic Category Theory - Problem solutions

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0.10

Let S be a set. The indiscrete topological space $I(S)$ is the space whose set of points is S and whose only open subsets are $\emptyset$ and S. To find a universal property satisfied by the space $I(S)$ proceed as follows. With this topology any map from a topological space to S is continuous.

Parroting the wording of the question, let us rephrase this in universal parlance. Define a function $i\colon S \to I(S)$, by $i(s) = s, s \in S$. Then $I(S)$ has the following property.

$$\begin{tikzcd}
S \arrow[r, "i"] & I(S) \\
& X \arrow[ul, dashed, "\forall f"] \arrow[u, "\bar{f}"']
\end{tikzcd}$$

For all topological spaces X and all functions $f\colon X \to S$ there exists a unique continuous map $\bar{f}\colon X \to I(S)$. What it says is all maps into an indiscrete space are continuous. It also says that given S, the universal property determines $I(S)$ and i, up to isomorphism.

0.11

The universal property that is satisfied by the pair $(ker(\theta), \iota)$ is depicted in the diagram below.

$$\begin{tikzcd}
ker(\theta) \arrow[r, "\iota"] & G \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\epsilon"'] & H \\
F \arrow[u, dashed, "\exists! \bar{f}"] \arrow[ur, "\forall f"']
\end{tikzcd}$$

The statement of the universal property is as follows. For any $f\colon F \to G$ such that $\theta \circ f = \epsilon \circ f$, there is a unique $\bar{f}\colon F \to ker(\theta)$ such that the diagram above commutes. That is $f = \iota \circ \bar{f}$.

0.13

(a) Choose $\phi(\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n a_i r^i$. Then ϕ with $\phi(x) = r$ is a homomorphism that satisfies additive and multiplicative properties. To prove uniqueness assume there is another homomorphism ψ, with $\psi(x) = r$. Then $\psi(\sum_{i=1}^n a_i x^i) = \sum_{i=1} a_i \psi(x) = \sum_{i=1} a_i r^i$ by properties of a homomorphism. So $\psi = \phi$.

(b) $\iota\colon \mathbb{Z}[x] \to A$ maps $\sum_{i=1}^n p_i x^i$ to $\sum_{i=1}^n p_i a^i$, using $\iota(x) = a$, the multiplicative property of a homomorphism to get $\iota(x^i) = \iota(x)^i$, and the additive property to get $\iota(p_i)\iota(x)^i = p_i \iota(x)^i$ remembering p_i is in $\mathbb{Z}$.

Going in the direction $A \to \mathbb{Z}[x]$ we know as provided in (b) that, taking $R = \mathbb{Z}[x]$, and $\phi = \iota'$, there exists a unique ring homomorphism such that $\iota'(a) = x$. So ι' maps $\sum_{i=1}^n p_i a^i$ to $\sum_{i=1}^n p_i x^i$ and $\iota' \circ \iota = 1_{\mathbb{Z}[x]}$. Also using definitions of ι and ι' easily yields $\iota \circ \iota' = 1_A$.

0.14

(a) For the triangles below to commute, we need, as stated in the question $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, dashed, "\forall f_1"'] & P \arrow[d, "\exists p_1"] \\ & X \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, dashed, "\forall f_2"'] & P \arrow[d, "\exists p_2"] \\ & Y \end{tikzcd}$$

Choosing $P = X \times Y$, p_1 and p_2 as below makes the triangles commute.

$$p_1\colon X \times Y \to X$$
$$p_2\colon X \times Y \to Y$$

(b) Proving uniqueness involves taking two arbitrary cones with the property stated in (a). Taking (P, p_1, p_2) and (P', p_1', p_2') we know from (a) that for all cones (V, f_1, f_2) there exists a unique linear map $f\colon V \to P'$ such that $p_1' \circ f = f_1$, $p_2' \circ f = f_2$. In this statement choose $V = P'$, then referring to the triangles in (a), observe there exists a $f\colon P \to P'$ such that $p_1' \circ f = p_1$, $p_2' \circ f = p_2$.

Comment The choice of P and p notation hinted very heavily that this is a projection of a product.

(c) We need to define the cocone (Q, q_1, q_2) with the property, for all cocones (V, f_1, f_2) there exists a unique linear map $f\colon Q \to V$ such that $f \circ q_1 = f_1$ and

$f \circ q_2 = f_2$. Choose $Q = X \times Y$, $q_1 \colon X \to X \oplus Y$, $q_2 \colon Y \to X \oplus Y$.

Comment This is the dual of the product in (b), the coproduct. Set equivalent is the disjoint union.

