Test 1

1) $f(x) = \begin{cases} cx & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

A) $P(0 < X 4) = \int_0^4 cx\,dx = \left[\frac{cx^2}{2}\right]_0^4$

$= \frac{c(4)^2}{2} - \frac{c(0)^2}{2} = 8c$

since $f(x) = 1$, $1 = 8c$, $c = 1/8$

B) $P(-1 \le X \le 1)$

$P(-1 \le X \le 0) = \emptyset$

$P(0 \le X \le 1) = \left[\frac{cx^2}{2}\right]_0^1 = \frac{c(1)^2}{2} - \emptyset$

$= \frac{c}{2} = \frac{1}{8} \cdot \frac{1}{2} = 1/16$

c) $P(X > 2)$ $\qquad P(X > 4) = \emptyset$

$P(2 < X \le 4) = \left(\frac{1}{8}\right)\frac{(4)^2}{2} - \frac{1}{8}\frac{(2)^2}{2}$

$= \frac{16}{16} - \frac{4}{16} = 12/16$

D) $P(X<3 \mid X>1)$

$P(X>4) = \emptyset$

$P(1 \le X \le 4) = \left.\frac{cx^2}{2}\right]_1^4 = \frac{c(4)^2}{2} - \frac{c(1)^2}{2}$

$= \frac{1}{8}\frac{16}{2} - \frac{1}{8}\frac{1}{2} = \frac{16}{16} - \frac{1}{16} = \frac{15}{16}$

$P(1 \le X \le 3) = \left.\frac{cx^2}{2}\right]_1^3 = \frac{1}{8}\frac{(3)^2}{2}$

$- \frac{1}{8}\frac{(1)^2}{2} = \frac{9}{16} - \frac{1}{16} = \frac{8}{16}$

$\frac{P(1 \le X \le 3)}{P(1 \le X \le 4)} = \frac{8/16}{15/16} = 8/15$

E) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_0^4 x \cdot (cx)\,dx = \int_0^4 cx^2\,dx$

$= c\left.\frac{x^3}{3}\right]_\emptyset^1 = c\frac{(4)^2}{3} - \emptyset = \frac{1}{8}\left(\frac{16}{3}\right)$

$\frac{1}{8} \cdot \frac{16}{3}$

$= \frac{16}{24}$

f) $Var(X) =$

$$E[X^2] = \int_0^4 x^2 \cdot c\underline{\underline{x}}\, dx$$

$$= c\frac{x^4}{4}\Big]_0^4 = \frac{\frac{1}{8}(4)^2}{4} - 0$$

$$= 16/32$$

$$E[X^2] - E[X]^2$$

$$= \frac{16}{32} - \left(\frac{16}{24}\right)^2 = .055$$

$$\sigma^X = \sqrt{.0275} = .2357$$

e) $$\int_0^x c x\, dt = c\frac{x^2}{2}\Big]_0^x$$

$$= \left(\frac{1}{8}\right)\frac{x^2}{2} - 0 = \frac{1}{16}x^2 \quad \text{if } 0 \le x \le 4$$

$$= F(x)$$

$F(X) = 0$ if $X < 0$

$F(X) = 1$ if $X > 4$

2) $f(x) = \begin{cases} 3/2\, x^2 & \text{if } -1 < x < 1 \\ \emptyset & \text{otherwise} \end{cases}$

$P(x < -1) = \emptyset$

a) $P(x < 0) = \int_{-1}^{0} f(x)\,dx$

$= \left(\frac{3}{2}\right)\frac{1}{3}x^3 \Big]_{-1}^{0} = \frac{1}{2}(0)^3 - \frac{1}{2}(-1)^3$

$= 0 + 1/2 = 1/2$

b) $P(-.5 < x < 1) = \frac{1}{2}x^3 \Big]_{-.5}^{1}$

$= \frac{1}{2}(1)^3 - \frac{1}{2}(-1/2)^3$

$= 1/2 + 1/4 = 3/4$

$P(x > 1) = \emptyset$

c) $P(x > .5 \mid x > .5) = 1$ for $P(-1 < x < 1)$

$\emptyset$ otherwise

d) $E[x] = \int_{-1}^{1} x\left(\frac{3}{2}\right)x^2\,dx =$

Page 5

$$\int_{-1}^{1} \tfrac{3}{2} x^3 dx = \tfrac{3}{2}\tfrac{1}{4} x^4 \Big]_{-1}^{1}$$

$$= \tfrac{3}{8} x^4 \Big]_{-1}^{1} = \tfrac{3}{8}(1)^4 - \tfrac{3}{8}(-1)^4$$

$$= 0$$

E) $Var(X) =$

$$E[X^2] = \int_{-1}^{1} x^2 \cdot \tfrac{3}{2}(x^3)^2 dx$$

$$= \int_{-1}^{1} \tfrac{3}{2} x^5 dx = \tfrac{1}{6}\tfrac{3}{2} x^6 \Big]_{-1}^{1}$$

$$= \tfrac{3}{12}(1)^6 - \tfrac{3}{12}(-1)^6 = 0$$

$$\sigma^x = \sqrt{0} = 0$$

F) $$\int_{0}^{x} \tfrac{3}{2} x^2 dx = \tfrac{3}{2}\tfrac{1}{3} x^3 \Big]_{-1}^{x}$$

$$\tfrac{1}{2} x^3 - \tfrac{1}{2}(-1)^3 = \tfrac{1}{2}x^3 + \tfrac{1}{2}$$

$$= \tfrac{1}{2}(x^3 + 1) \text{ for } -1 < x < 1$$

$$= F(x)$$

$F(x) = 0$ for $x < -1$
$F(x) = 1$ for $x > 1$

3) 1) $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2-x & \text{if } 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

