

motivating eigenvalues and eigenvectors

slides: ajbc.io/misc/eigen.pdf

dynamic systems: foxes and rabbits



 F_t

population of foxes at year t

 R_t

population of rabbits at year t



$$F_{t+1} = 0.6F_t + 0.2R_t$$



$$R_{t+1} = -0.2F_t + 1.1R_t$$

foxes die without food



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more rabbits means
more support for foxes

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foxes kill off rabbits

rabbits multiply without predators



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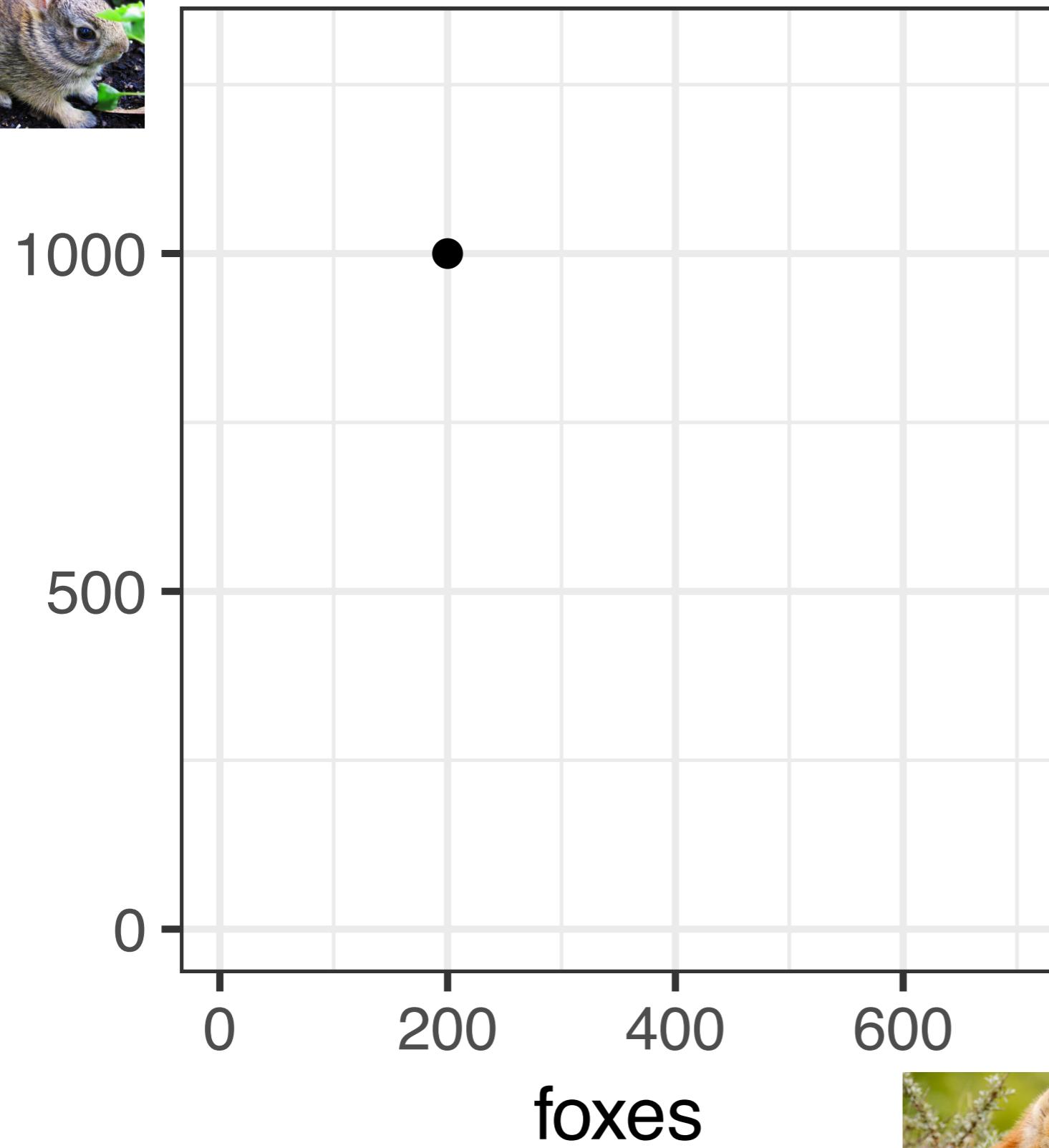
$$\begin{bmatrix} F_{t+1} \\ R_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} F_t \\ R_t \end{bmatrix}$$

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$$\vec{p}_{t+1} = A \vec{p}_t$$



rabbits

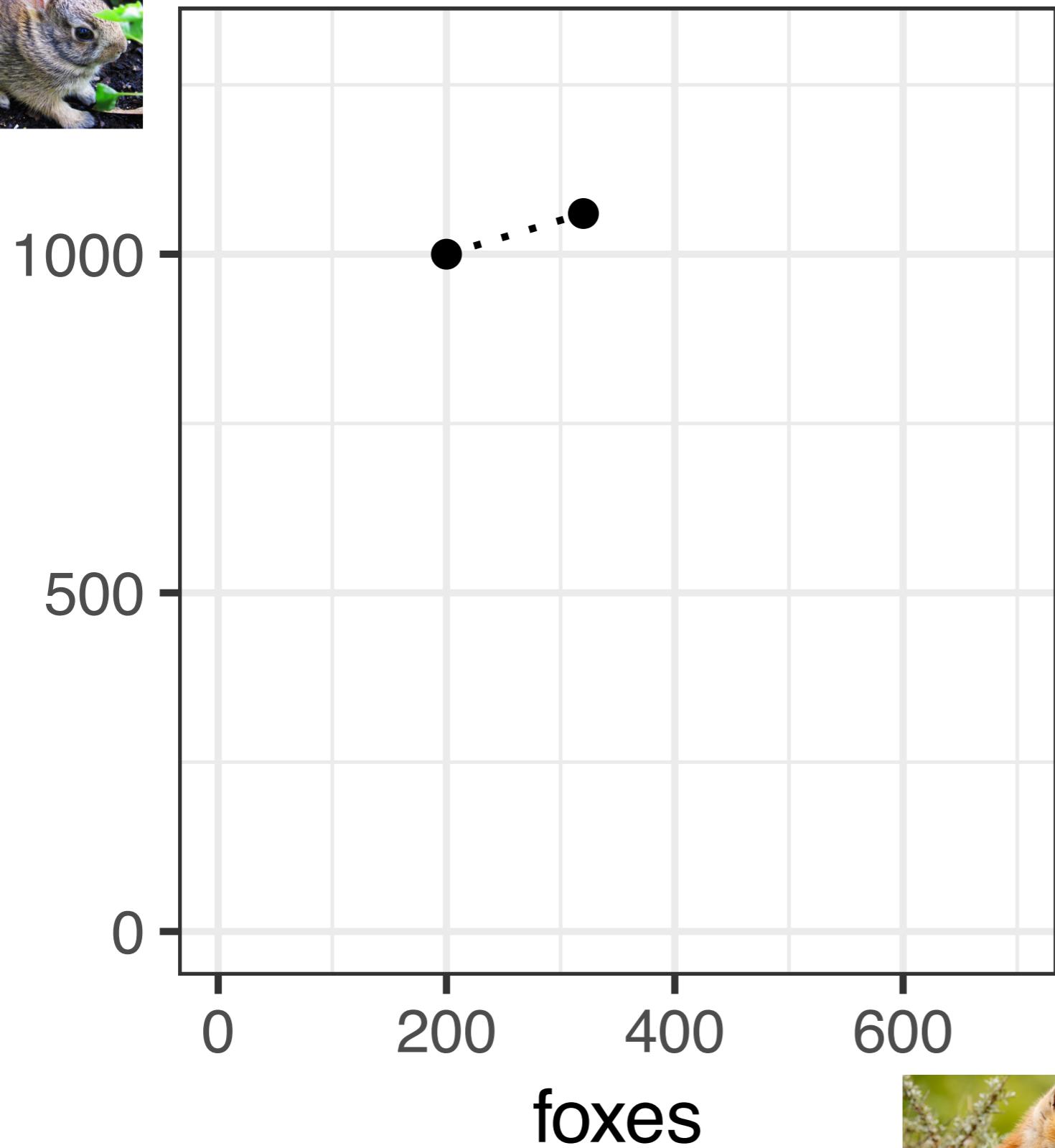


$$\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$$





rabbits



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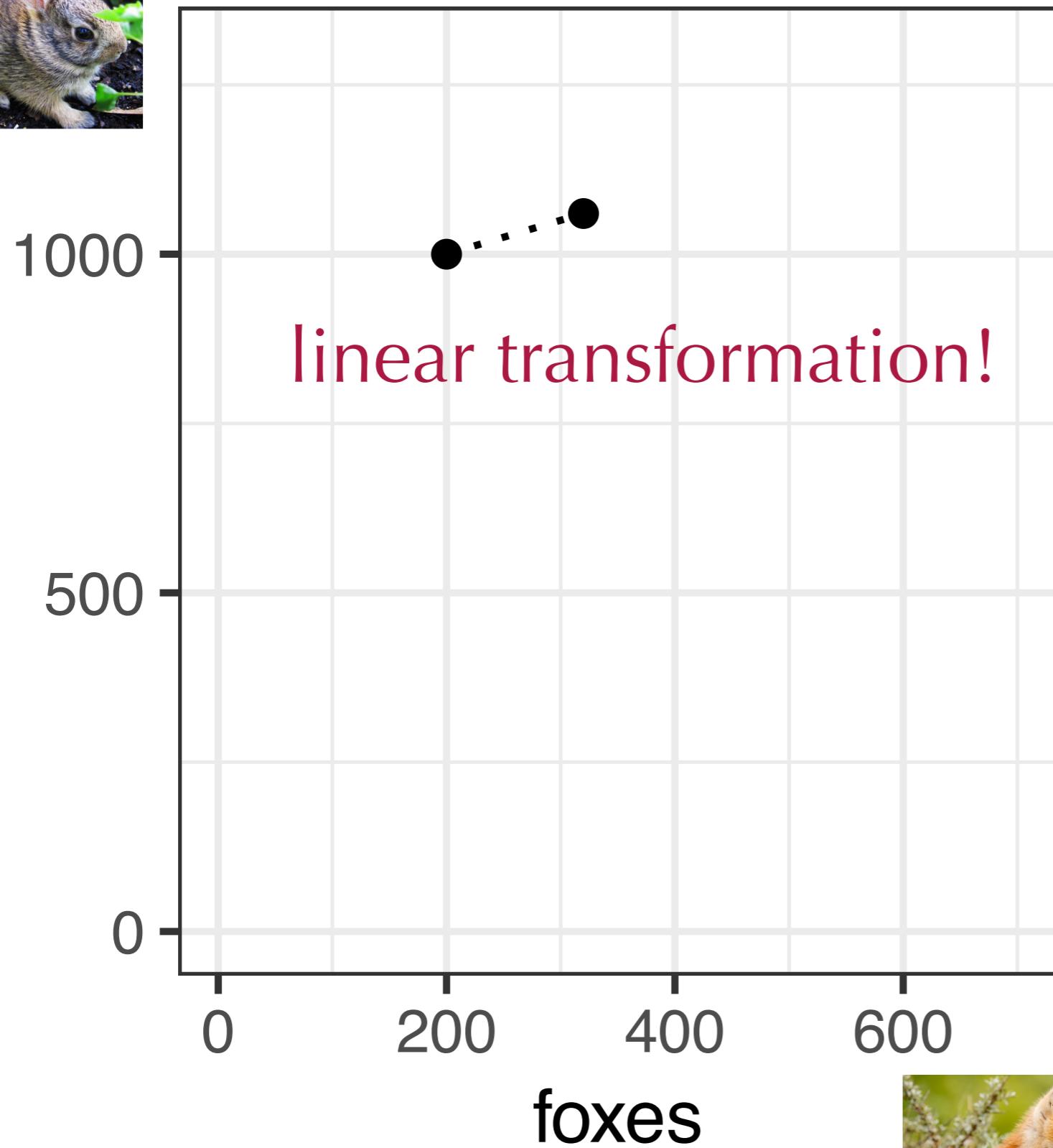
$$\begin{bmatrix} F_1 \\ R_1 \end{bmatrix} =$$

$$\begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix} \begin{bmatrix} F_0 \\ R_0 \end{bmatrix}$$

