

# motivating eigenvalues and eigenvectors

slides: [ajbc.io/misc/eigen.pdf](http://ajbc.io/misc/eigen.pdf)

# dynamic systems: foxes and rabbits



 $F_t$ 

population of foxes at year  $t$

 $R_t$ 

population of rabbits at year  $t$



$$F_{t+1} = 0.6F_t + 0.2R_t$$



$$R_{t+1} = -0.2F_t + 1.1R_t$$

foxes die without food



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rabbits multiply without predators

foxes die without food



foxes multiply with increased food supply

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foxes die without food



foxes multiply with increased food supply

$$F_{t+1} = 0.6F_t + 0.2R_t$$



foxes kill off rabbits

rabbits multiply without predators

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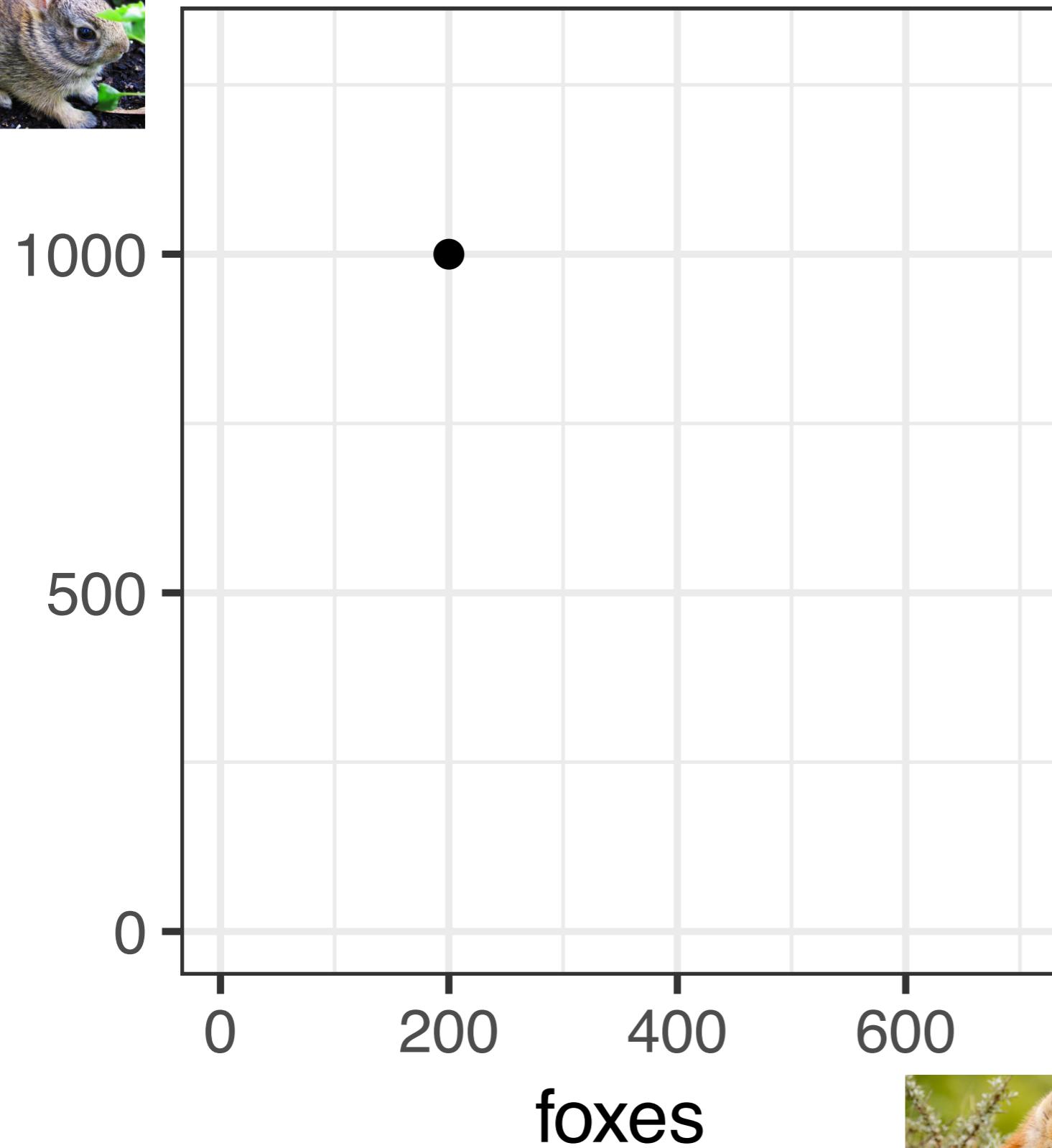
$$\begin{bmatrix} F_{t+1} \\ R_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} F_t \\ R_t \end{bmatrix}$$

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$$\vec{p}_{t+1} = A \vec{p}_t$$



rabbits

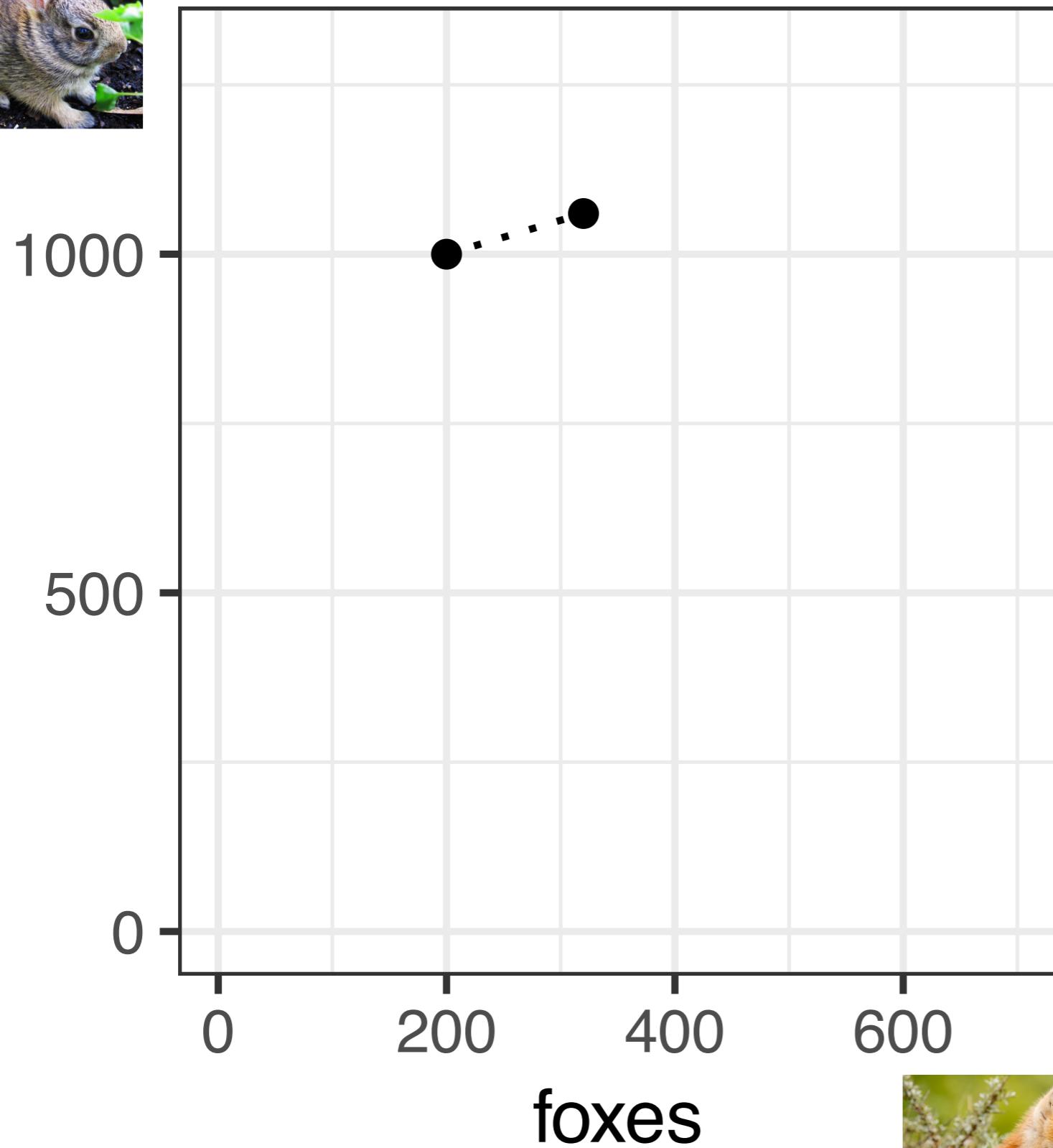


$$\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$$





rabbits



$$\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$$

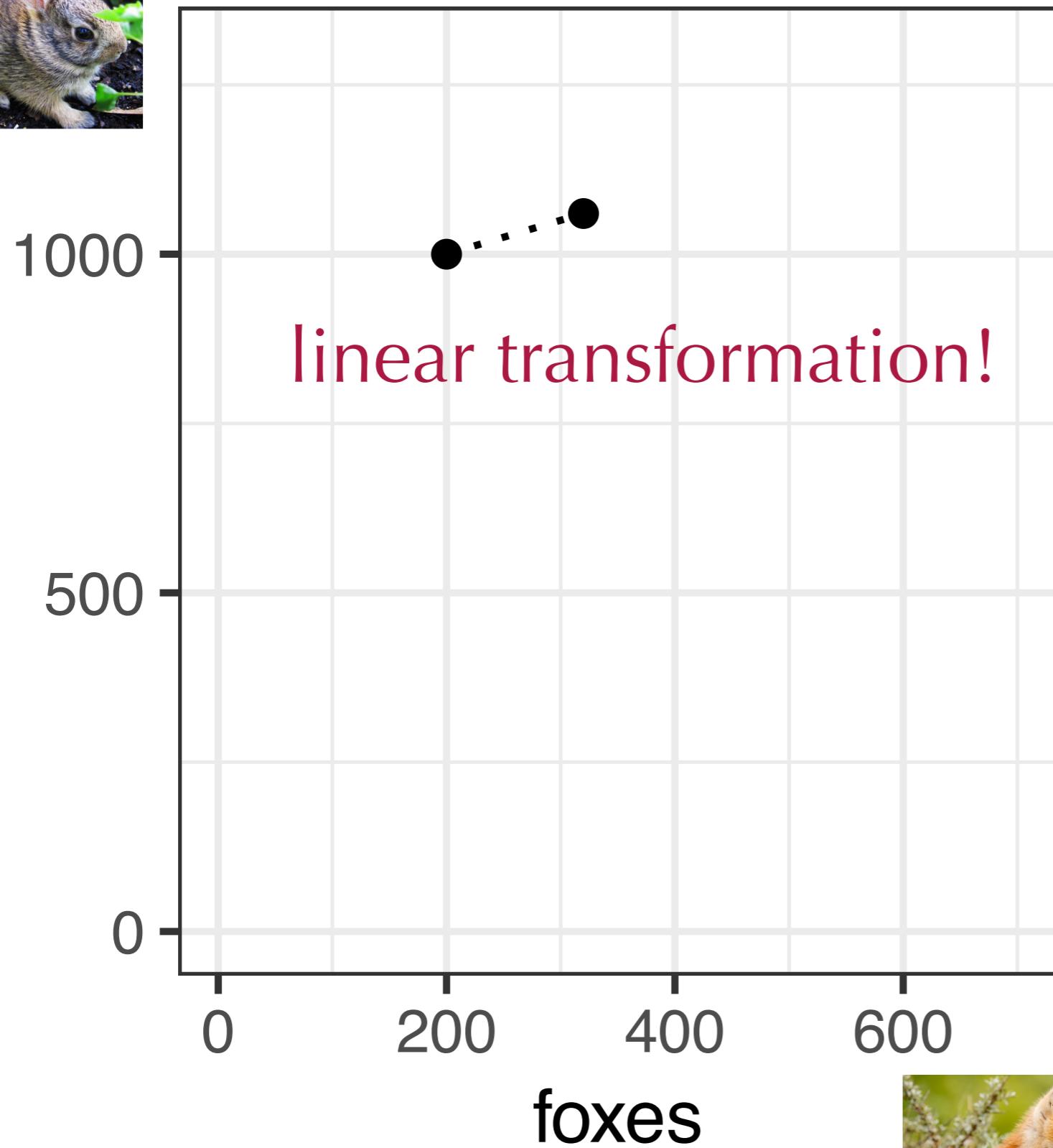
$$\begin{bmatrix} F_1 \\ R_1 \end{bmatrix} =$$

$$\begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} F_0 \\ R_0 \end{bmatrix}$$

