motivating eigenvalues and eigenvectors

slides: ajbc.io/eigen

dynamic systems: foxes and rabbits

F_t population of foxes at year t

R_t population of rabbits at year t

$$F_{t+1} = 0.6F_t + 0.2R_t$$

$$R_{t+1} = -0.2F_t + 1.1R_t$$

foxes die without food

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rabbits multiply without predators

foxes die without food

foxes multiply with increased food supply

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foxes multiply with increased food supply

$$F_{t+1} = 0.6F_t + 0.2R_t$$

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foxes kill off rabbits

rabbits multiply without predators

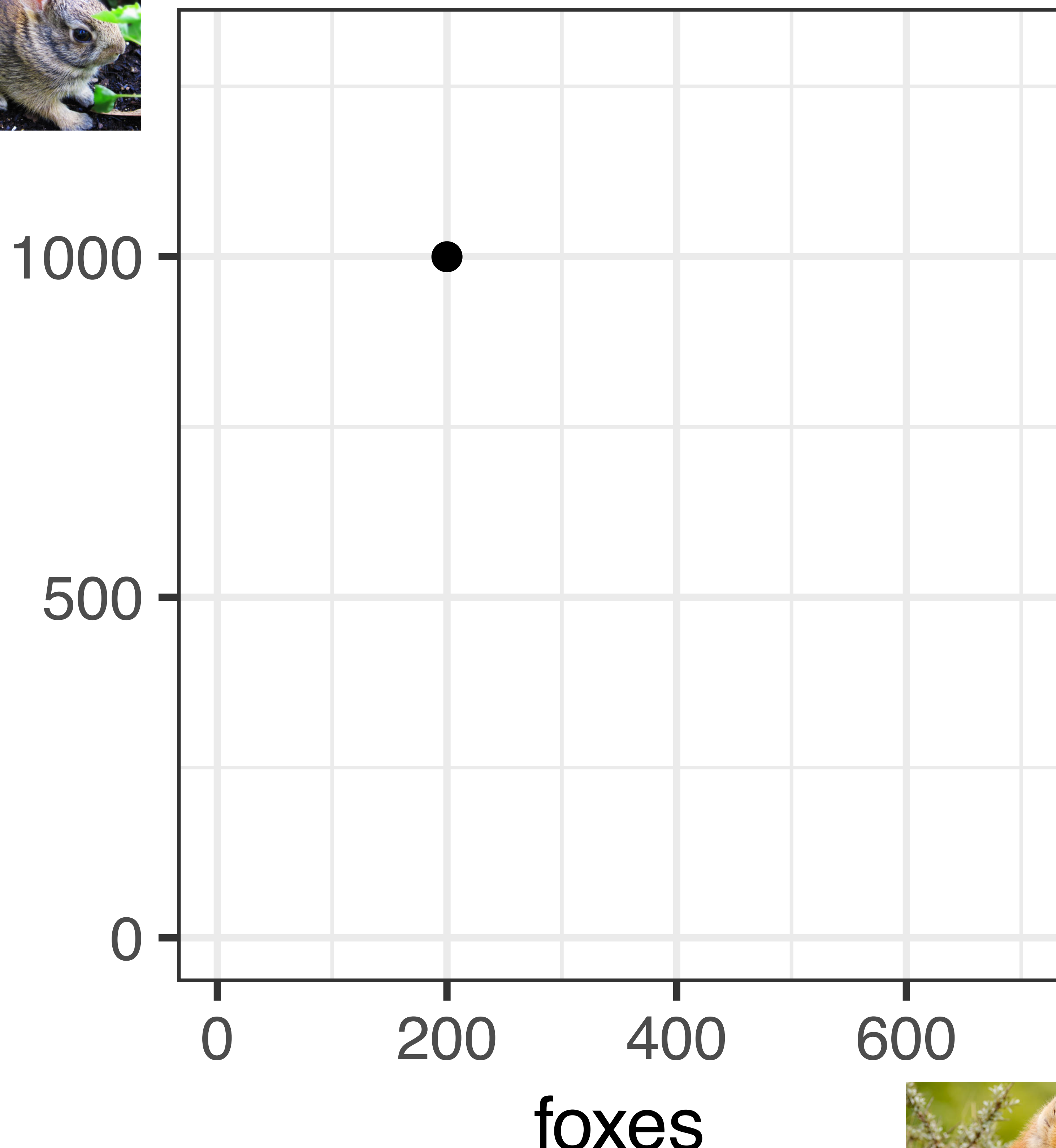
$$F_{t+1} = 0.6F_t + 0.2R_t$$

$$R_{t+1} = -0.2F_t + 1.1R_t$$

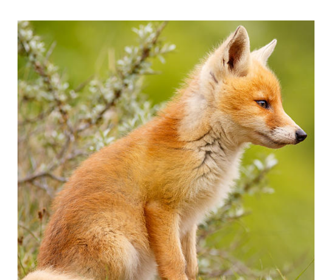
$$\begin{bmatrix} F_{t+1} \\ R_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} F_t \\ R_t \end{bmatrix}$$

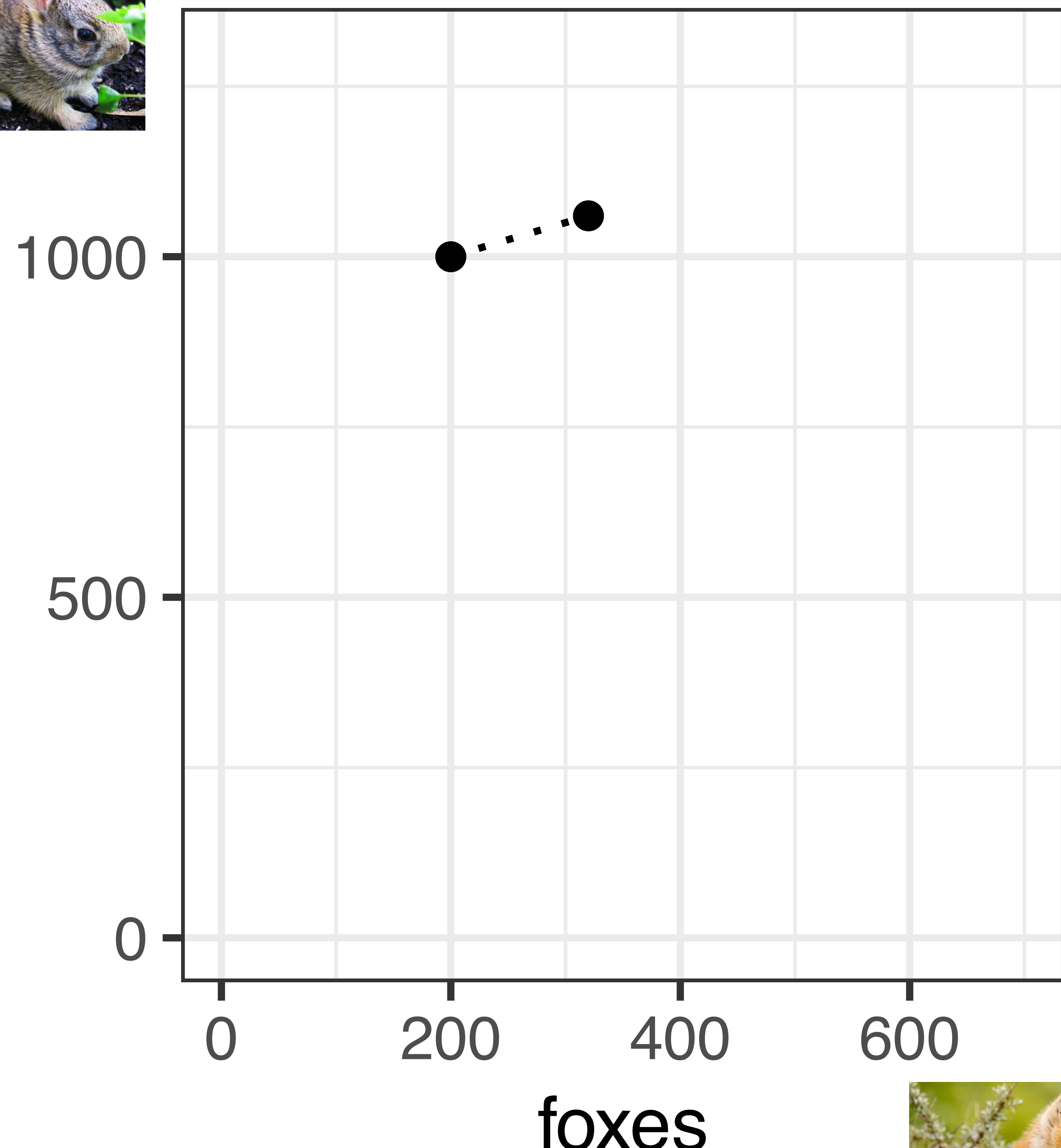
$$\begin{bmatrix} F_{t+1} \\ R_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} F_t \\ R_t \end{bmatrix}$$

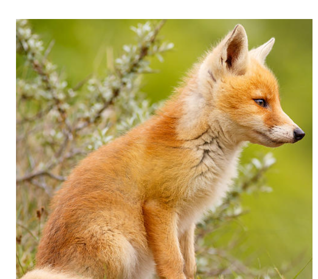
$$\vec{p}_{t+1} = A \vec{p}_t$$



$$\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$$

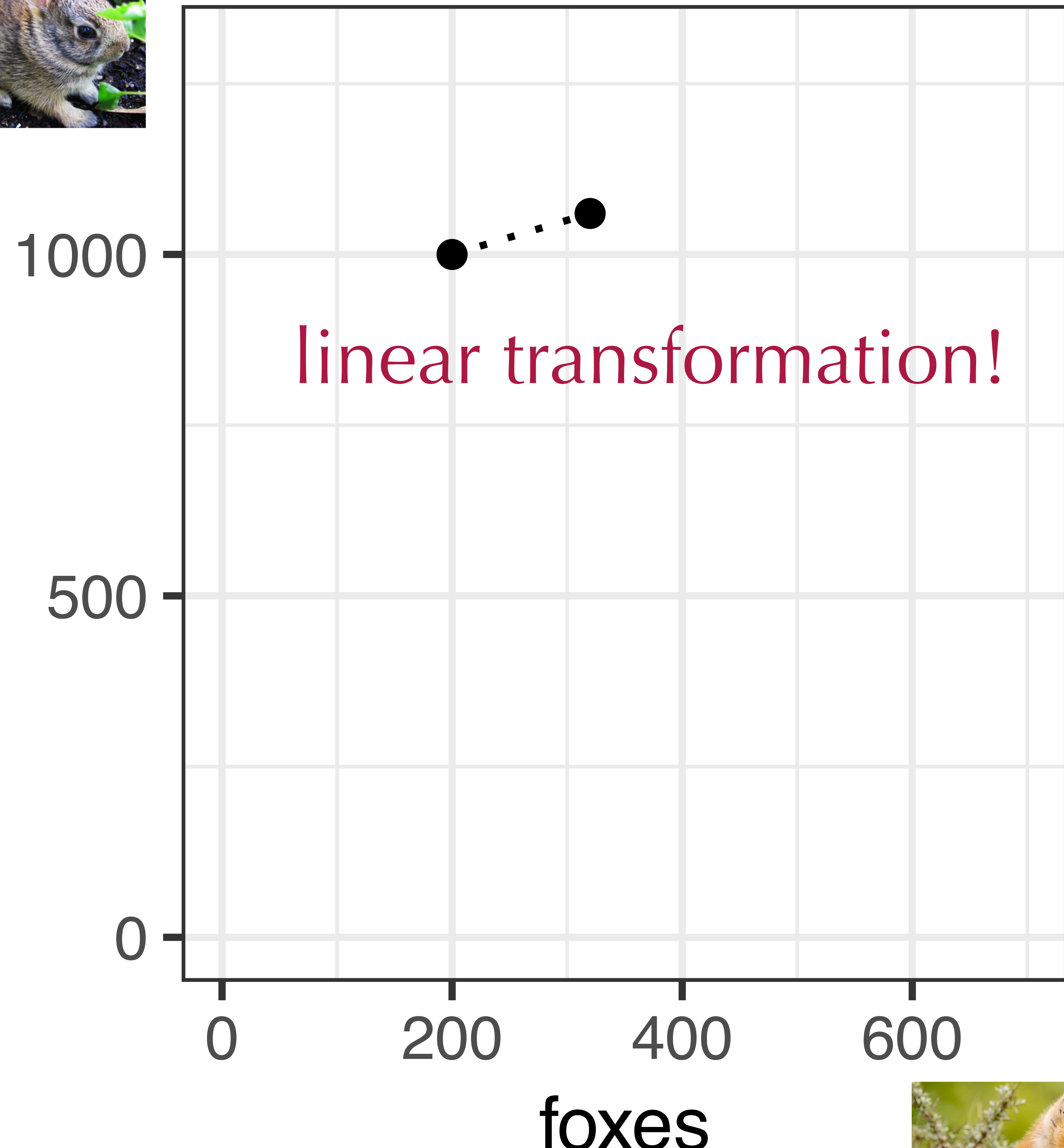


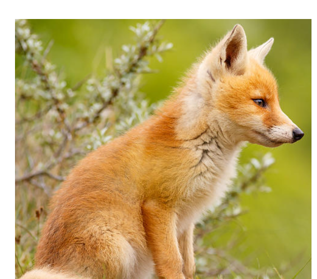




$$\begin{bmatrix}F_0\\R_0\end{bmatrix} = \begin{bmatrix}200\\1000\end{bmatrix}$$

$$\begin{bmatrix}F_1\\R_1\end{bmatrix} = \begin{bmatrix}0.6 & 0.2\\-0.2 & 1.1\end{bmatrix}\begin{bmatrix}F_0\\R_0\end{bmatrix} = \begin{bmatrix}320\\1060\end{bmatrix}$$