$$= \begin{bmatrix} 320 \\ 1060 \end{bmatrix}$$

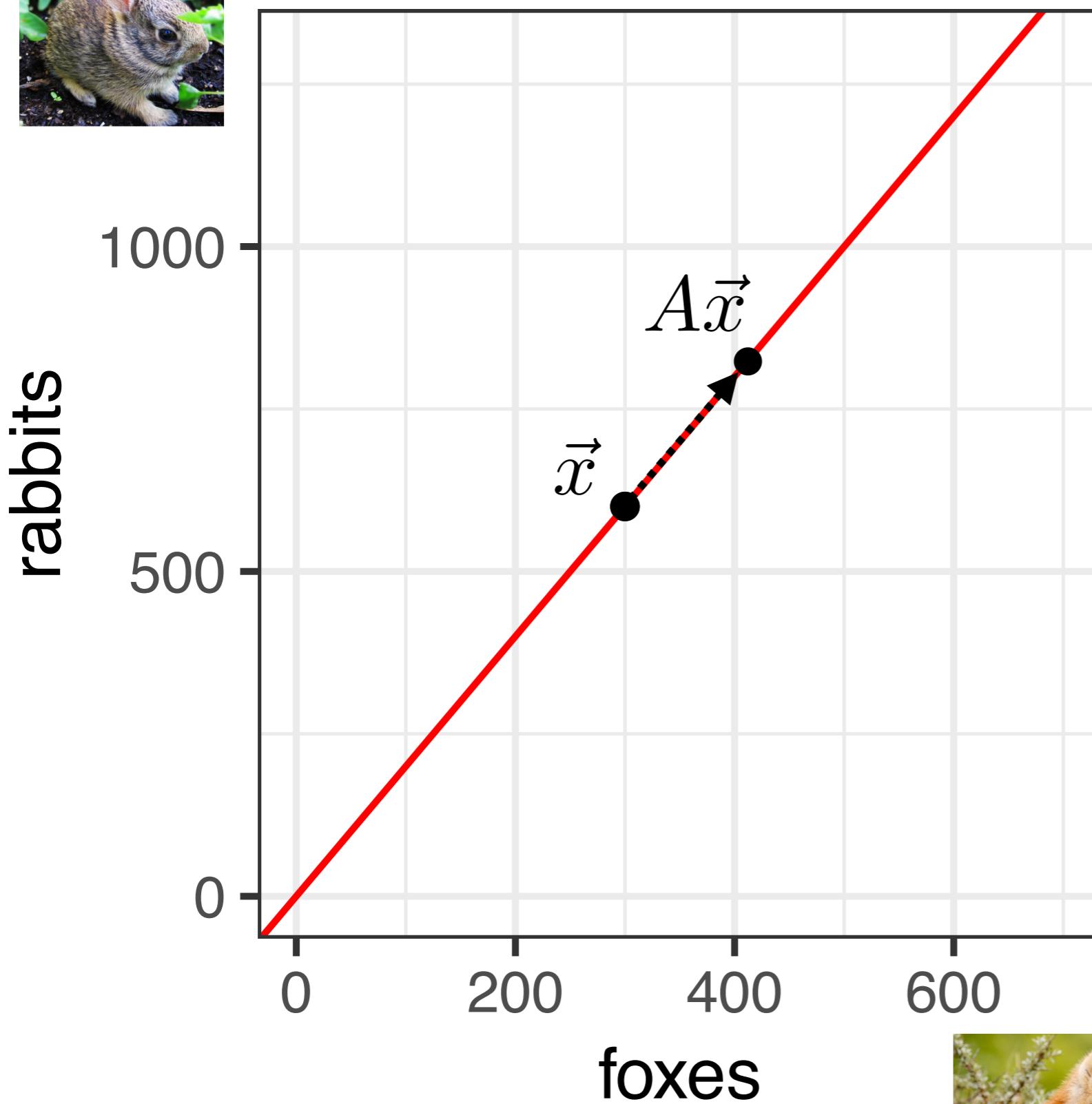


rabbits



$$\begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$$

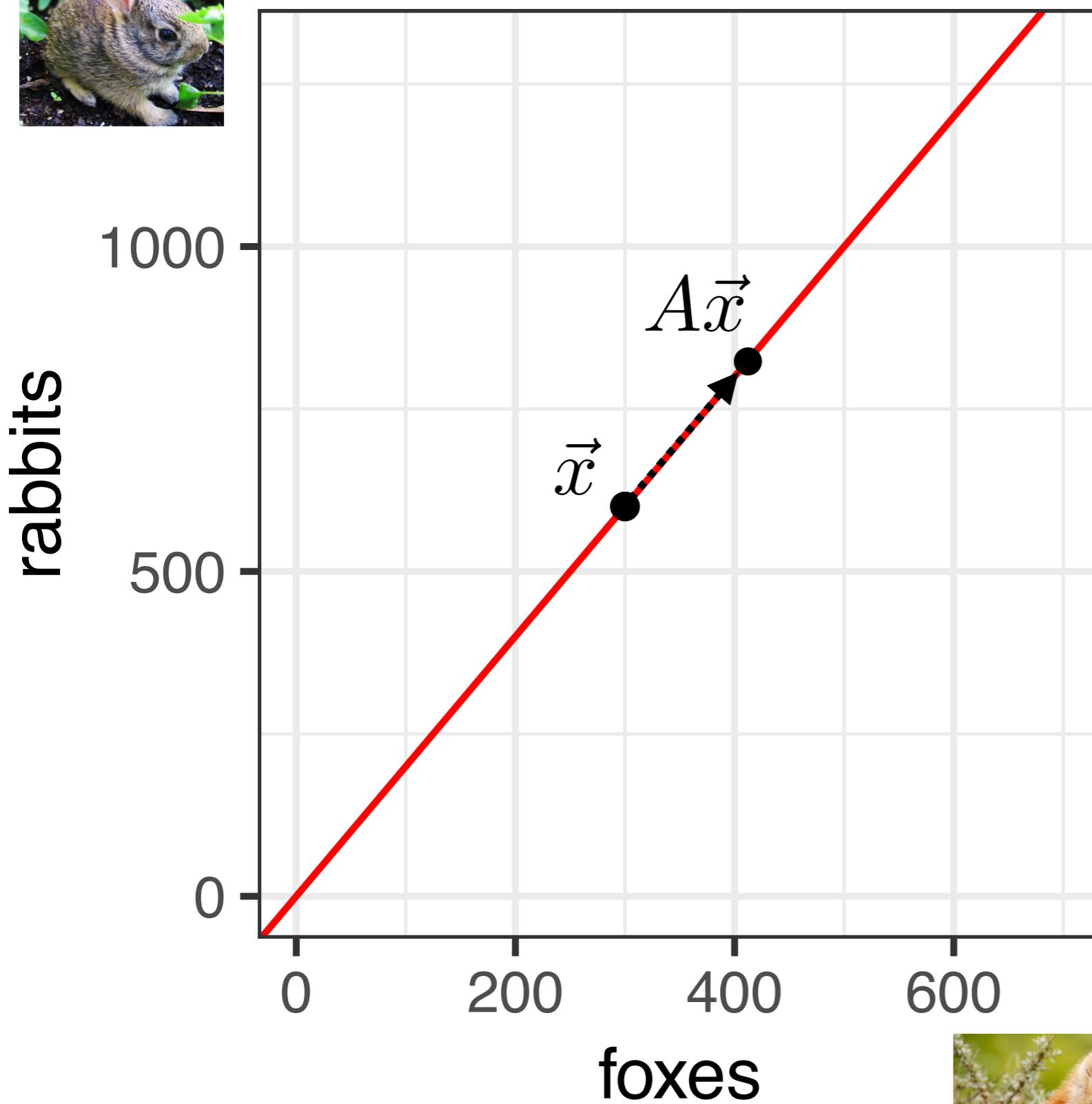
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imagine a special
case where

$$A\vec{x} = \lambda\vec{x}$$



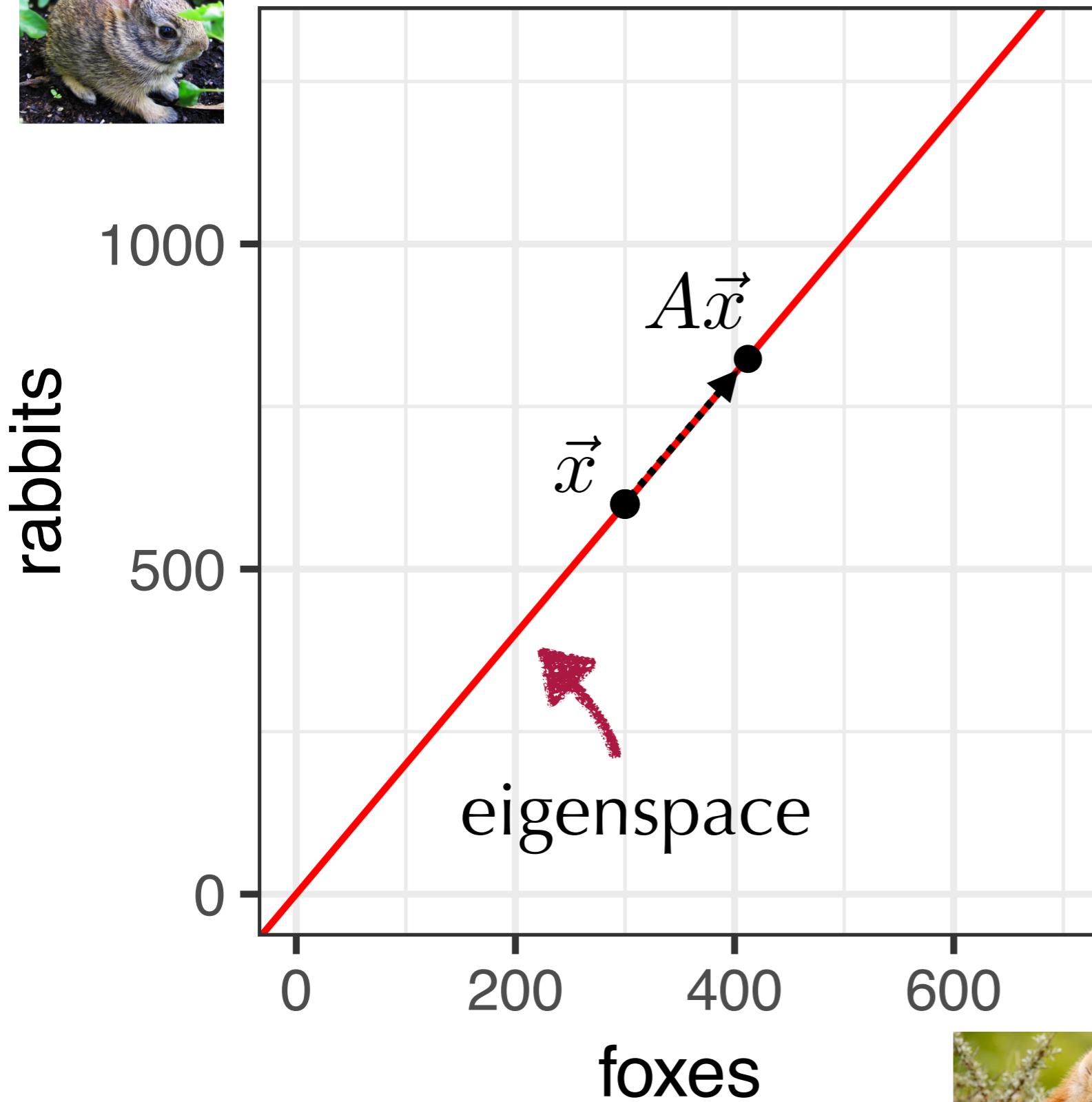


imagine a special
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scalar

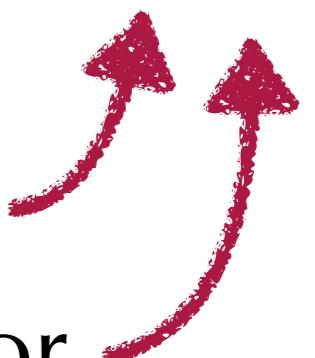




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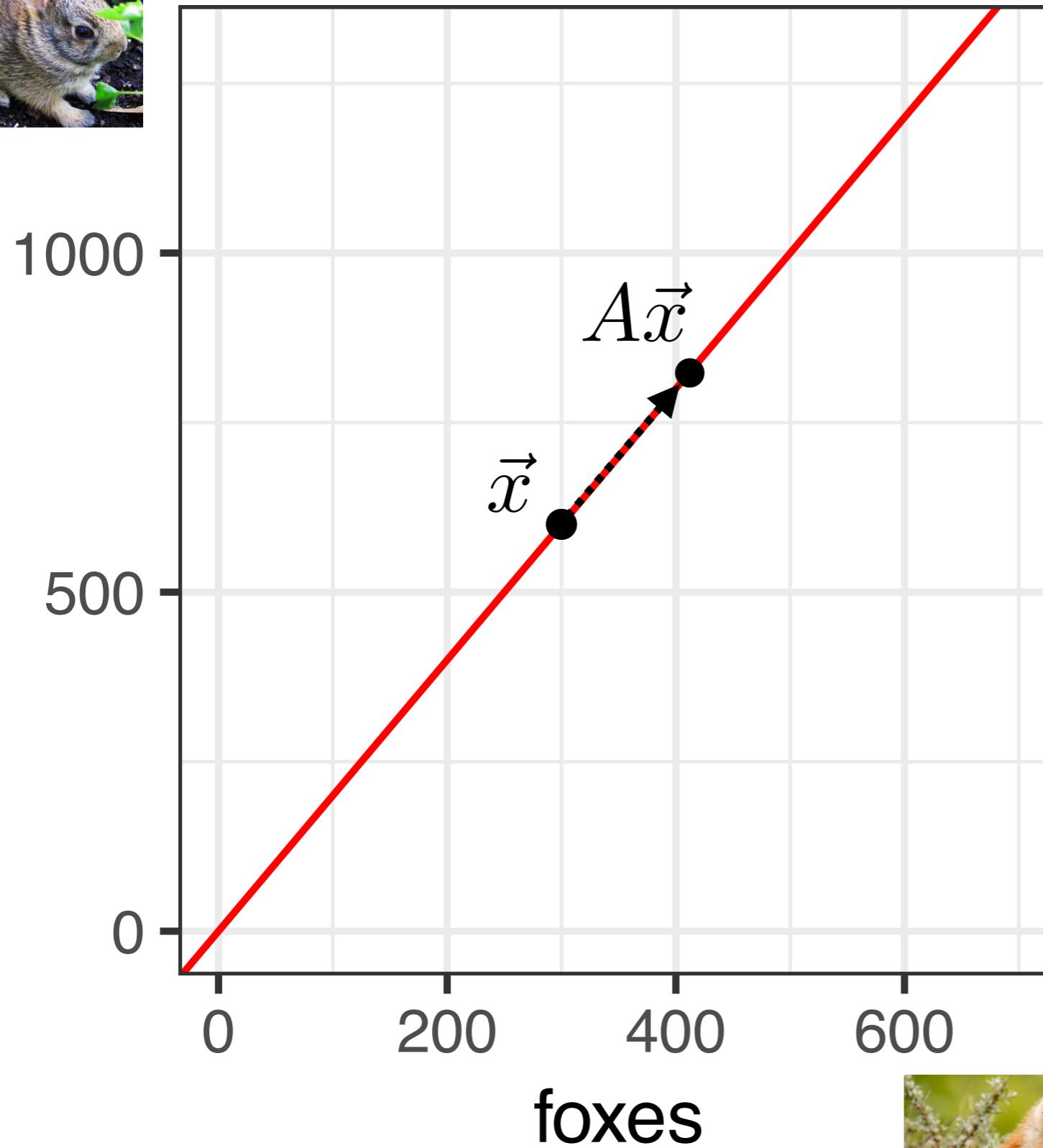
$$A\vec{x} = \lambda\vec{x}$$

