CS4150 Theory 6 Solutions

Fall 2023

October 2023

Quiz A

Question 1

Solution: We start with $GCD(F_{k+1}, F_k)$ meaning we will recurse to $GCD(F_k, F_{k+1}\%F_k)$. Since $F_{k+1} = F_k + F_{k-1}$, then $F_{k+1}\%F_k = F_{k-1}$. Therefore $GCD(F_k, F_{k+1}\%F_k) = GCD(F_k, F_{k-1})$. We can recursively continue this process until we get to $GCD(F_1, F_0)$ which will return. Since on each level we are decreasing the value of F_k by one, we end up with k iterations.

Question 2

Solution: Consider if $d > 1$. Then $GCD(x, y) = d \Rightarrow d|x$ and $d|y$. This implies we can rewrite $x = d \cdot q$ and $y = d \cdot q'$. By substituting these in to the equation $x \equiv 1 \mod y$ to get $d \cdot q \equiv 1 \mod d \cdot q'$. Then it must mean that $dq = dq'z + 1, z \in \mathrm{Z}^+$. Let $z = 1$, then $dq - dq' = 1 \Rightarrow d \cdot (q - q') = 1$. Since $d, q, q' \in \mathrm{Z}^+$, it must be that $q - q' = d = 1$. However, this contradicts our original statement that $d > 1$. Therefore, it must be true that: If $x \equiv 1 \mod y \Rightarrow GCD(x, y) = 1$

Question 3

Solution:

i)

If $a \equiv b \mod N$ and $x \equiv y \mod N$, then since both are $\mod N$ they adhere to normal rules of multiplication, meaning we may write $ax \equiv by \mod N$.

ii)

False with counter example $a = 2, b = 2, x = 7, y = 2, N = 5$. Note that $2 \equiv 2 \mod 5$ and $7 \equiv 2 \mod 5$ but $2^7 \not\equiv 2^2 \mod 5 \Rightarrow 3 \not\equiv 4 \mod 5$

Question 4

Solution: Consider $(X + Y) \cdot (X - Y)$. If we expand we get $X^2 + Y^2 + XY - XY = X^2 + Y^2$. Since we can find squares in $O(n)$ time, we do $2 \cdot O(n) = O(n)$ operations to multiply numbers $(X + Y)$ and $(X - Y)$. Therefore we have done better than $O(n \cdot \log^3(n))$. Note: a cool generalization of this is that for any number J, J can be factored into computations involving only squares or additions.

Question 5

Solution: Let $a = 3, b = 6$. $ab \equiv 0 \mod 18 \Rightarrow 18 \equiv 0 \mod 18$, but $3 \not\equiv 0 \mod 18$ and $6 \not\equiv 0 \mod 18$

Question 6

Solution Since $GCD(c, N) = 1$ then c and N are co-prime. This means by the cancellation law of congruence: $cx \equiv cy \mod n \Rightarrow x \equiv y \mod n$

Alternate Solution $cx \equiv cy \mod N \Rightarrow cx - cy \equiv 0 \mod N \Rightarrow c(x-y) \equiv 0 \mod N \Rightarrow N|c(x-y)$. If $GCD(c, N) = 1$, then c and N don't share any factors, meaning if $N|c(x-y) \Rightarrow N|(x-y)$ since $x-y$ must be some multiple of N. $N|(x-y) \Rightarrow x-y \equiv 0 \mod N \Rightarrow x \equiv y \mod N$

Theorem: Cancellation Law of Congruence
$ca \equiv cb \mod n \Rightarrow a \equiv b \mod \frac{n}{d}$ where $d = gcd(c, n)$.

$pf.$
$n|ca - cb \Rightarrow ca - cb = qn, \exists q \in \mathrm{Z}(*)$
$\Rightarrow c(a-b) = qn, \exists q \in \mathrm{Z}$
From the hypothesis $d = gcd(c, n)$
$c = dr$ and $n = ds$ with $gcd(r, s) = 1(**)$
plugging $*$ into $**$ we get $dr(a-b) = qds$
$\Rightarrow r(a-b) = qs$ since $gcd(r, s) = 1$
$\Rightarrow s|(a-b)$ by Euclid's lemma ($a|bc$ and $gcd(a, b) = 1 \Rightarrow a|c$)
$\Rightarrow a \equiv b \mod s$ (since $n = ds \Rightarrow s = \frac{n}{d}$)
$\Rightarrow a \equiv b \mod \frac{n}{d}$

Question 7

Solution: Since $N \equiv 7 \mod 11$ then $N = 11k + 7$. Try integers $k = 1 \rightarrow 20$ until one works for both equations. $k = 11$ works, $N = 11 \cdot 11 + 7 = 128$. Then $128 \equiv 7 \mod 11$ and $128 \equiv 11 \mod 13$ (note that $13 * 9 = 117 \Rightarrow 128 - 117 = 11$)

Question 8

Solution:

a)

If one digit is incorrect we will be off by some number $i \in [0, 9]$ which all have unique values $\mod 11$. Therefore, the total sum is off by $|a_i - i| \leq 9$ which will have unique value $\mod 11$.

b)

If two indices are swapped and produce the same result then $a_i + 2a_{i+1} \equiv a_{i+1} + 2a_i \mod 11$. If we rearrange the equation to get $a_i + 2a_{i+1} - (a_{i+1} + 2a_i) \equiv 0 \mod 11 \Rightarrow a_{i+1} - a_i \equiv 0 \mod 11$. However, note that the largest value that $a_{i+1} - a_i$ can be is 9, and since $GCD(11, 9) = 1$, then there can only be unique values for the sum if two numbers are swapped.