$$= \begin{bmatrix} 320 \\ 1060 \end{bmatrix}$$

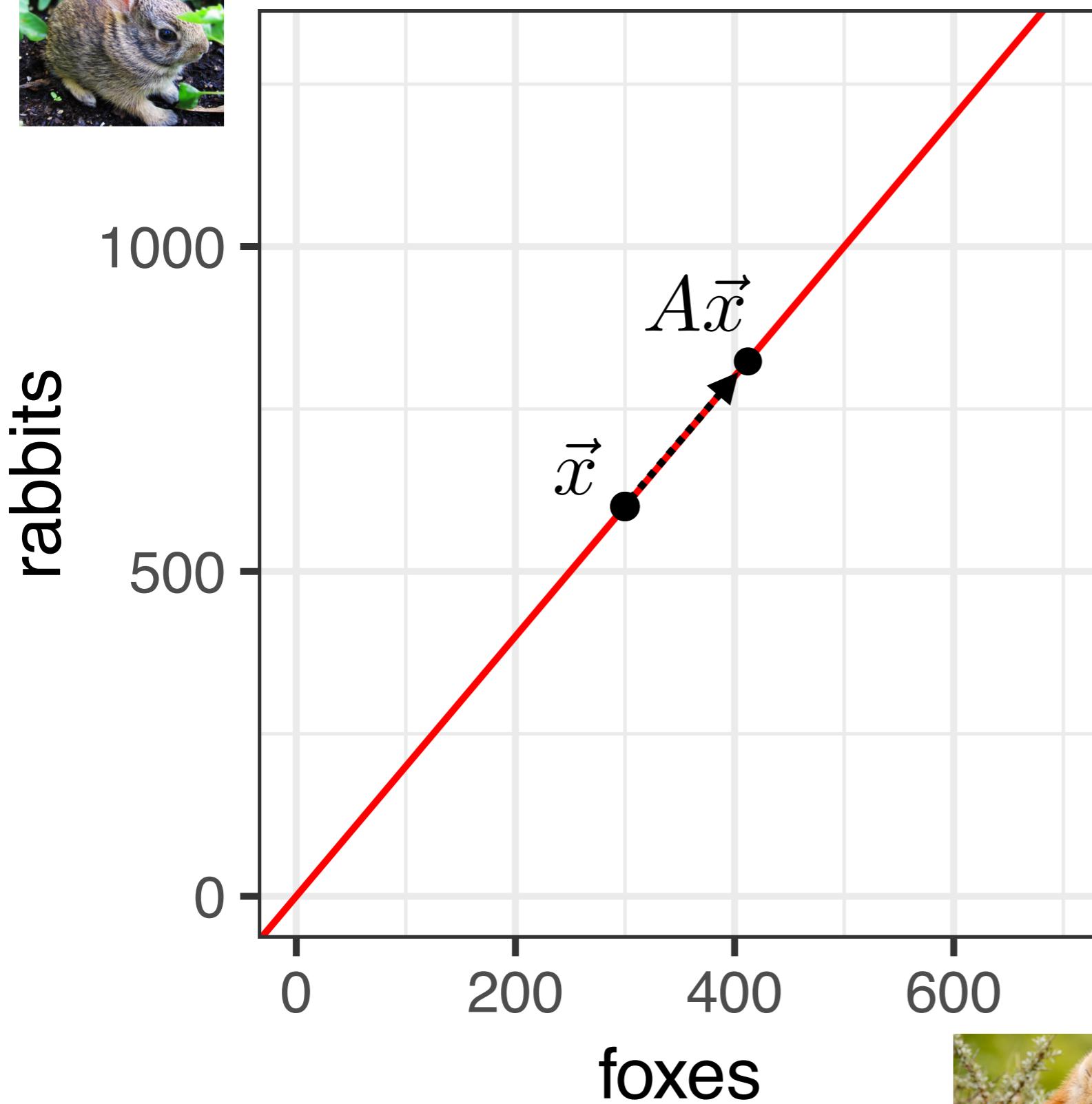


rabbits



$$\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$$

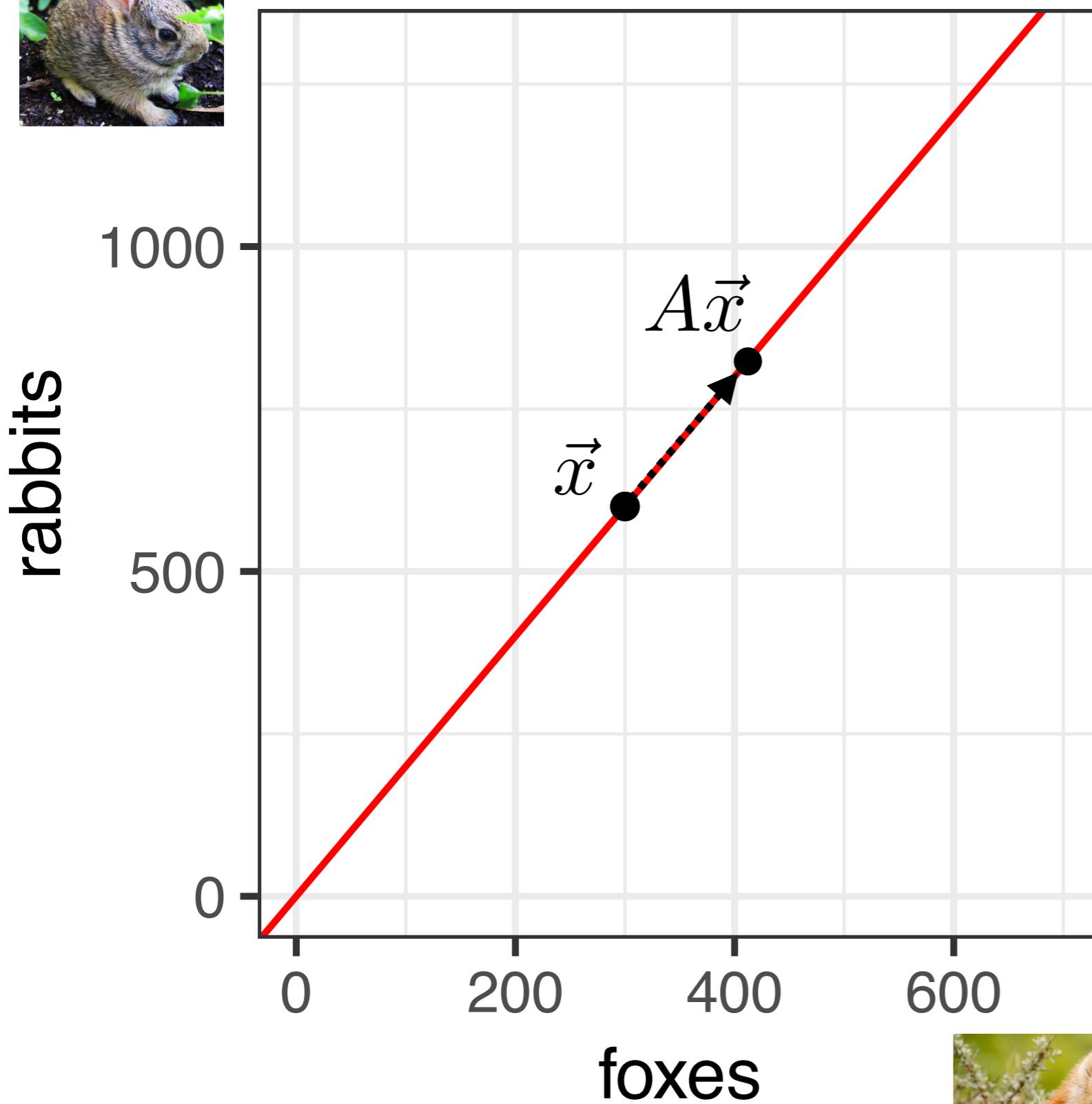
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imagine a special  
case where