$$\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$$

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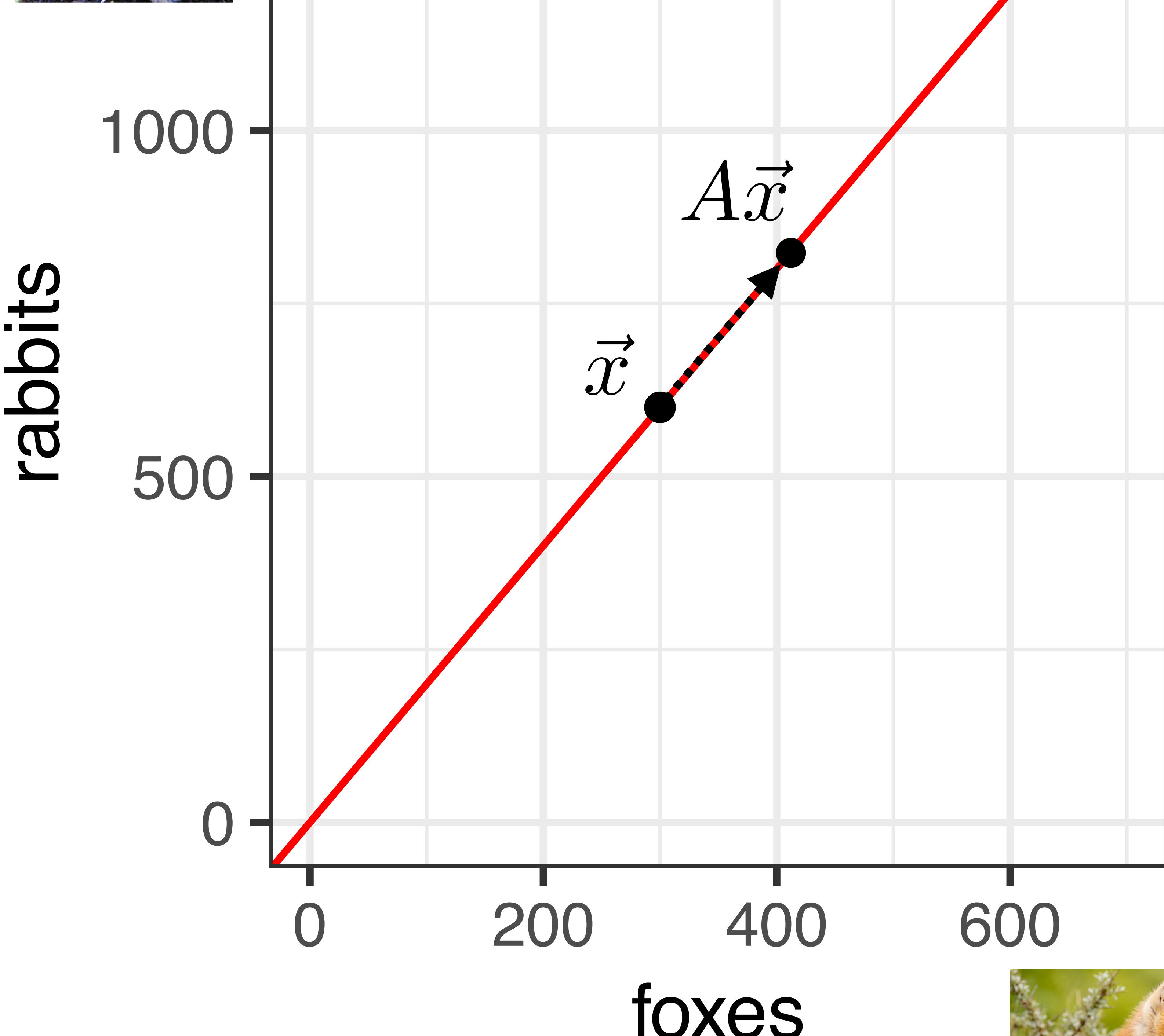
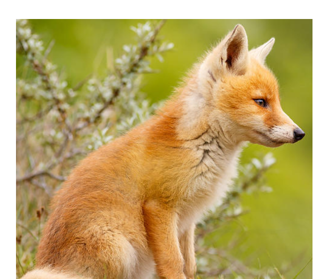
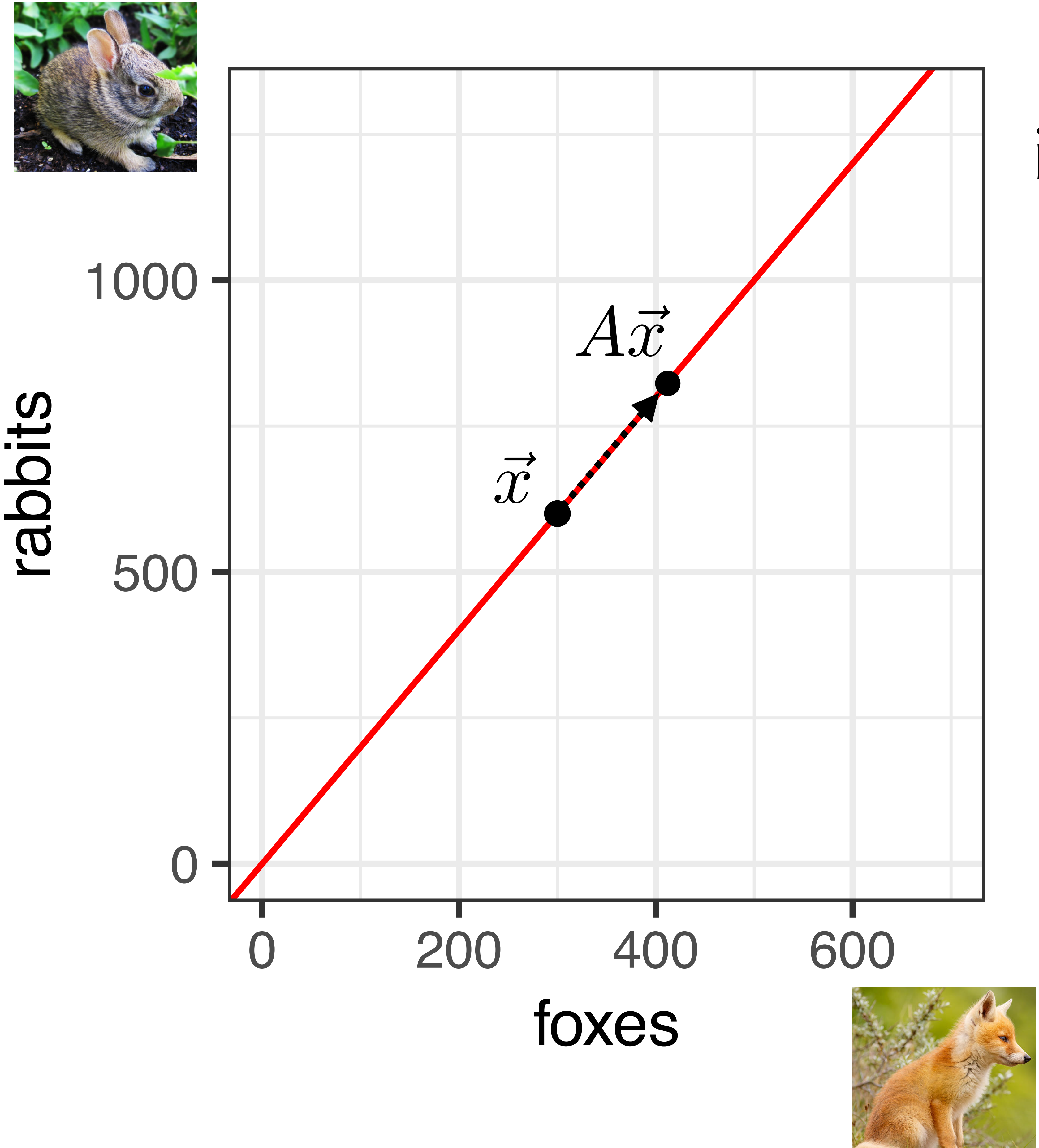