1.2.25

(a) Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a functor. We are given that for each A in $\mathcal{A}$ there is a morphism $F^A : \mathcal{B} \to \mathcal{C}$ defined on objects B in $\mathcal{B}$ by $F^A = F(A, B)$ and on maps g in $\mathcal{B}$ by $F_A(g) = F(1_A, g)$. We need to prove F^A is a functor.

First, we need to show $F^A(g \circ \bar{g}) = F^A(g) \circ F^A(\bar{g})$.

$$\begin{aligned} F^A(g \circ \bar{g}) &= F(1_A, g \circ \bar{g}) \\ &= F(1_A, g) \circ F(1_A, \bar{g}) \\ &= F^A(g) \circ F^A(\bar{g}) \end{aligned}$$

The second step above uses our formula from composition of a product category derived Ex 1.1.14.

We also need

$$F^A(1_B) = F(1_A, 1_B) = 1_C$$

The identity maps because F is a functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$. So F^A is a functor.

Apply analogous reasoning for F_B.

(b) We are given $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor. The question asks us to show for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$

$$F^A(B) = F_B(A) \tag{1}$$

and if $f : A \to A'$ in $\mathcal{A}$ and $g : B \to B'$ in $\mathcal{B}$ then

$$F^{A'}(g) \circ F_B(f) = F_{B'}(f) \circ F^A(g).$$

In the following answers recall that:

$$\begin{aligned} F^A(g) &= F(1_A, g) \\ F_B(f) &= F(1_B, f) \end{aligned}$$

Equation (1) is verified by basic checking. Consider the second equation above along with the diagram below.

$$\begin{array}{ccc} (A,B) & \xrightarrow{F_B(f)} & (A',B) \\ \downarrow{\scriptstyle F^A(g)} & & \downarrow{\scriptstyle F^{A'}(g)} \\ (A,B') & \xrightarrow{F_{B'}(f)} & (A',B') \end{array}$$

We know from Exercise 1.1.14 that in the product category represented by $\mathcal{A}\times\mathcal{B}$, maps compose in the following manner

$$(f,g)\circ(f',g') = (ff',gg')$$

We also know from the axioms of our functor F that different strings of maps under F from $F(A,B)$ to $F(A',B')$ are equal.[1] So the above square commutes and we have the required equality

$$F^{A'}(g)\circ F_B(f) = F_{B'}(f)\circ F^A(g)$$

(c) We need to prove there is a unique functor F, satisfying the conditions in (a.). Take families of functors F^A and F_B as in (b), which satisfy the below

- If $f\colon A\to A'$ in $\mathcal{A}, g\colon B\to B'$ in $\mathcal{B}$, then $F^{A'}(g)\circ F_B(f) = F_{B'}(f)\circ F^A(g)$
- $F^A(B) = F_B(A)$ if $A\in\mathcal{A}, B\in\mathcal{B}$,

To begin, write

$$\begin{aligned} F &= F^A(g)\circ F_B(f) \text{ for morphisms,} \\ F &= F^A(B) \text{ for objects.} \\ &= F_B(A) \\ &= F(A,B) \end{aligned}$$

We need to prove that F is a functor. We are given in this question that F_A,

[1]See Remarks 1.2.2 of the Leitner text

$A \in \mathcal{A}$ and F_B, $B \in \mathcal{B}$ are functors.

$$\begin{aligned} F(f \circ \bar{f}, g \circ \bar{g}) &= F_A(g \circ \bar{g}) \circ F_B(f \circ \bar{f}) \\ &= F_A(g) \circ F_A(\bar{g}) \circ F_B(f) \circ F_B(\bar{f}) \\ &= F_A(g) \circ F_{B'}(f) \circ F_A(\bar{g}) \circ F_B(f) \text{ using result from (b.)} \\ &= F(f, g) \circ F(\bar{f}, \bar{g}) \end{aligned}$$

Also,

$$\begin{aligned} F(1_A, 1_B) &= F_A(1_A) \circ F_B(1_B) \\ &= 1_{F_A(A)} \circ 1_{F_B(B)} \end{aligned}$$

So functions compose under the functor F, the identity maps, and all objects are mapped. So we have established F exists and is a functor. We still need to determine uniqueness. Given the property in (a)

$$\begin{aligned} F^A(B) &= F_B(A) \\ &= F(A, B) \end{aligned}$$

Fixing an $A \in \mathcal{A}, B \in \mathcal{B}$ our functor maps the object (A, B) to $F(A, B)$. So for each object mapping there is a unique F^A and F_B that interlock to produce $F(A, B)$. Put another way, fix an object in $\mathcal{C}, F(A, B)$. Then out of our two families of functors $(F^A)_{A \in \mathcal{A}}$, $(F_B)_{B \in \mathcal{B})}$ there is only one choice in each family to yield the desired object $F(A, B)$.