$$A\vec{x} = \lambda\vec{x}$$



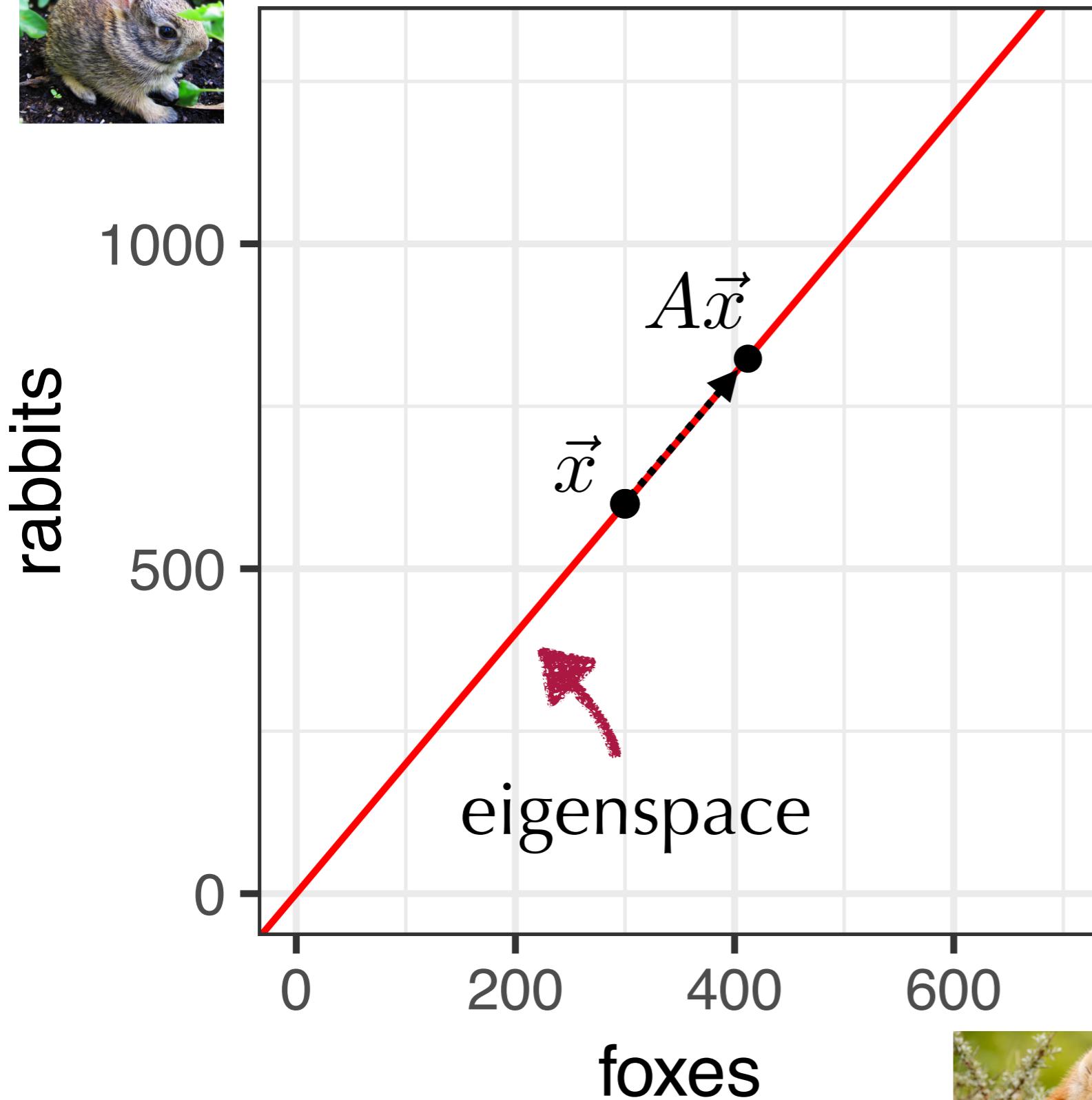


imagine a special  
case where

$$A\vec{x} = \lambda\vec{x}$$

scalar





imagine a special  
case where

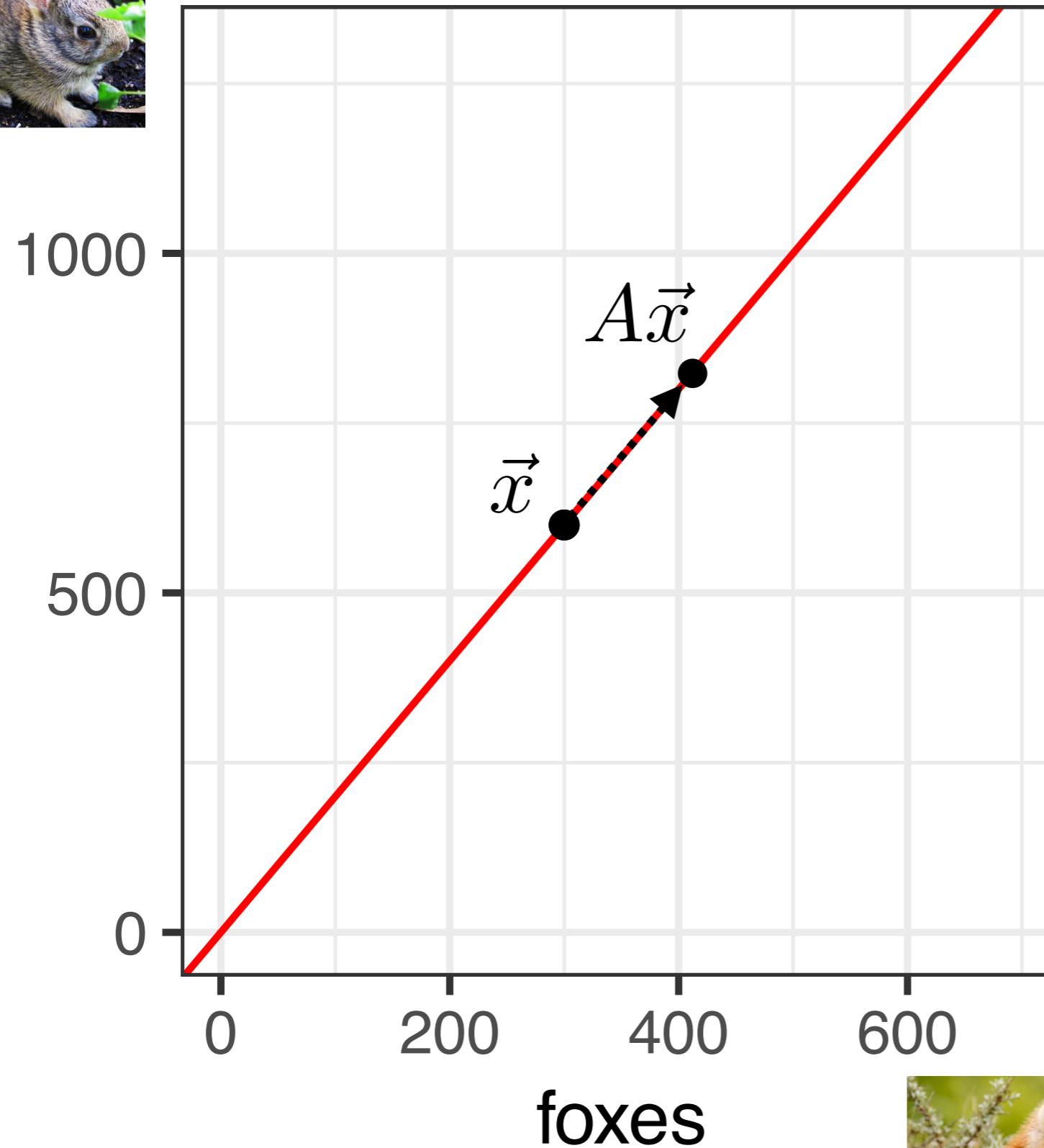
$$A\vec{x} = \lambda\vec{x}$$

eigenvalue  
eigenvector





rabbits

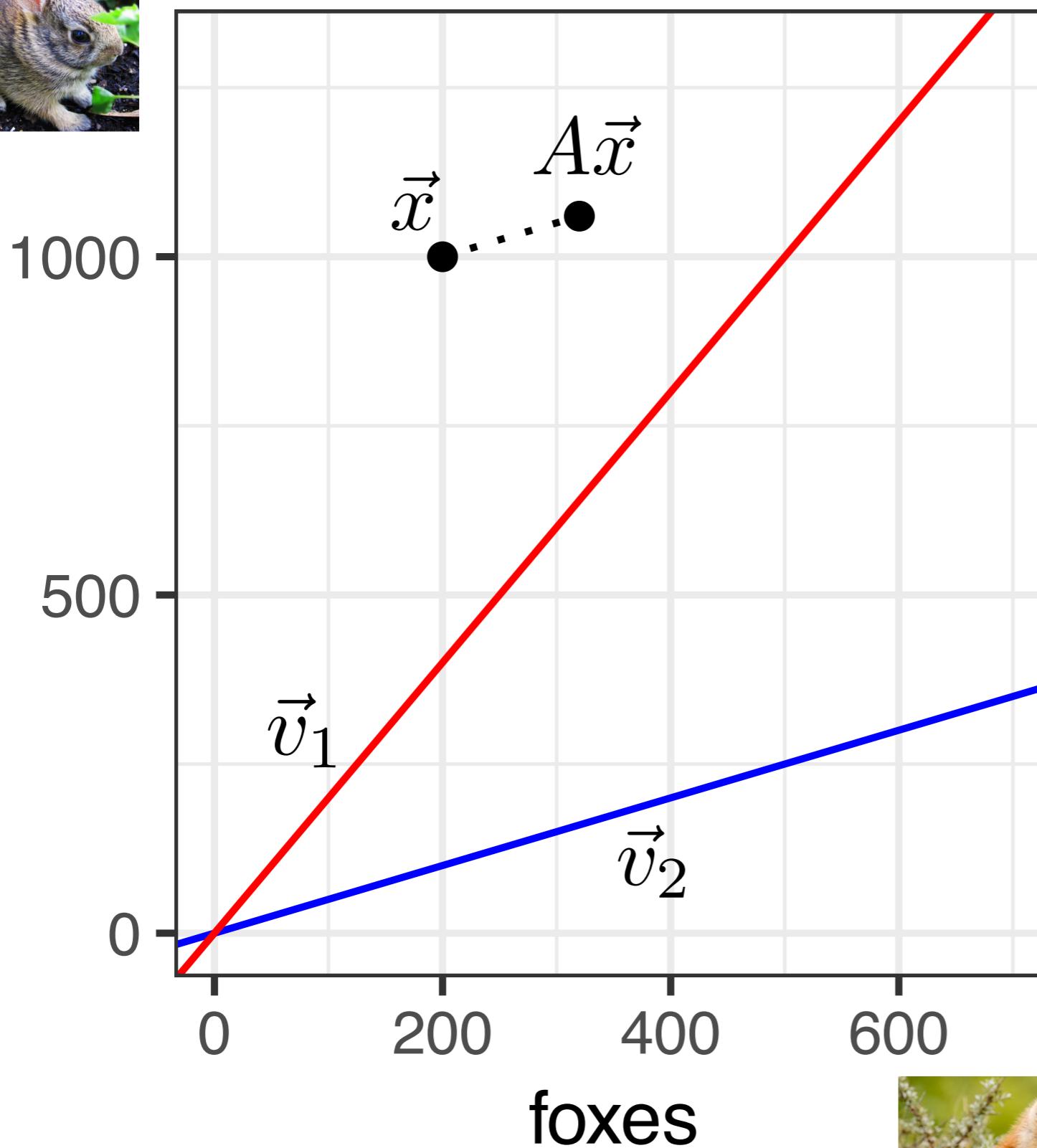


we can think of  
eigenvectors as  
**a set of basis vectors**





rabbits

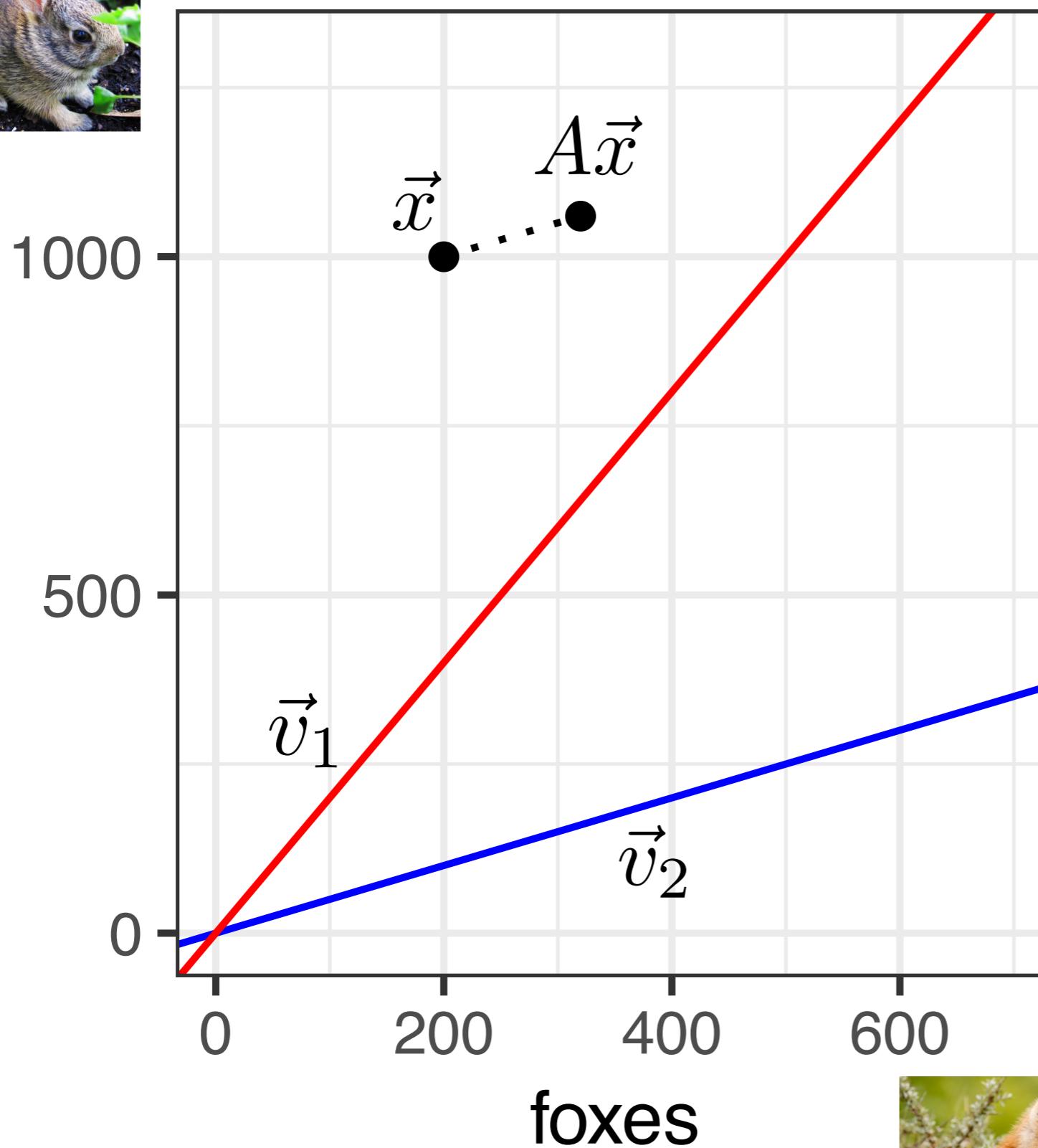


$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$



rabbits

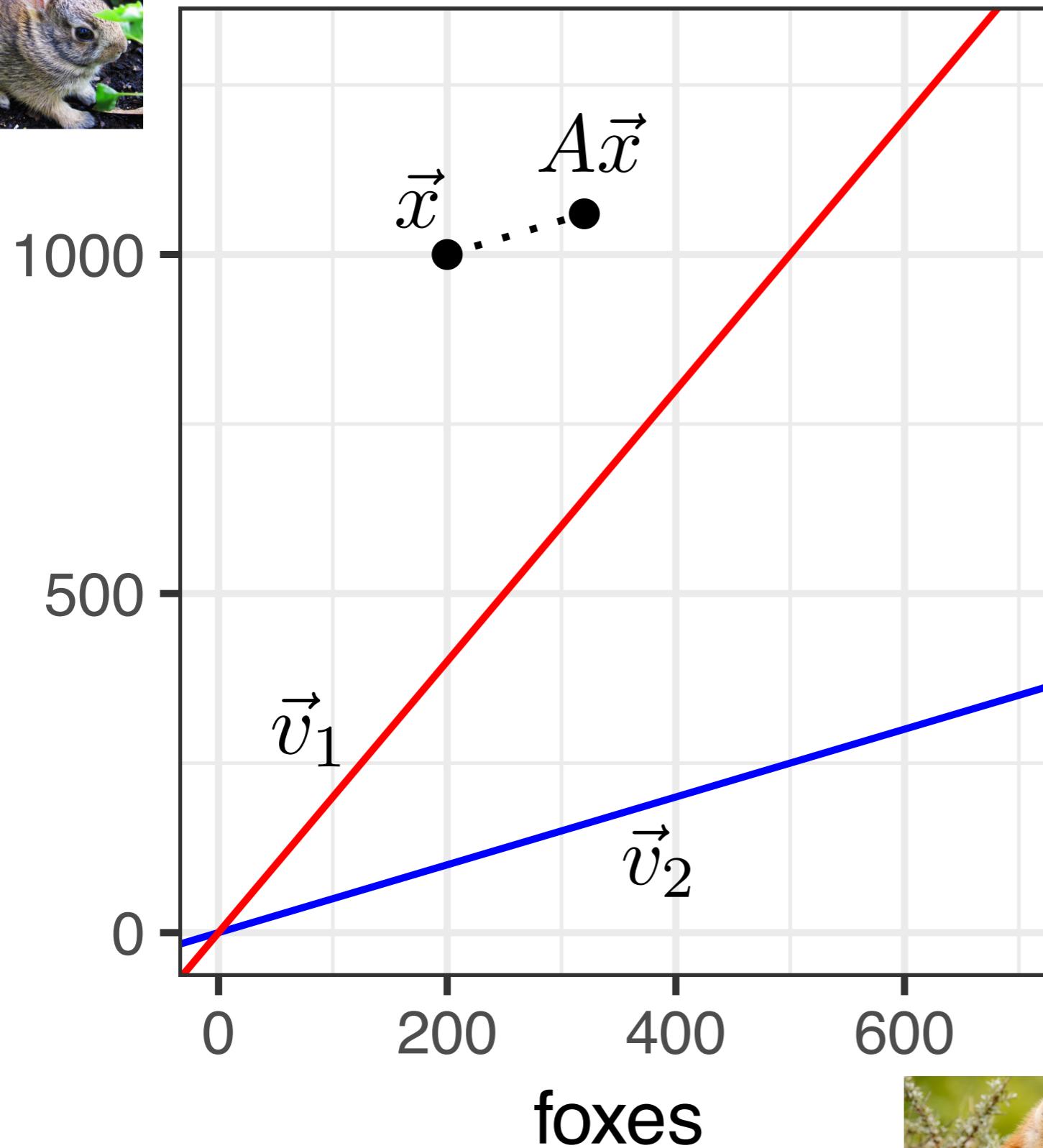


$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$

$$\vec{x} = a\vec{v}_1 + b\vec{v}_2$$



rabbits



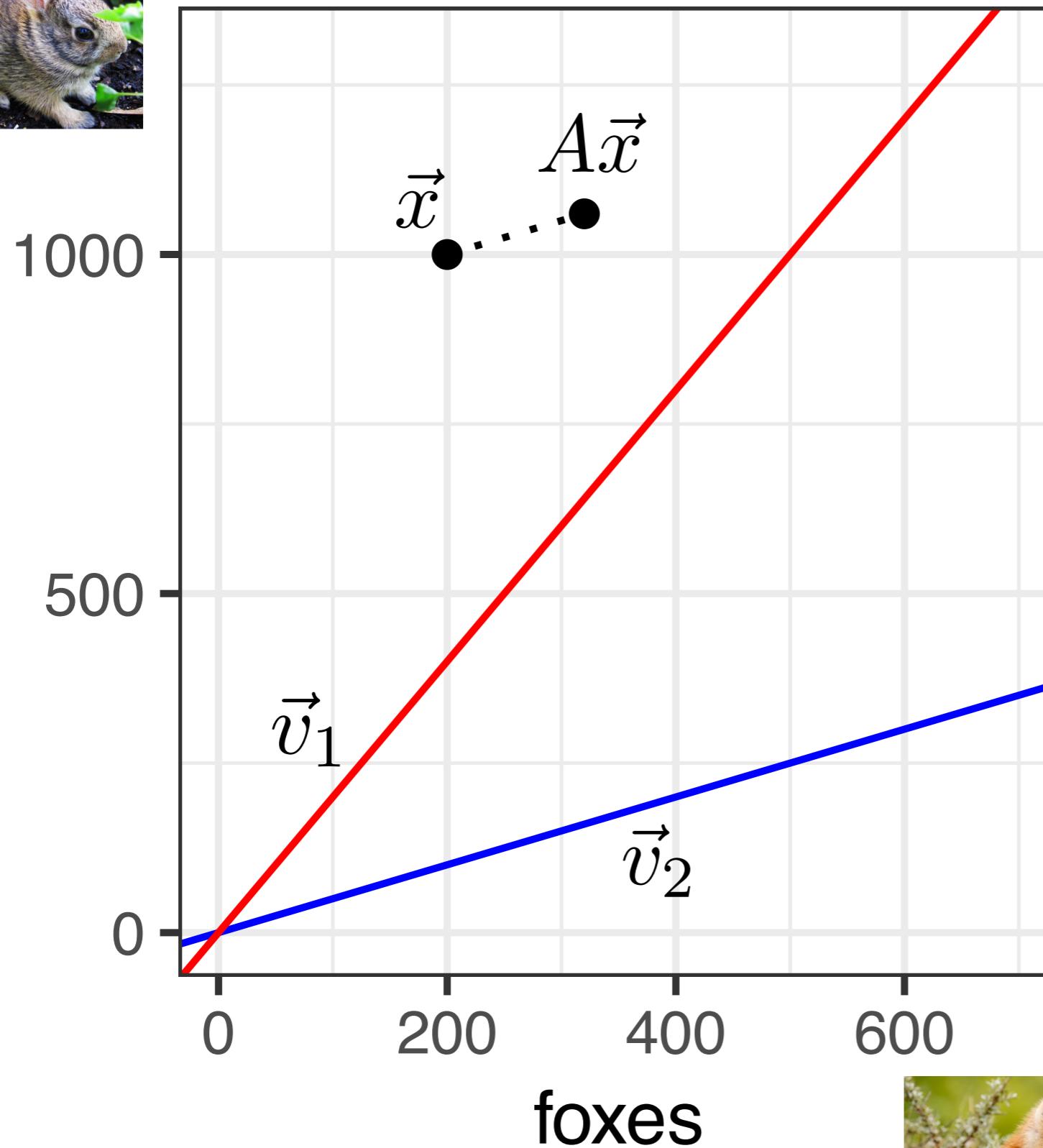
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rabbits