eigenvalue
eigenvector





rabbits

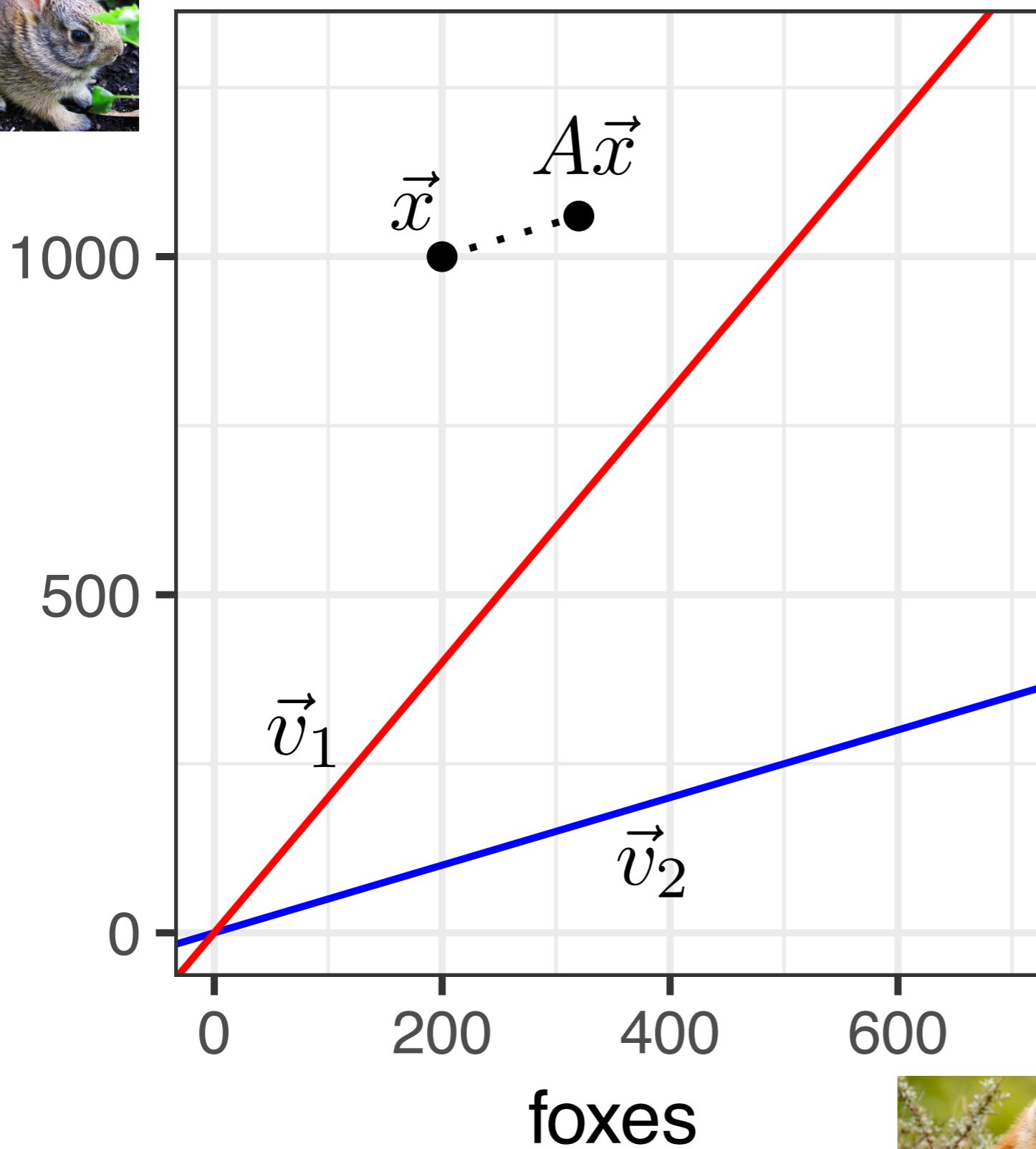


we can think of
eigenvectors as
a set of basis vectors





rabbits

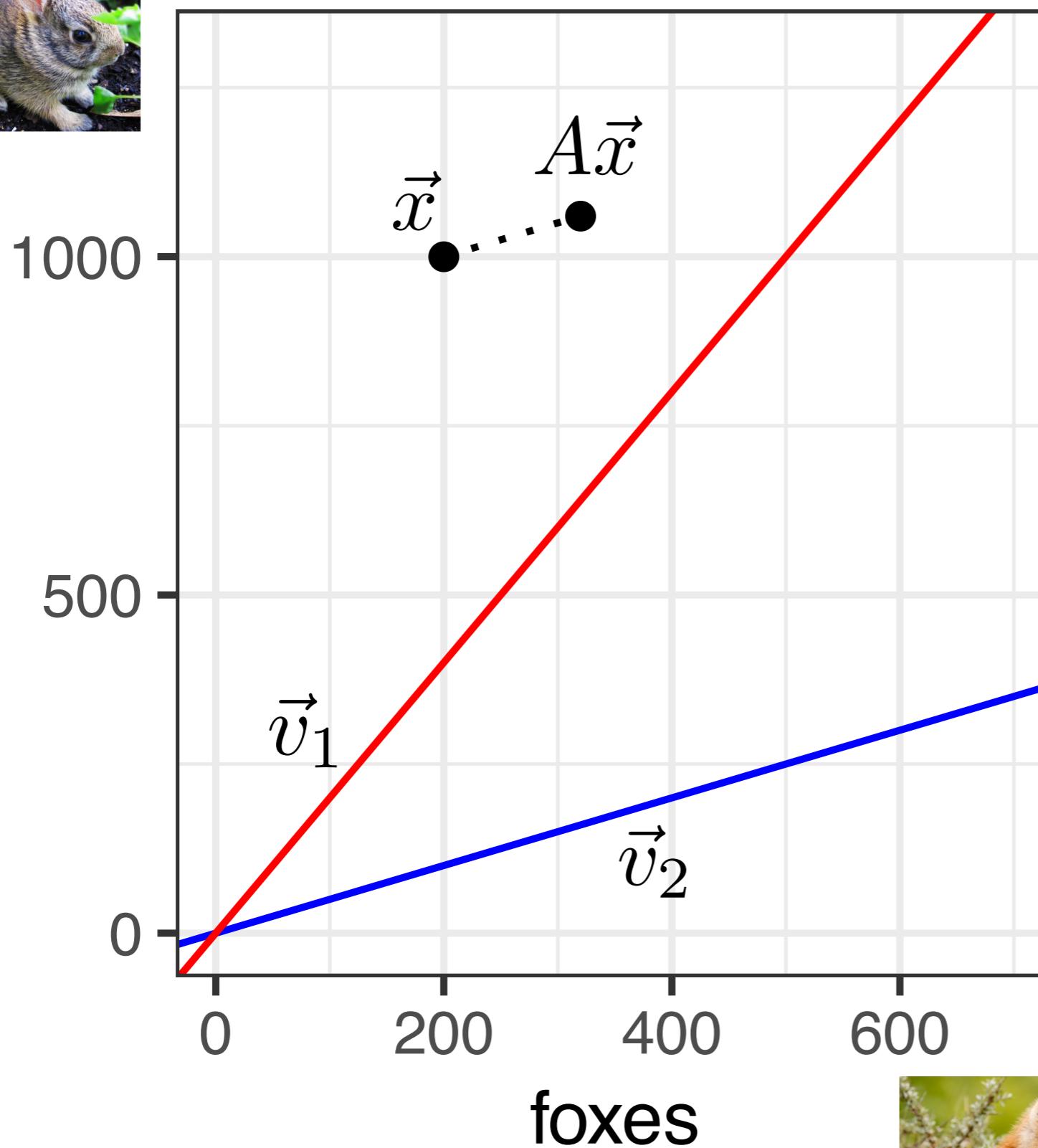


$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$



rabbits

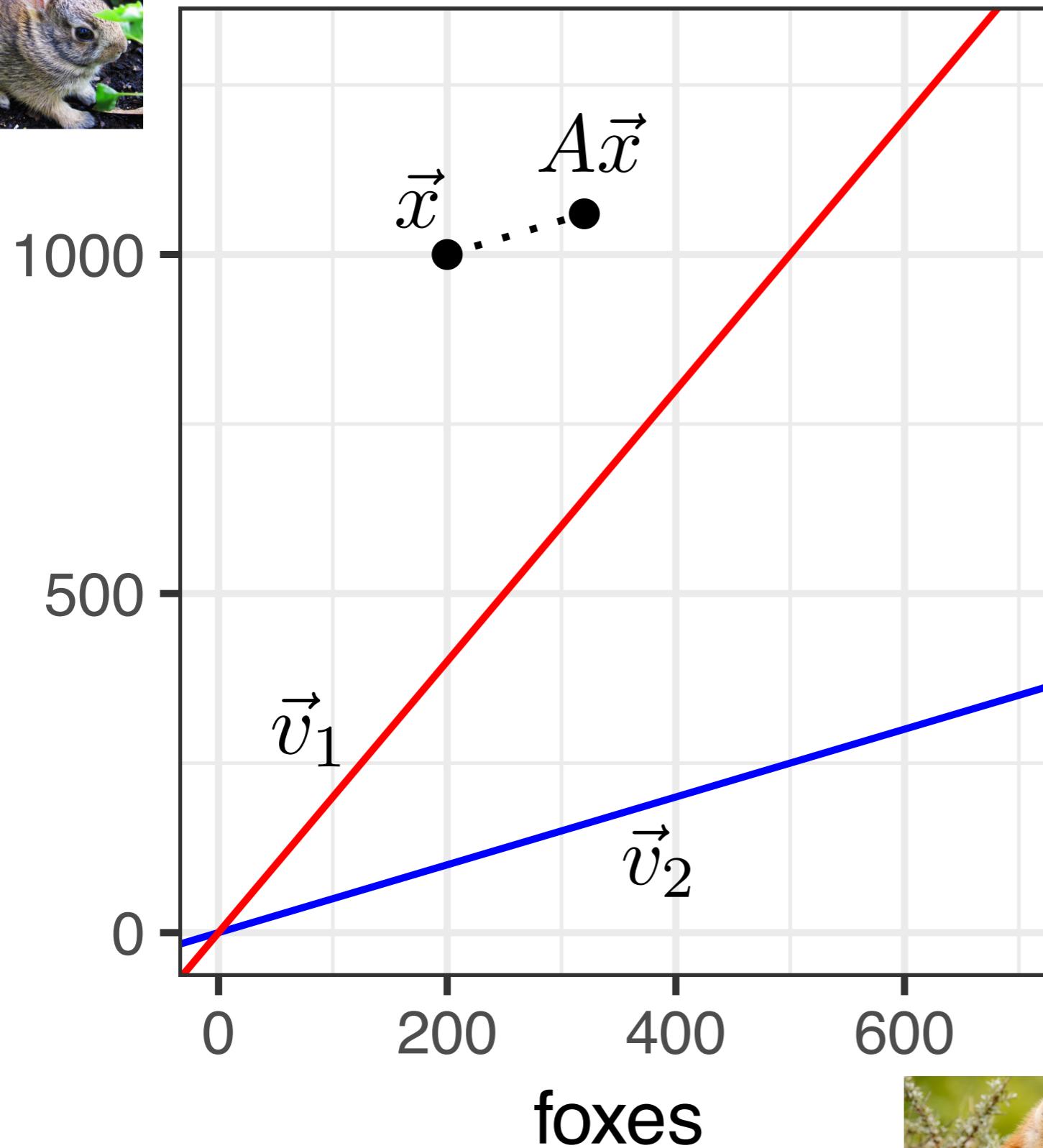


$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
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$$\vec{x} = a\vec{v}_1 + b\vec{v}_2$$



rabbits



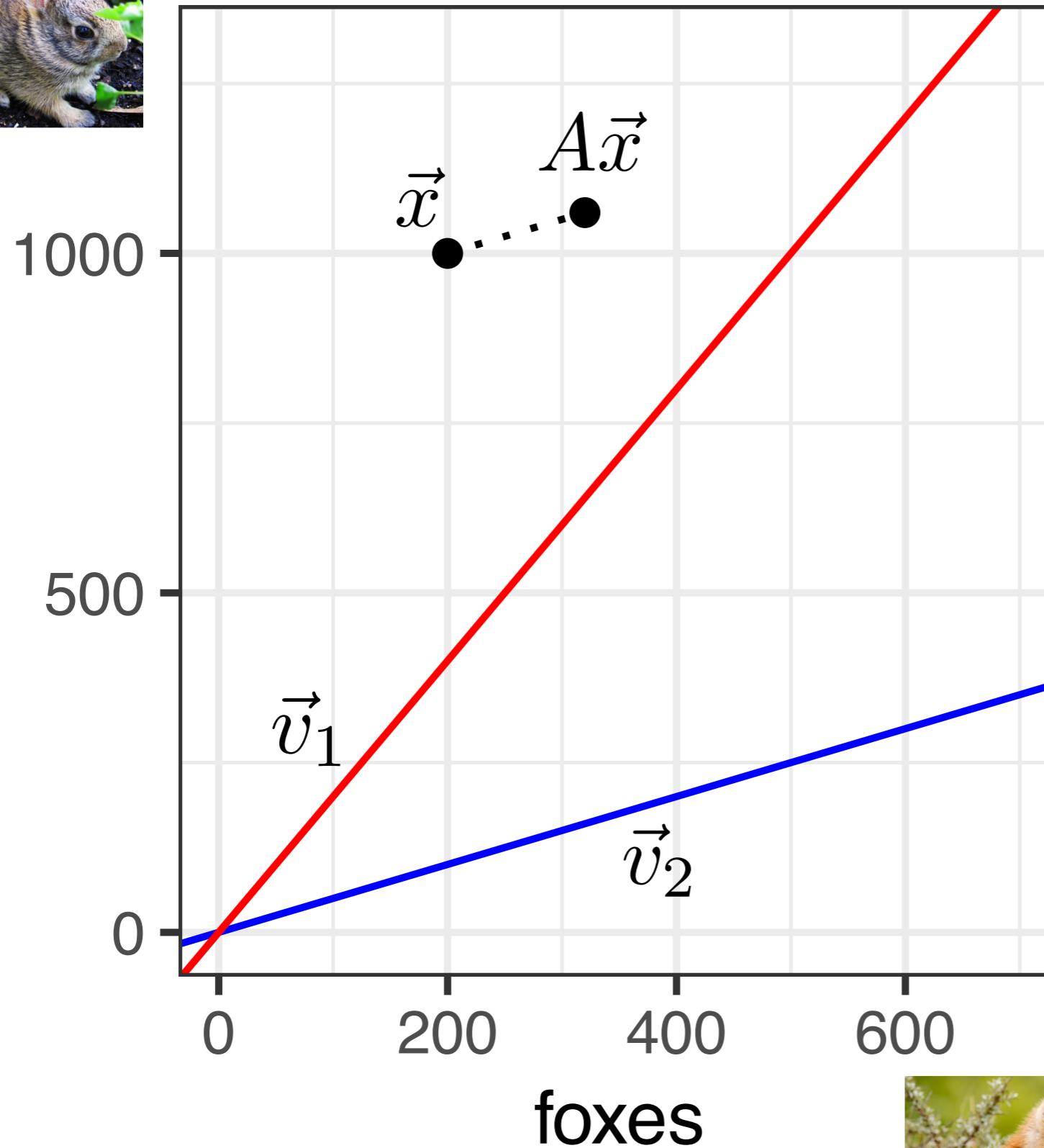
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rabbits



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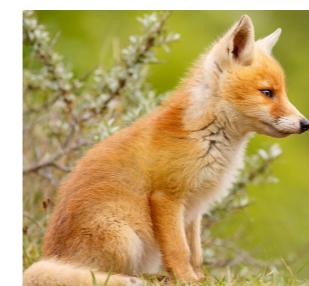
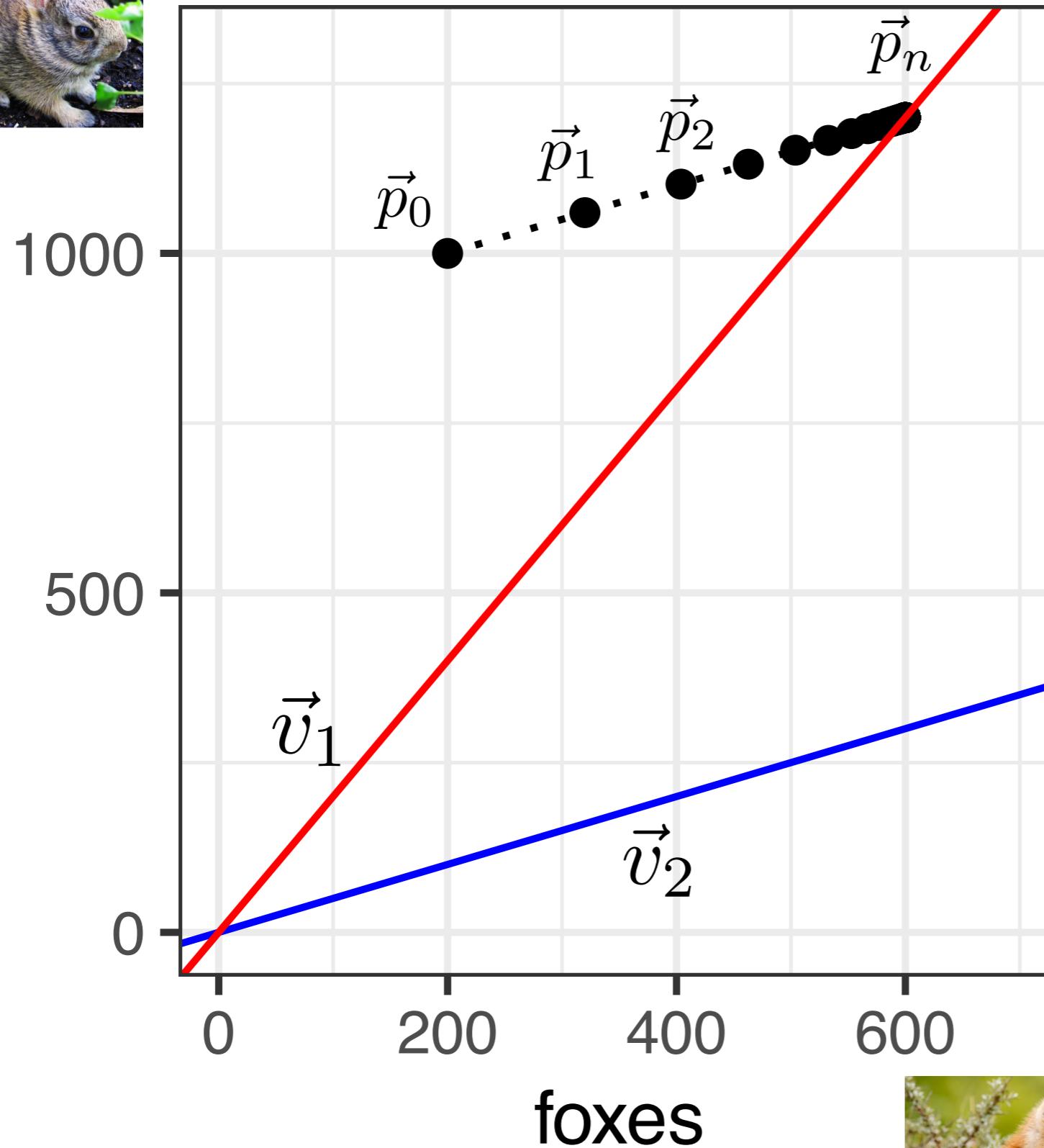
$$\vec{x} = a\vec{v}_1 + b\vec{v}_2$$