imagine a special case where

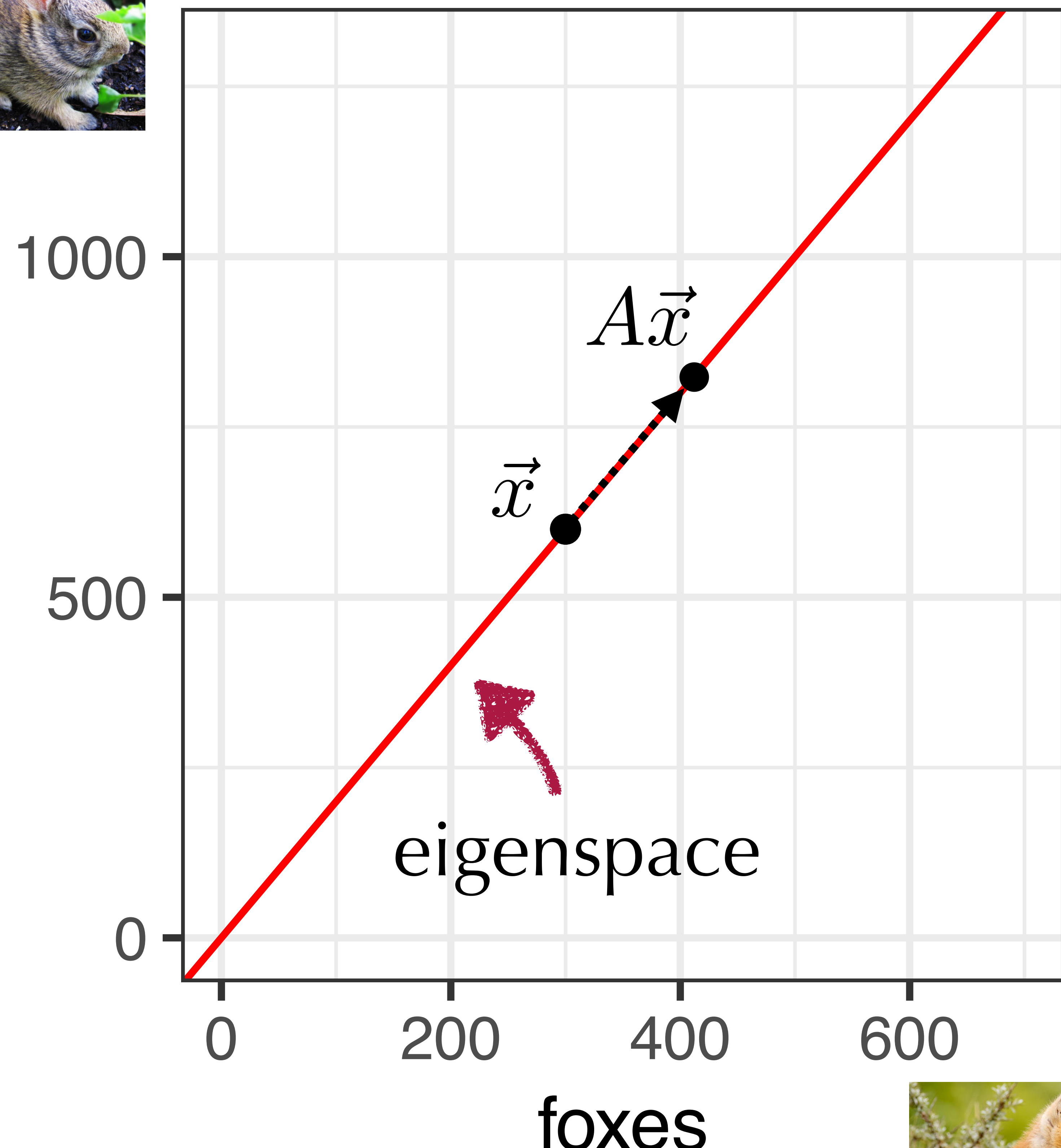
$$A\vec{x} = \lambda\vec{x}$$

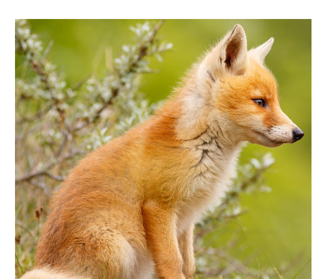


imagine a special case where

$$A\vec{x} = \lambda\vec{x}$$

scalar



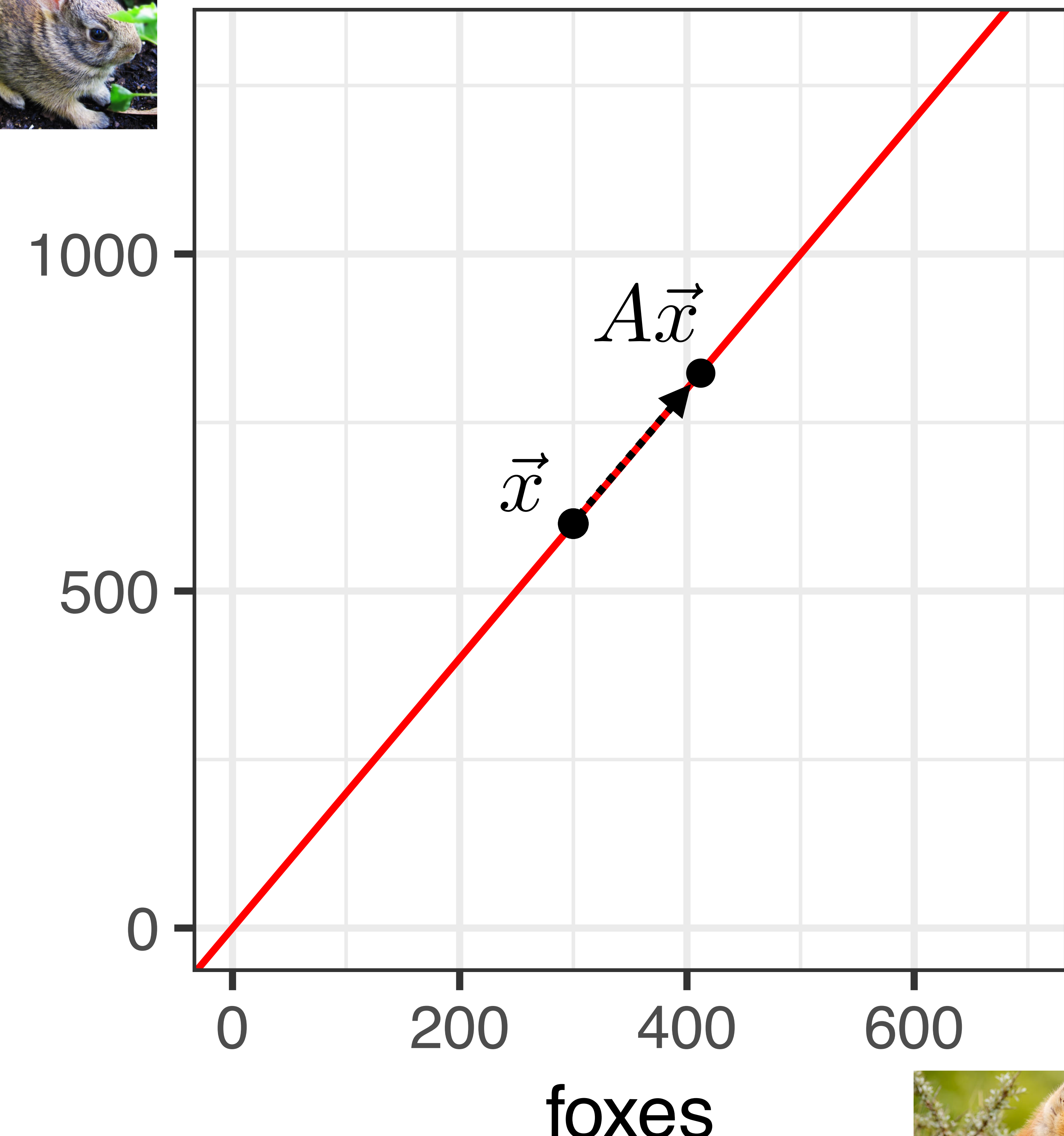


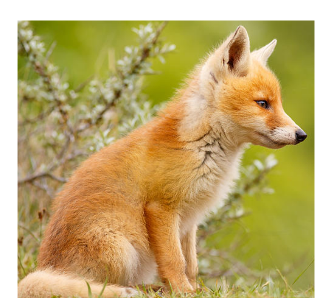
imagine a special case where

$$A\vec{x} = \lambda\vec{x}$$

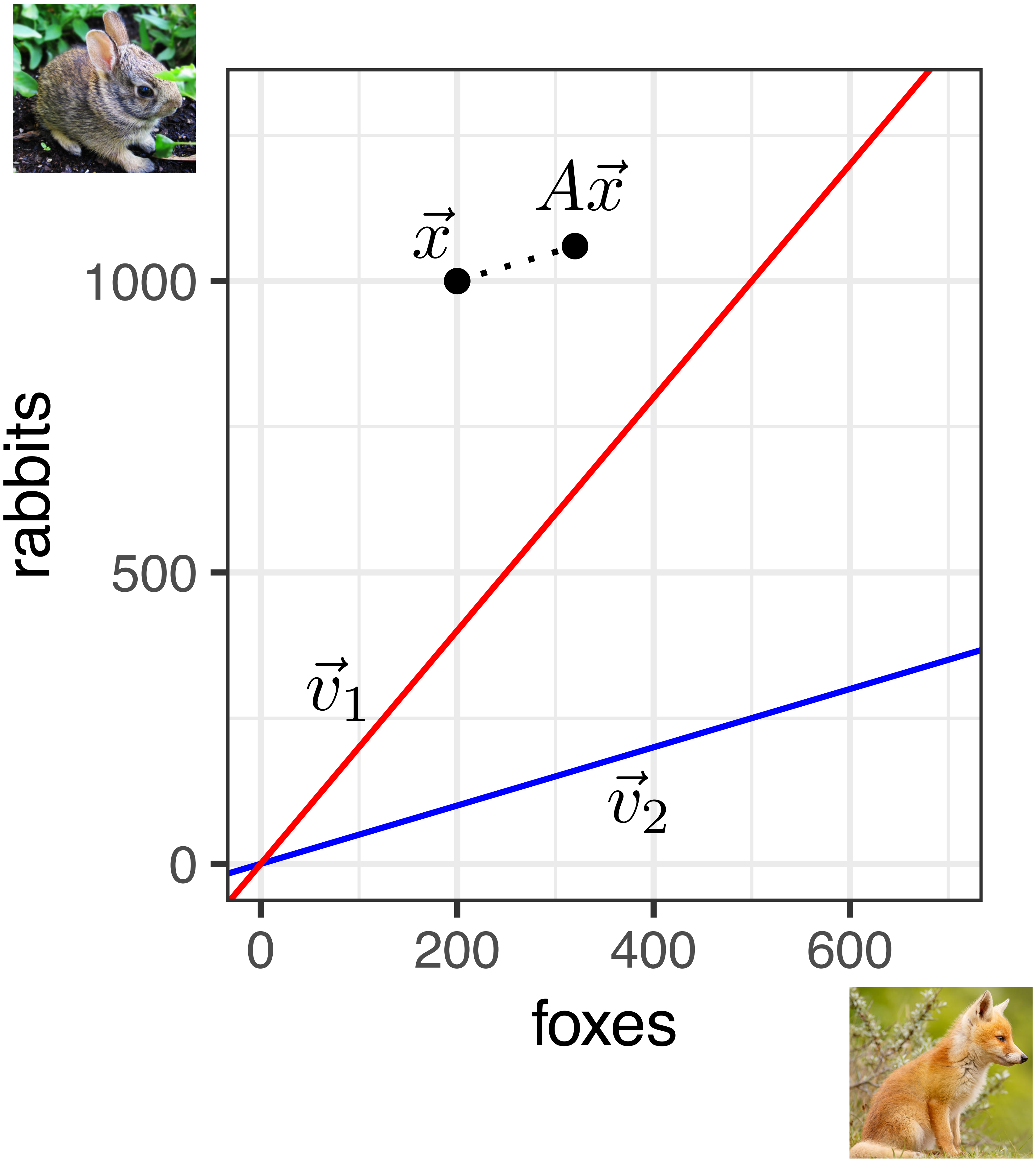
eigenvalue

eigenvector

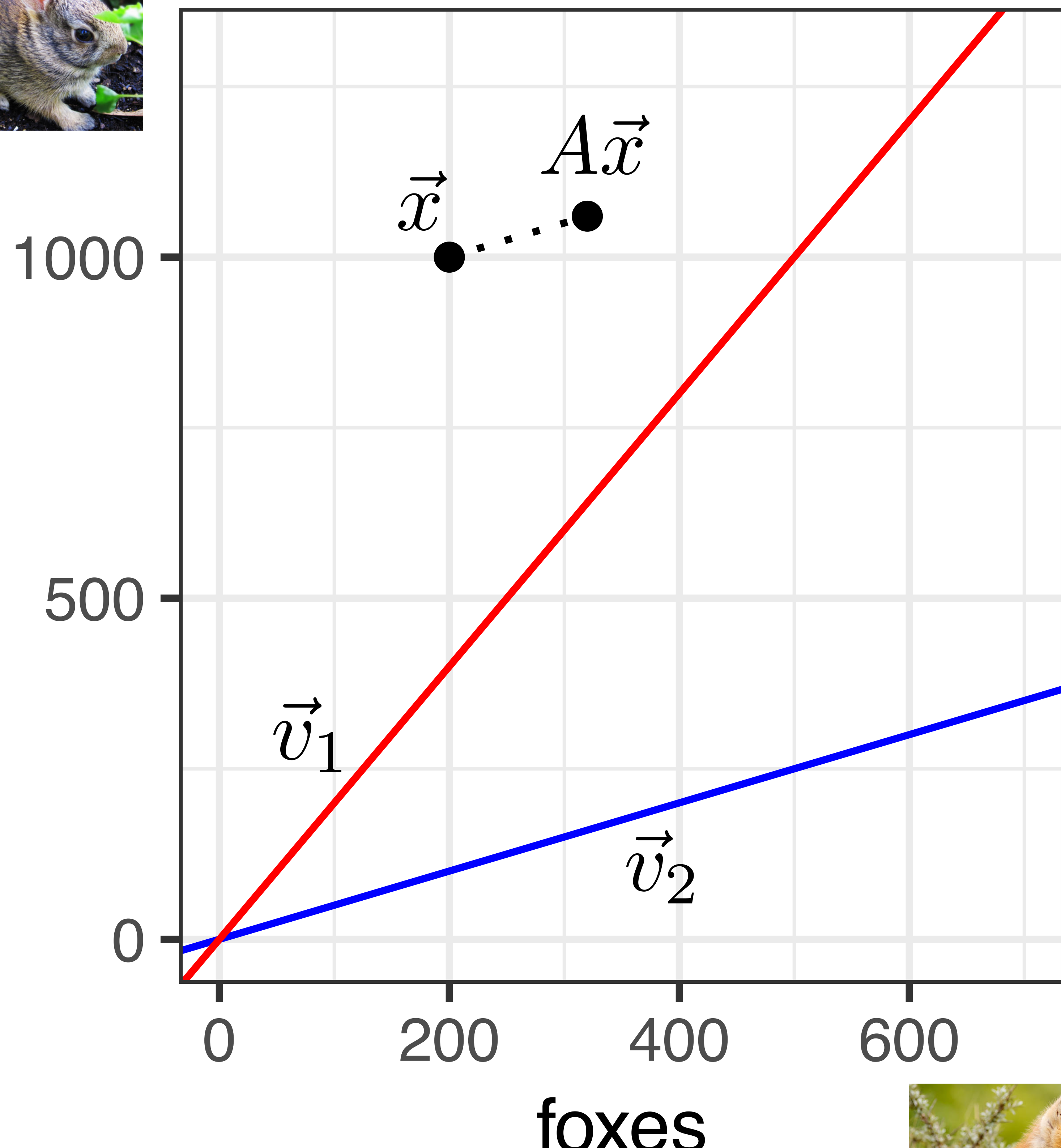


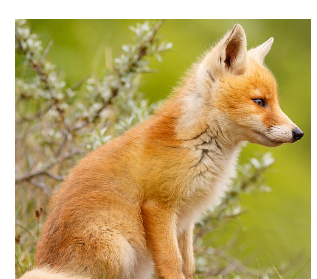


we can think of
eigenvectors as
a set of basis vectors



$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$





$$A\vec{v}_1 = \lambda_1\vec{v}_1$$
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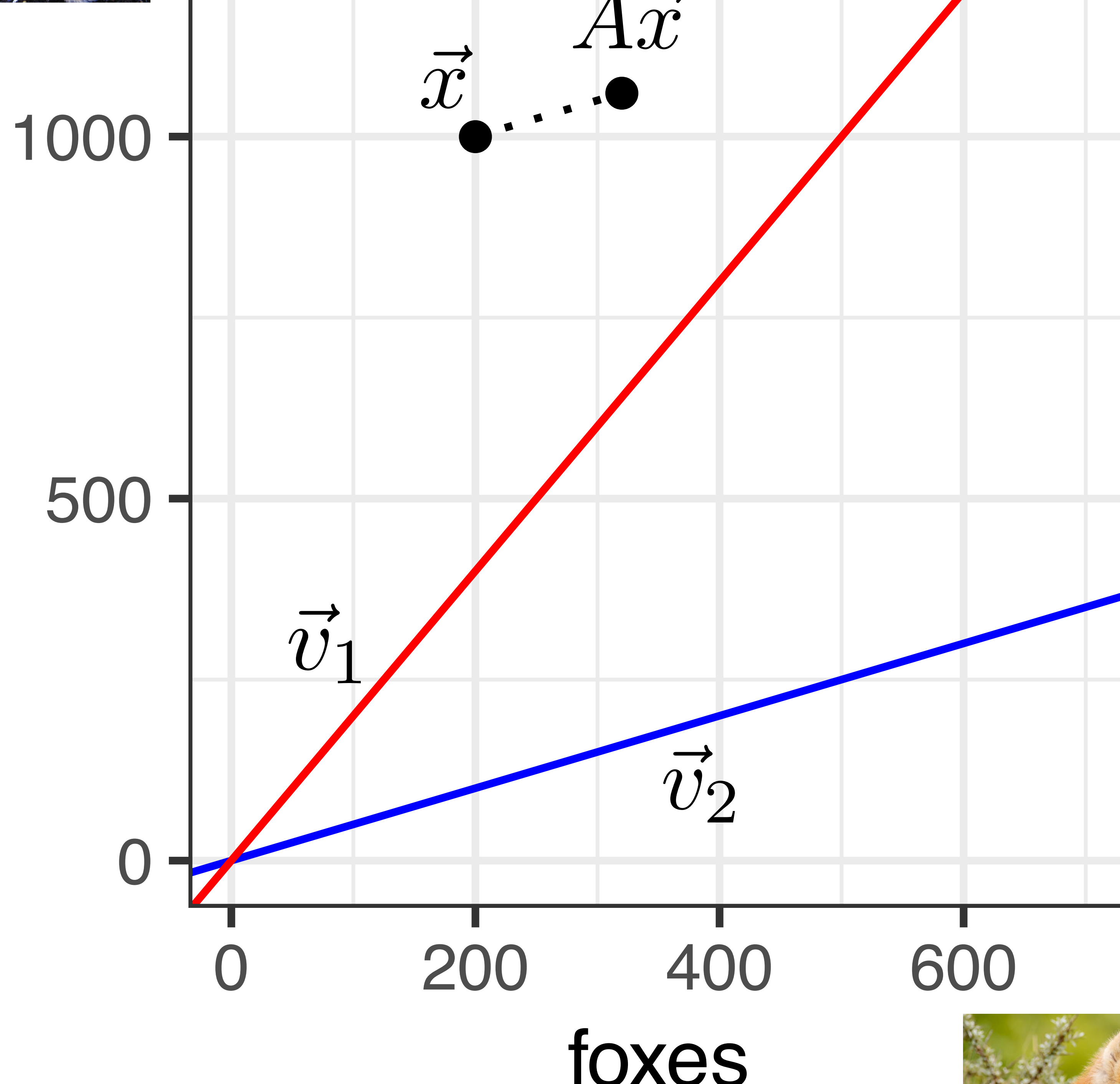
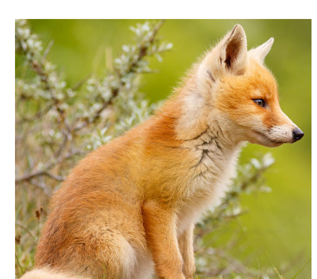
$$\vec{x} = a\vec{v}_1 + b\vec{v}_2$$





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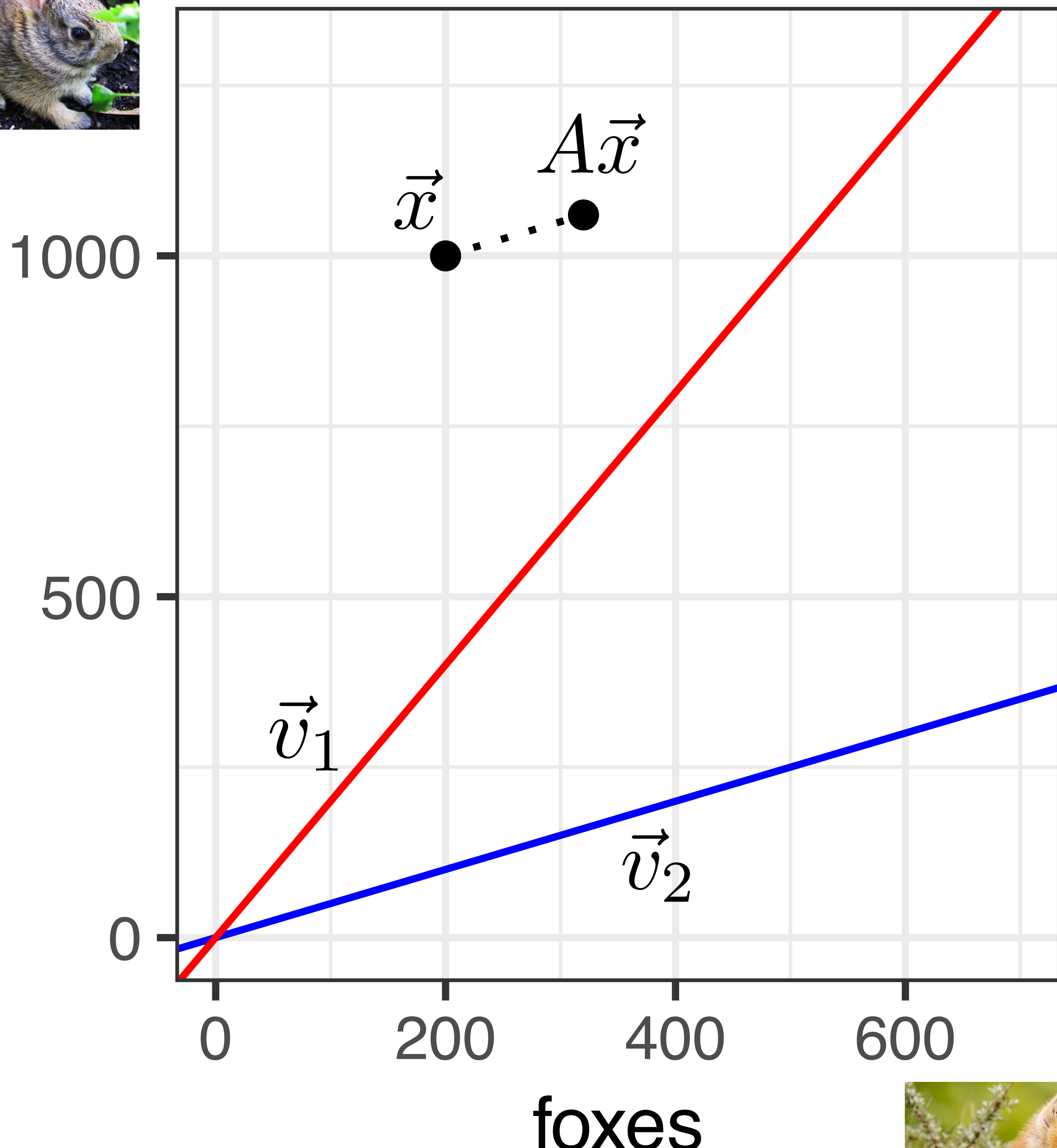

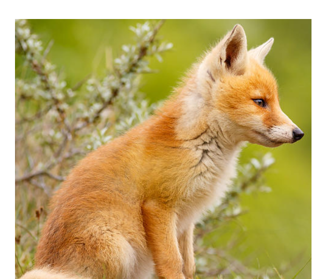

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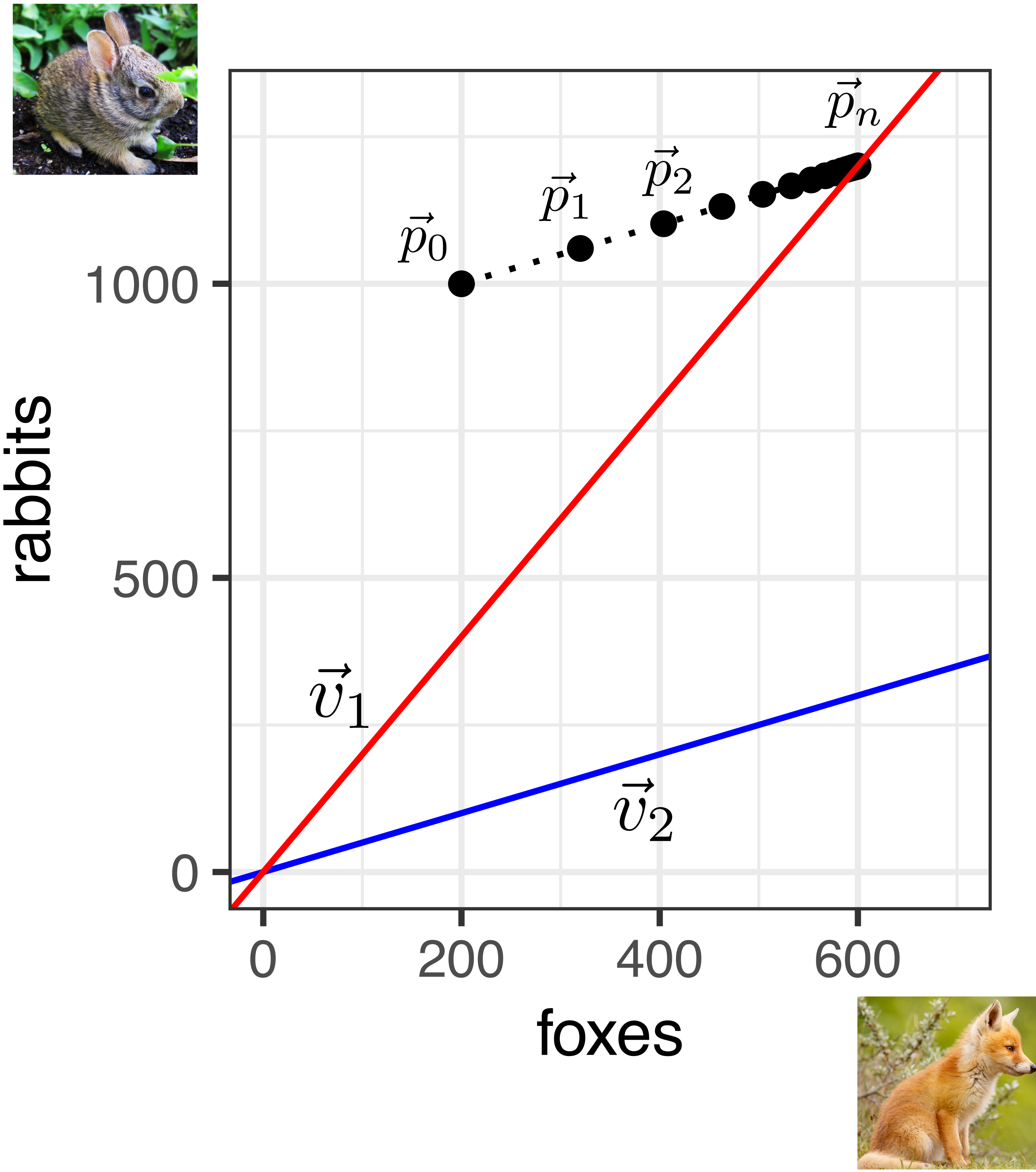