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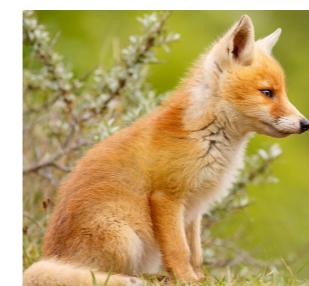
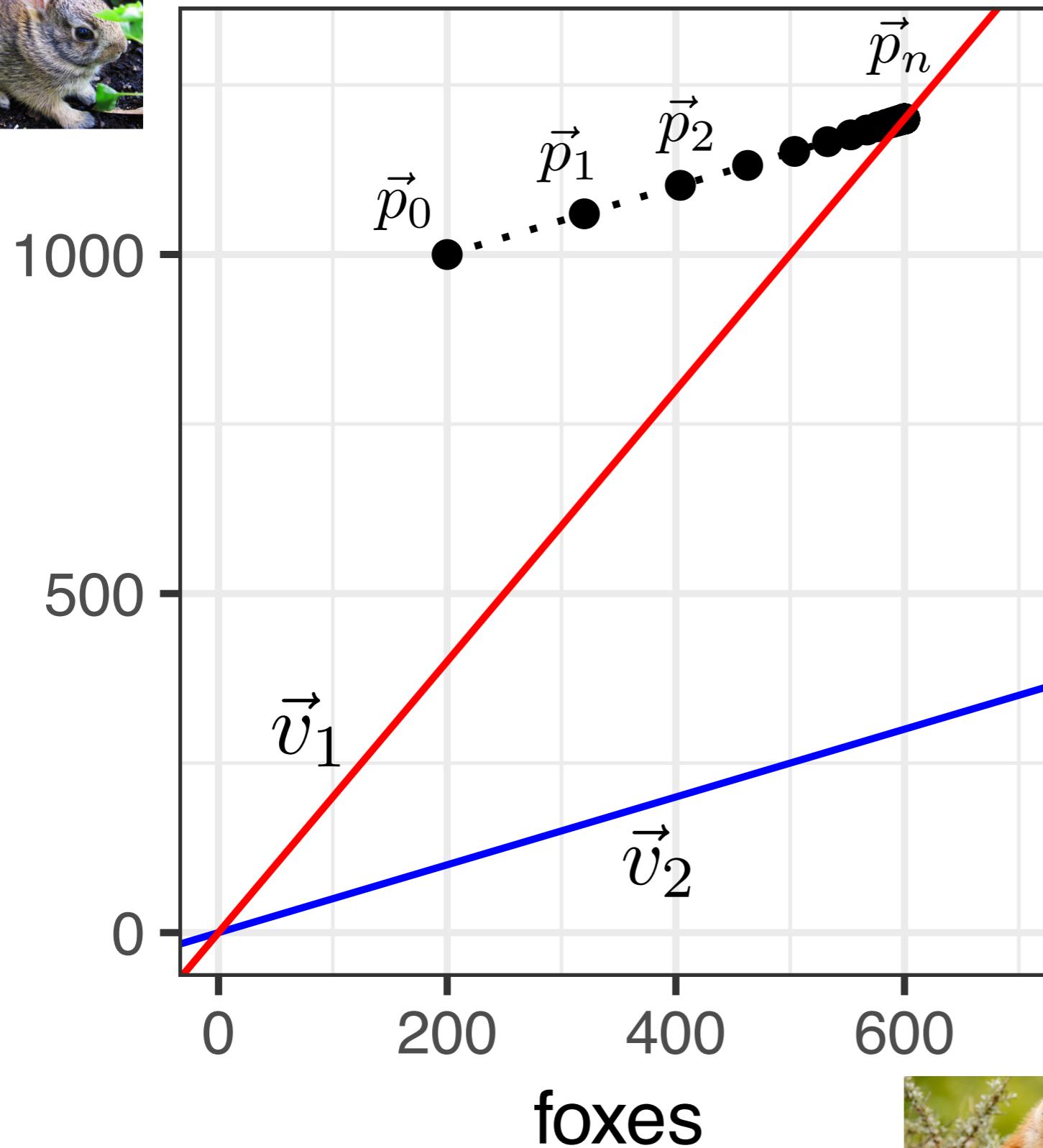
$$\vec{x} = a\vec{v}_1 + b\vec{v}_2$$

$$A\vec{x} = Aa\vec{v}_1 + Ab\vec{v}_2$$

$$= \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2$$



rabbits



$$\vec{p}_1 = A\vec{p}_0$$

$$\vec{p}_2 = A\vec{p}_1 = A^2\vec{p}_0$$

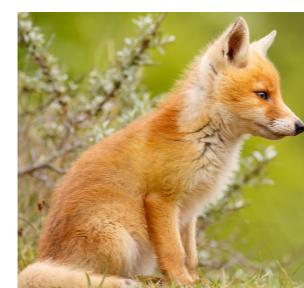
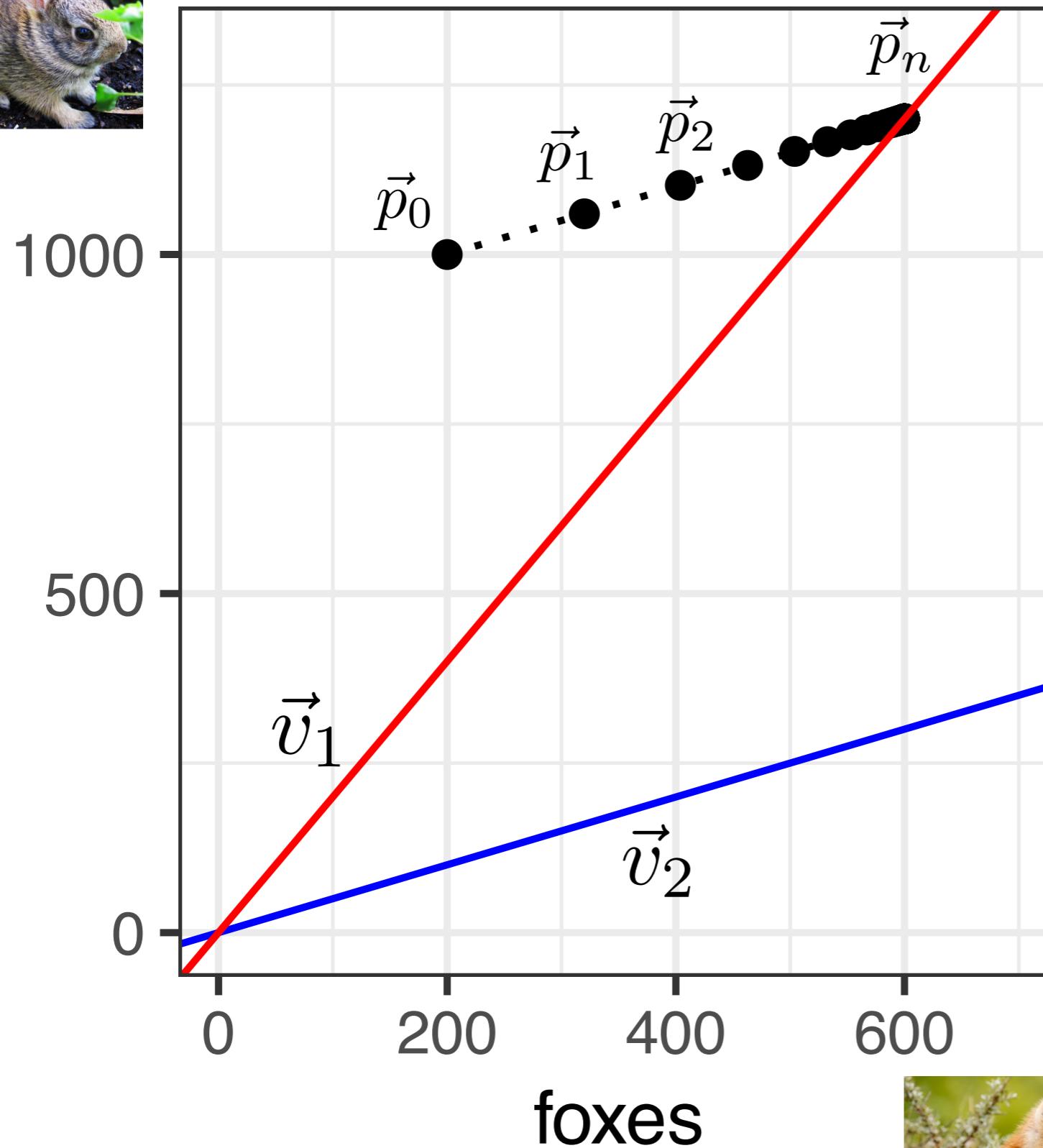
$$\vec{p}_3 = A\vec{p}_2 = A^3\vec{p}_0$$

⋮

$$\vec{p}_n = A\vec{p}_{n-1} = A^n\vec{p}_0$$



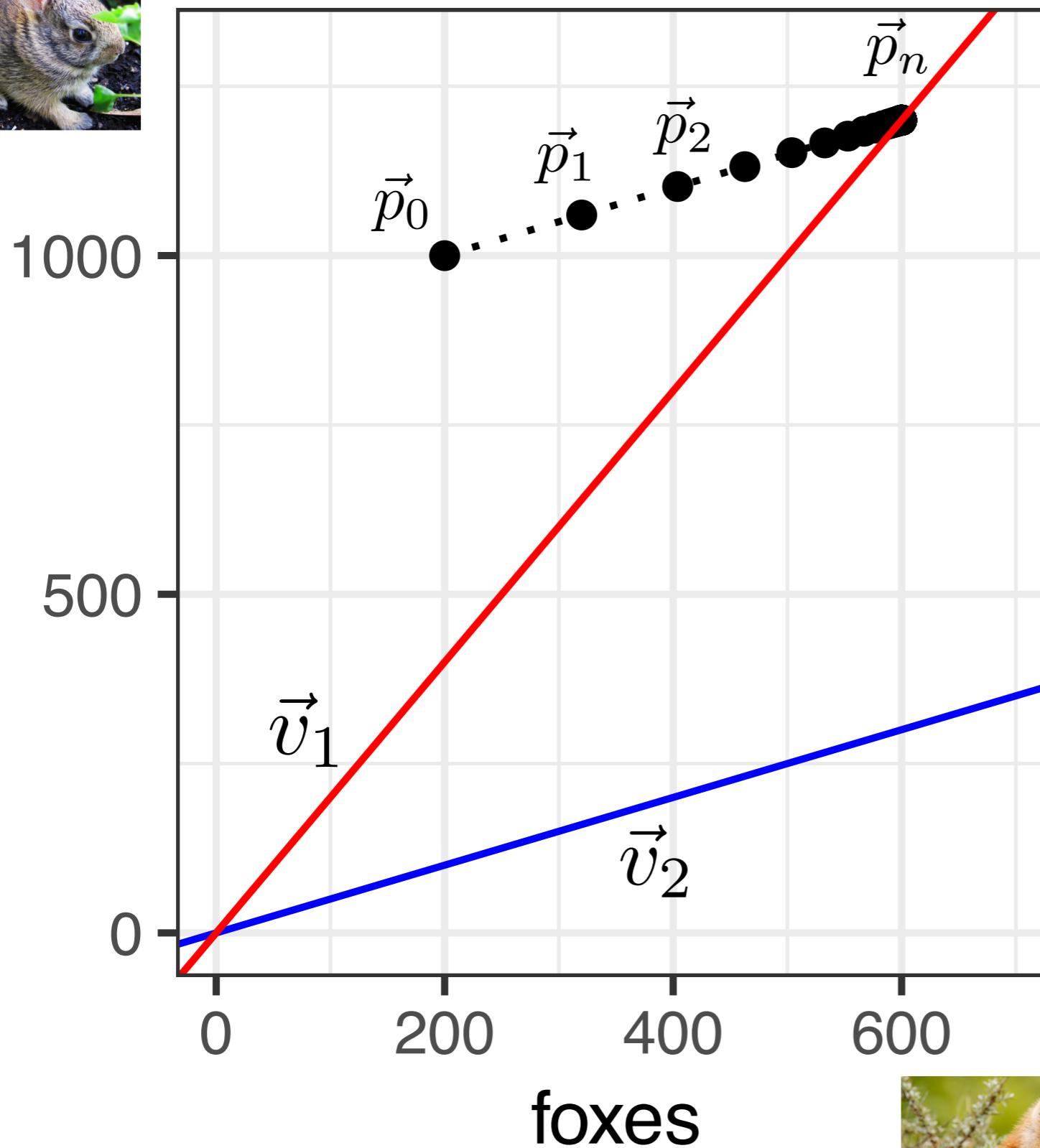
rabbits



$$\begin{aligned}A\vec{x} &= Aa\vec{v}_1 + Ab\vec{v}_2 \\&= \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2\end{aligned}$$



rabbits



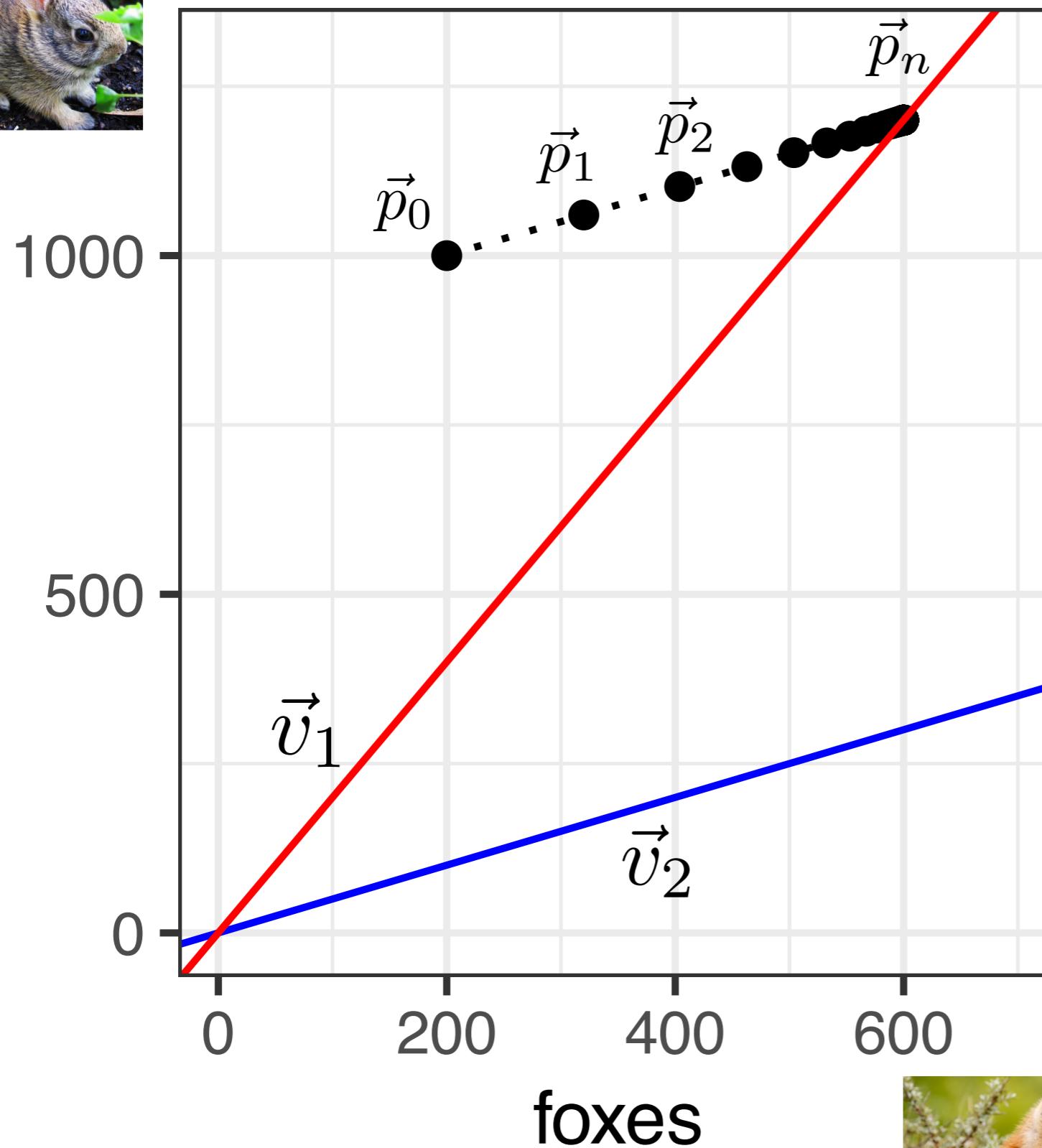
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$$\begin{aligned}A^n\vec{x} &= \\&\lambda_1^n a\vec{v}_1 + \lambda_2^n b\vec{v}_2\end{aligned}$$





rabbits



$$A\vec{x} = Aa\vec{v}_1 + Ab\vec{v}_2$$
$$= \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2$$

