

Leinster - Basic Category Theory - Selected problem solutions for Chapter 2

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2.1.16

(a) Interesting adjoint functors to G -sets.

The trivial group functor I sends a set to a \mathbf{G} -set with the trivial action $gx = x$.
Interesting functors

Orbit functor sends a G -set with underlying set elements a of A to:

$$A_G = \{g \cdot a, g \in G\}$$

Fixed point functor sends a G -set with underlying set elements a of A to:

$$A^G = \{a \text{ such that } g \cdot a = a \text{ for all } g \in G, a \in A\}$$

Fixed point functor - right adjoint Morphisms in a G -set are functions on the underlying set, where f commutes with g for every $g \in G$.

There is a bijection for each $A \in \mathbf{Set}$ and $B \in [G, \mathbf{Set}]$ as follows

$$\begin{aligned} [G, \mathbf{Set}](I(A), B) &\rightarrow \mathbf{Set}(A, B^G) \\ \psi &\mapsto \bar{\psi} \end{aligned}$$

$\bar{\psi}$ sends each element a of A to $\psi(a)$ if $g \cdot a = a$, otherwise it sends a to $\psi(\emptyset)$.

$$\begin{aligned} \mathbf{Set}(A, B^G) &\rightarrow [G, \mathbf{Set}](I(A), B) \\ \phi &\mapsto \bar{\phi} \end{aligned}$$

ϕ sends each $a \in A$ in the underlying set of the G -set to the G -set $(g, \bar{\phi}(a)), g \in G$.

Orbit functor - left adjoint There is a bijection for each $A \in [G, \mathbf{Set}]$ and $B \in \mathbf{Set}$ as follows

$$\begin{aligned} \mathbf{Set}(A_G, B) &\rightarrow [G, \mathbf{Set}](A, I(B)) \\ \psi &\mapsto \bar{\psi} \end{aligned}$$

So each morphism in \mathbf{Set} sends the set formed by the orbits of an element a of A , call this a_G , to $\psi(a_G)$, where ψ is a function of sets. Choose a G -set morphism $\bar{\psi} = \psi$, where $\bar{\psi}$ commutes with g for every g in G .

$$\begin{aligned} [G, \mathbf{Set}](A, I(B)) &\rightarrow \mathbf{Set}(A_G, B) \\ \phi &\mapsto \bar{\phi} \end{aligned}$$

Choose $\bar{\phi}$ to be a disjoint union of each orbit of a in A , $\bar{\phi}(a) = \coprod \{\phi(g \cdot a), g \in G\}$

2.1.17

Write $\mathcal{O}(X)$ for the poset of open subsets of a topological space X ordered by inclusion.

$$\Delta : \mathbf{Set} \rightarrow [\mathcal{O}(X)^{op}, \mathbf{Set}]$$

Write \mathcal{P} for the presheaf functor category, and $P \in \mathcal{P}$ for the functor which maps $\mathcal{O}(X)^{op}$ to \mathbf{Set} . Take open sets U, V , such that $U \subseteq V$ in X . A presheaf consists of

- restriction maps, $P(V) \rightarrow P(U)$, these are morphisms which enforce some sort of ordering of the mapped sets,
- and the actual mapped sets $P(U), P(V)$ which are called sections.

Since the question specifies a constant presheaf, by definition, the restriction maps of ΔA are identity maps. And the sections are just the A . Specifically $\Delta A(U) = A$ for subsets U of X , and $\Delta A(\rightarrow) = 1_A$ for morphisms.

Write $\Gamma P = P(X)$ for the **global** sections functor which takes an element of \mathcal{P} to a \mathbf{Set} .

We are required to show a bijection:

For A in \mathbf{Set} and B in \mathcal{P}

$$\mathbf{Set}(A, \Gamma B) \rightarrow \mathcal{P}(\Delta A, B)$$

and

$$\mathcal{P}(\Delta A, B) \rightarrow \mathbf{Set}(A, \Gamma B)$$

The maps between the presheaf functors in \mathcal{P} are natural transformations. Natural transformations are a collection of maps $\alpha_A: \{\Delta A(A) \rightarrow B(A)\}_{A \in \mathcal{A}}$. For $U \subseteq V \subseteq X$ we have the commuting square:

$$\begin{array}{ccccc} \Delta A(X) & \xrightarrow{1_A} & \Delta A(V) & \xrightarrow{1_A} & \Delta A(U) \\ \downarrow \alpha_X & & \downarrow \alpha_V & & \downarrow \alpha_U \\ B(X) & \xrightarrow{B(f)} & B(V) & \xrightarrow{B(f)} & B(U) \end{array} \quad (1)$$

Recall $\Delta A(\cdot) = A$. Then the morphism in \mathbf{Set} is represented by α_X above. As visible from the figure above this corresponds one to one with each α_A in \mathcal{A} , so the bijection holds. Dually using the exact same reasoning Π , the left adjoint of Δ is the presheaf evaluation at the empty set, $\Pi(P) = P(\emptyset)$.

