Leitner - Basic Category Theory - Problem solutions Adam Barber
0.10:

Let S be a set. The indiscrete topological space $I(S)$ is the space whose set of points is S and whose only open subsets are $\emptyset$ and S. To find a universal property satisfied by the space $I(S)$ proceed as follows. With this topology any map from a topological space to S is continuous.

Parroting the wording of the question, let us rephrase this in universal parlance. Define a function $i : S \to I(S)$, by $i(s) = s, s \in S$. Then $I(S)$ has the following property.

$$\begin{tikzcd} S \arrow[r, "i"] & I(S) \\ & X \arrow[ul, dashed, "\forall f"] \arrow[u, "\overline{f}"'] \end{tikzcd}$$

For all topological spaces X and all functions $f : X \to S$ there exists a unique continuous map $\overline{f} : X \to I(S)$. What it says is all maps into an indiscrete space are continuous. It also says that given S, the universal property determines $I(S)$ and i, up to isomorphism.

0.11

The universal property that is satisfied by the pair $(ker(\theta), \iota)$ is depicted in the diagram below.

$$\begin{tikzcd} ker(\theta) \arrow[r, "\iota"] & G \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\epsilon"'] & H \\ F \arrow[u, dashed, "\exists! \overline{f}"] \arrow[ur, "\forall f"'] & & \end{tikzcd}$$

The statement of the universal property is as follows. For any $f : F \to G$ such that $\theta \circ f = \epsilon \circ f$ there is a unique $\overline{f} : F \to ker(\theta)$ such that the diagram above commutes. That is $f = \iota \circ \overline{f}$

0.13:

(a)

Choose $\phi(\sum_{i=1}^{n} a_i x^i) = \sum_{i=1}^{n} a_i r^i$. Then ϕ with $\phi(x) = r$ is a homomorphism that satisfies additive and multiplicative properties. To prove uniqueness assume there is another homomorphism ψ, with $\psi(x) = r$. Then $\psi(\sum_{i=1}^{n} a_i x^i) = \sum_{i=1} a_i \psi(x) = \sum_{i=1} a_i r^i$ by properties of a homomorphism. So $\psi = \phi$.

(b)

ι: $\mathbb{Z}[x] \to A$ maps $\sum_{i=1}^{n} p_i x^i$ to $\sum_{i=1}^{n} p_i a^i$, using $\iota(x) = a$, the multiplicative property of a homomorphism to get $\iota(x^i) = \iota(x)^i$, and the additive property to get $\iota(p_i)\iota(x)^i = p_i \iota(x)^i$ remembering p_i is in $\mathbb{Z}$.

Going in the direction $A \to \mathbb{Z}[x]$ we know as provided in (b) that, taking $R = \mathbb{Z}[x]$, and $\phi = \iota'$, there exists a unique ring homomorphism such that $\iota'(a) = x$. So ι' maps $\sum_{i=1}^{n} p_i a^i$ to $\sum_{i=1}^{n} p_i x^i$ and $\iota' \circ \iota = 1_{\mathbb{Z}[x]}$. Also using definitions of ι and ι' easily yields $\iota \circ \iota' = 1_A$.