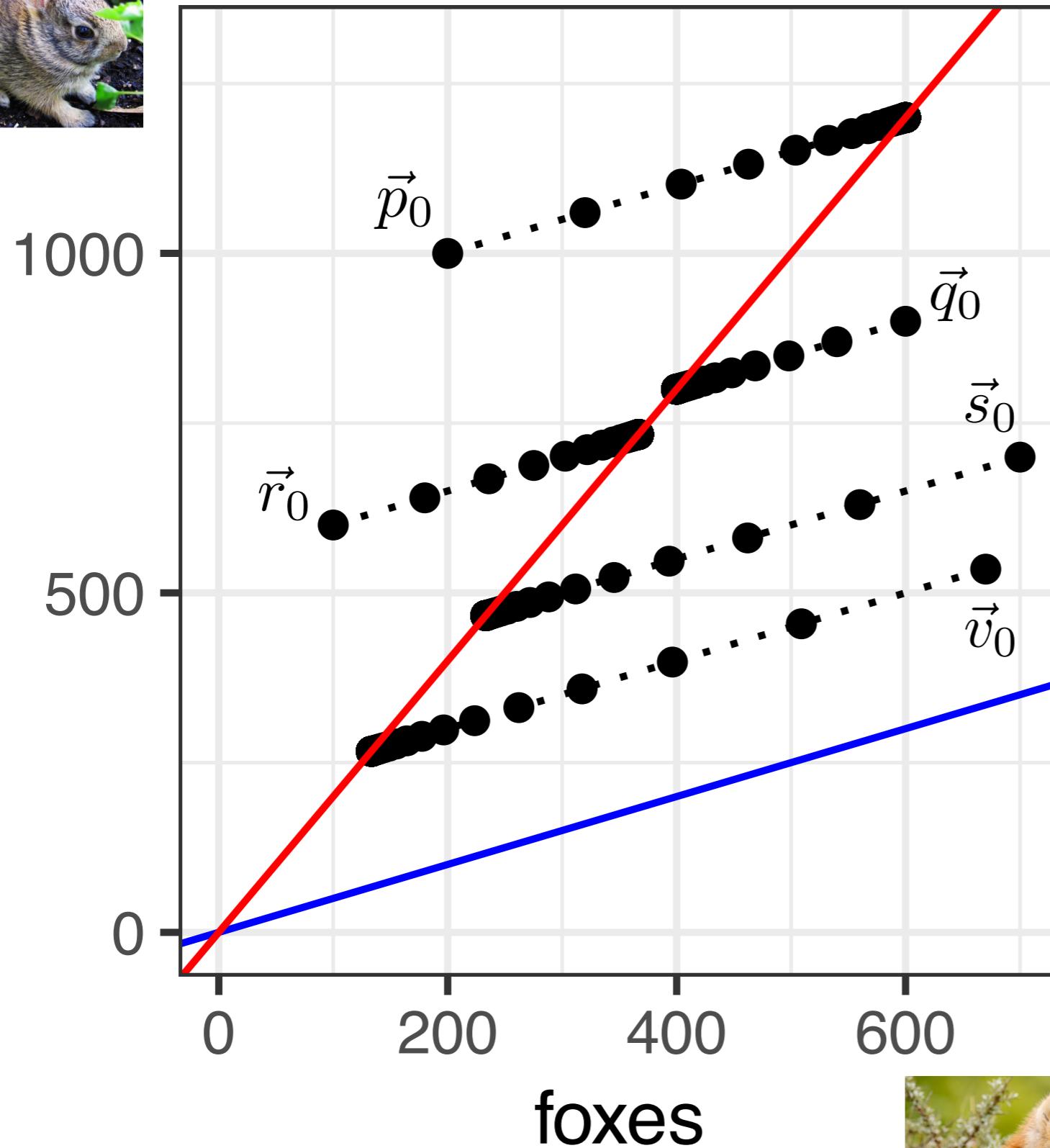
hard

$$A^n \vec{x} =$$
$$\lambda_1^n a\vec{v}_1 + \lambda_2^n b\vec{v}_2$$

easy



rabbits



$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix}$$

$$\lambda_1 = 1.0$$

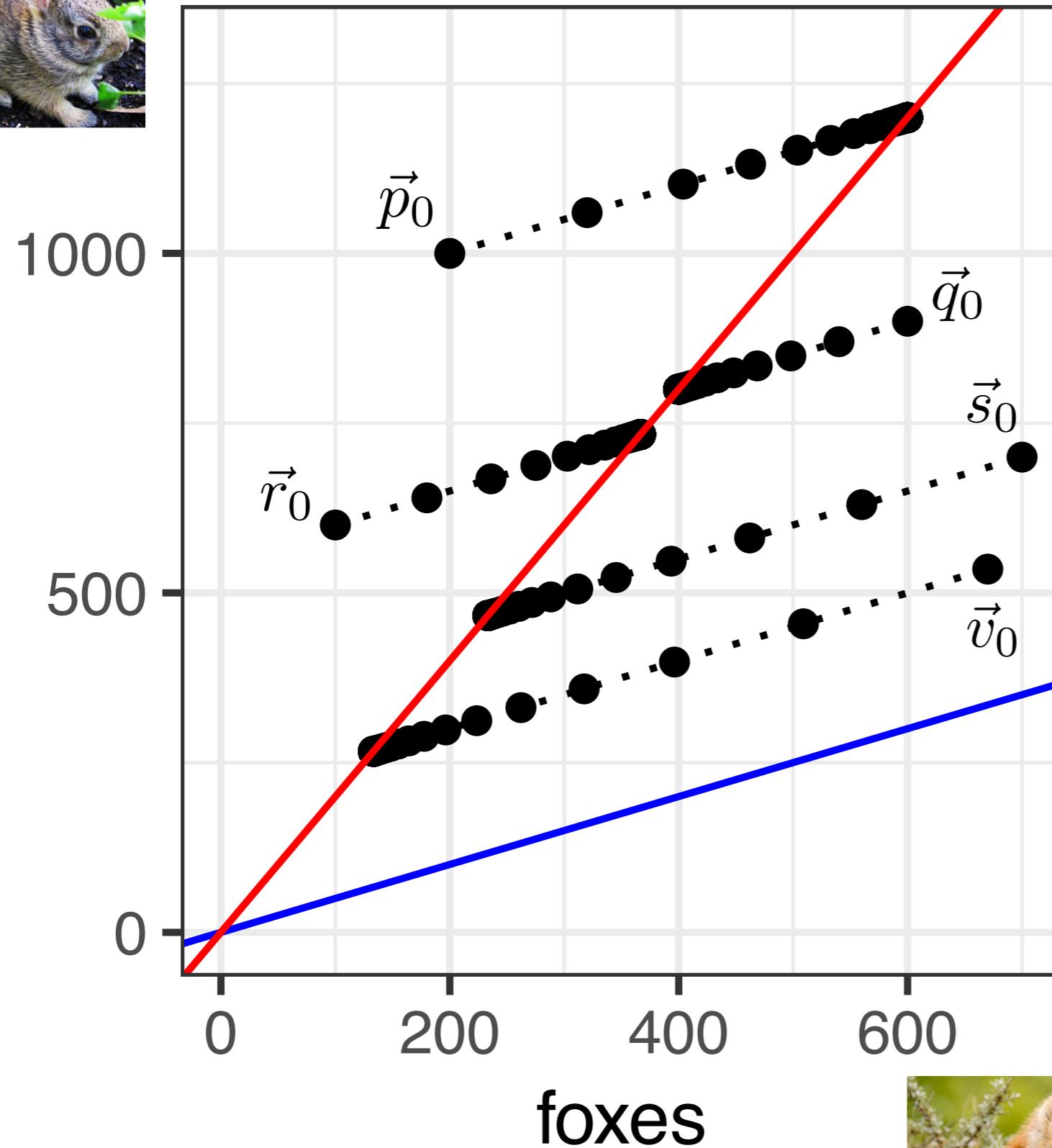
$$\vec{v}_1 \approx [-0.447 \quad -0.894]$$

$$\lambda_2 = 0.7$$

$$\vec{v}_2 \approx [-0.894 \quad -0.447]$$



rabbits



eigenvectors and eigenvalues help us understand the system's dynamics

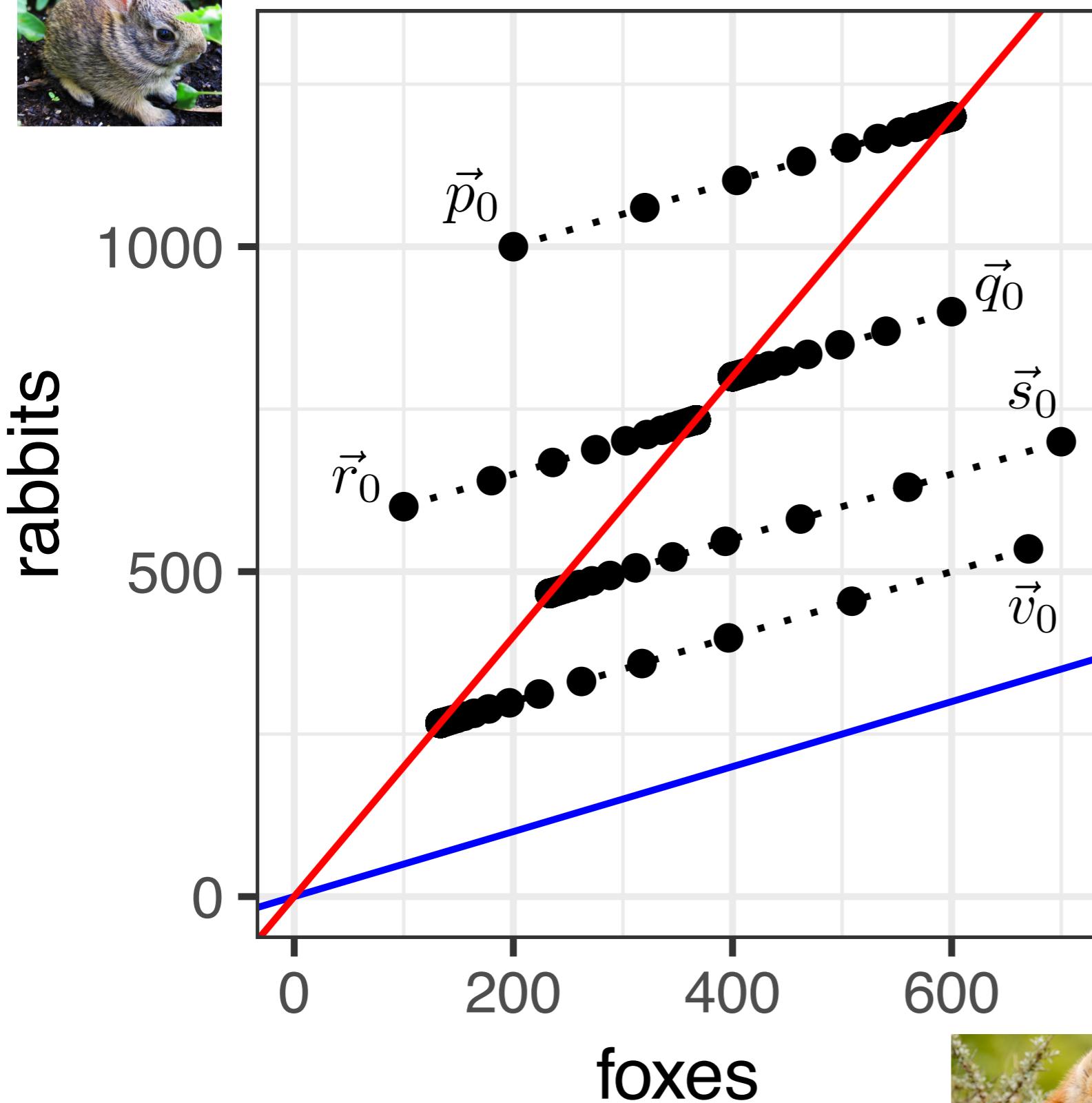
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$$\lambda_1 = 1.0$$

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$$\lambda_2 = 0.7$$

$$\vec{v}_2 \approx \begin{bmatrix} -0.894 & -0.447 \end{bmatrix}$$



rabbits

1000

500

0

0

200

400

600

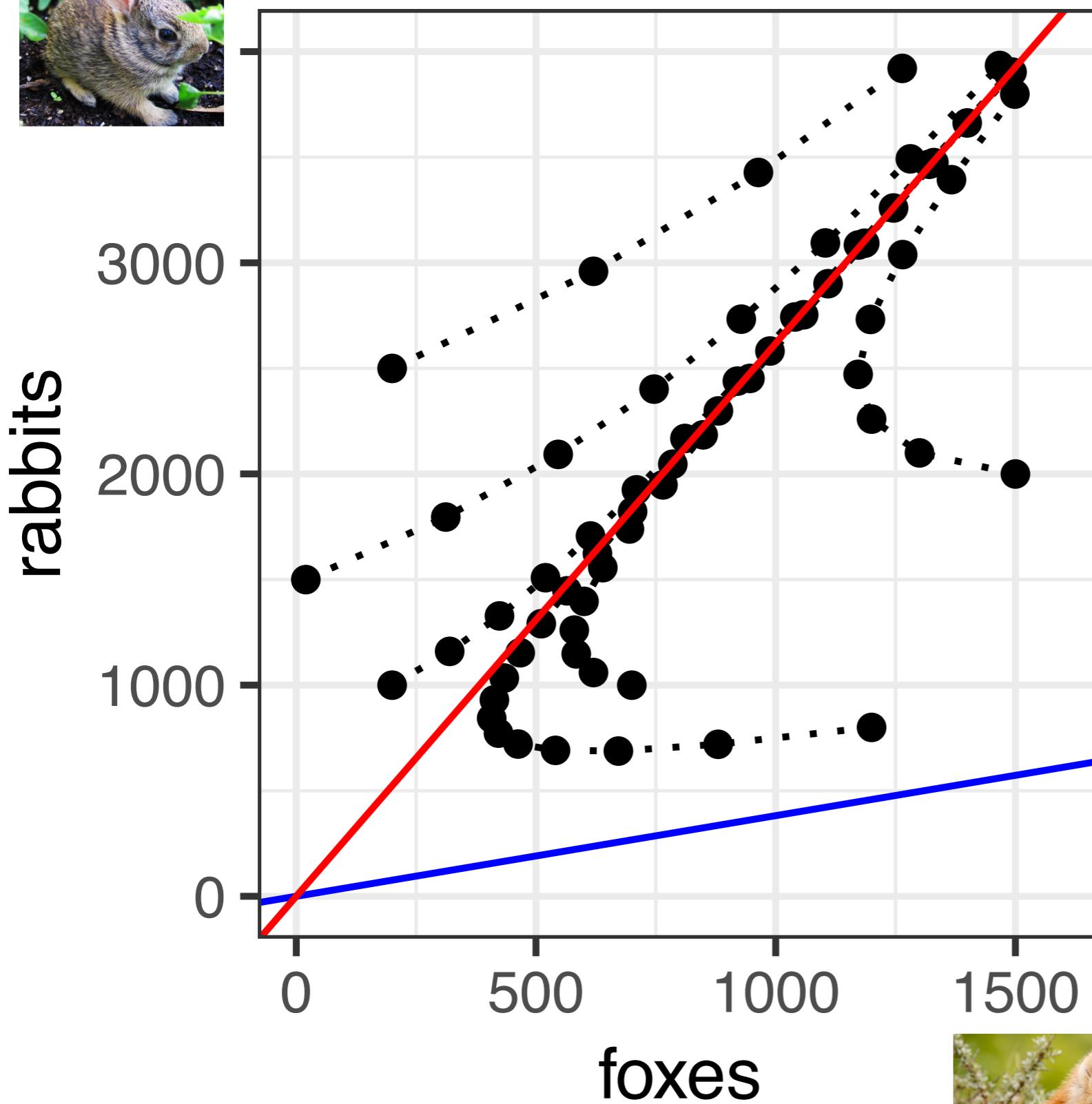
foxes

?