For the left adjoint to Π , Λ , and for A in \mathbf{Set} and B in \mathcal{P} , we need to show a bijection between:

$$\mathcal{P}(\Lambda A, B) \leftrightarrow \mathbf{Set}(A, \Pi(B))$$

To try and cobble together a definition of the presheaf functor Λ , start with the naturality diagram representing morphisms in \mathcal{P} :

$$\begin{array}{ccc} \Lambda(U) & \xrightarrow{A(f)} & \Lambda(\emptyset) \\ \downarrow \alpha_U & & \downarrow \alpha_{\emptyset} \\ B(U) & \xrightarrow{B(f)} & B(\emptyset) \end{array}$$

Note that $\Pi(B) = B(\emptyset)$. Start by choosing $\Lambda(\emptyset) = A$, so the morphism in \mathbf{Set} is α_{\emptyset} . Our choice of Λ needs to make this diagram commute for all U in $\mathcal{O}(X)^{op}$. For $U \neq \emptyset$ we could try $\Lambda(U) = A$, however to force the square above to commute with this choice, will impose some structure on the presheaf B . Rather, try setting $\Lambda(U) = \emptyset$ for $U \neq \emptyset$. Choosing the initial object \emptyset of $\mathcal{O}(X)^{op}$, means there is one map out of the top LHS of the square in the above diagram, and the square commutes as required.

We also have

$$\mathcal{P}(A, \nabla B) \leftrightarrow \mathbf{Set}(\Gamma A, B)$$

∇ , the right adjoint to Γ can be obtained dually, by swapping \mathbf{Set} with \mathbf{Set}^{op} and $\mathcal{O}(X)^{op}$ with $\mathcal{O}(X)$. This is simply a relabelling which has the effect of reversing the chain of adjoint functors stated in the question. We then apply analogous reasoning, take $\nabla(U) = \{*\}$, for $U \neq X$, and $\nabla(X) = B$.

2.2.11

The full subcategory where η_a is an isomorphism

2.2.13

(a) We have sets S, T , a function $f: S \rightarrow T$. $P(S)$ denotes the set of all subsets of S . The functor f^* takes elements of T to their inverse under f . Looking for left and right adjoints of f^* . We can immediately see the left adjoint of f^* is f from below.

$$P(S)(A, f^{-1}(B)) \cong P(T)(f(A), B) \quad (2)$$

Now to find the right adjoint of f^* , G below:

$$P(T)(A, G(B)) \cong P(S)(f^{-1}(A), B)$$

Dualising

$$P(T)^{op}(G(B), A) \cong P(S)^{op}(B, f^{-1}(A)) \quad (3)$$

Equation (3) is (2) up to an isomorphism. So we choose $G = f$ here. In fact the power set P is self adjoint. Loosely, this means we have an isomorphism between the opposite category and the original category. We use this isomorphism to get a representation of G in $P(T)$, with the right adjoint sending B to $\overline{f(B)}$.

Because f is a bijection, elements in T that are not in $\overline{f(B)}$ must have elements in B as their preimage. So $\overline{f(B)}$ consists of all sets of T where $f^{-1}(T) \subseteq B$. This could indeed be empty. In summary the left adjoint of f^* is

$$\{f(s), s \in S\}$$

and the right adjoint

$$\{t \in T : f^{-1}(t) \subseteq S\}$$

(b) We are asked to interpret, in light of the results in (a.), the unit $\eta: 1_T \rightarrow G \circ F$, and counit $\epsilon: F \circ G \rightarrow 1_S$.

Right adjoint

- η takes as argument the predicate $S(x)$. Applying F the inverse image functor returns $R(x, y)$ implying $S(x)$. Applying G to the result of F in the previous step, returns $S'(x)$ implied by $R(x, y)$ for all y .
- ϵ takes as argument the predicate $R(x, y)$. Applying G returns $S(x)$ implied by $R(x, y)$ for all y . Applying F the inverse image functor in turn returns the predicate $R'(x, y)$.

Considered as sets rather than predicates:

- G takes a set $R(x, y)$ and returns $S(x)$ with preimage in $R(x, y)$ for all y .
- F^* takes a set $S(x)$ and returns the $R(x, y)$ implying $S(x)$
- F takes a set $R(x, y)$ and returns $S(x)$ with preimage in $R(x, y)$ for at least one y .

Left adjoint

- η takes as argument the predicate $R(x, y)$. Applying F returns $S(x)$ implied by a $R(x, y)$ for some y . Applying G the inverse image functor returns the predicate $R'(x, y)$ implying $S(x)$ in the previous step.
- ϵ takes as argument the predicate $S(x)$. Applying G the inverse image functor returns the predicate $R(x, y)$ implying $S(x)$. Applying F in turn returns $S'(x)$ implied by $R(x, y)$ for some y .