$$A\vec{x} = Aa\vec{v}_1 + Ab\vec{v}_2$$

$$= \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2$$



rabbits



$$\vec{p}_1 = A\vec{p}_0$$

$$\vec{p}_2 = A\vec{p}_1 = A^2\vec{p}_0$$

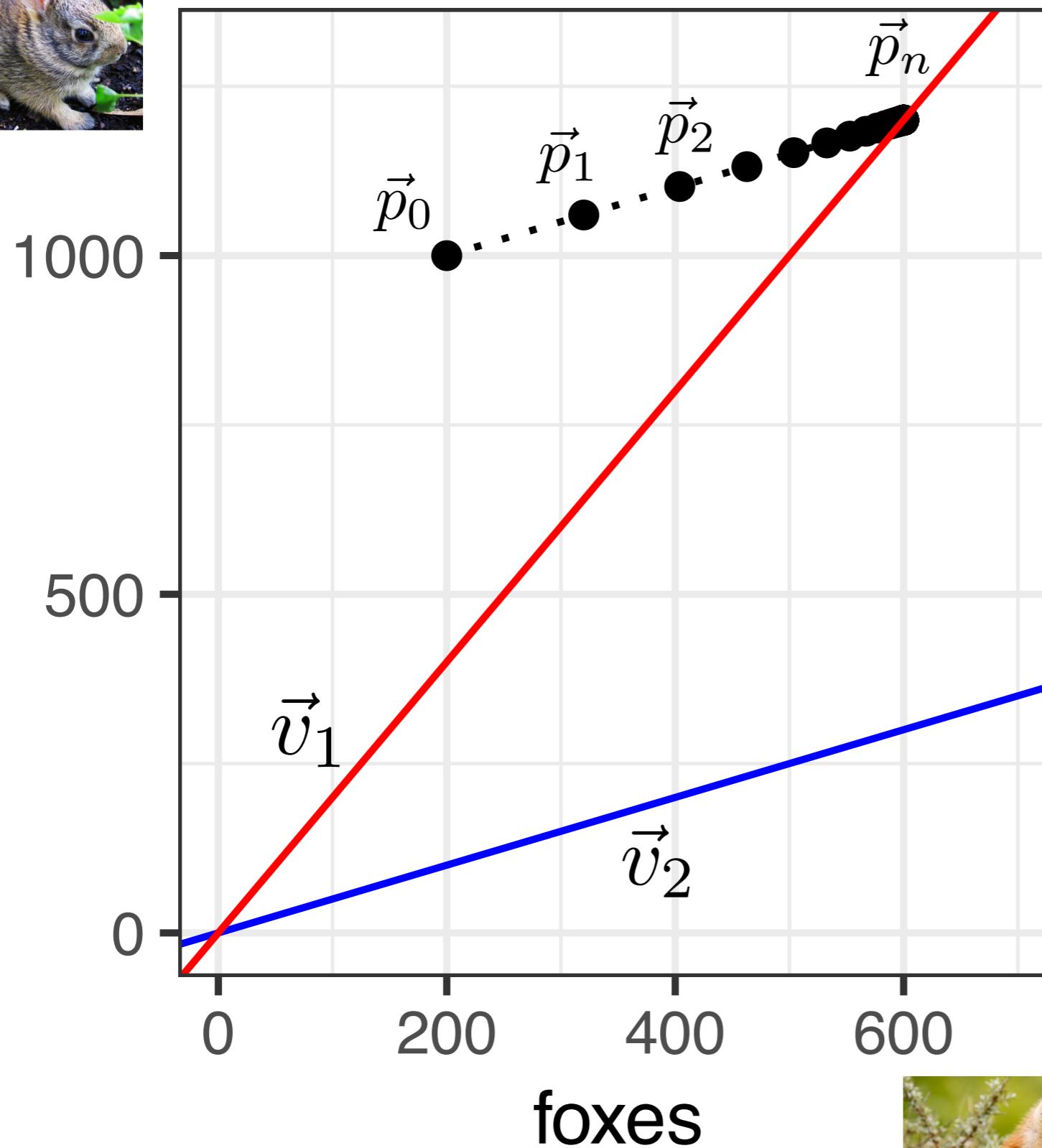
$$\vec{p}_3 = A\vec{p}_2 = A^3\vec{p}_0$$

⋮

$$\vec{p}_n = A\vec{p}_{n-1} = A^n\vec{p}_0$$



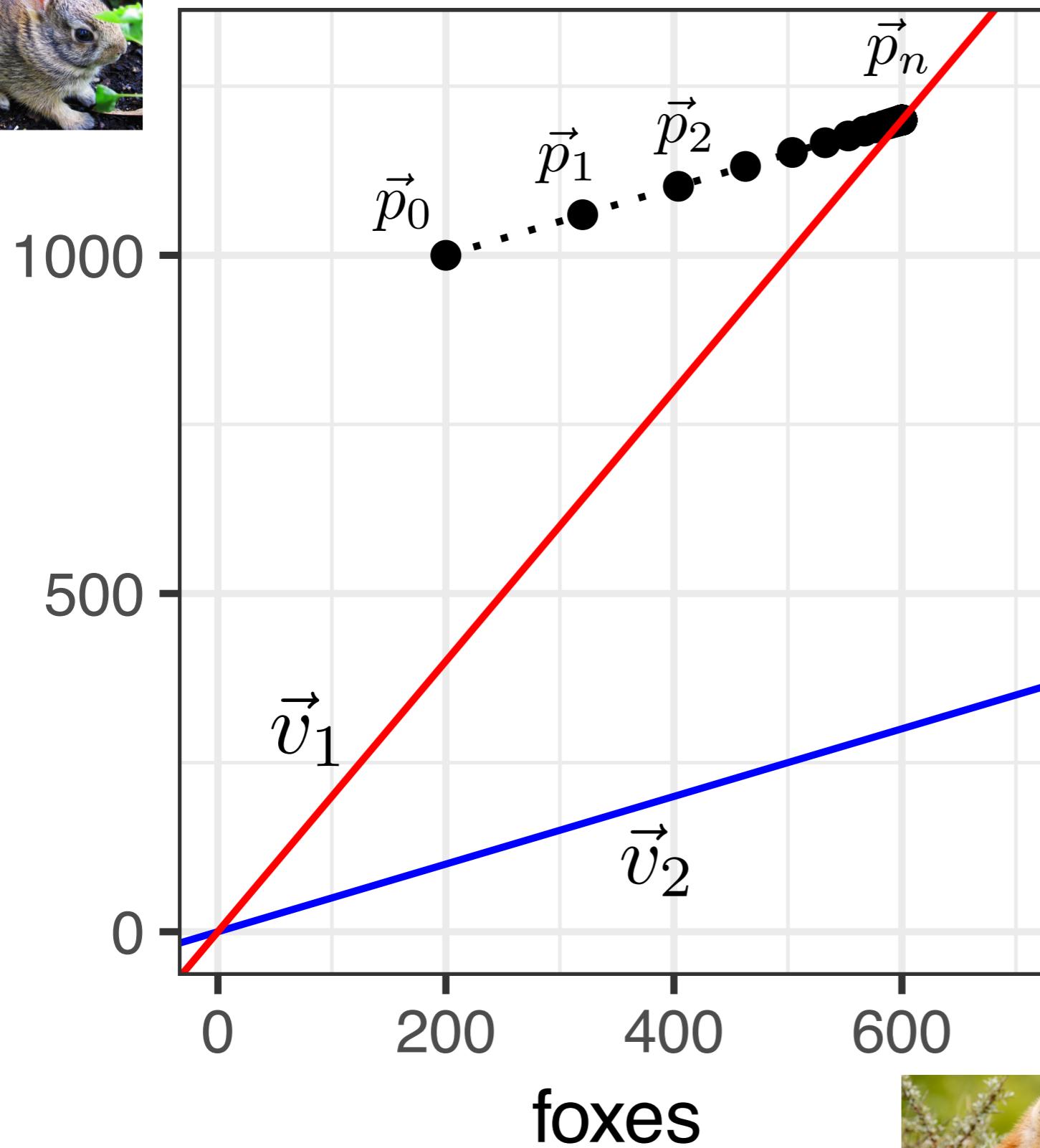
rabbits



$$\begin{aligned}A\vec{x} &= Aa\vec{v}_1 + Ab\vec{v}_2 \\&= \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2\end{aligned}$$

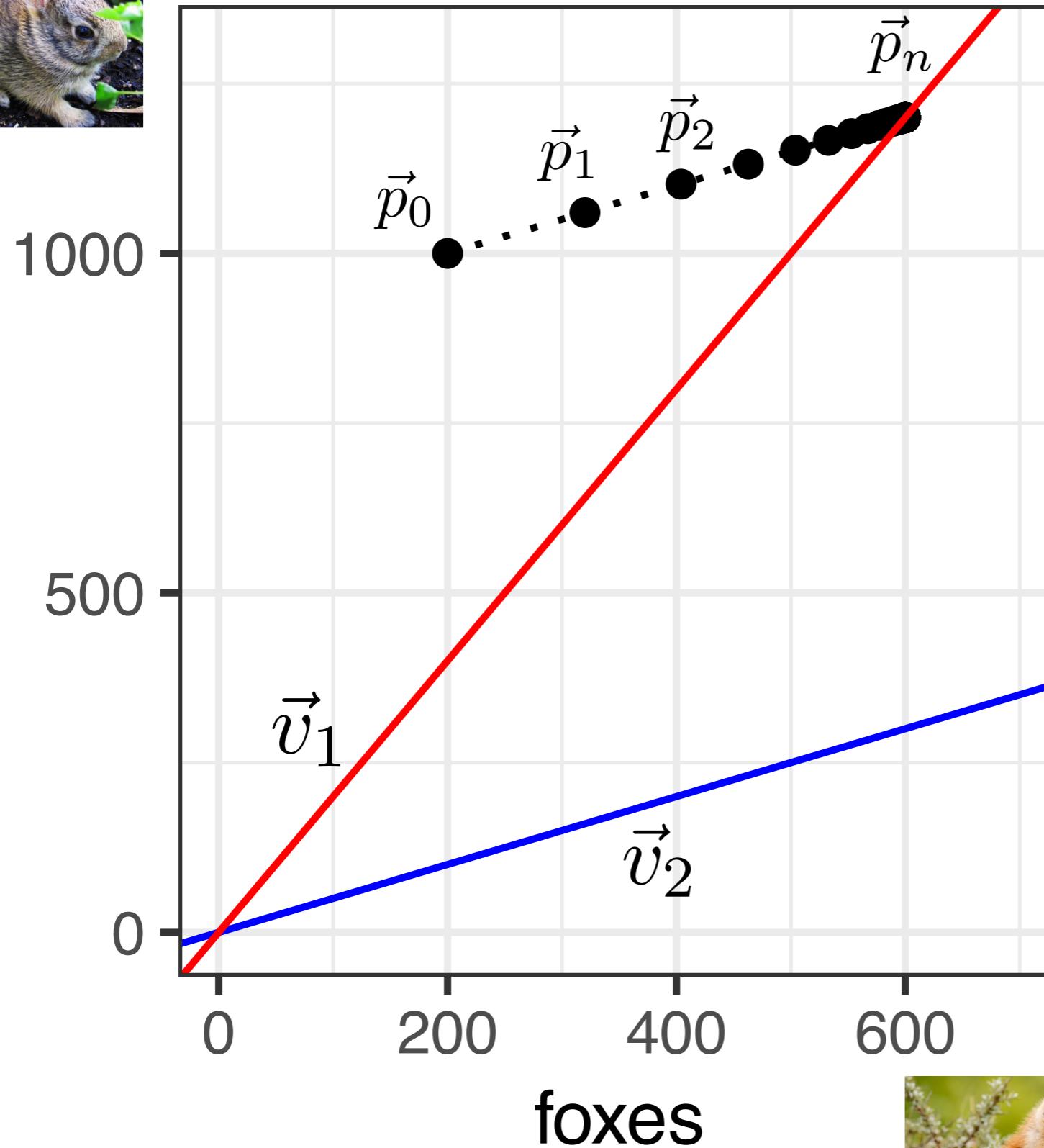


rabbits





rabbits



$$A\vec{x} = Aa\vec{v}_1 + Ab\vec{v}_2 \\ = \lambda_1 a\vec{v}_1 + \lambda_2 b\vec{v}_2$$