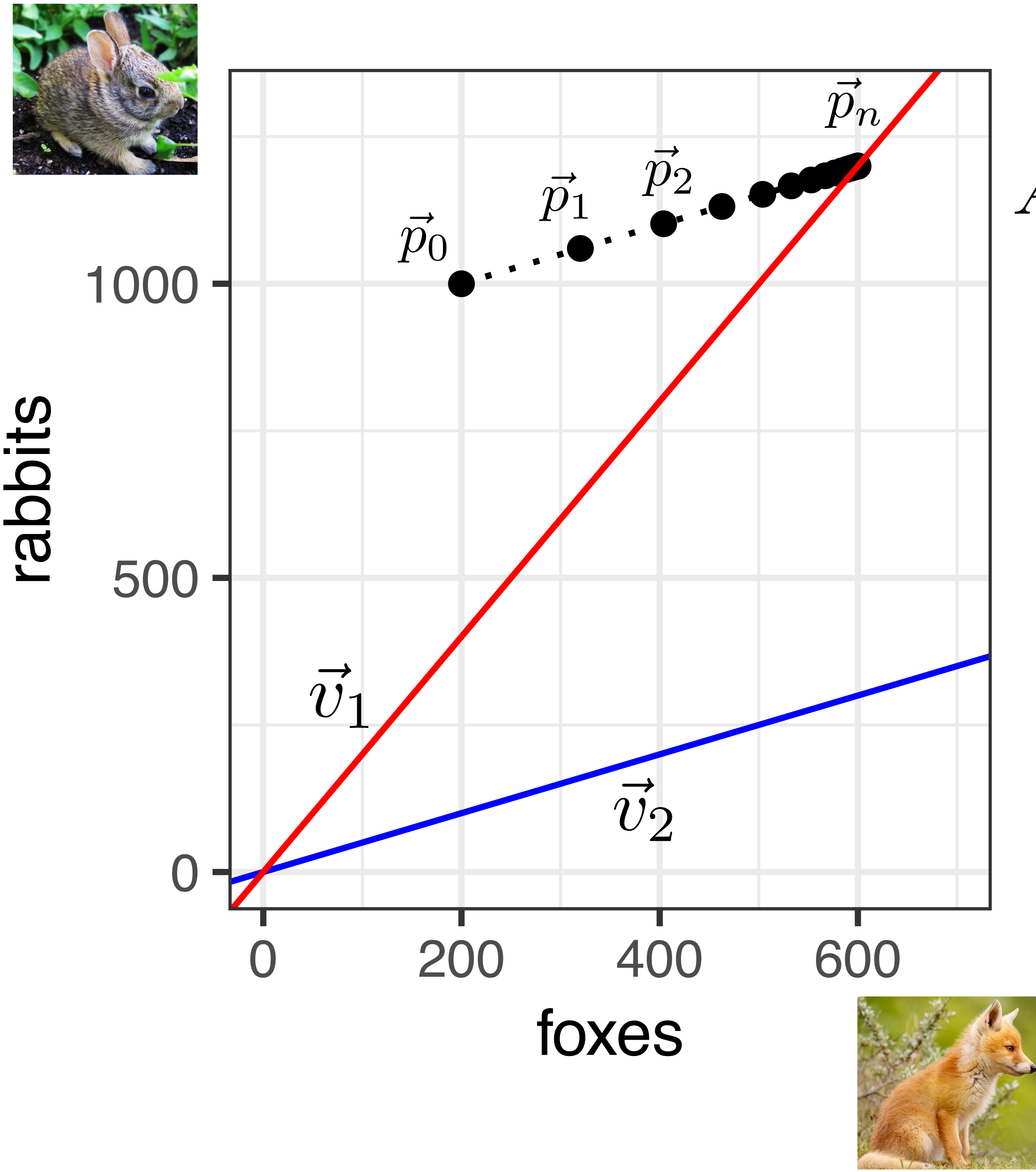
$$\vec{p}_1 = A\vec{p}_0$$

$$\vec{p}_2 = A\vec{p}_1 = A^2\vec{p}_0$$

$$\vec{p}_3 = A\vec{p}_2 = A^3\vec{p}_0$$

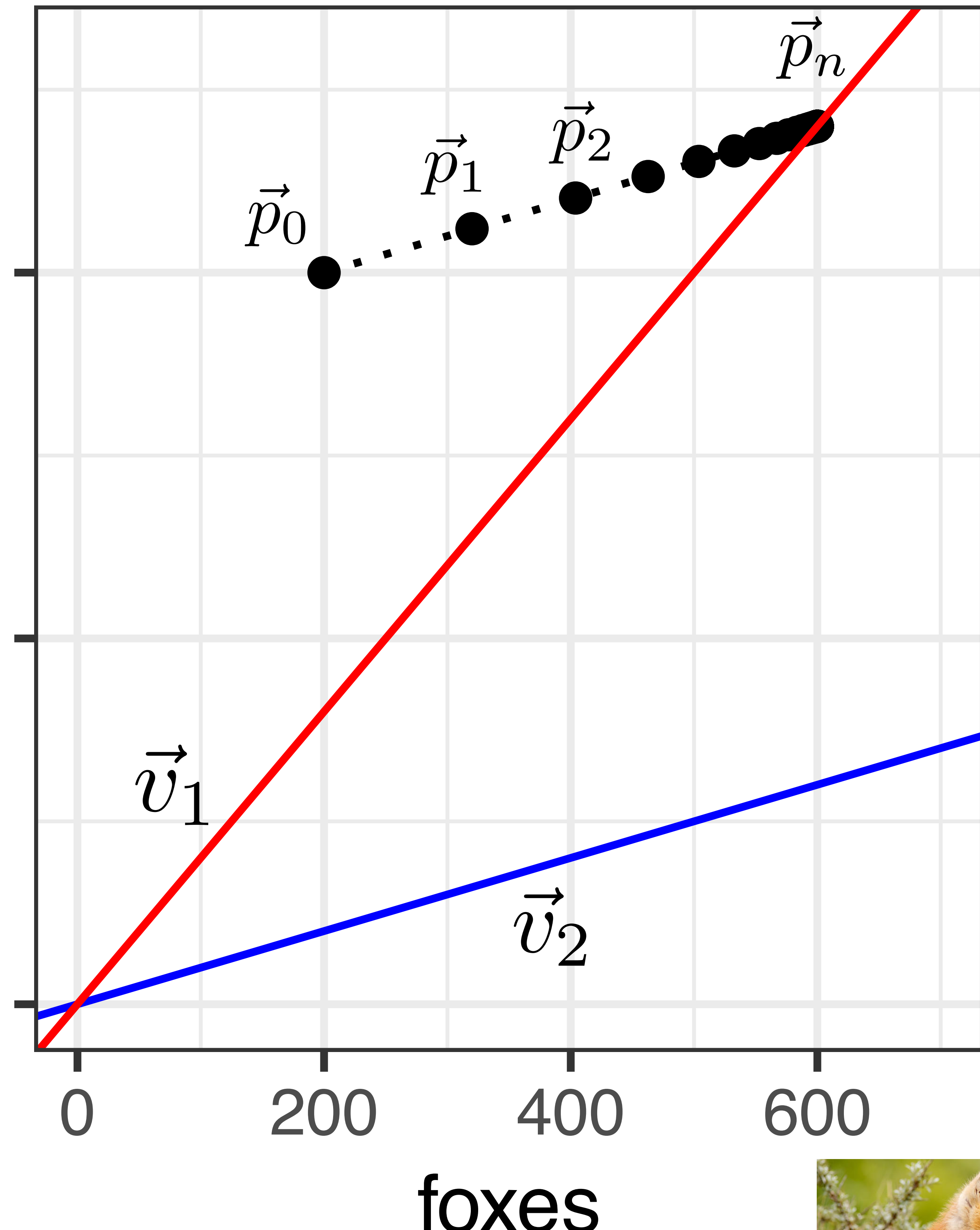
$$\vdots$$

$$\vec{p}_n = A\vec{p}_{n-1} = A^n\vec{p}_0$$



$$A\vec{x} = Aa\vec{v}_1 + Ab\vec{v}_2$$

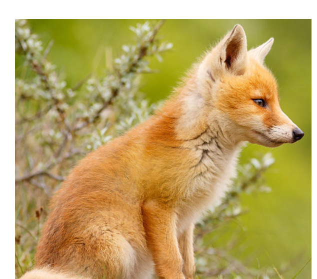
$$= \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2$$

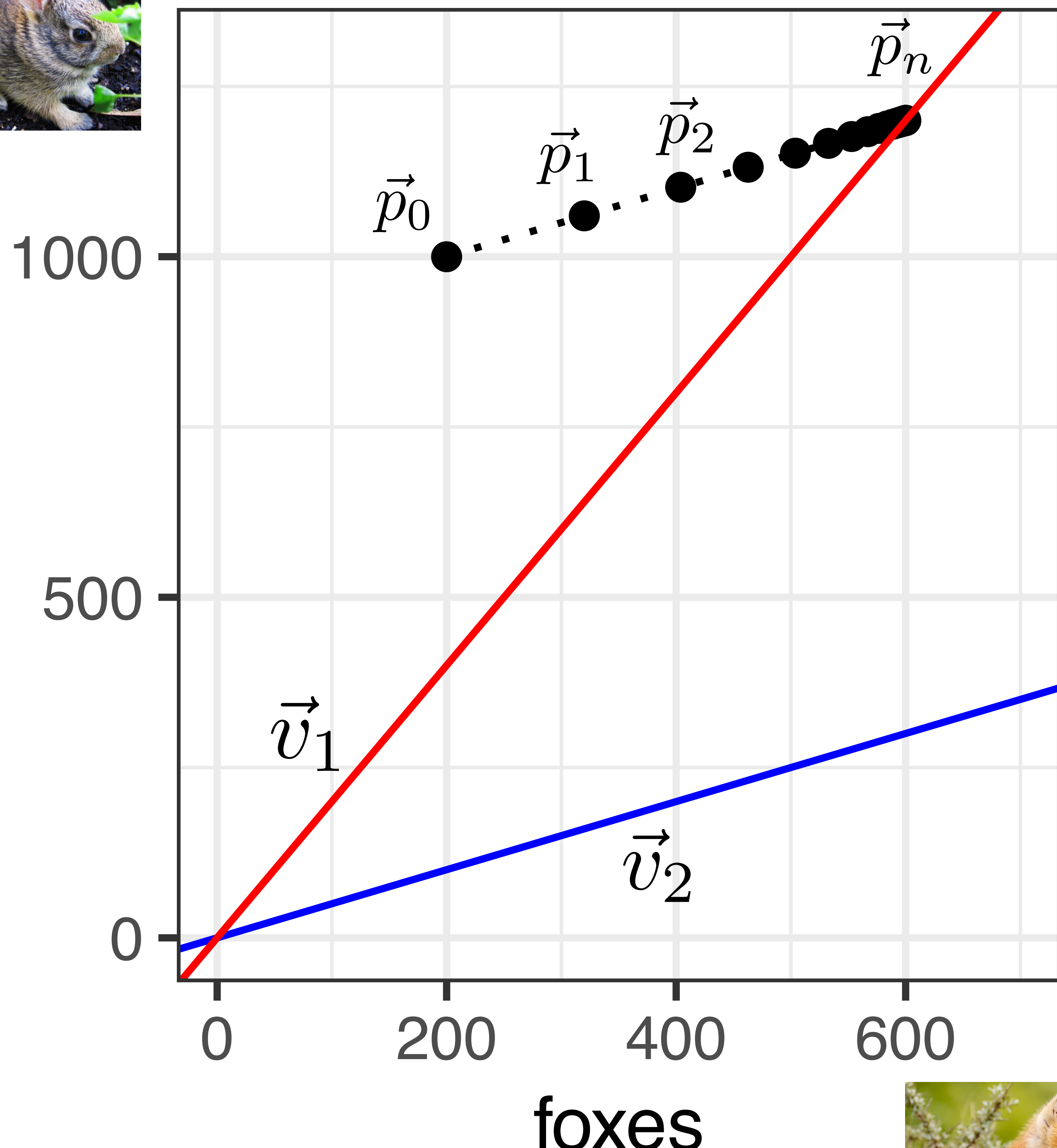


$$A\vec{x} = Aa\vec{v}_1 + Ab\vec{v}_2$$

$$= \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2$$

$$A^n\vec{x} = \lambda_1^n a\vec{v}_1 + \lambda_2^n b\vec{v}_2$$





$$A\vec{x} = Aa\vec{v}_1 + Ab\vec{v}_2$$

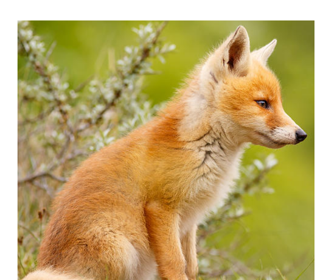
$$= \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2$$