“rabbits multiply”

$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.2 \end{bmatrix}$$



“rabbits multiply”

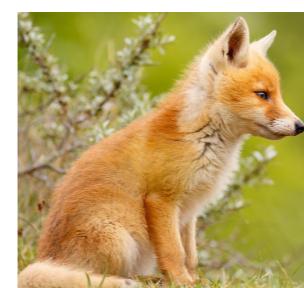
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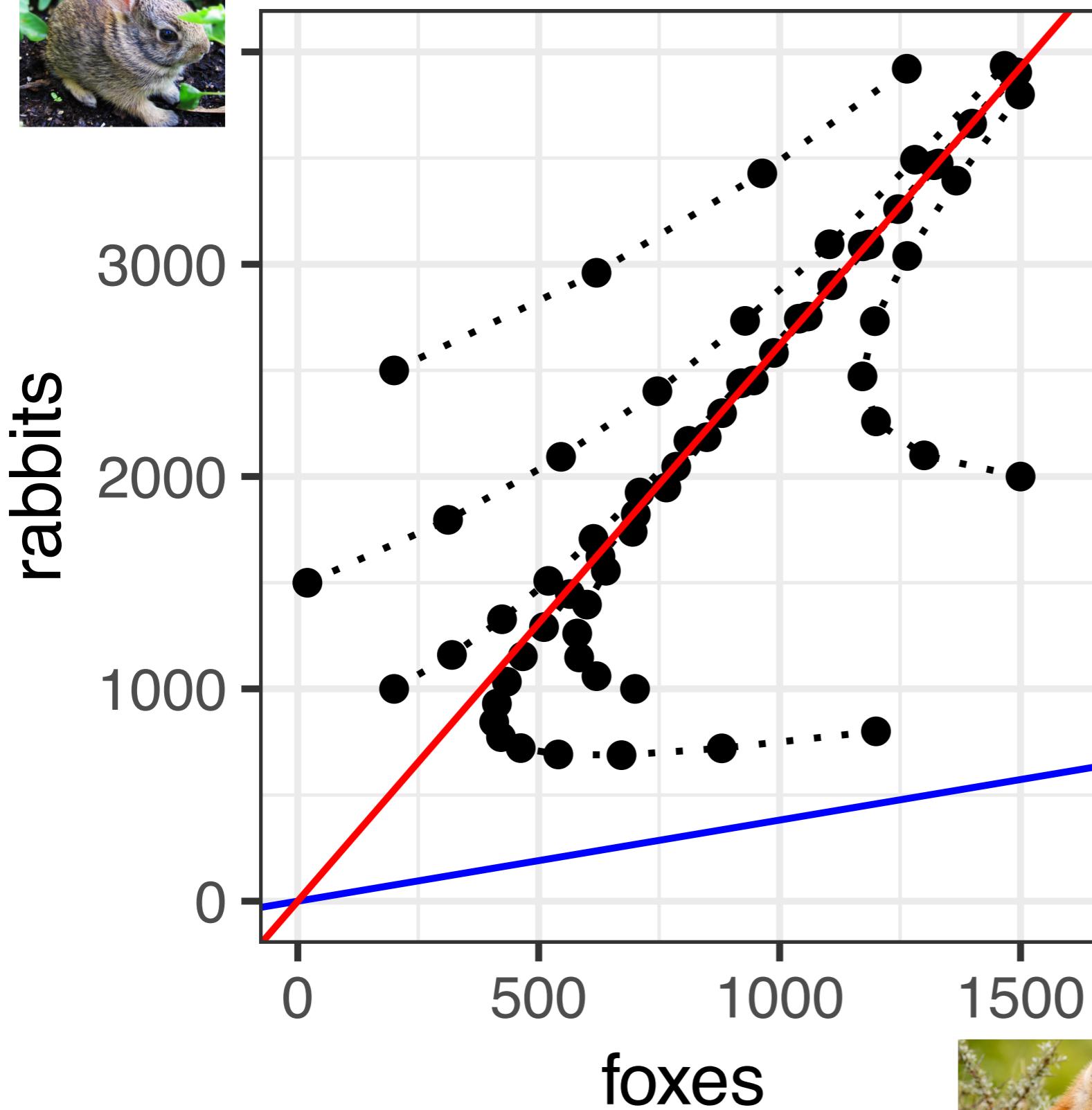
$$\lambda_1 \approx 1.124$$

$$\vec{v}_1 \approx [-0.357 \quad -0.934]$$

$$\lambda_2 \approx 0.676$$

$$\vec{v}_2 \approx [-0.934 \quad -0.357]$$





“rabbits multiply”

$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.2 \end{bmatrix}$$

$$\lambda_1 \approx 1.124$$

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rabbits

4000

2000

0

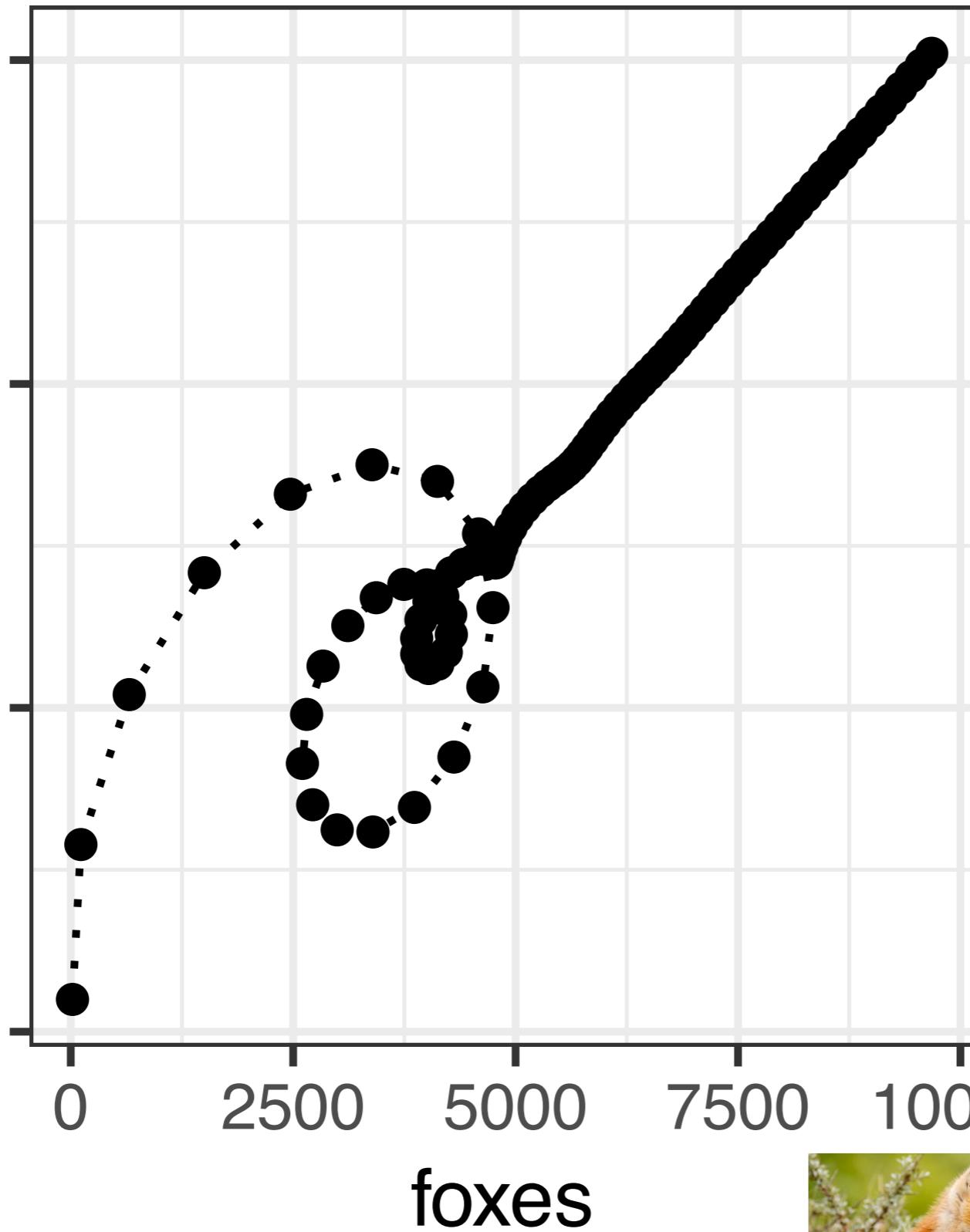
0

2500

foxes

7500

1000



$$A = \begin{bmatrix} 0.7 & 0.5 & 0 \\ -0.2 & 0.8 & 0.5 \\ 0 & -0.2 & 1.2 \end{bmatrix}$$

introducing “grass”  
into the model



A close-up photograph of a red fox sitting in a grassy field. The fox is facing right, with its head turned slightly towards the camera. Its fur is a rich reddish-orange color. The background is a soft-focus green, suggesting a natural, outdoor setting.

# Takeaway?

we need to be  
able to compute

$$A^n$$

to understand  
system dynamics

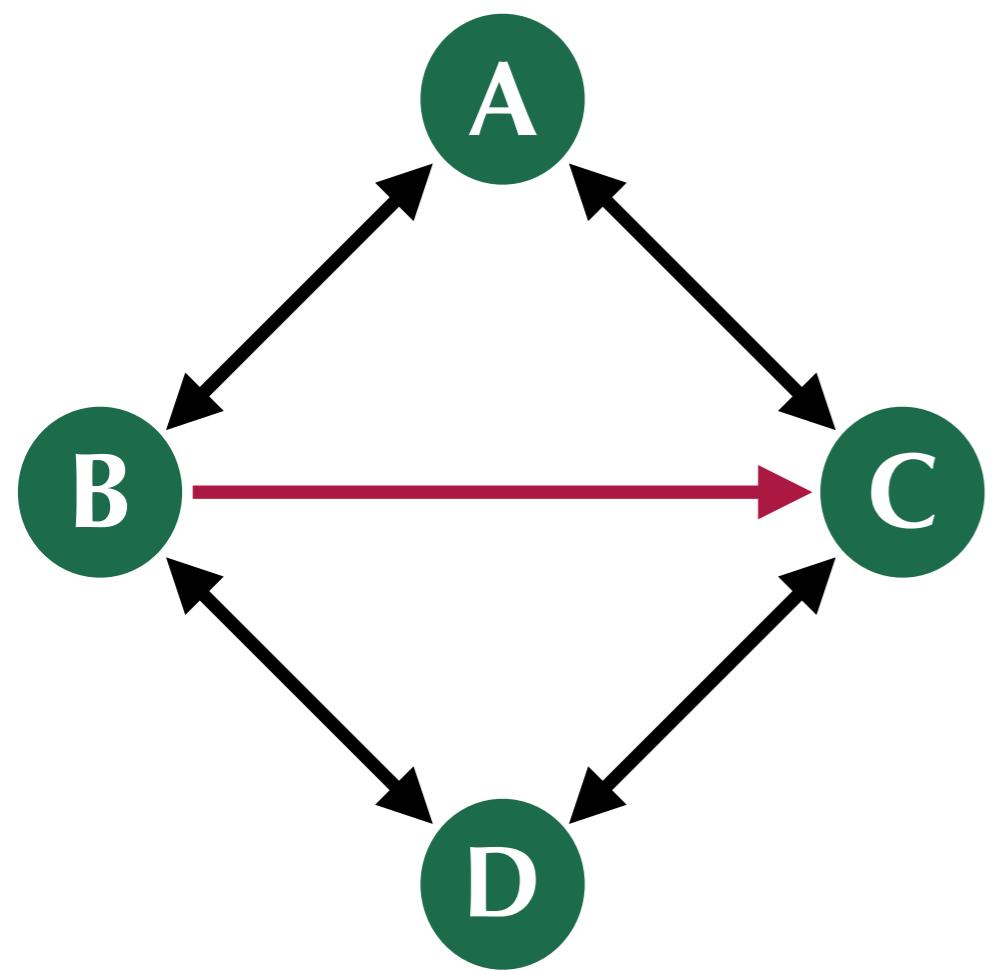
A close-up photograph of a red fox sitting in a grassy field. The fox is facing right, with its head turned slightly to look over its shoulder. Its fur is a rich reddish-orange color. The background is a soft-focus green, suggesting a natural outdoor setting.

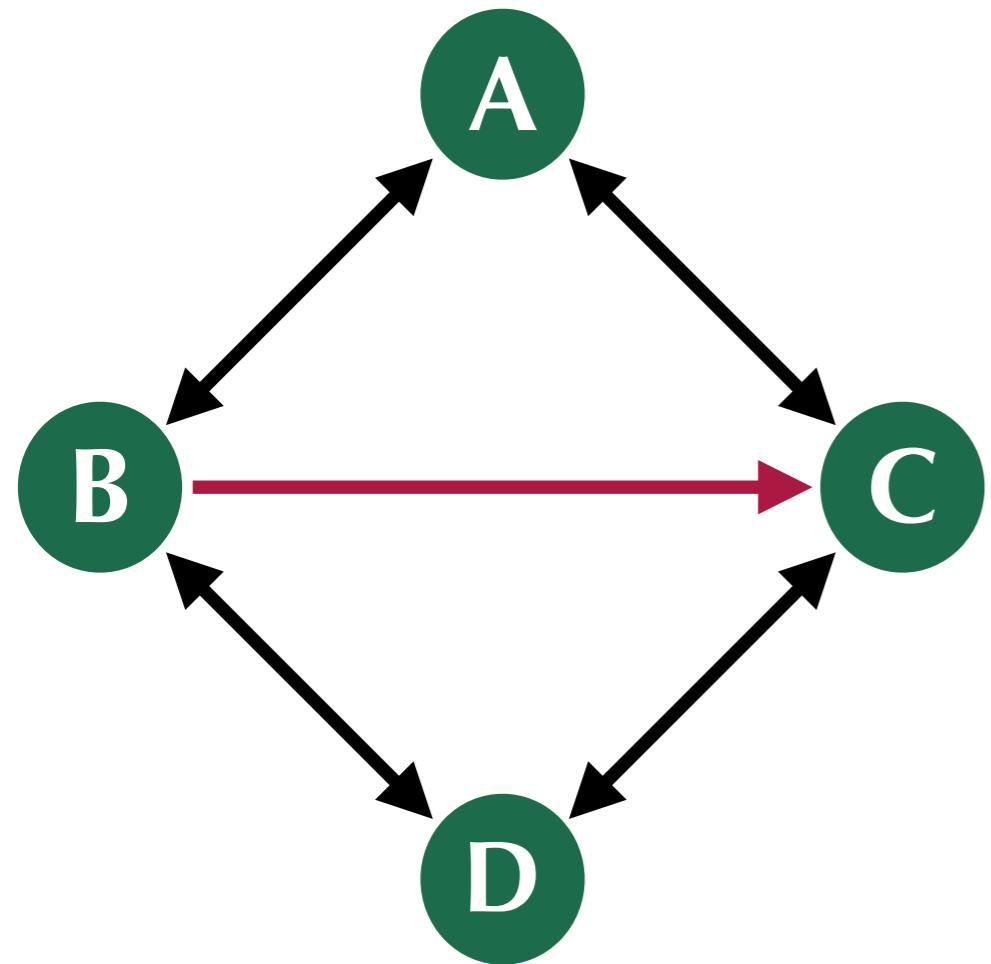
# Takeaway?