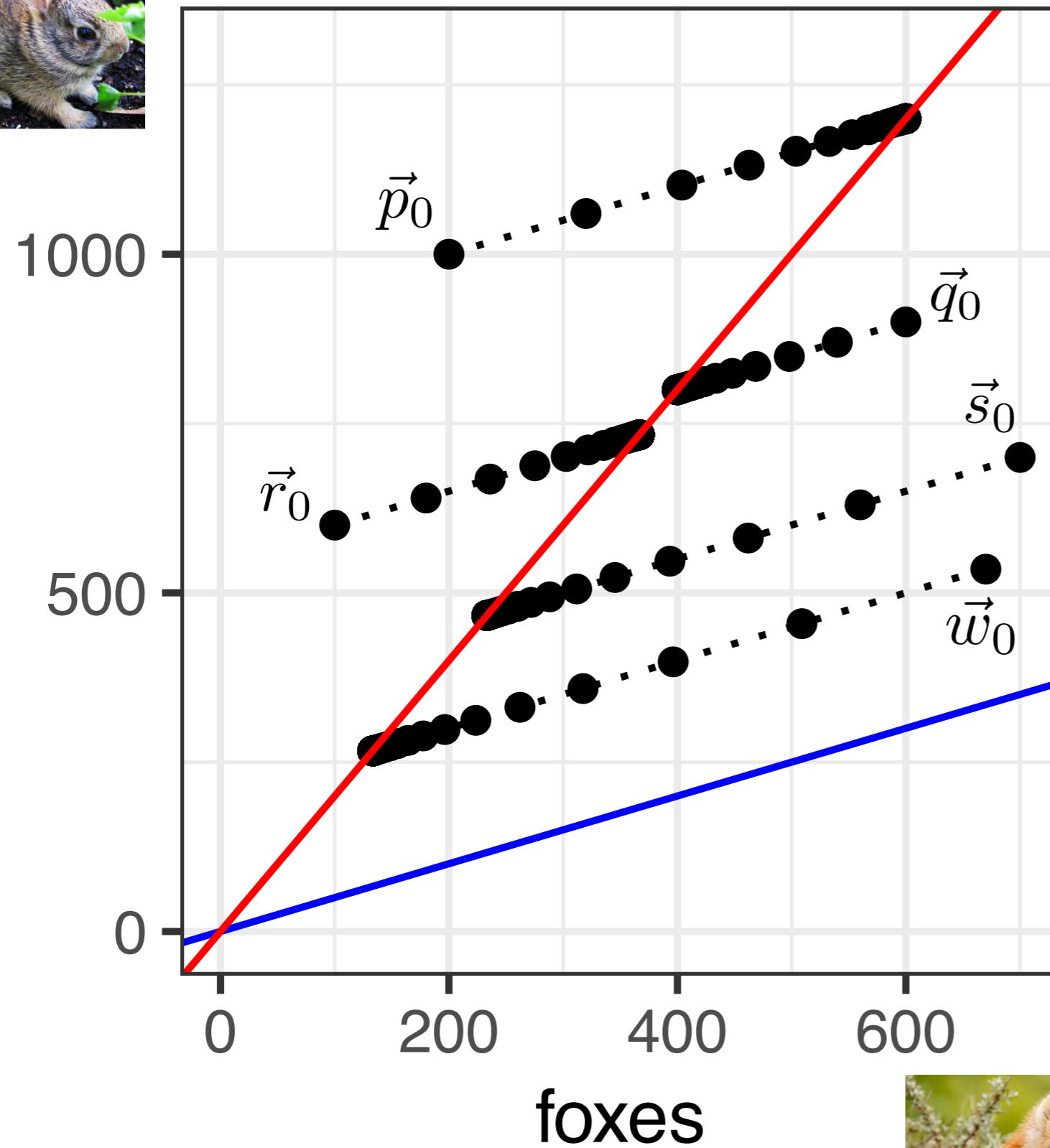
hard

$$A^n\vec{x} = \\ \lambda_1^n a\vec{v}_1 + \lambda_2^n b\vec{v}_2$$

easy



rabbits



$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.1 \end{bmatrix}$$

$$\lambda_1 = 1.0$$

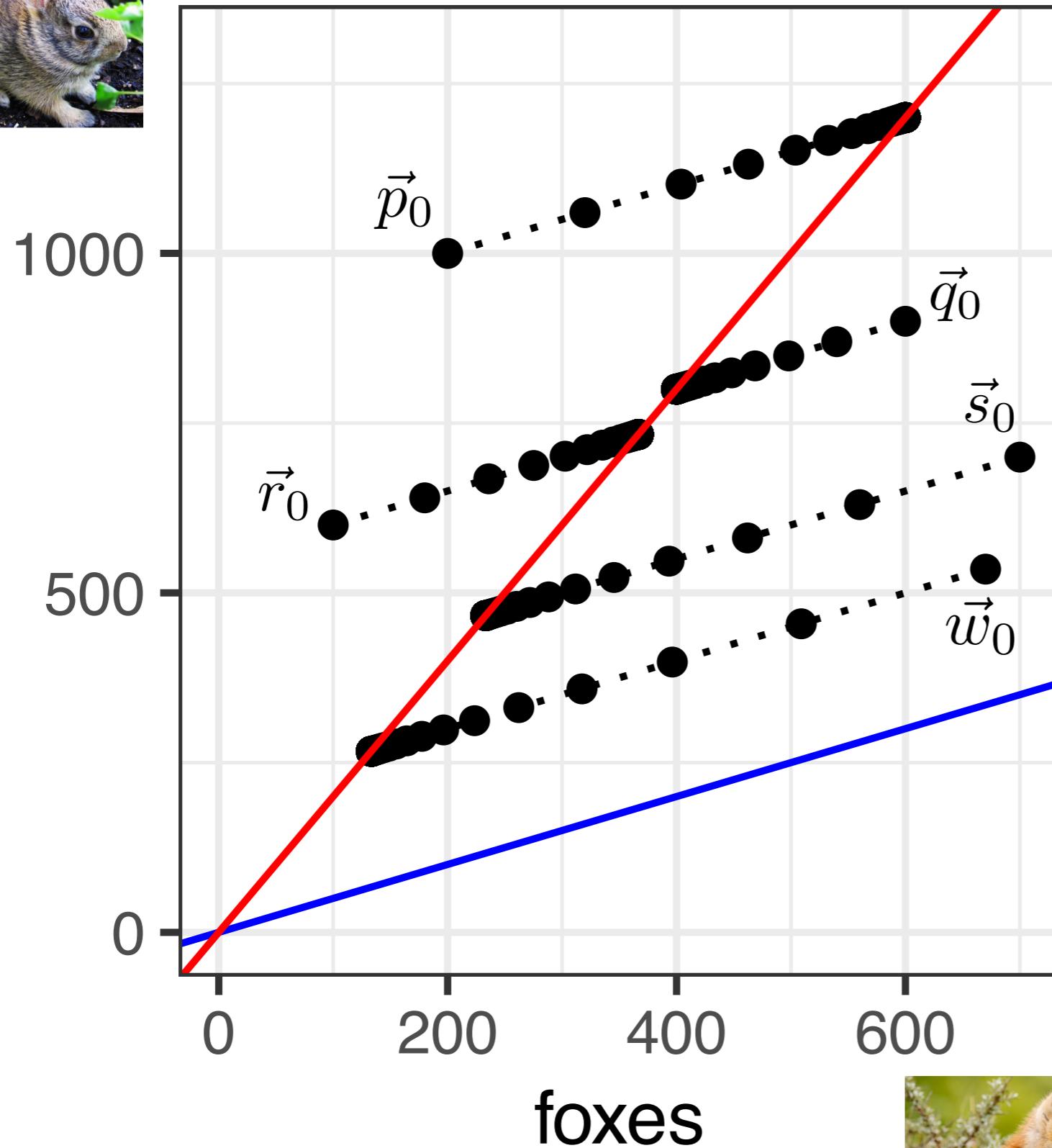
$$\vec{v}_1 \approx [-0.447 \quad -0.894]$$

$$\lambda_2 = 0.7$$

$$\vec{v}_2 \approx [-0.894 \quad -0.447]$$



rabbits



eigenvectors and
eigenvalues help us
understand the
system's dynamics

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$$\lambda_1 = 1.0$$

$$\vec{v}_1 \approx [-0.447 \quad -0.894]$$

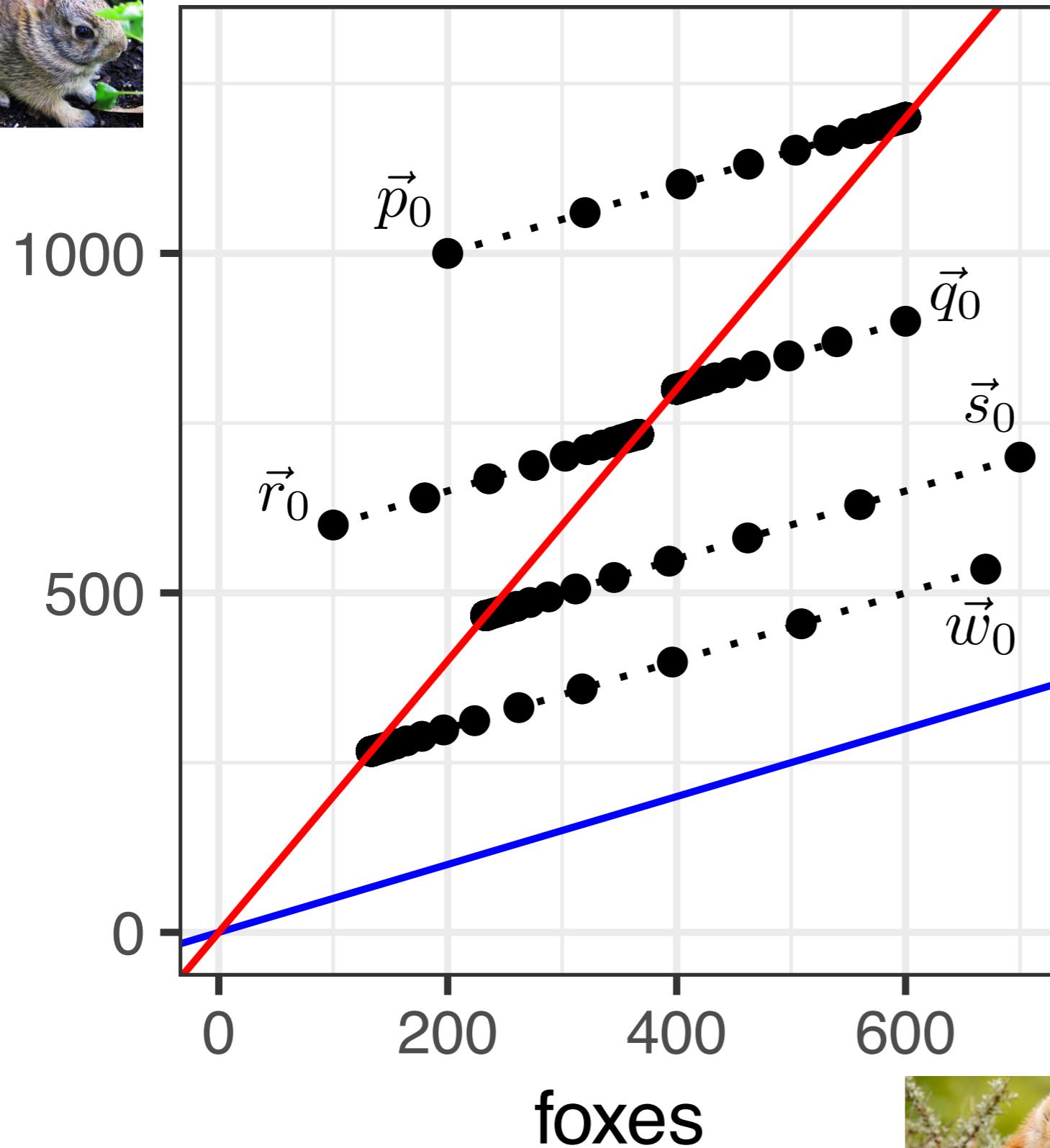
$$\lambda_2 = 0.7$$

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rabbits



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$$\lambda_1 = 1.0$$

$$\vec{v}_1 \approx \begin{bmatrix} -0.447 & -0.894 \end{bmatrix}$$

$$\lambda_2 = 0.7$$

$$\vec{v}_2 \approx \begin{bmatrix} -0.894 & -0.447 \end{bmatrix}$$



rabbits

1000

500

0

0

200

400

600

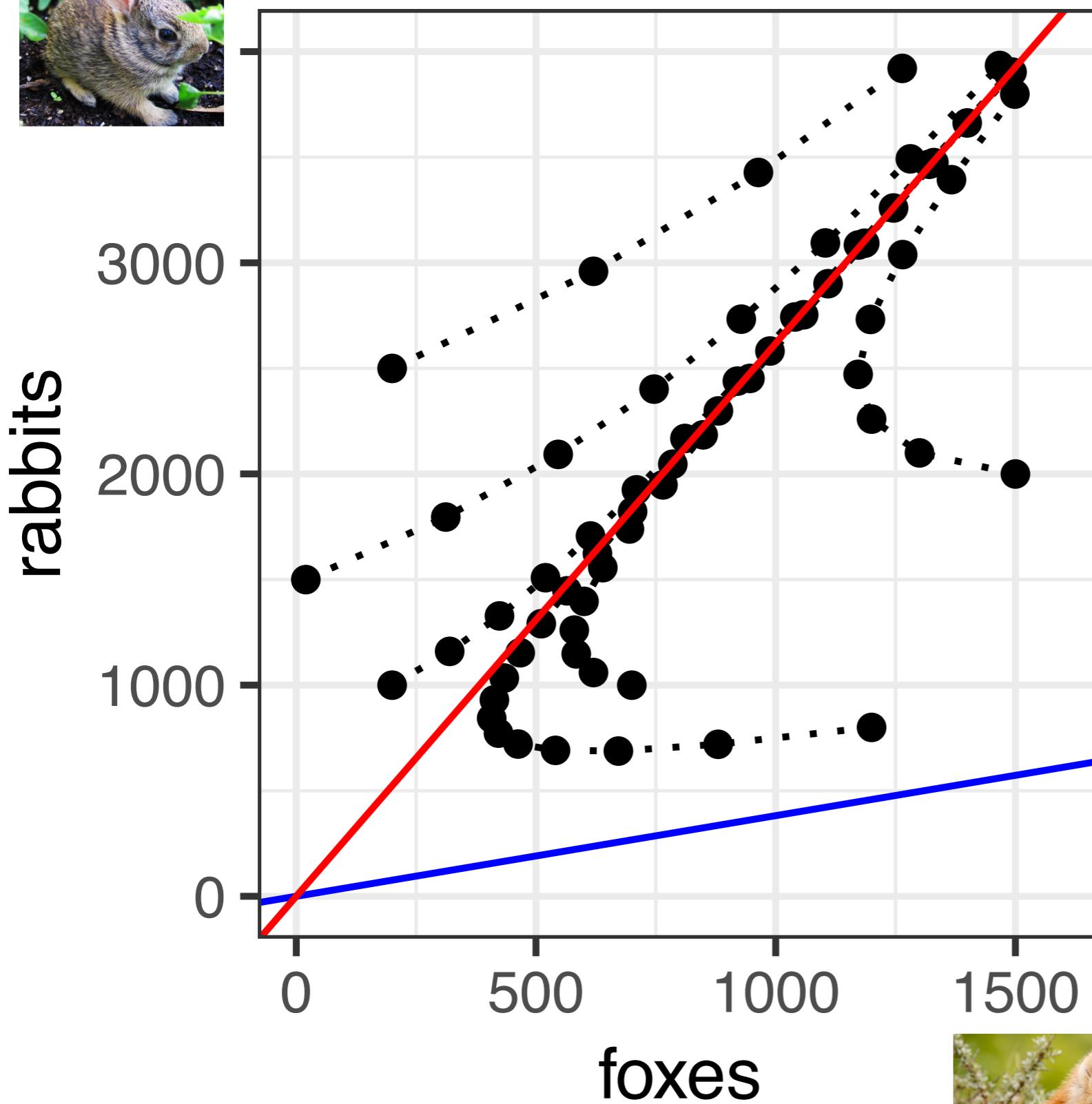
foxes

?



“rabbits multiply”

$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.2 \end{bmatrix}$$



"rabbits multiply"

$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.2 \end{bmatrix}$$

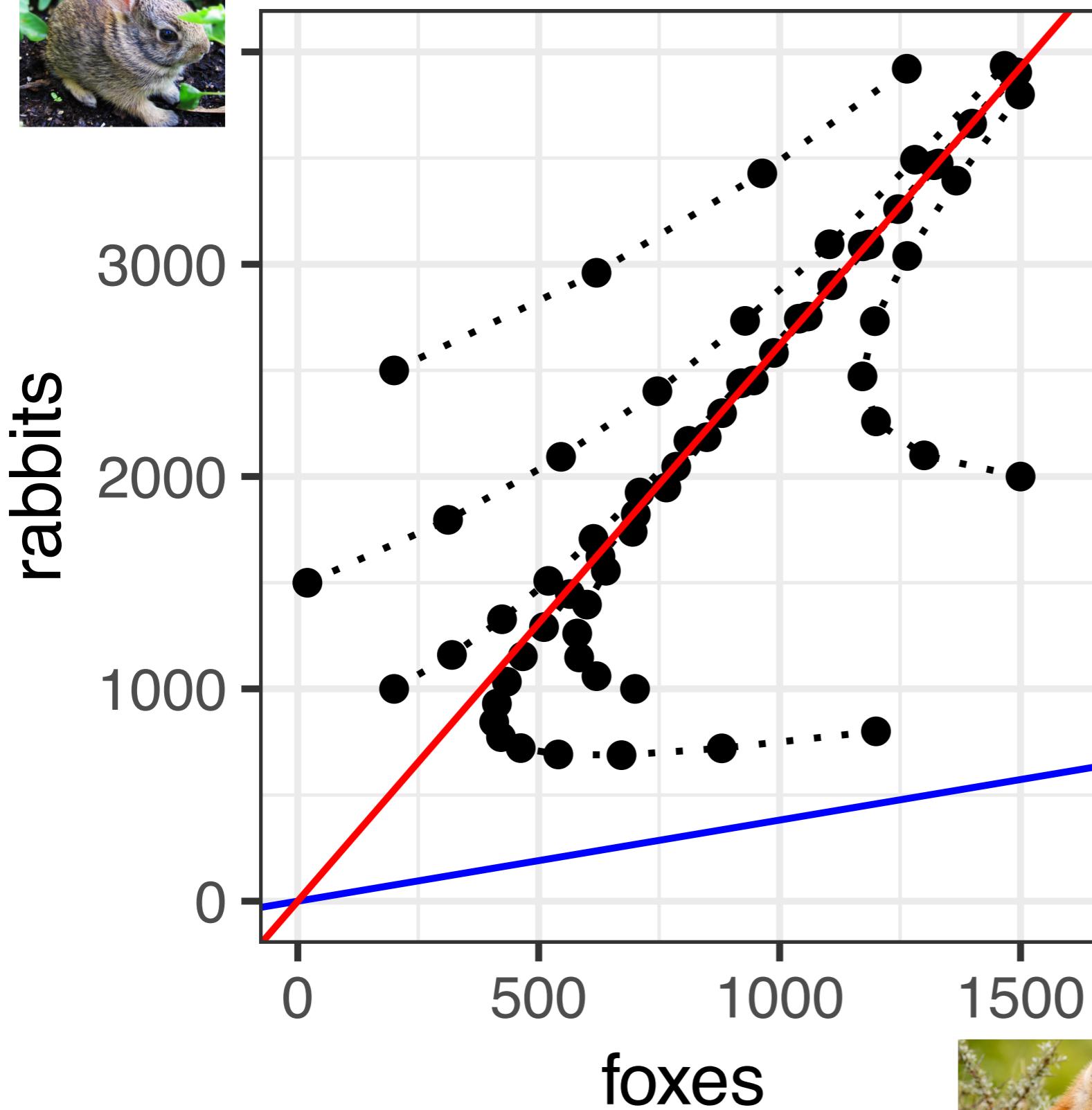
$$\lambda_1 \approx 1.124$$

$$\vec{v}_1 \approx [-0.357 \quad -0.934]$$

$$\lambda_2 \approx 0.676$$

$$\vec{v}_2 \approx [-0.934 \quad -0.357]$$





“rabbits multiply”