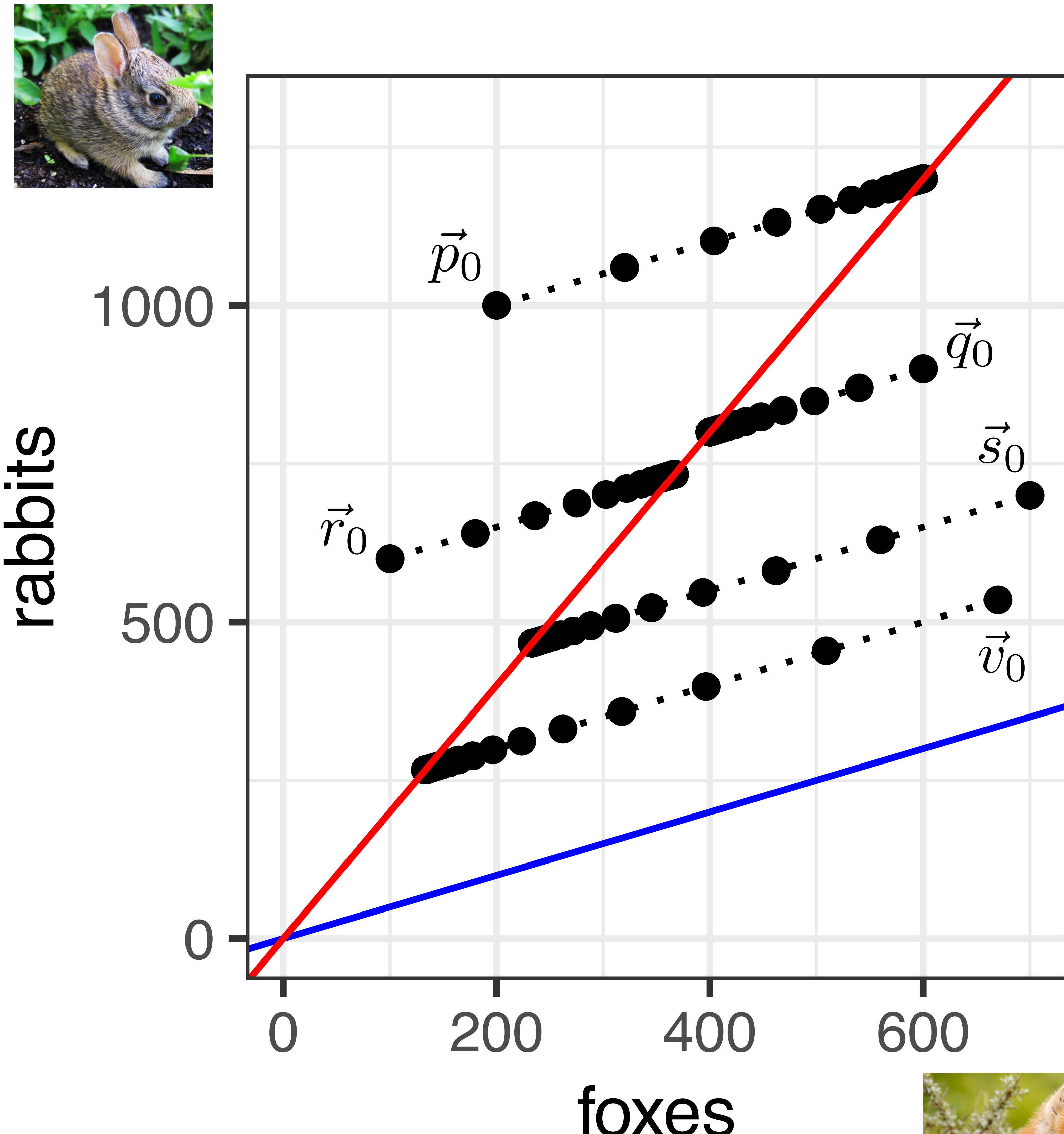
hard

$$A^n\vec{x} =$$

$$\lambda_1^n a\vec{v}_1 + \lambda_2^n b\vec{v}_2$$

easy

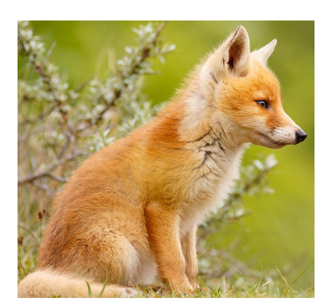


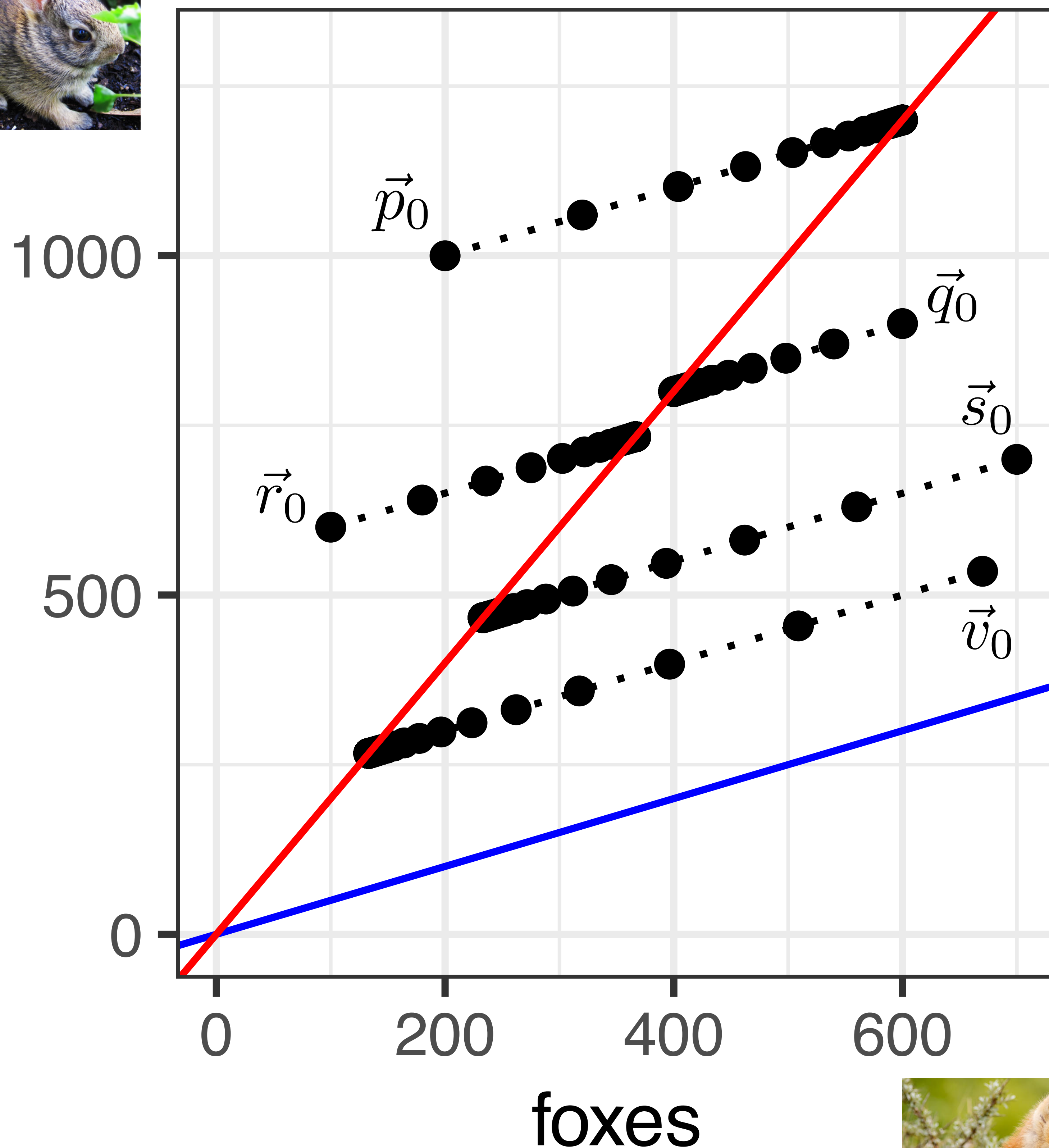


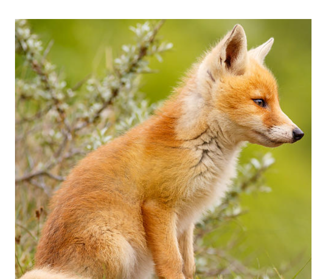
$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix}$$

$$\lambda_1 = 1.0$$
$$\vec{v}_1 \approx \begin{bmatrix} -0.447 & -0.894 \end{bmatrix}$$

$$\lambda_2 = 0.7$$
$$\vec{v}_2 \approx \begin{bmatrix} -0.894 & -0.447 \end{bmatrix}$$







eigenvectors and eigenvalues help us understand the system's dynamics

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$\lambda_2 = 0.7$

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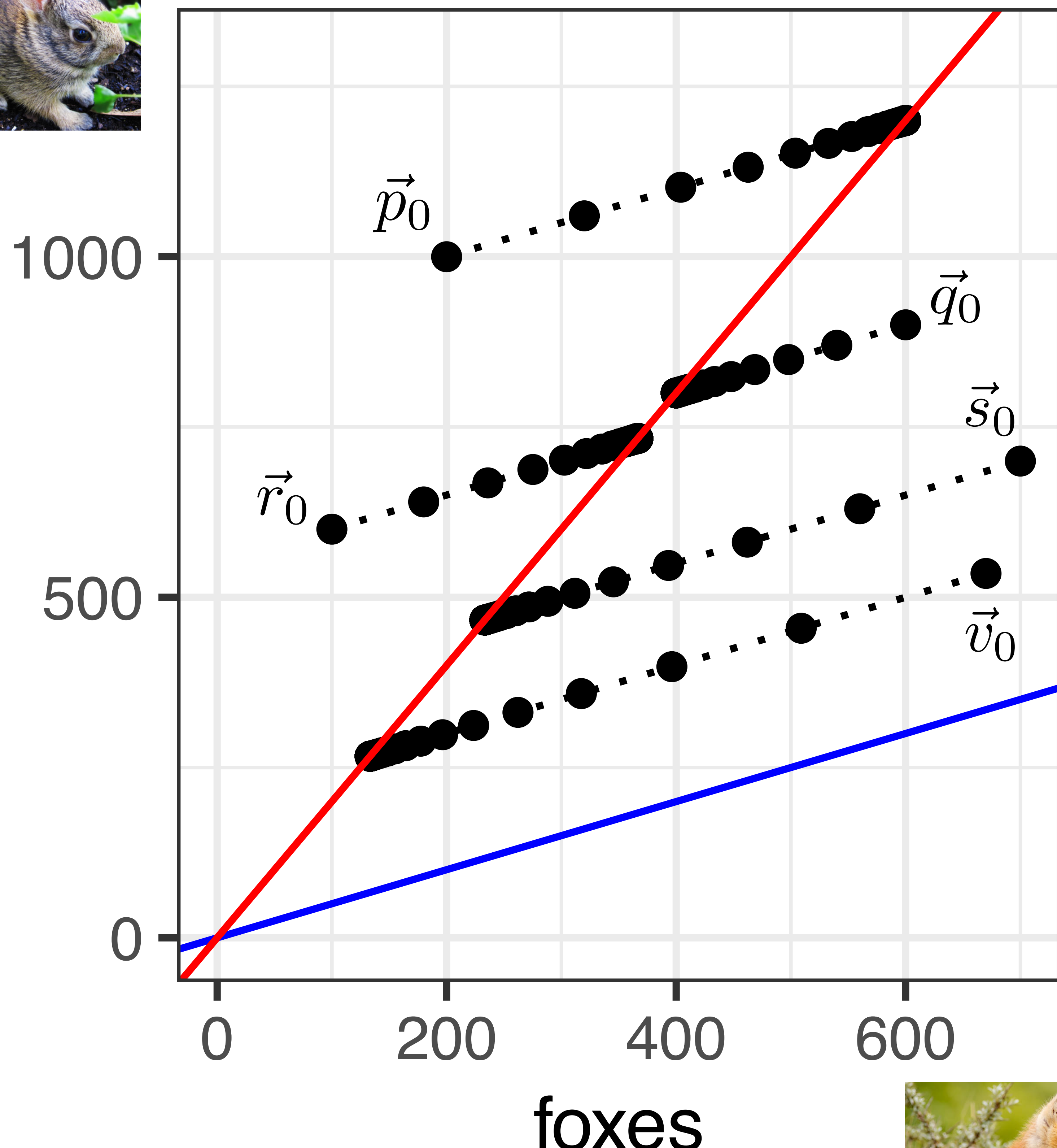

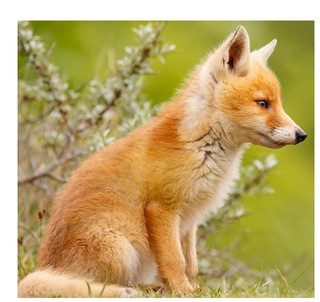

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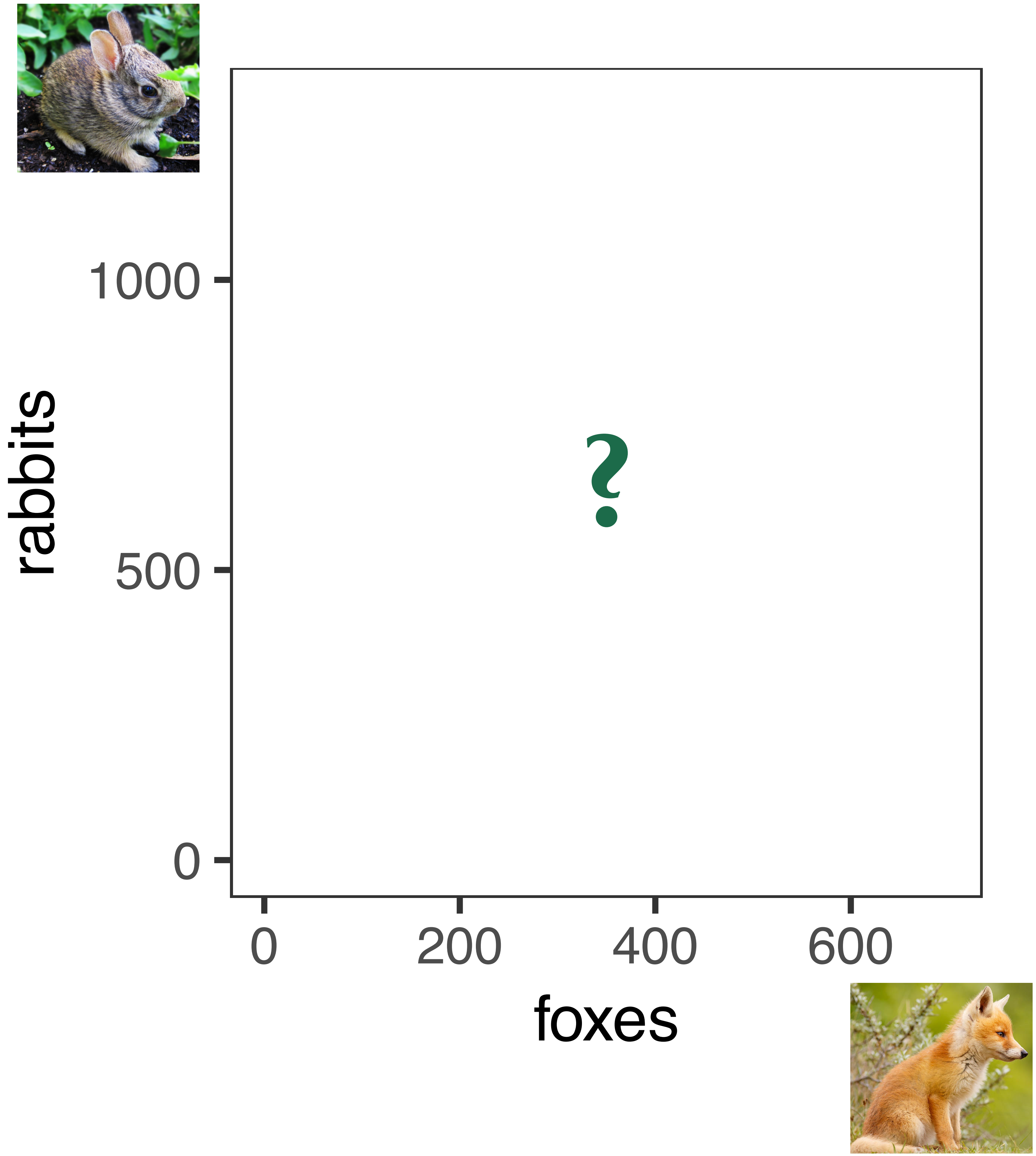
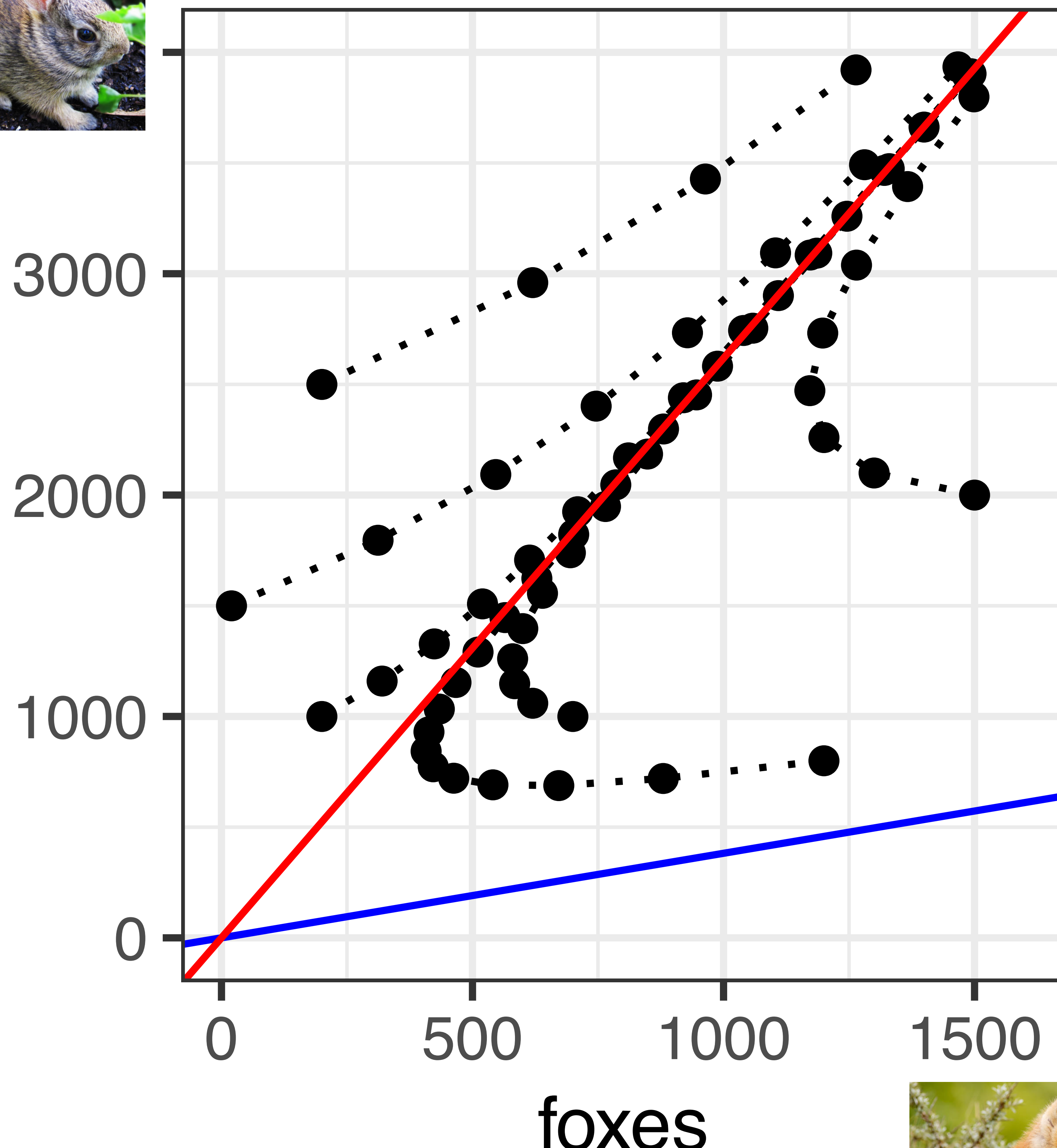
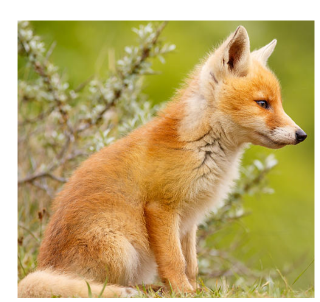