eigenvectors and  
values help us:

compute  $A^n$  quickly

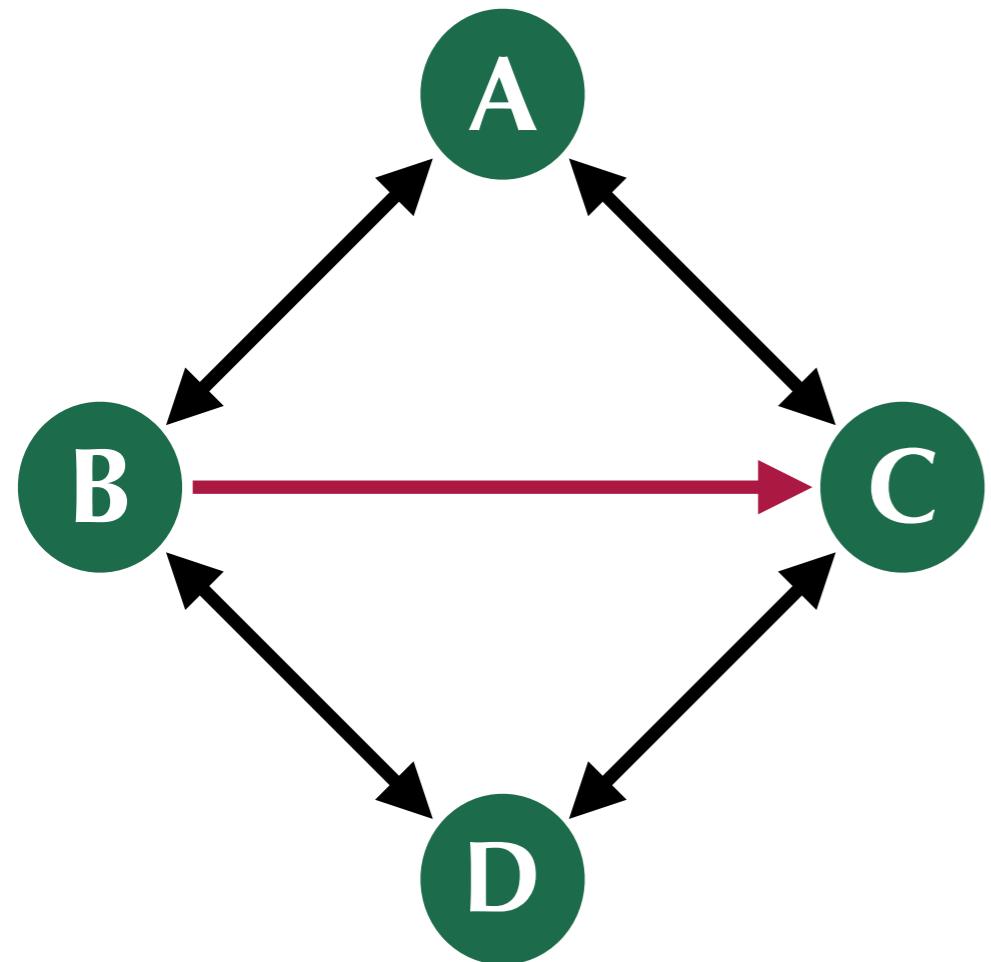
understand  
convergence to  
steady states





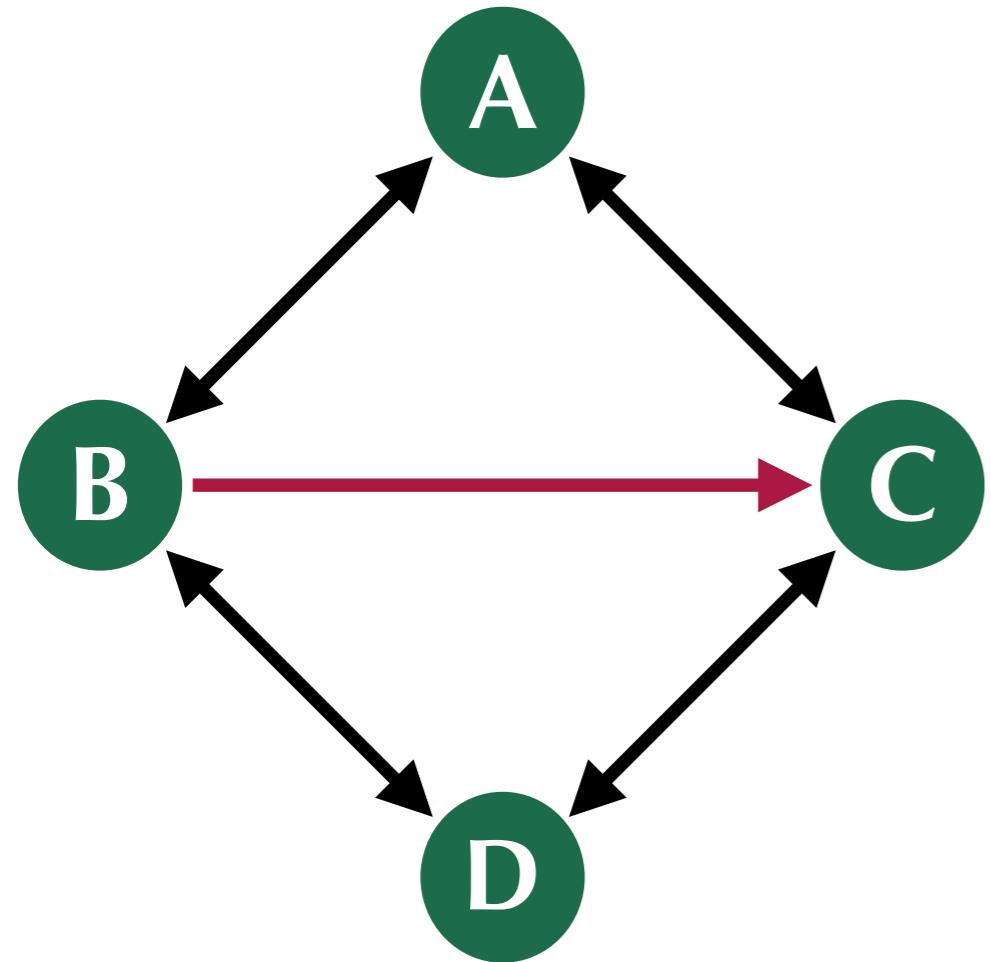
## Adjacency Matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



## Adjacency Matrix

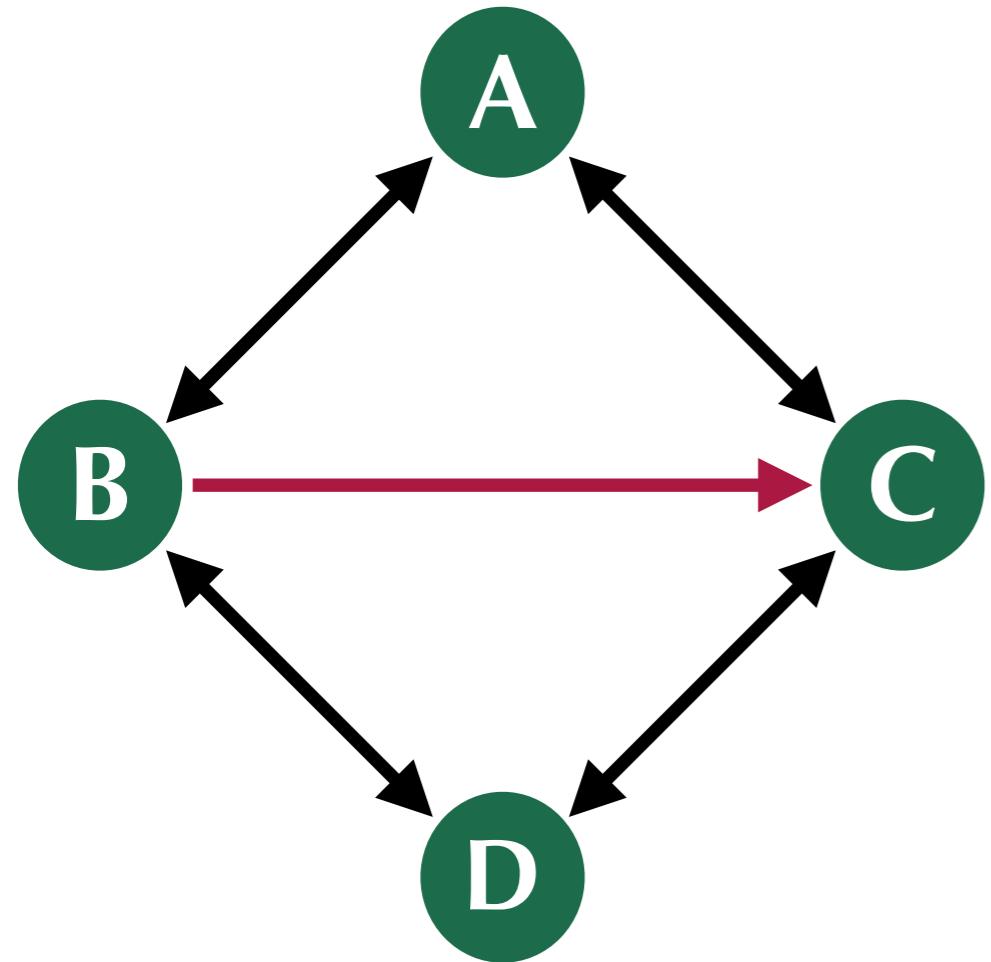
$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{l} A \\ B \\ C \\ D \end{array}$$



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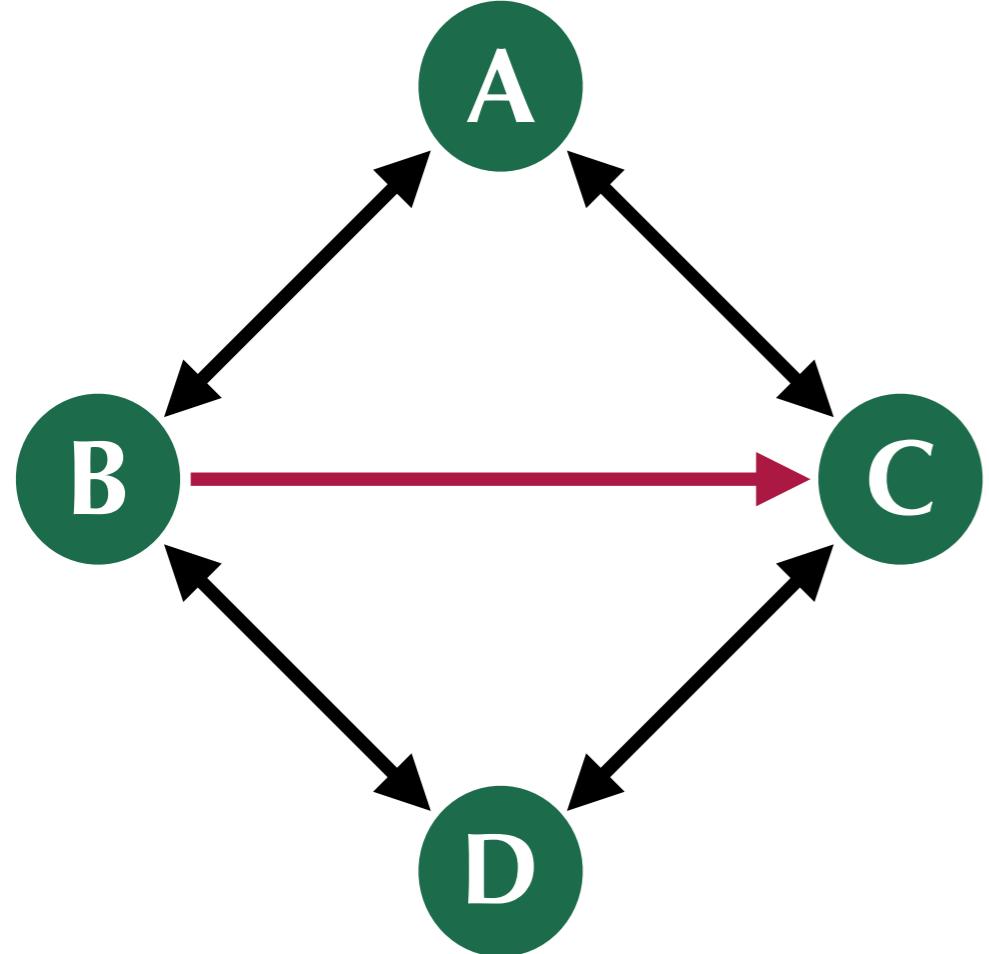
$m_{ij} = 1$  iff there's an edge from  $i$  to  $j$