$\boxed{P(1<X<3) = 1/2}$

$P(X>2) = \emptyset$

$P(1<X<2) = \int_1^2 (2-x)\,dx$

$= 2x - \frac{1}{2}x^2 \Big]_1^2 = \left[2(2) - \frac{1}{2}(2)^2\right] - \left[2(1) - \frac{1}{2}(1)^2\right]$

$= (4-2) - (2 - 1/2)$

$= \frac{4}{2} - 3/2 = \frac{1}{2}$

B) $P(X>.5) = 7/8$

$P(X>2) = \emptyset$

$P(.5<X<1) = \int_{.5}^{1} x\,dx = \frac{x^2}{2}\Big]_{.5}^{1}$

$= \frac{(1)^2}{2} - \frac{(1/2)^2}{2} = 1/2 - \frac{1}{8} = 3/8$

$P(1 \leq X < 2) = 1/2$

$P(X > .5) = 1/2 + 3/8 = 7/8$

c) $P(X > 1 \mid X > .5)$

$P(X > .5) = 7/8$

$P(X > 1 \cap X > .5) = P(X > 1)$

$P(X > 2) = \emptyset$

$P(1 < X < 2) = 1/2$

$$P(X > 1 \mid X > .5) = \frac{P(X > 1 \cap X > .5)}{P(X > .5)} = \frac{1/2}{7/8}$$

$$= (1/2)(8/7) = \frac{8}{14} = \frac{4}{7}$$

D) $$E[X] = \int_0^1 x \cdot x \, dx = \frac{x^3}{3}\Big]_{\emptyset}^{1}$$

$1/3 - \emptyset = 1/3$ if $\emptyset < X < 1$

$$\int_1^2 x(2-x)\,dx = \int_1^2 (2x - x^2)\,dx$$

$$= \frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\Big]_1^2 = \left((2)^2 - \frac{(2)^3}{3}\right)$$

$$- \left((1)^2 - \frac{(1)^3}{3}\right) = \left(\overset{12/3}{4} - \frac{8}{3}\right) - \left(1 - 1/3\right)$$

$$= 4/3 - 2/3 = 2/3 \text{ , if } 1 \le x < 2$$

e) ~~E[X]~~ variance (X) for $0 \le x \le 1$

$$= \int_0^1 (x^2)^2\,dx = \int_0^1 x^4\,dx = \frac{1}{5}x^5\Big]_0^1$$

$$= 1/5$$

$$E[X^2] - E[X]^2 = 1/5 - (1/3)^2 = 1/5 - 1/9$$

$$= \frac{9}{45} - \frac{5}{45} = \frac{4}{45}$$

$$= \sigma_X = \sqrt{4/45}$$

VARIANCE (X) for $1<X<2$

$$\int_{1}^{2} x^2(2-x)\,dx = \int_{0}^{2}(2x^2 - x^3)\,dx$$

$$= \frac{2}{3}x^3 - \frac{x^4}{4}\Big]_1^2$$

$$= \left(\frac{2}{3}(2)^3 - \frac{(2)^4}{5}\right) - \left(\frac{2}{3}(1)^3 - \frac{(1)^4}{5}\right)$$

$$= \frac{2}{3}(8) - \frac{16}{5} - \left(\frac{2}{3} - \frac{1}{5}\right)$$

$$= \frac{80}{15} - \frac{48}{15} - \left(\frac{10}{15} + \frac{3}{15}\right)$$

$$= 25/15$$

$$\sigma_X^2 = \sqrt{25/15}$$

$\frac{16}{3}$

$\frac{3}{15}$

$$F(X) = \int_0^X x\,dx = \frac{X^2}{2}\Big]_0^X$$

$$\frac{X^2}{2} - 0 = \frac{X^2}{2} \text{ for } 0<X<1$$

$$F(x) = \int_1^x (2-x)\,dx =$$

$$2x - \frac{x^2}{2}\Big]_1^x = \left(2x - \frac{x^2}{2}\right) - \left(2 - 1/2\right)$$

$$\boxed{= 2x - \frac{x^2}{2} - 3/2} \quad \text{for } 1 \le x \le 1$$

$F(x) = 0$ for $x < 0$
$F(x) = 1$ for $x > 1$

4) $E[X] = 4 = 1/\lambda$

5) $\lambda = 1/4$

$$f(x) = \begin{cases} 1/4\, e^{-1/4 x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$