$$A = \begin{bmatrix} 0.6 & 0.2 \\ -0.2 & 1.2 \end{bmatrix}$$

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rabbits

4000

2000

0

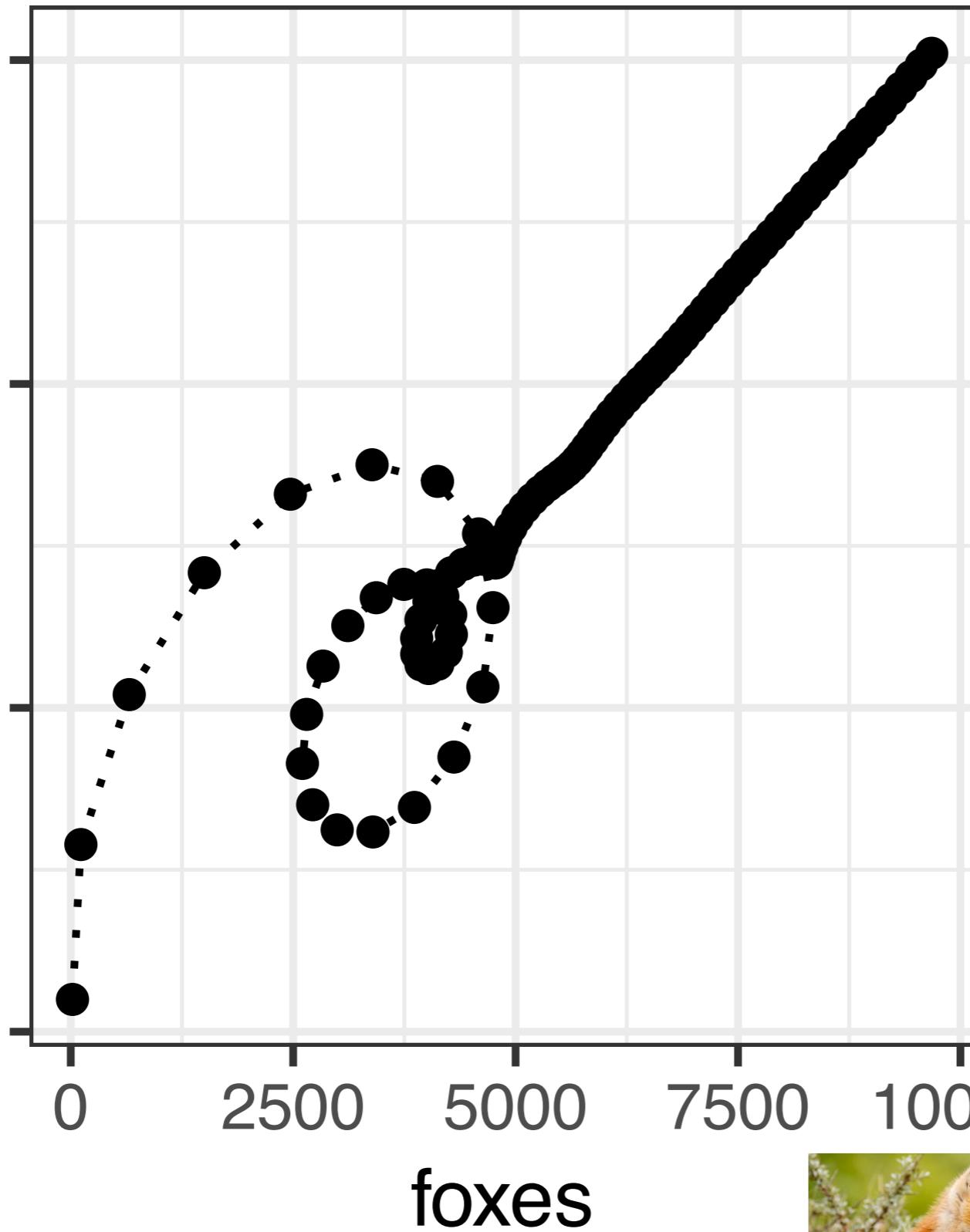
0

2500

foxes

7500

10000



A =

$$A = \begin{bmatrix} 0.7 & 0.5 & 0 \\ -0.2 & 0.8 & 0.5 \\ 0 & -0.2 & 1.2 \end{bmatrix}$$

introducing "grass"
into the model



A close-up photograph of a red fox sitting in a grassy field. The fox is facing right, with its head turned slightly towards the camera. Its fur is a rich reddish-orange color. The background is a soft-focus green, suggesting a natural, outdoor setting.

Takeaway?

we need to be
able to compute

$$A^n$$

to understand
system dynamics

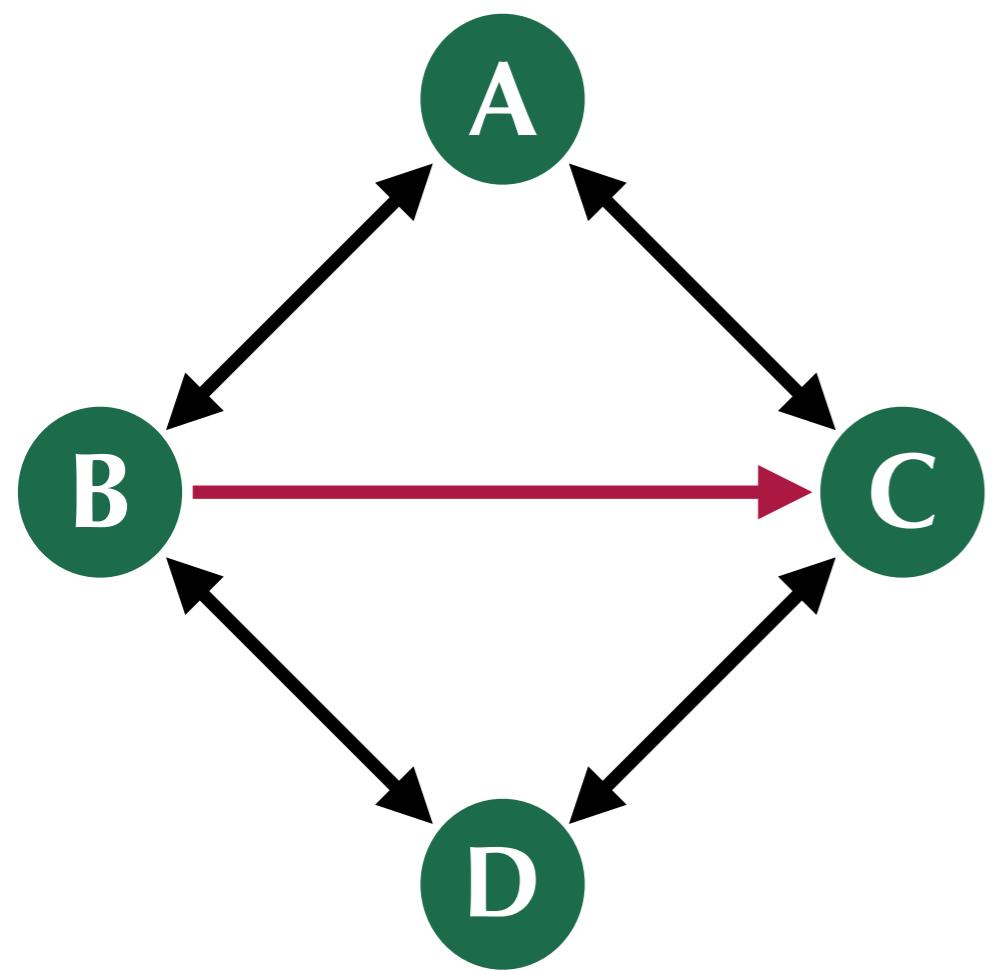
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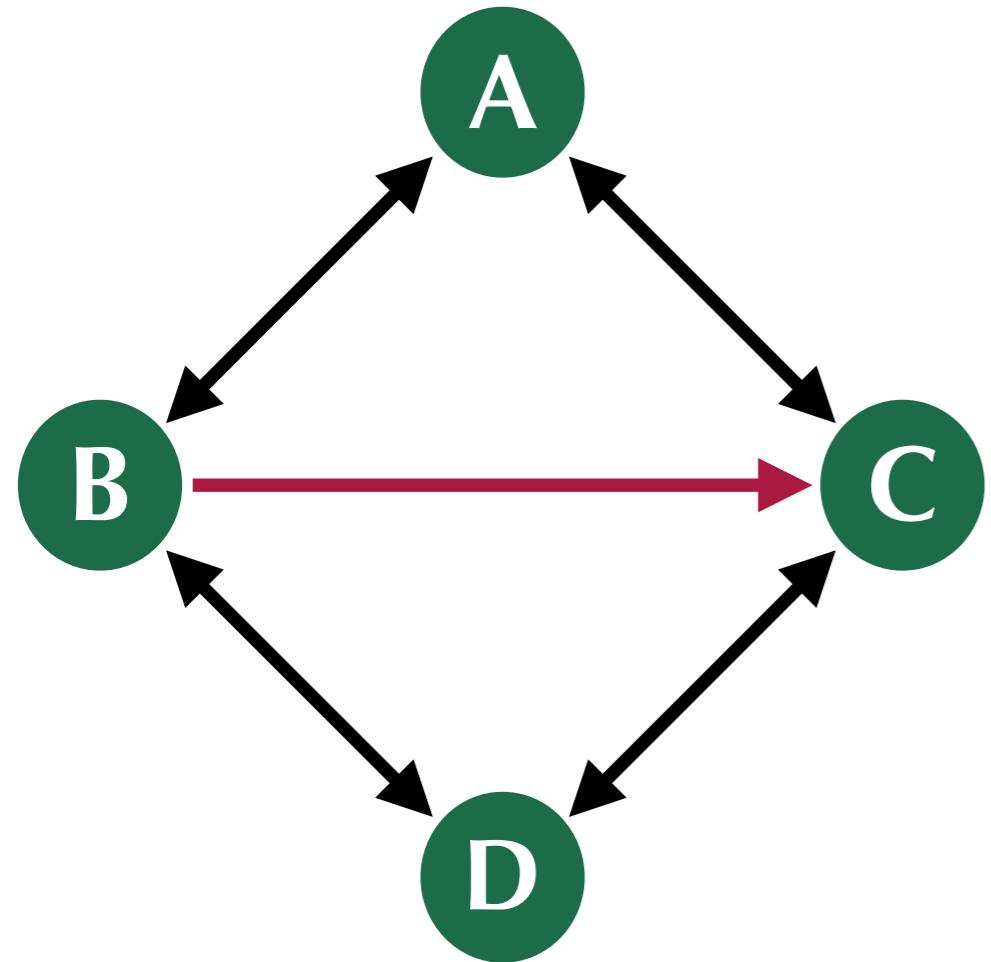
Takeaway?

eigenvectors and
values help us:

compute A^n quickly

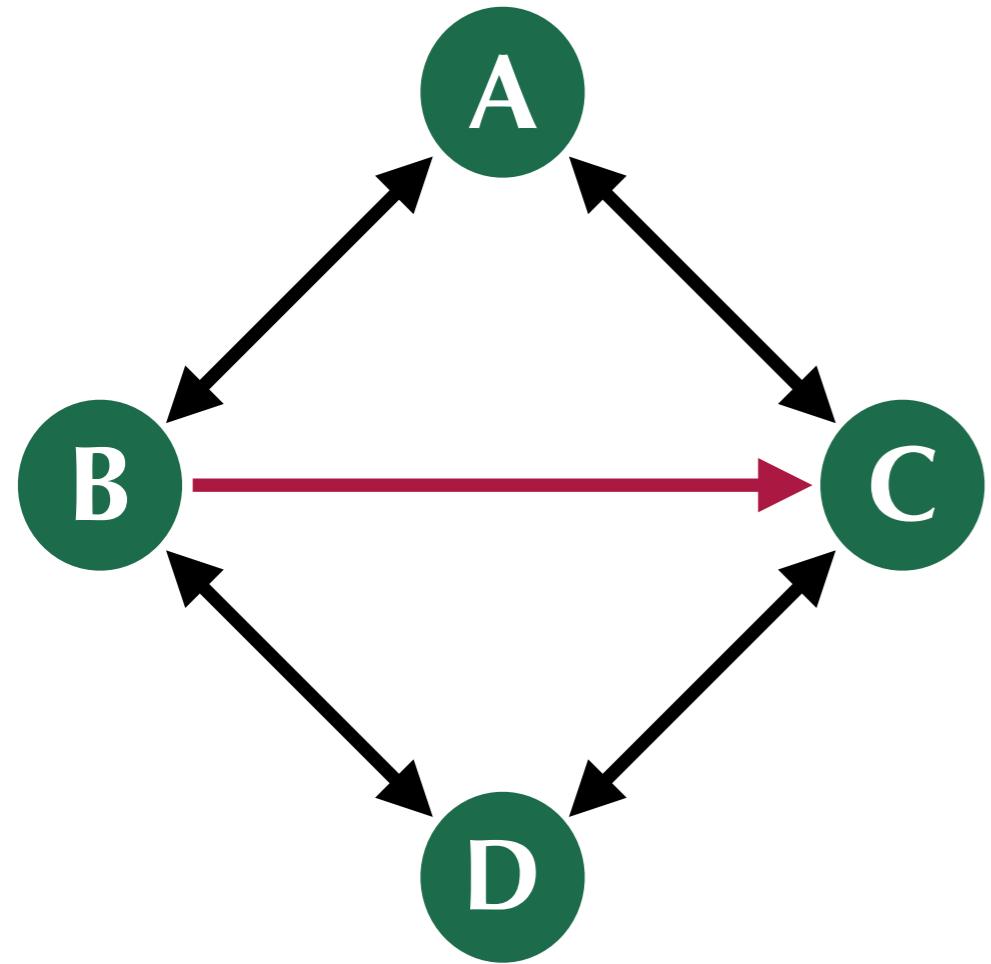
understand
convergence to
steady states





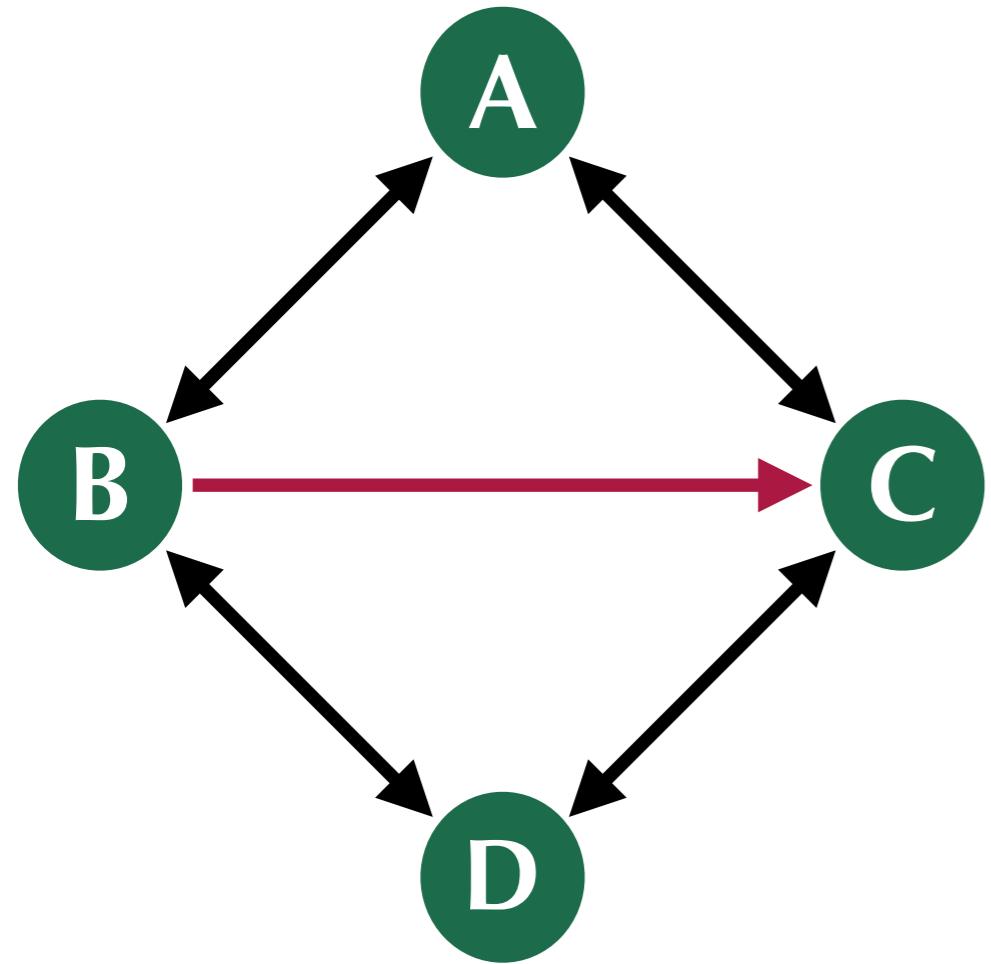
Adjacency Matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Adjacency Matrix