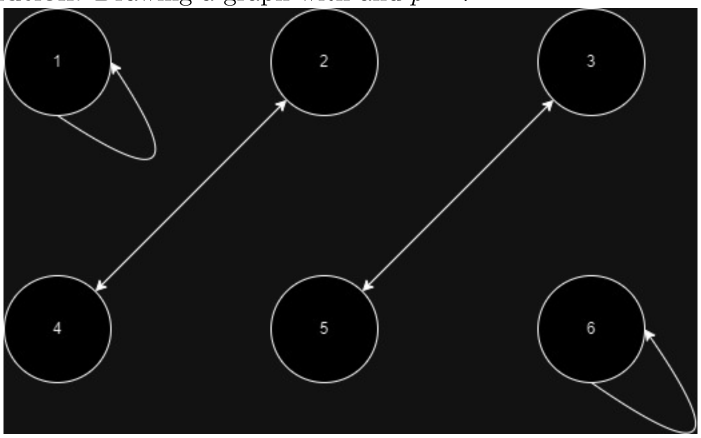
Quiz B

Question 1

Solution: $x^2 \equiv 1 \mod p \Rightarrow x^2 - 1 \equiv 0 \mod p \Rightarrow (x+1) \cdot (x-1) \equiv 0 \mod p$. From Quiz A, we know that iff $ab \equiv 0 \mod p \Rightarrow a \equiv 0 \mod p$ and $b \equiv 0 \mod p$. Therefore, $(x+1) \cdot (x-1) \equiv 0 \mod p \Rightarrow (x+1) \equiv 0 \mod p$ and $(x-1) \equiv 0 \mod p$. Let's start with $(x-1) \equiv 0 \mod p \Rightarrow x \equiv 1 \mod p$. Since $x < p$ the only value that satisfies this equation is $x = 1$. Now for $(x+1) \equiv 0 \mod p \Rightarrow x \equiv -1 \mod p$, but we can't have x be equivalent to a negative modulo a number, so we take the next common multiple (just add the modulus to the negative until it is positive). This means that $x \equiv -1 + p \mod p \Rightarrow x \equiv p - 1 \mod p$. Similarly as before, since $x < p$ the only value that satisfies this equation is $x = p - 1$

Question 2

Solution: Drawing a graph with and $p = 7$



Question 3

Solution: $(p-1)! \equiv p-1 \mod p \Rightarrow 1 \cdot 2 \cdot \ldots \cdot (p-2) \cdot (p-1) \equiv p-1 \mod p$. Note that in the previous question we saw that the only numbers who don't have inverses were 1 and $p-1$. This means all values $2, \ldots, p-2$ will have an inverse with another number in range $[2, p-2]$. This means that $1 \cdot 2 \cdot \ldots \cdot (p-2) \cdot (p-1) \equiv p-1 \mod p \Rightarrow 1 \cdot (p-1) \cdot \prod_{i=1}^{k=\frac{p-3}{2}} 1 \equiv p-1 \mod p \Rightarrow p-1 \equiv p-1 \mod p$.

Question 4

Solution: Note that if N is composite, there exist some factors $ab = N$

Case $a = b$

If $a = b \Rightarrow a^2 = N \equiv 0 \mod N$. Now, note that $a^2 - a$ is a multiple of a, which means $a^2 - a \equiv 0 \mod N$. Given our original equation: $(N-1)! \equiv 0 \mod N \Rightarrow 1 \cdot 2 \cdot \ldots \cdot (a-1) \cdot a \cdot \ldots \cdot (N-1) \equiv 0 \mod N$. Then we can pull $(a-1) \cdot a = a^2 - a$ out of the expression to get $(a^2 - a) \cdot [1 \cdot \ldots \cdot (a-2) \cdot (a+1) \cdot \ldots \cdot (N-1)] \equiv 0 \mod N$. Note that any multiple of $a^2 - a \equiv 0 \mod N$, so if we let $k = 1 \cdot \ldots \cdot (a-2) \cdot (a+1) \cdot \ldots \cdot (N-1)$, then $(a^2 - a) \cdot k \equiv 0 \mod N$.

Case $a \neq b$

This case follows similar logic, since $a < N$ and $b < N$, then $a, b \in [1, N-1]$. Therefore, $(N-1)! \equiv 0 \mod N \Rightarrow 1 \cdot \ldots \cdot a \cdot b \cdot \ldots \cdot (N-1) \equiv 0 \mod N \Rightarrow a \cdot b \cdot [1 \cdot \ldots (a-1) \cdot (b+1) \cdot \ldots \cdot (N-1)] \equiv 0 \mod N$. Let $k = 1 \cdot \ldots (a-1) \cdot (b+1) \cdot \ldots \cdot (N-1)$. Then, $a \cdot b \cdot k \equiv 0 \mod N$ since $ab = N$ and any multiple of N will satisfy the equation.

Therefore, by cases $a = b$ and $a \neq b$, we have shown $(N-1)! \equiv 0 \mod N$ if $N > 4$ and N is composite.

Question 5

Solution: This will be very slow and resource intensive for large numbers. For example RSA uses numbers that are 4096 bits. This means that we would have to calculate $(2^{4096})!$ which is not efficient at all. A probabilistic test will be much faster with a high probability of success.

Fun note: For example Fermat's test $a^{p-1} \equiv 1 \mod p$ can be done for 10 primes will probability of failure $< \frac{1}{2^{10}}$ meaning the probability of success is $> 1 - \frac{1}{2^{10}} = 0.9990234375$