"rabbits multiply"

$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.2 \end{bmatrix}$$





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$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.2 \end{bmatrix}$$

$$\lambda_1 \approx 1.124$$

$$\vec{v}_1 \approx \begin{bmatrix} -0.357 & -0.934 \end{bmatrix}$$

$$\lambda_2 \approx 0.676$$

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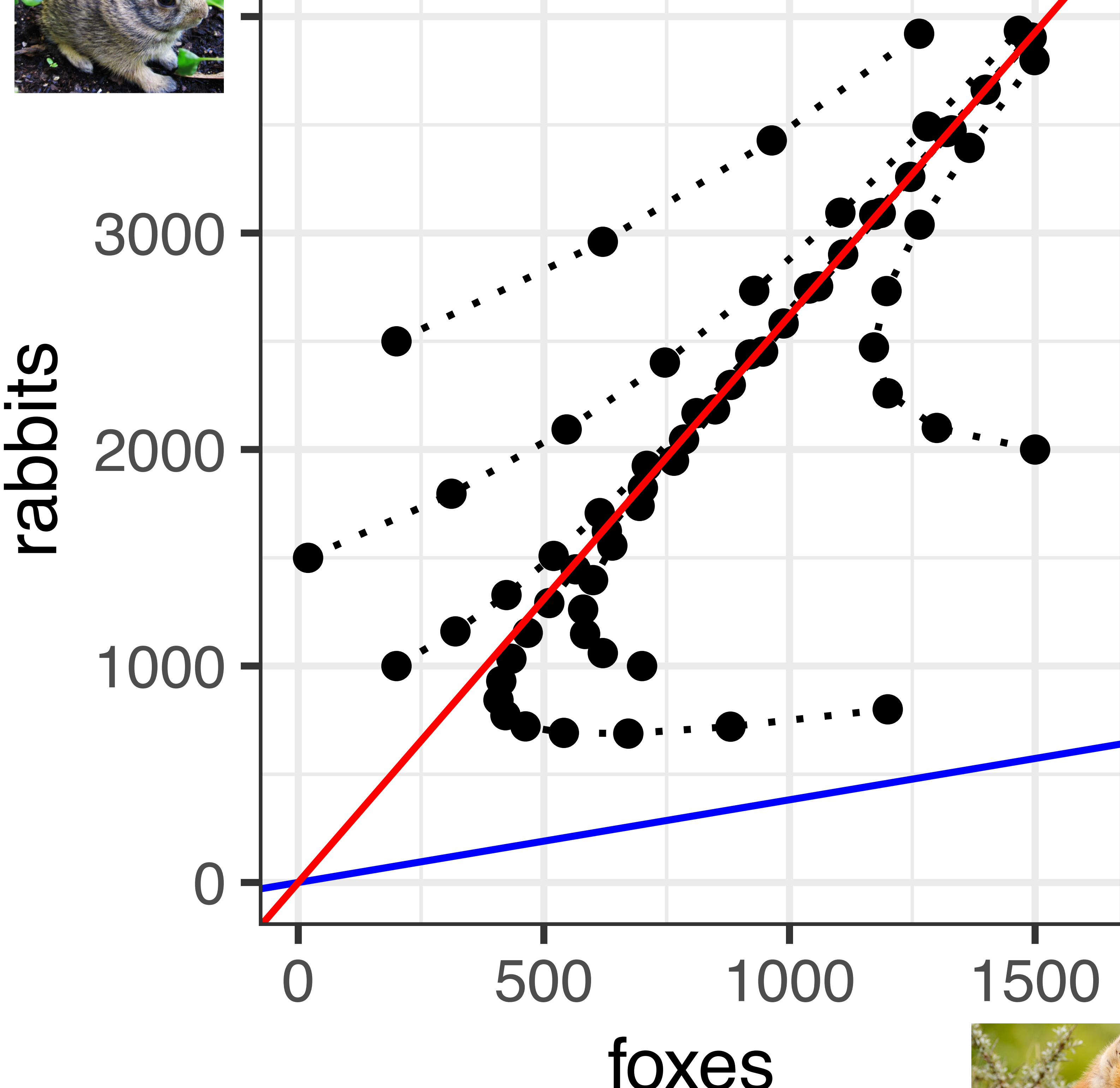

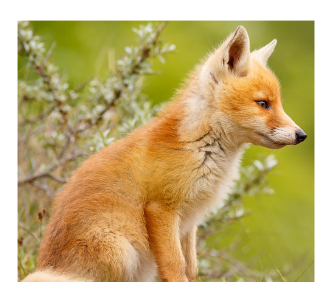

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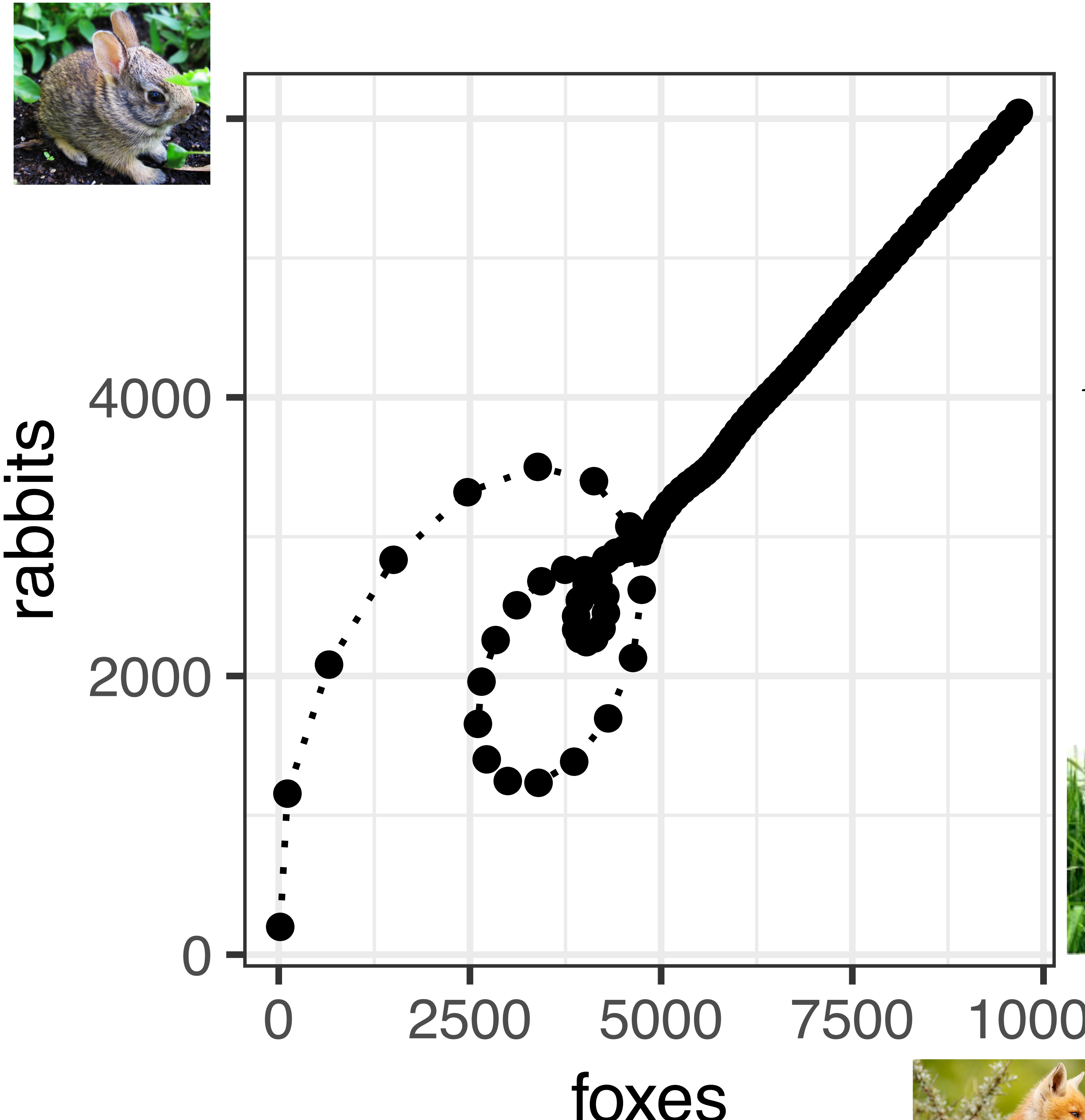
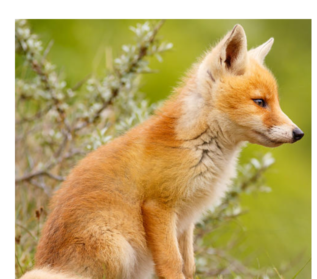


$$A = \begin{bmatrix} 0.7 & 0.5 & 0 \\ -0.2 & 0.8 & 0.5 \\ 0 & -0.2 & 1.2 \end{bmatrix}$$

introducing “grass” into the model



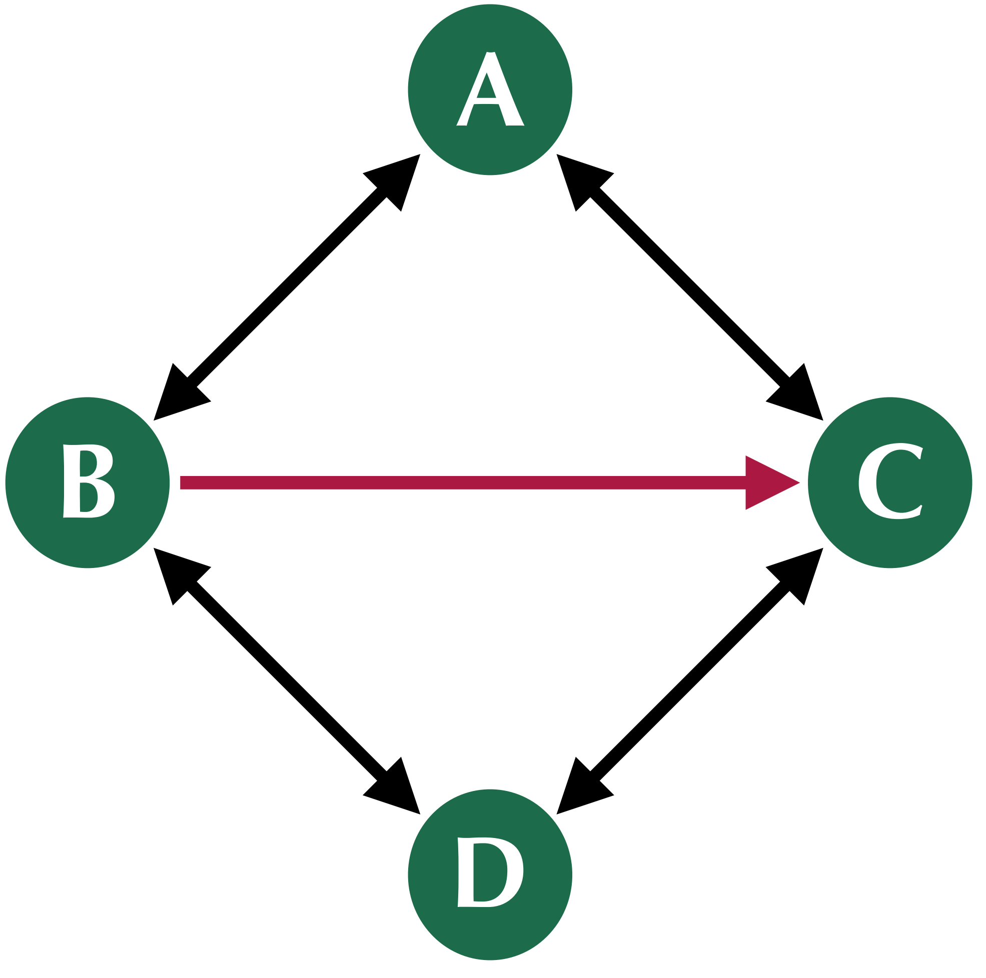

Takeaway?
we need to be
able to compute
A^n
to understand
system dynamics

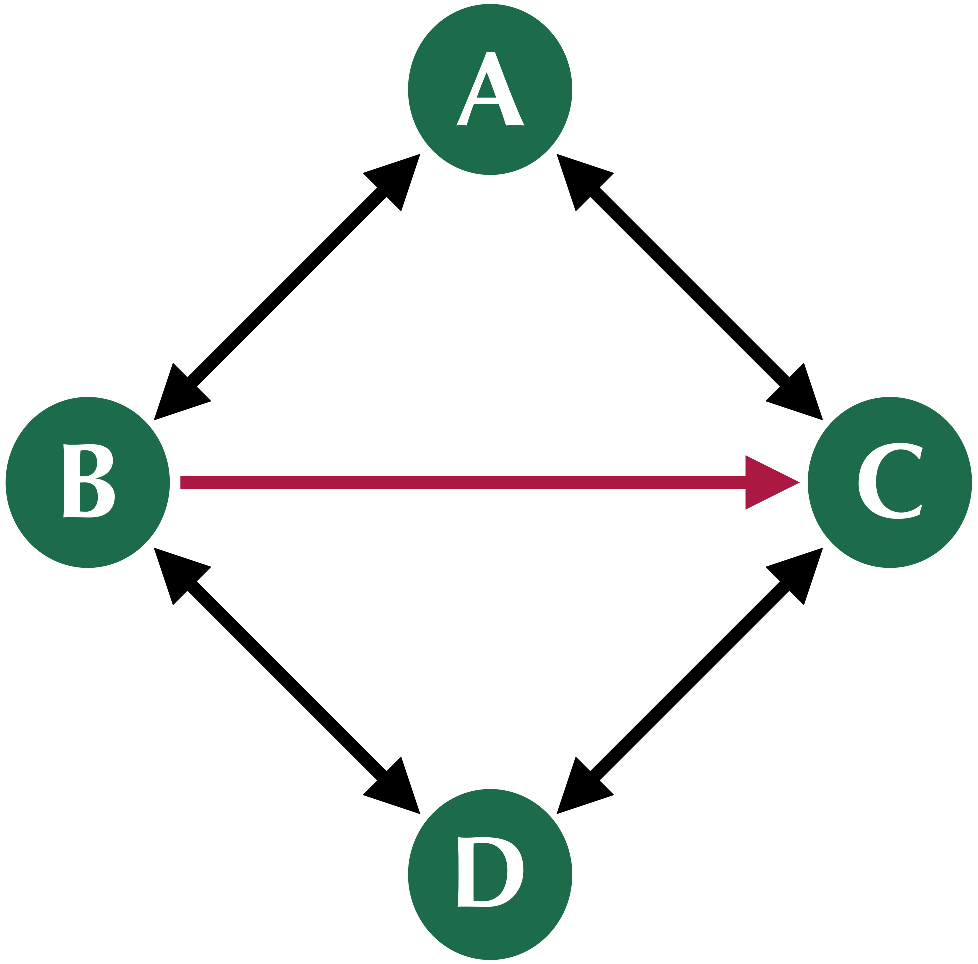
Takeaway?