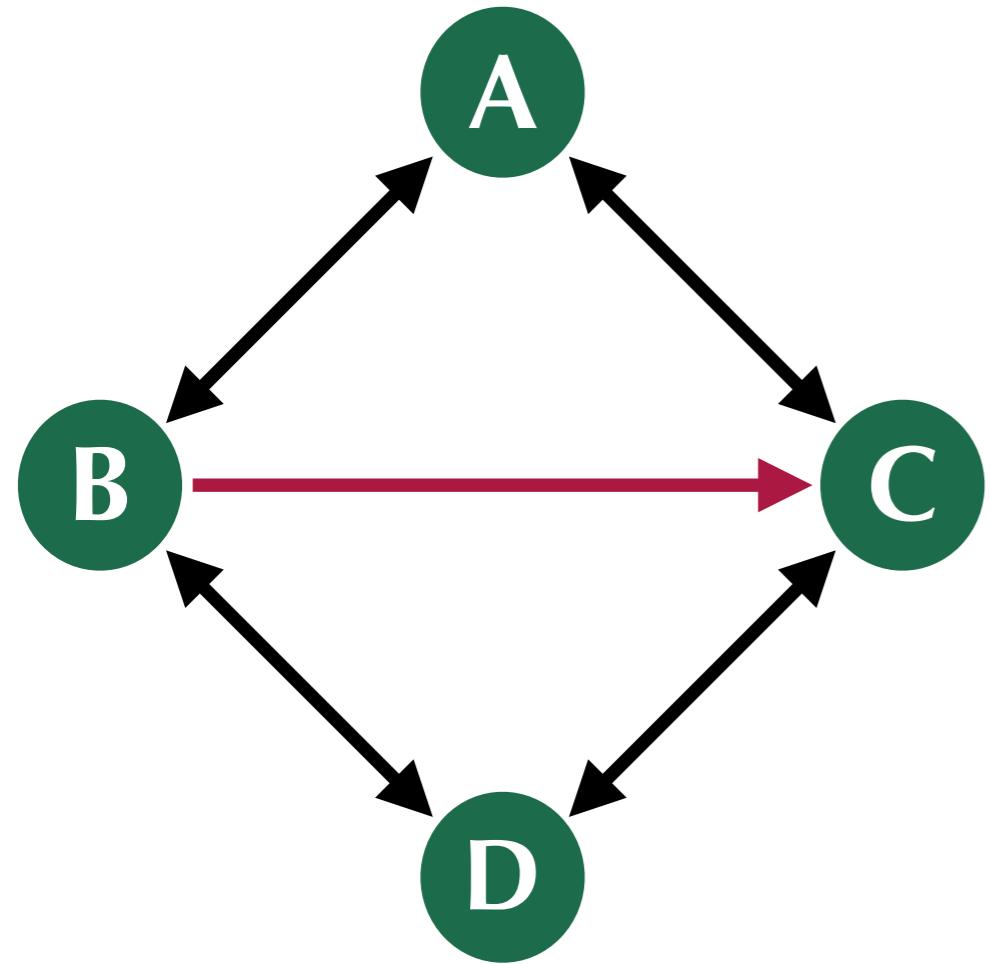
$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{l} A \\ B \\ C \\ D \end{array}$$



Adjacency Matrix

$$M = \begin{bmatrix} & \text{A} & \text{B} & \text{C} & \text{D} \\ \text{A} & 0 & 1 & 1 & 0 \\ \text{B} & 1 & 0 & 1 & 1 \\ \text{C} & 1 & 0 & 0 & 1 \\ \text{D} & 0 & 1 & 1 & 0 \end{bmatrix}$$

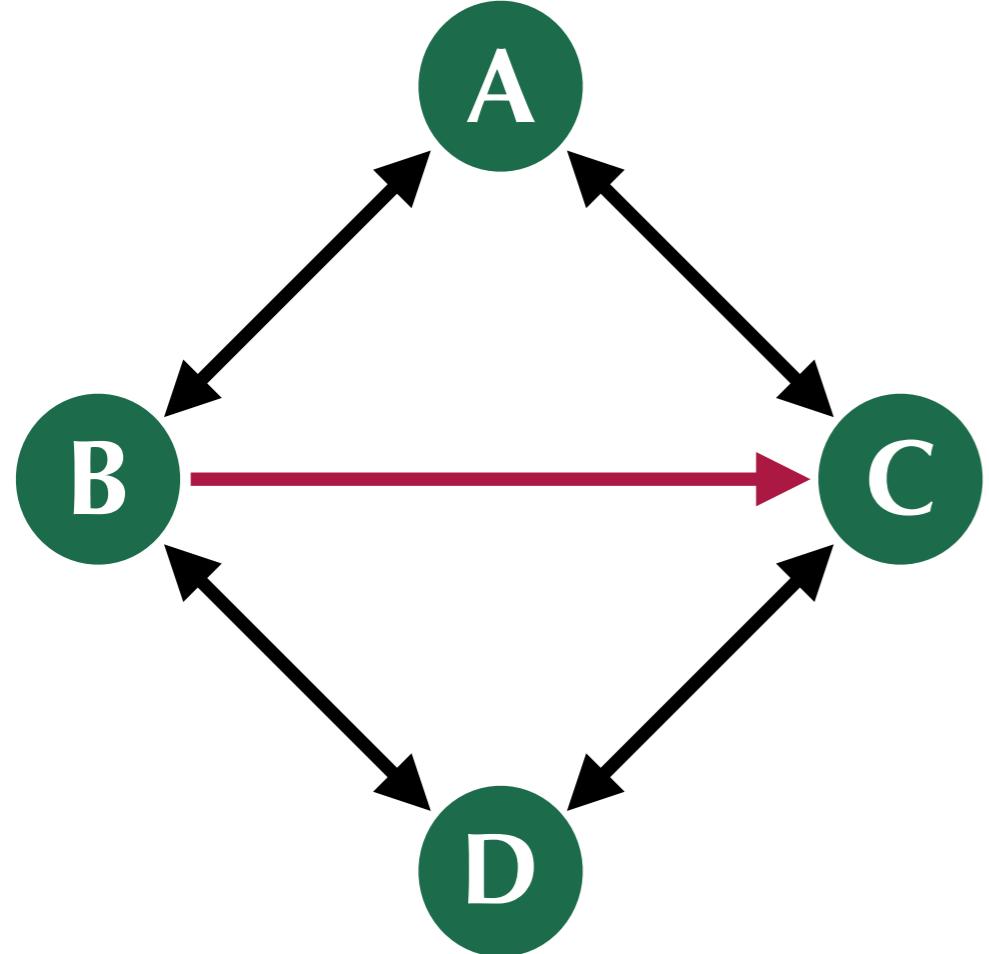
$m_{ij} = 1$ iff there's an edge from i to j



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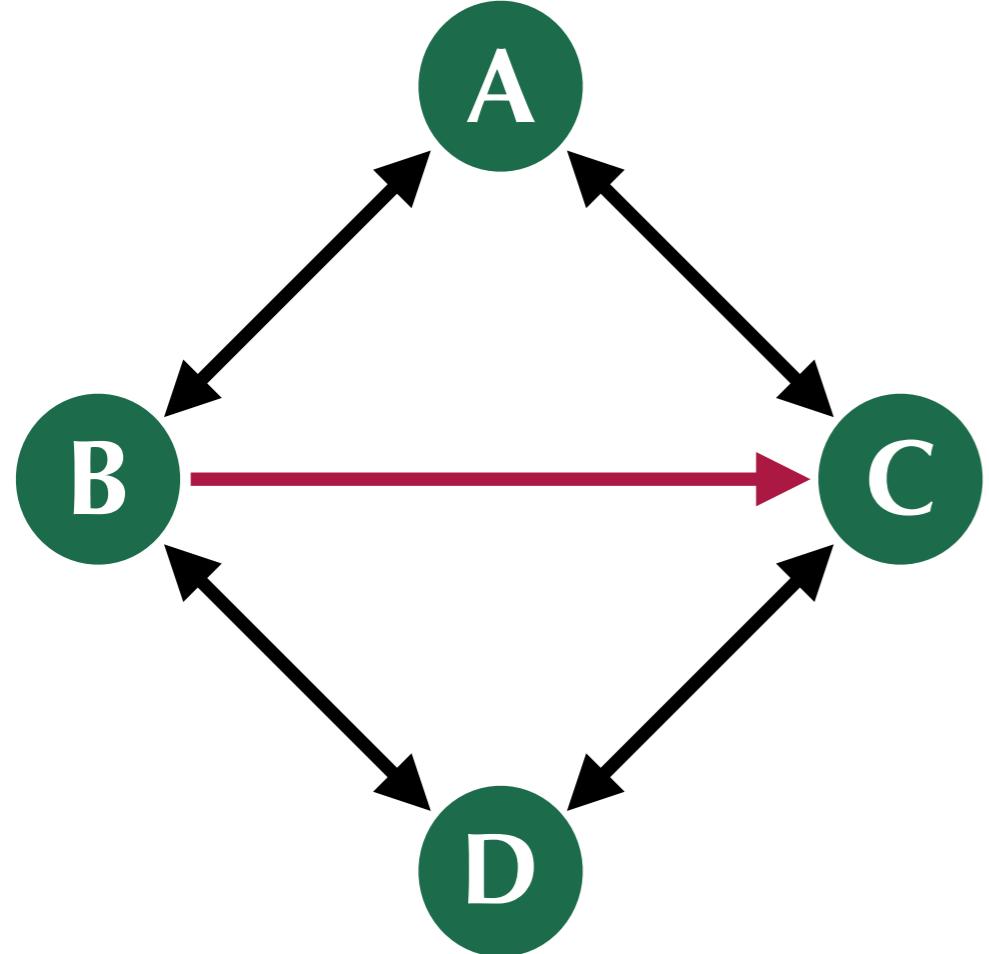
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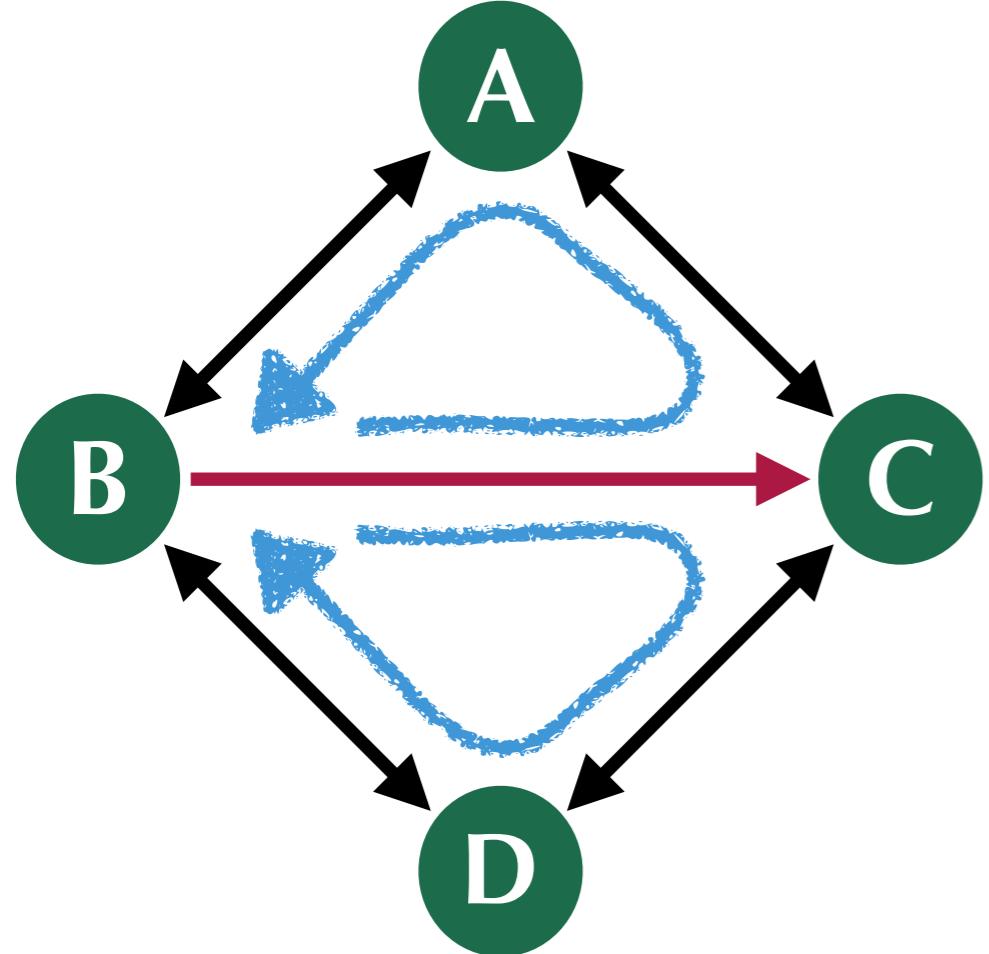
M^n is a matrix where $m_{ij}^{(n)}$ represents the number of n -step paths from i to j



$$M^3 = \begin{bmatrix} A & B & C & D \\ 1 & 4 & 4 & 1 \\ 4 & 2 & 4 & 4 \\ 4 & 0 & 2 & 4 \\ 1 & 4 & 4 & 1 \end{bmatrix} \begin{array}{l} A \\ B \\ C \\ D \end{array}$$