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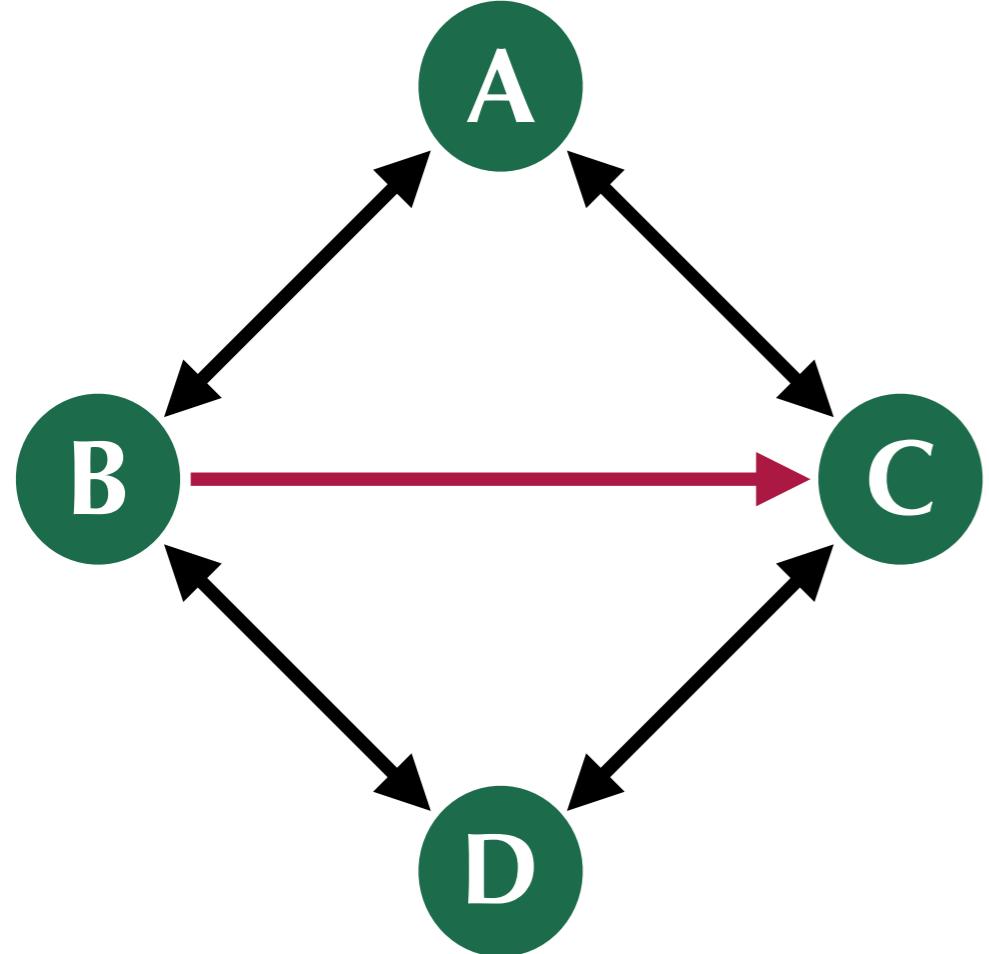
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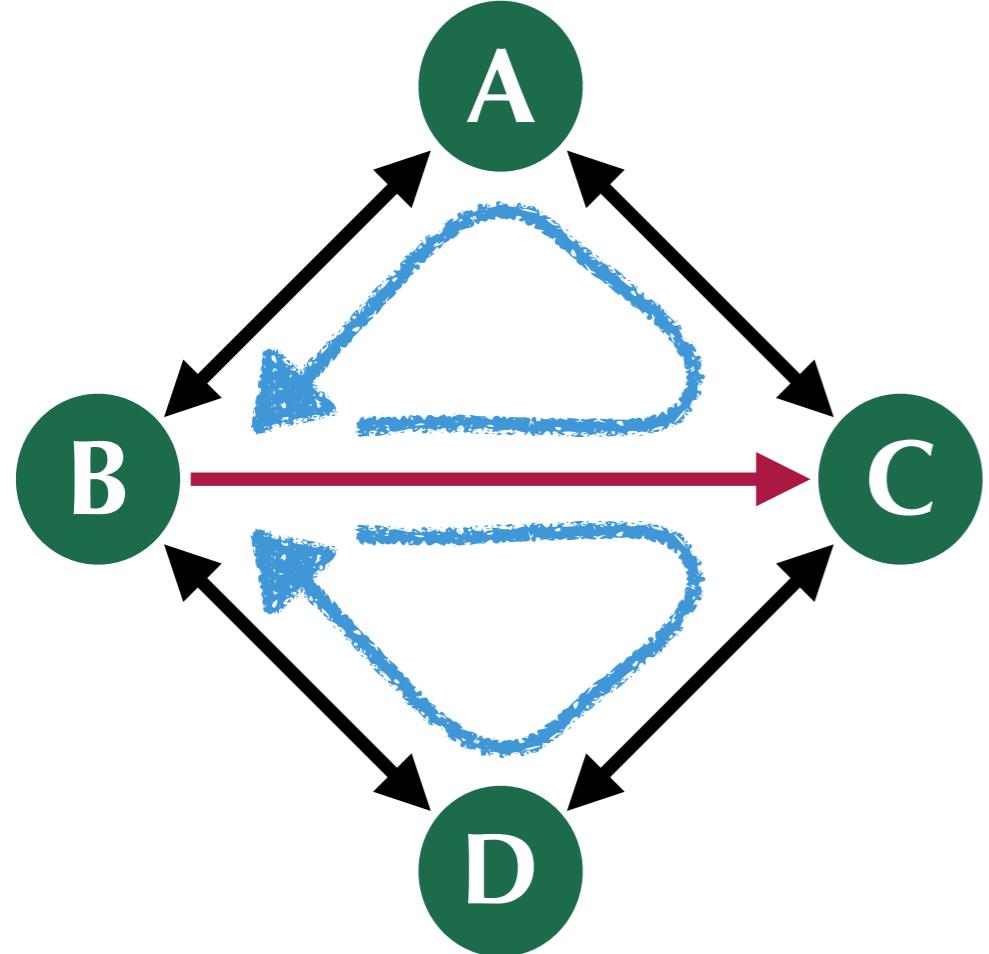
$M^n$  is a matrix where  $m_{ij}^{(n)}$  represents the number of  $n$ -step paths from  $i$  to  $j$



$$M^3 = \begin{bmatrix} A & B & C & D \\ 1 & 4 & 4 & 1 \\ 4 & 2 & 4 & 4 \\ 4 & 0 & 2 & 4 \\ 1 & 4 & 4 & 1 \end{bmatrix} \begin{array}{l} A \\ B \\ C \\ D \end{array}$$



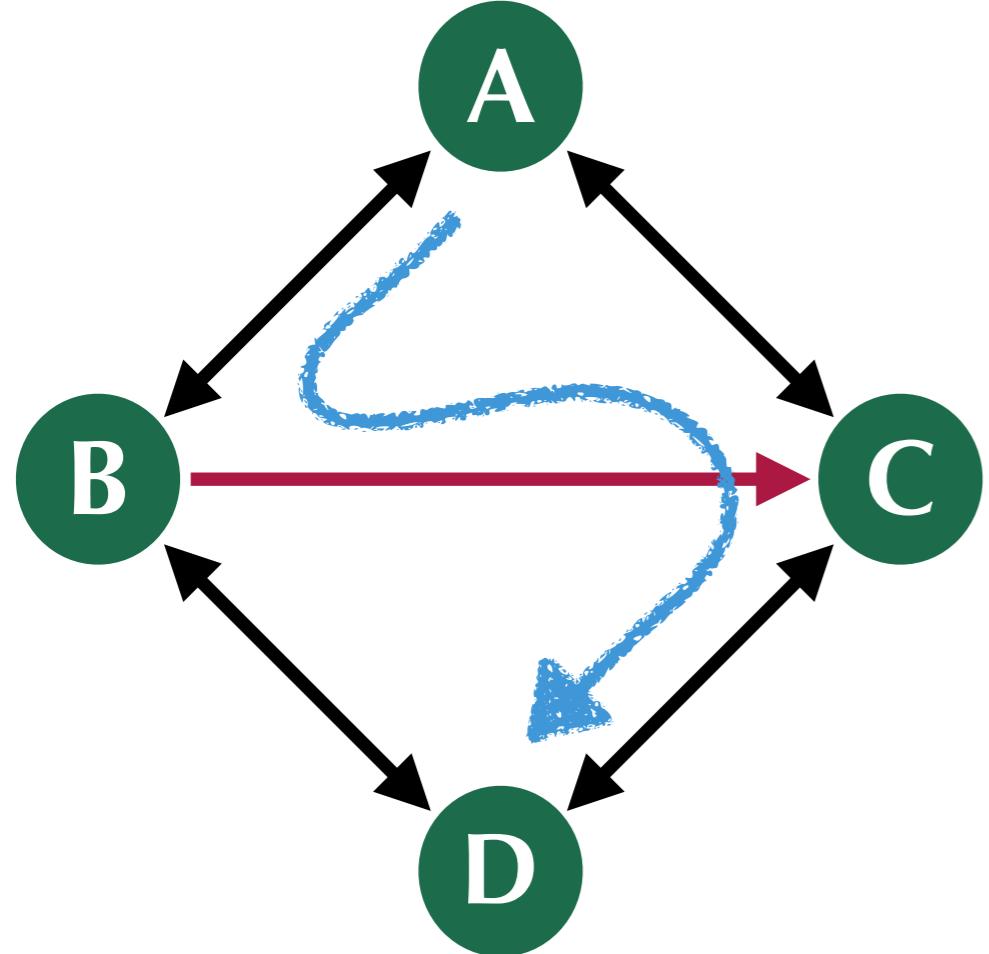
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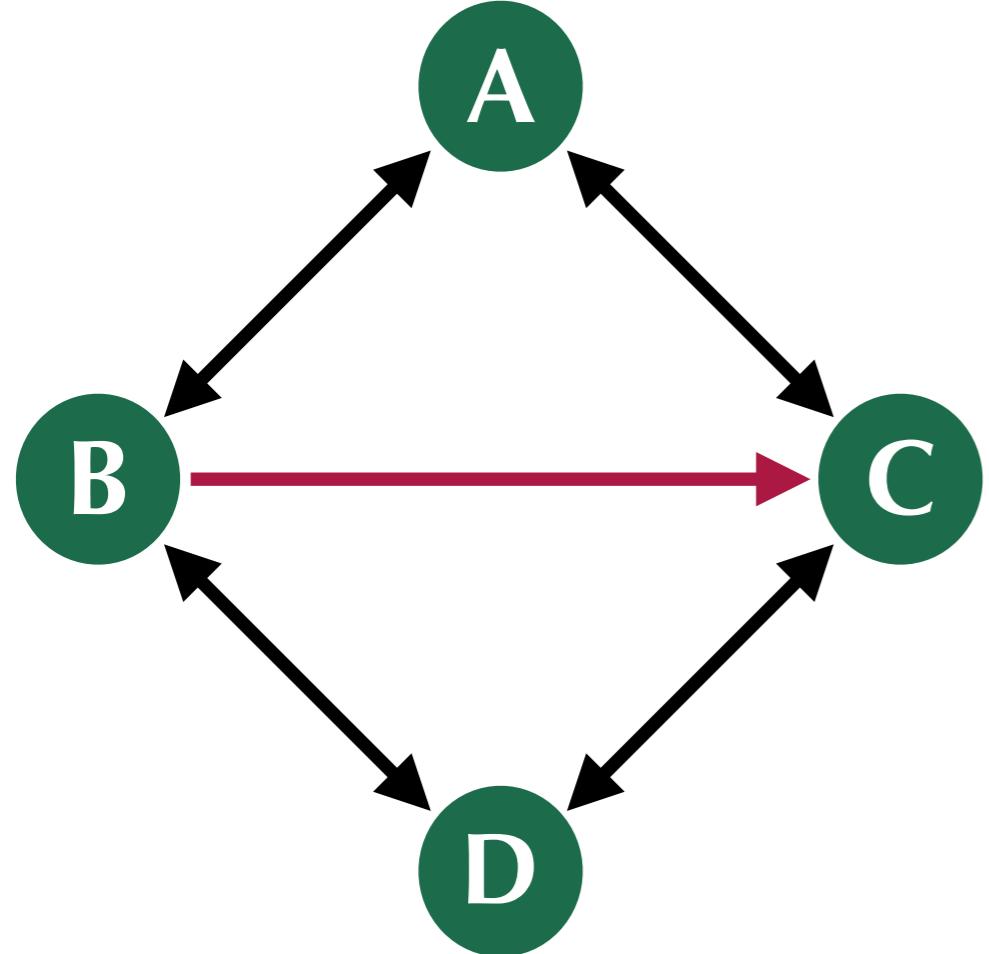
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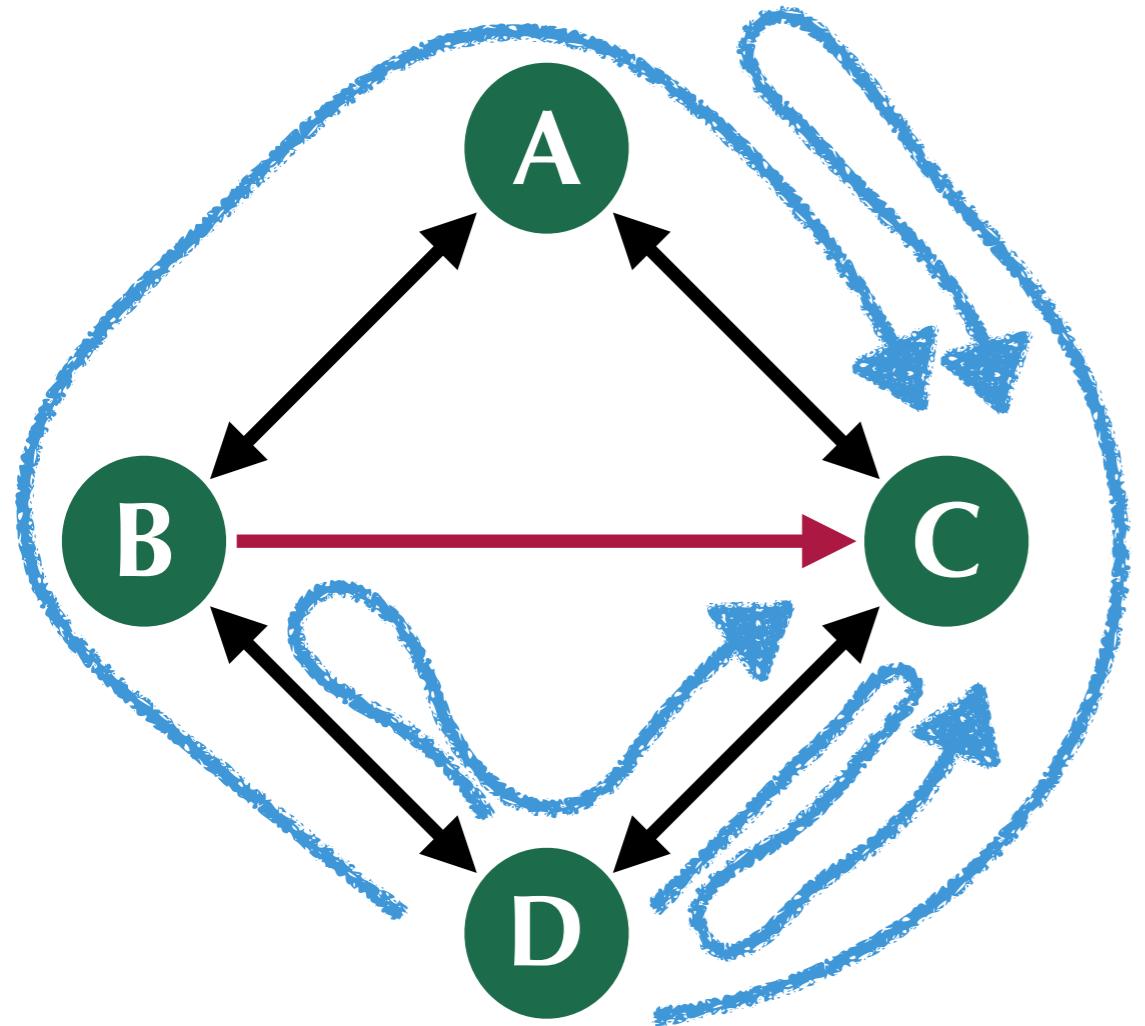


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The matrix  $M^3$  is shown with columns labeled A, B, C, and D. The entry in the third column and fourth row is highlighted with a blue circle.



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# Takeaway?

we need to be able to compute

$$M^n$$

to understand networks



# Takeaway?

eigenvectors and eigenvalues  
help us compute  $M^n$  efficiently



**eigenvectors** and **eigenvalues** help us:

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compute matrix powers efficiently/quickly

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understand how quickly systems converge

**eigenvectors** and **eigenvalues** help us:

compute matrix powers efficiently/quickly

understand how quickly systems converge

see if a matrix is invertible

## how do we find them?

**Definition:** Given  $A_{n \times n}$ , say  $\lambda$  is an eigenvalue for  $A$  if  $A\vec{v} = \lambda\vec{v}$  for some non-zero vector  $\vec{v}$ .

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called an *eigenvalue* of  $A$   
corresponding to  $\lambda$

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$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$

$$[A - \lambda I]\vec{v} = \vec{0}$$

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non-zero

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