eigenvectors and values help us:

compute A^n quickly

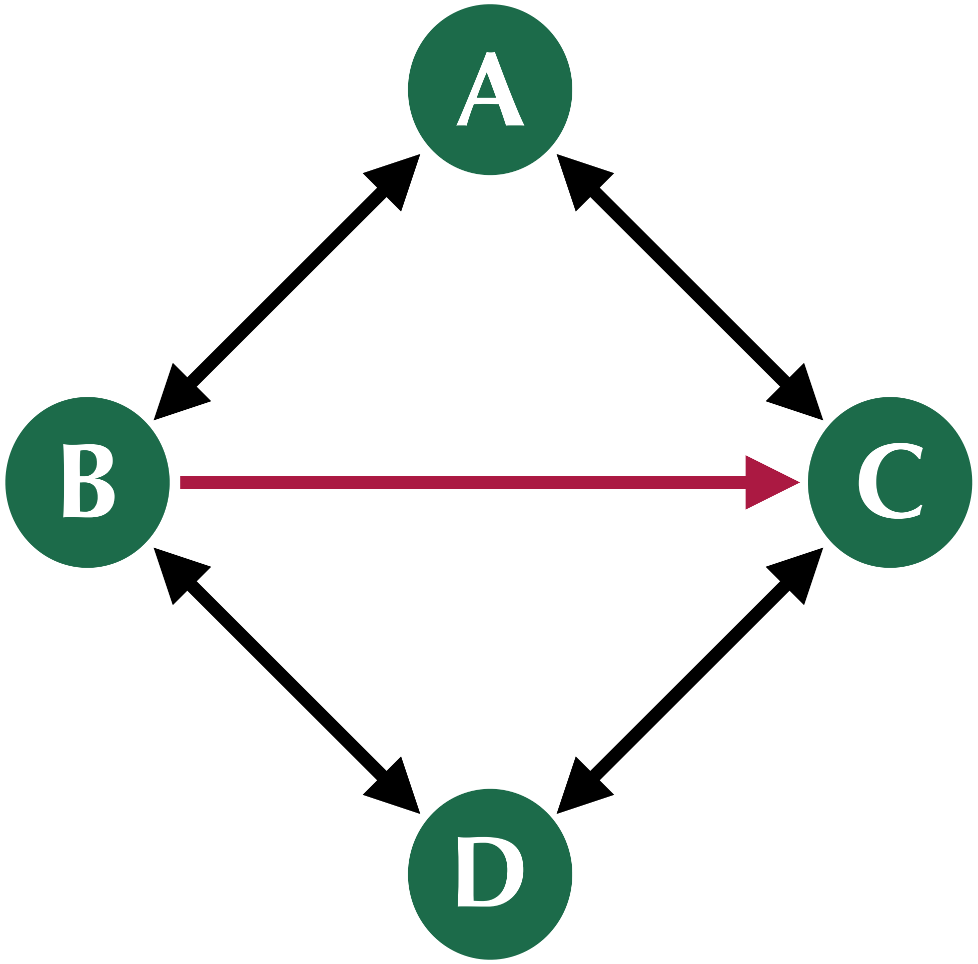
understand convergence to steady states


A
B
C
D



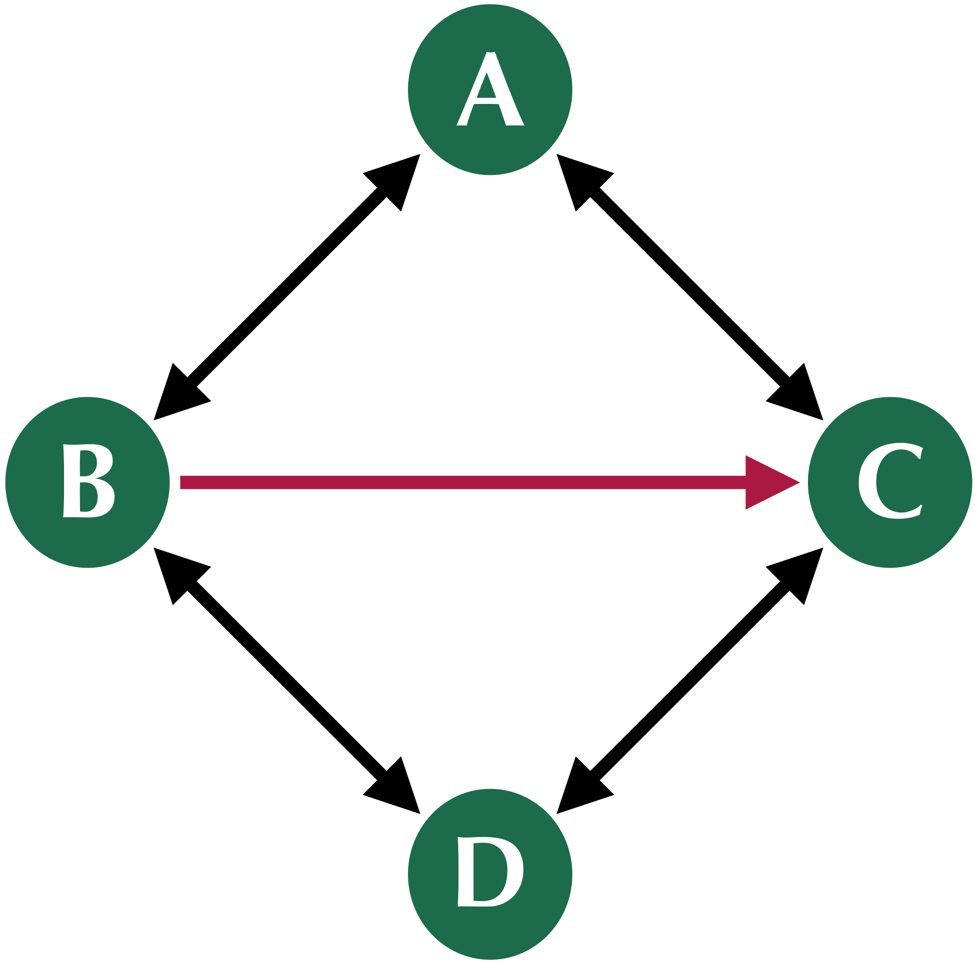
Adjacency Matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Adjacency Matrix

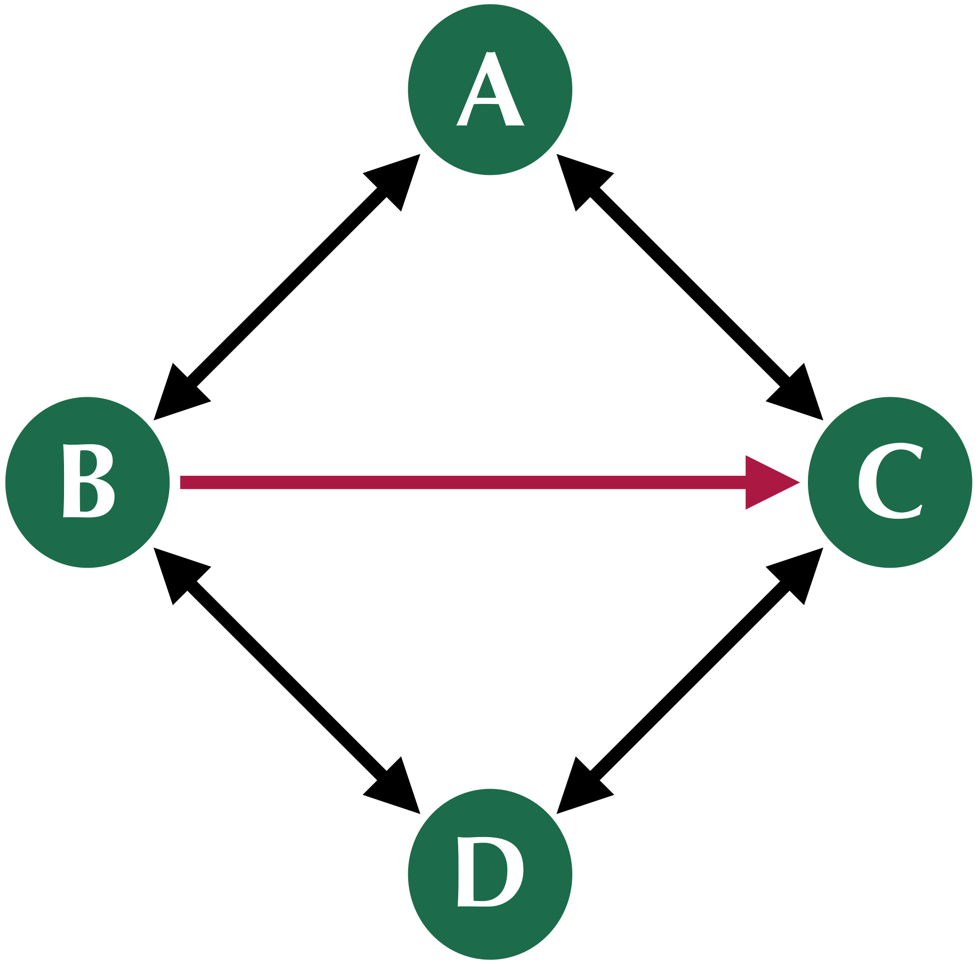
$$M = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array} \end{array}$$



Adjacency Matrix

$$M = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array}$$

$m_{ij} = 1$ iff there's an edge from *i* to *j*



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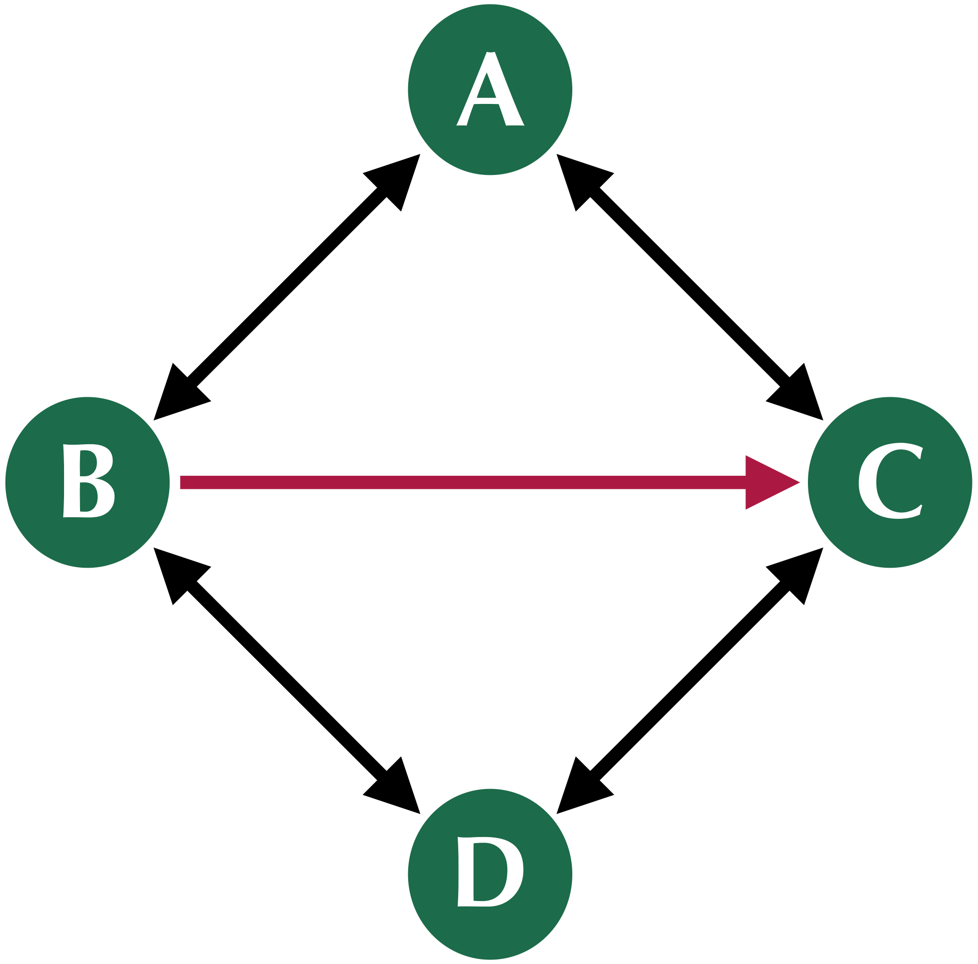

Adjacency Matrix

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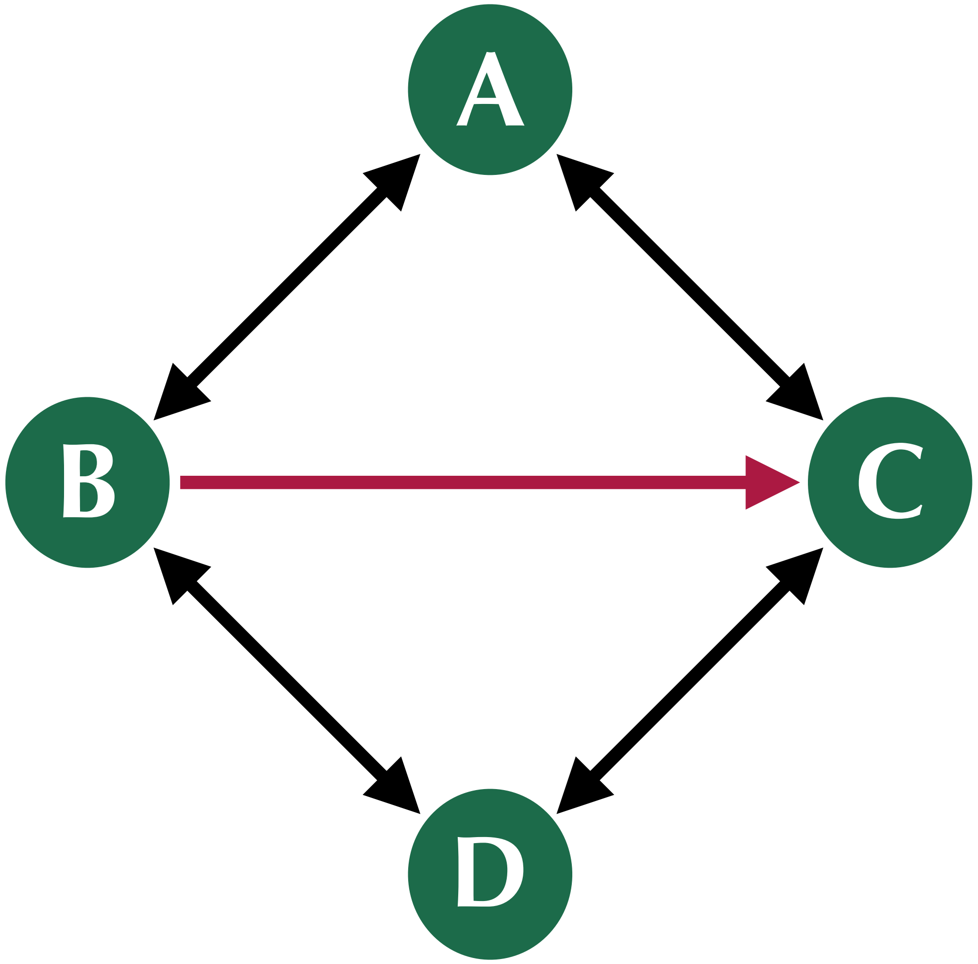
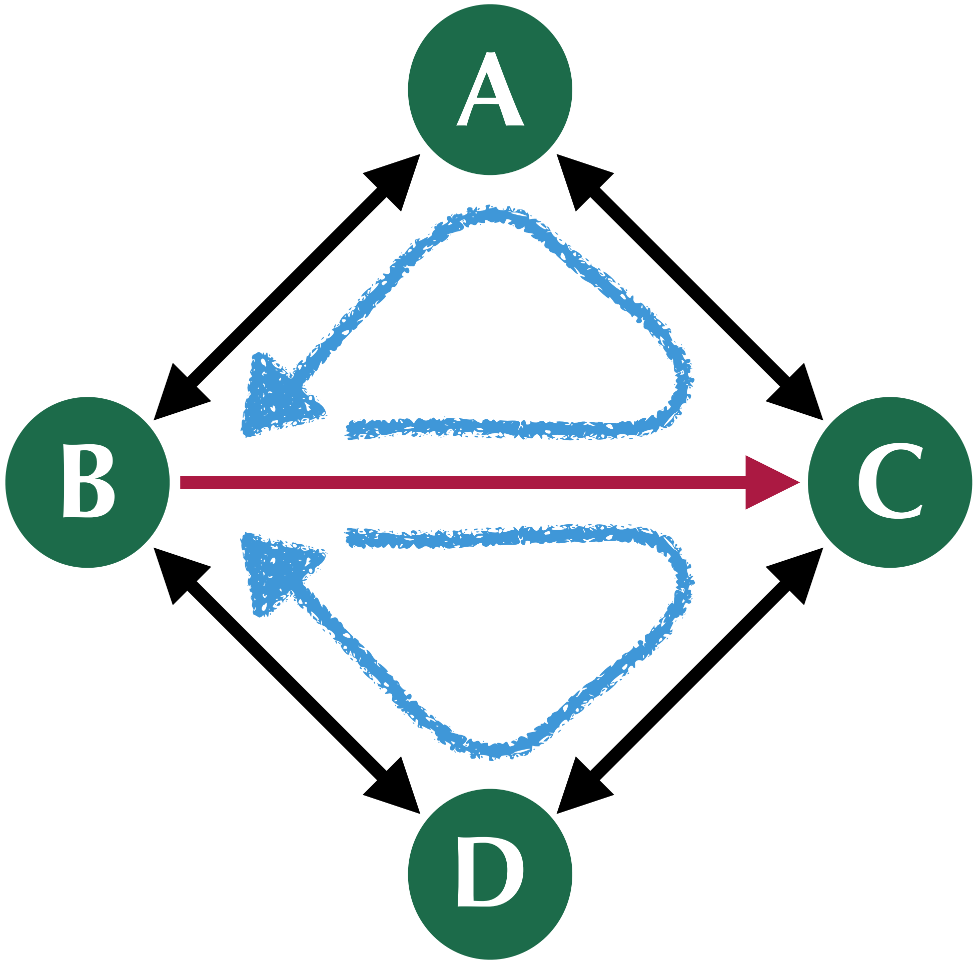
$m_{ij} = 1$ iff there's an edge from *i* to *j*