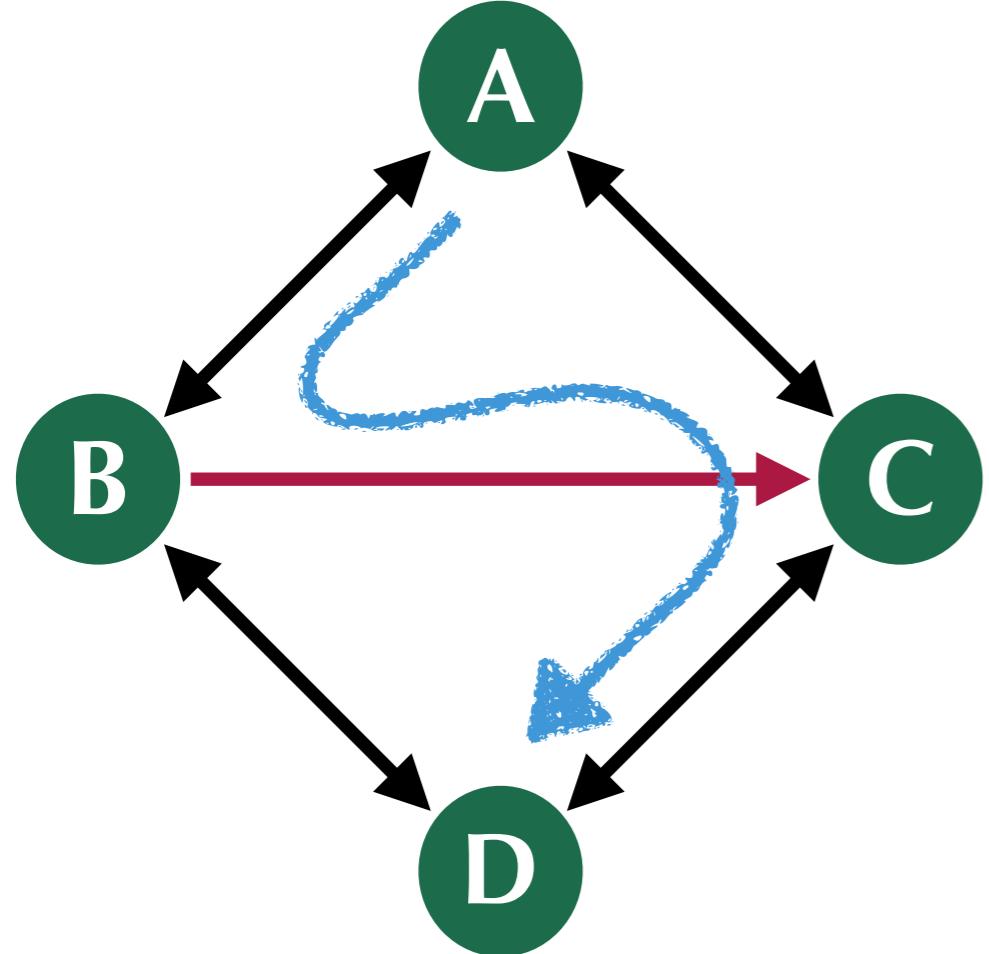
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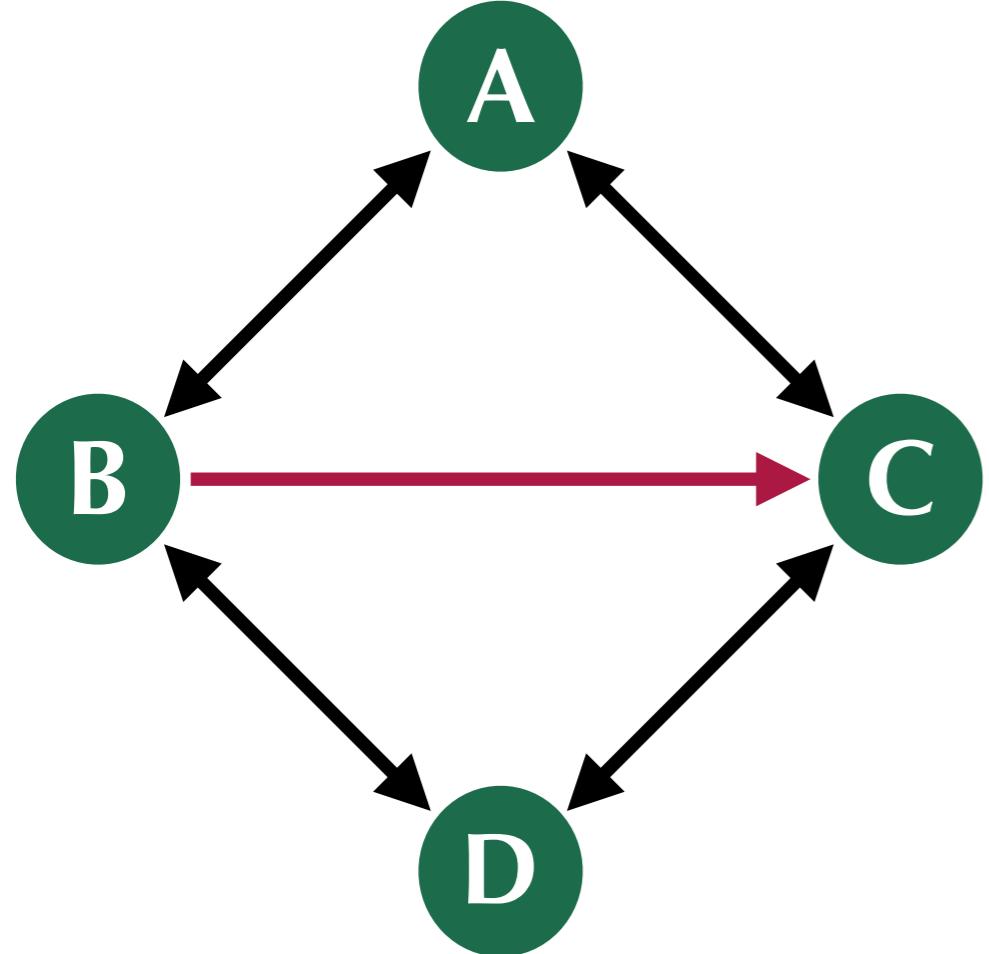
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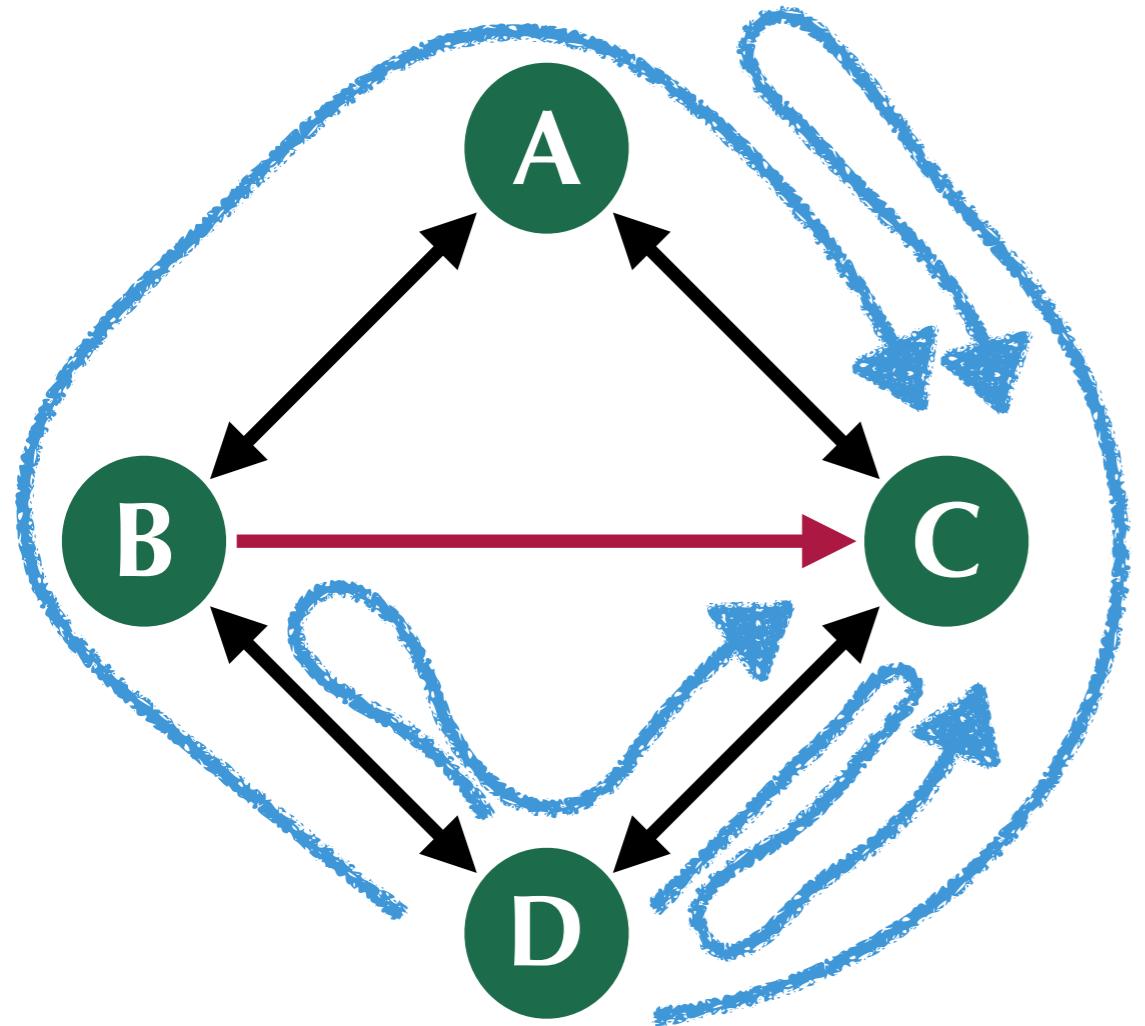


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The matrix M^3 is shown with columns labeled A, B, C, and D. The entry in the third column and fourth row is highlighted with a blue circle.



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M^n is a matrix where $m_{ij}^{(n)}$ represents the number of n -step paths from i to j

Takeaway?

we need to be able to compute

$$M^n$$

to understand networks



Takeaway?

eigenvectors and eigenvalues
help us compute M^n efficiently



eigenvectors and **eigenvalues** help us:

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compute matrix powers efficiently/quickly

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understand how quickly systems converge

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see if a matrix is invertible

how do we find them?

Definition: Given $A_{n \times n}$, say λ is an eigenvalue for A if $A\vec{v} = \lambda\vec{v}$ for some non-zero vector \vec{v} .

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called an *eigenvector* of A
corresponds to λ

how do we find them?

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$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$

$$[A - \lambda I]\vec{v} = \vec{0}$$

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non-zero

how do we find them?

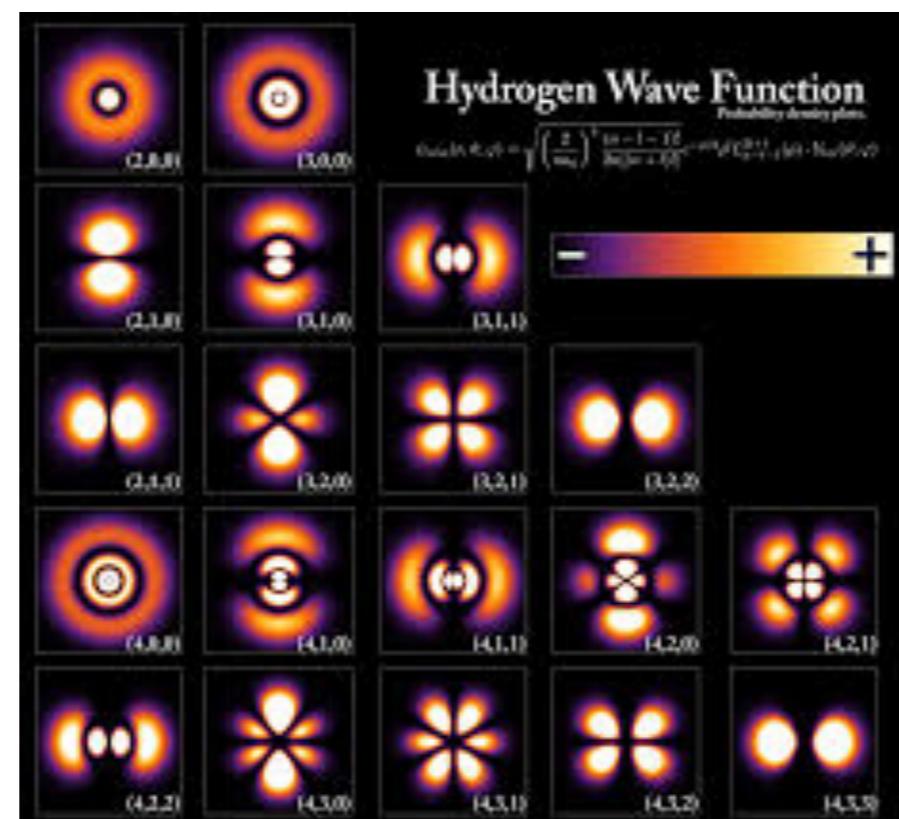
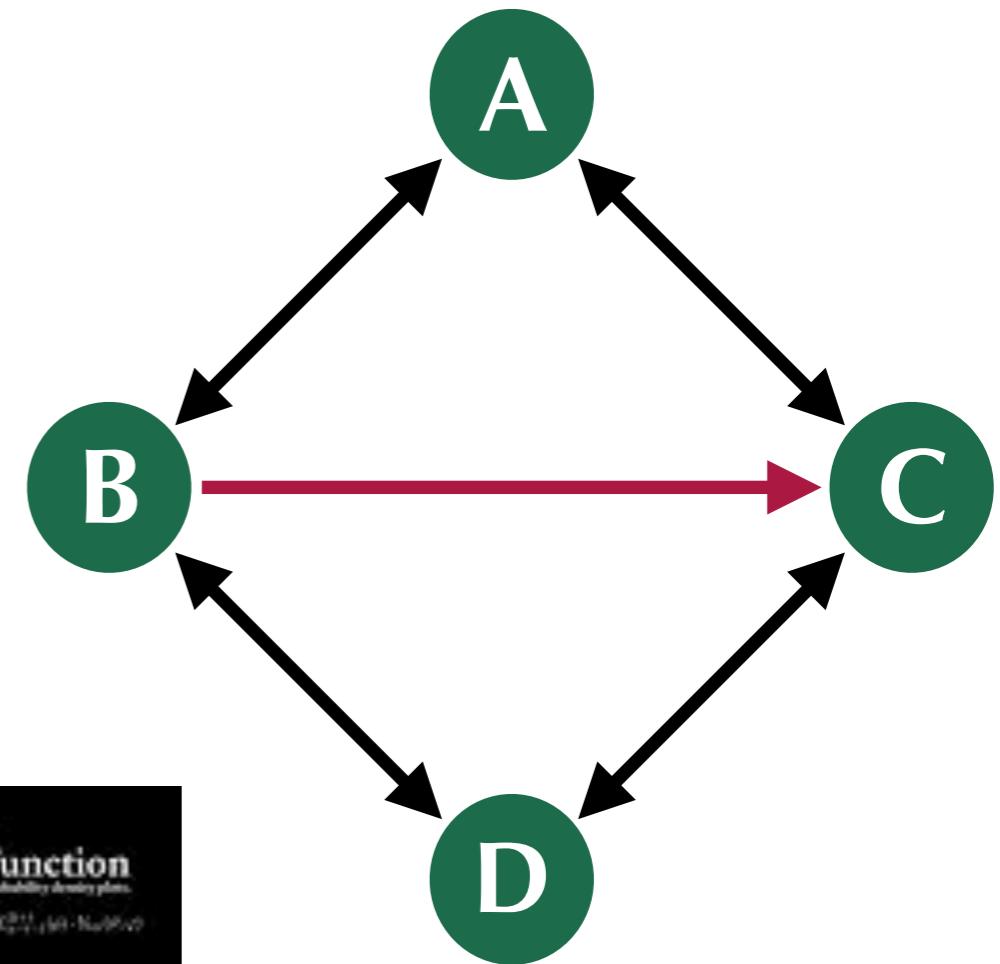
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$$|[A - \lambda I]| = 0$$



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