M^n is a matrix where $m_{ij}^{(n)}$ represents the number of *n*-step paths from *i* to *j*



$$M^3 = \begin{array}{cc} \begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{array} & \\ \begin{bmatrix} 1 & 4 & 4 & 1 \\ 4 & 2 & 4 & 4 \\ 4 & 0 & 2 & 4 \\ 1 & 4 & 4 & 1 \end{bmatrix} & \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array} \end{array}$$

M^n is a matrix where $m_{ij}^{(n)}$ represents the number of *n*-step paths from *i* to *j*



$$M^3 = \begin{array}{c c} \begin{matrix} \;\;A & \;\;B & \;\;C & \;\;D \end{matrix} & \\ \begin{bmatrix} 1 & 4 & 4 & 1 \\ 4 & 2 & 4 & 4 \\ 4 & 0 & 2 & 4 \\ 1 & 4 & 4 & 1 \end{bmatrix} & \begin{matrix} A \\ B \\ C \\ D \end{matrix} \end{array}$$

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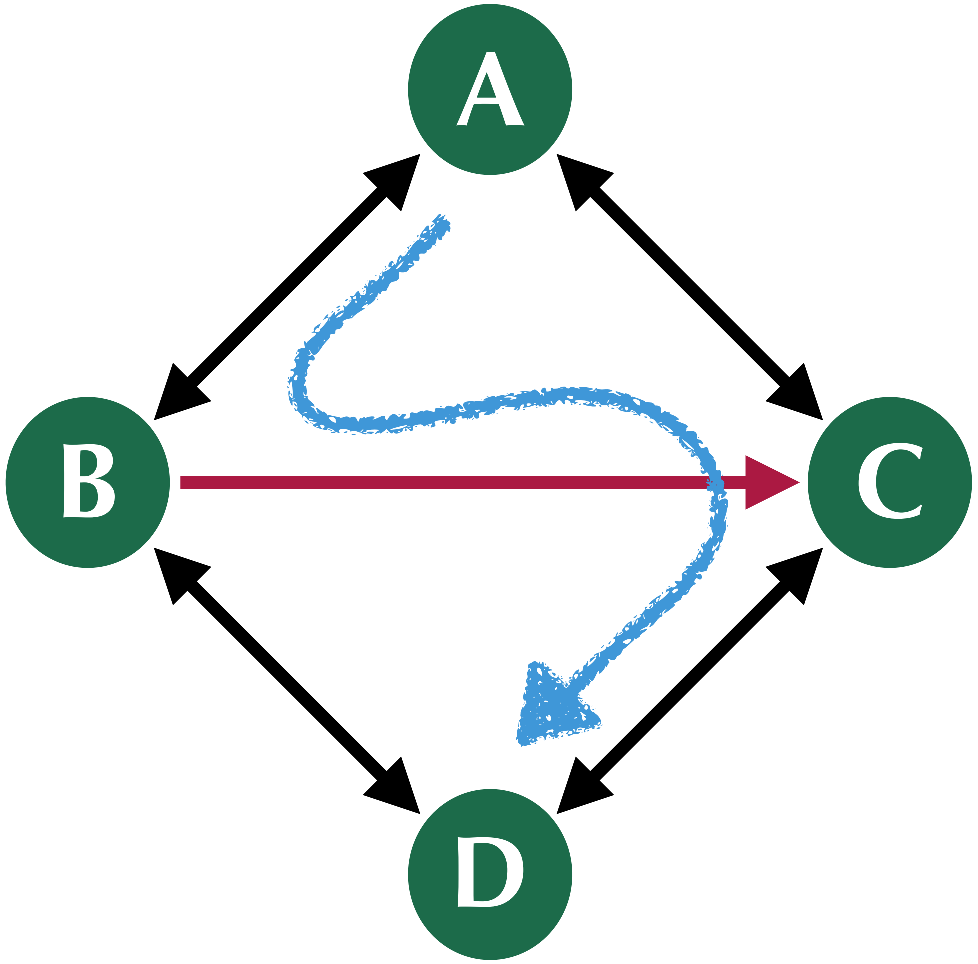

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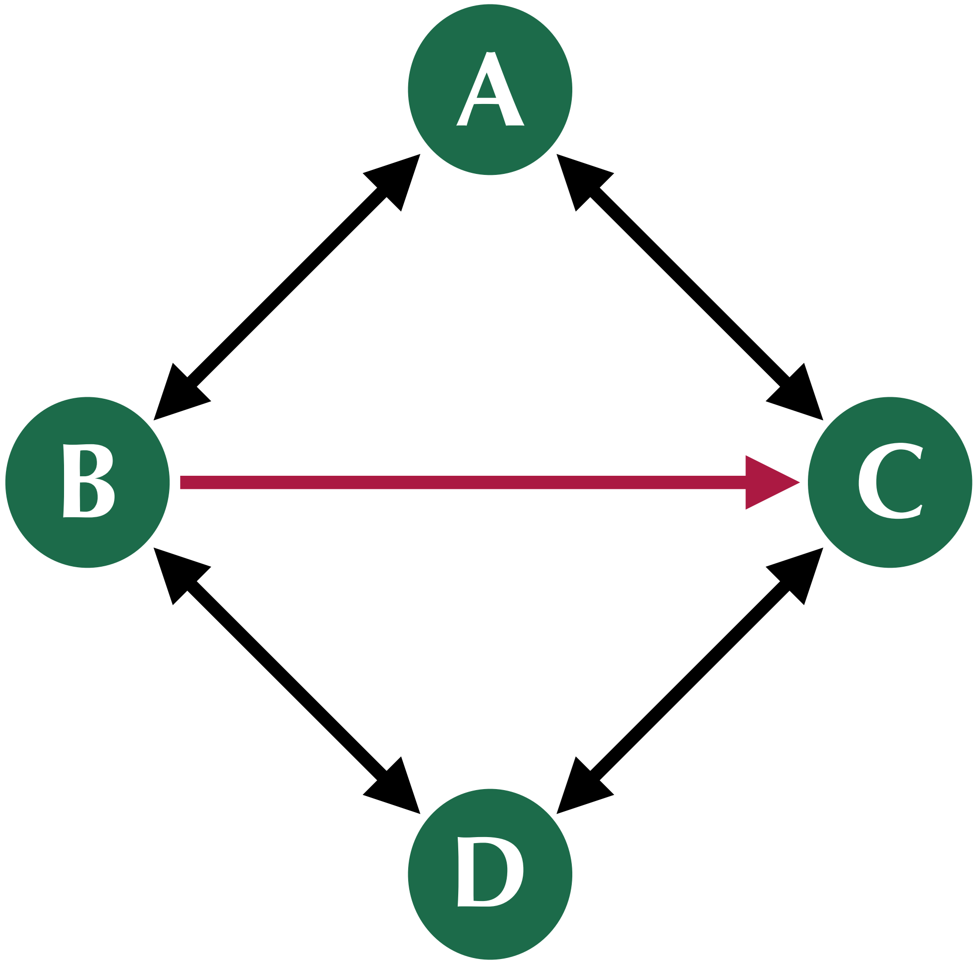

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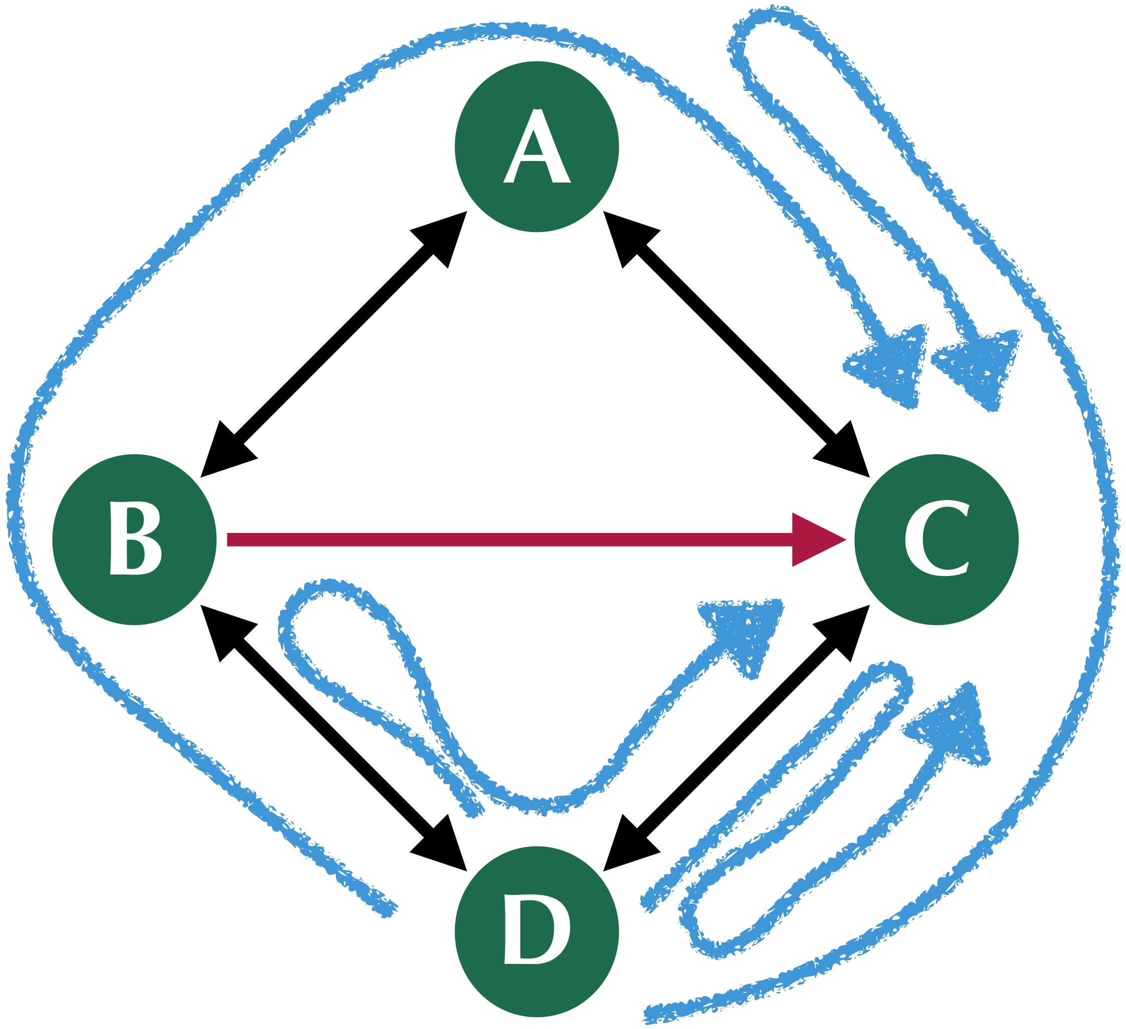

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Takeaway?

we need to be able to compute

$$M^n$$

to understand networks

Takeaway?

eigenvectors and eigenvalues help us compute M^n efficiently

eigenvectors and **eigenvalues** help us:

eigenvectors and **eigenvalues** help us:

compute matrix powers efficiently/quickly

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understand how quickly systems converge

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understand how quickly systems converge

see if a matrix is invertible

how do we find them?

Definition: Given $A_{n\times n}$, say λ is an eigenvalue for A if $A\vec{v} = \lambda\vec{v}$ for some non-zero vector $\vec{v}$.

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called an *eigenvalue* of A
corresponding to λ

how do we find them?

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$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$

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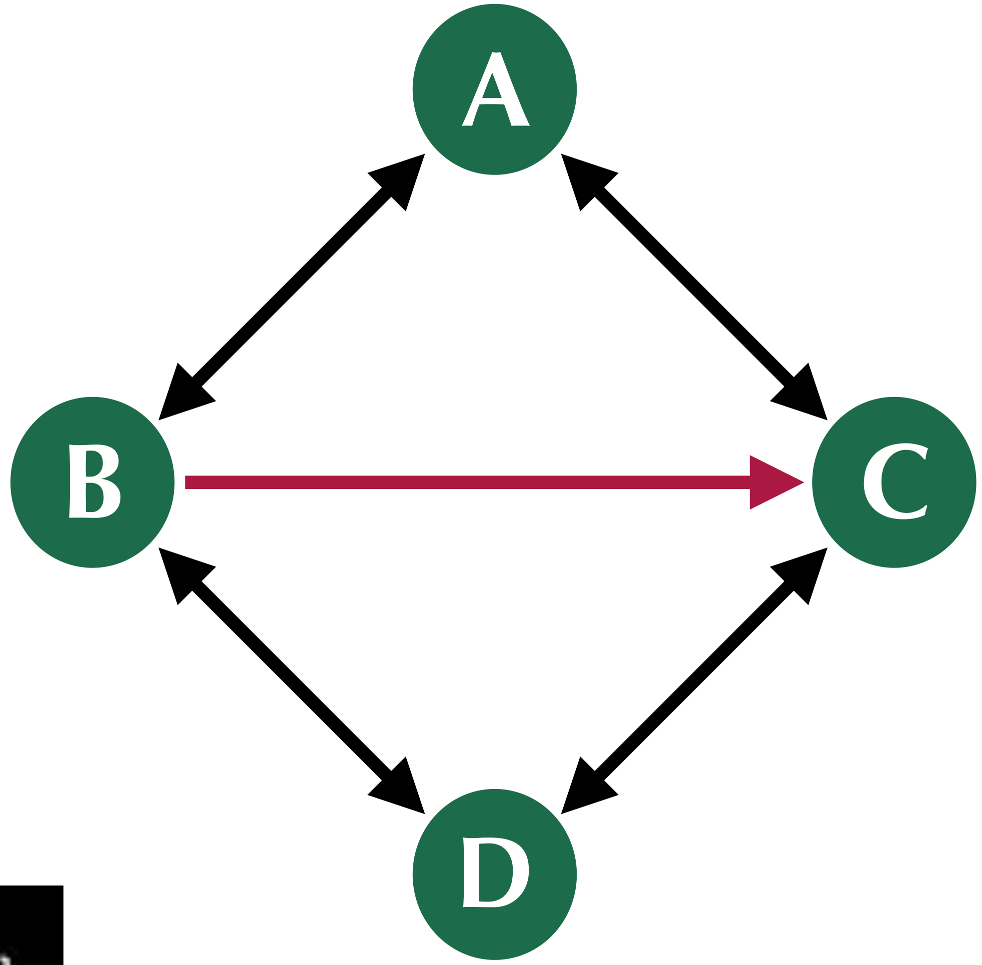
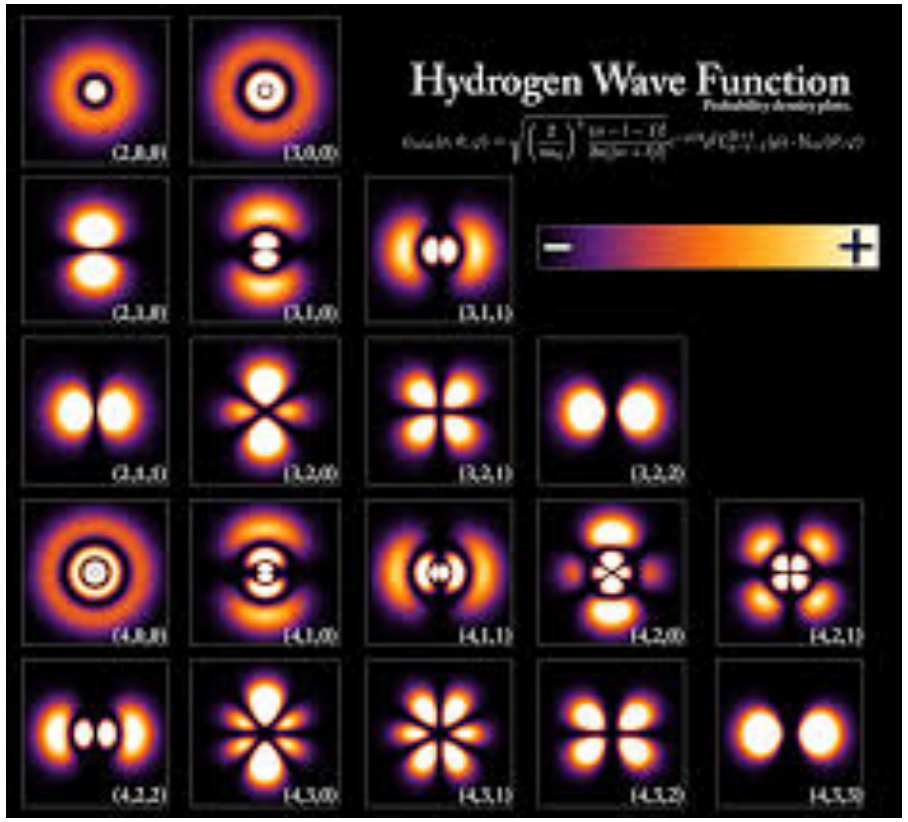

A
B
C
D


Hydrogen Wave Function
Probability density plots.