Algebraic Topology Notes

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1 Monday, September 13

Logistics

- Textbooks: Peter May's "Concise Course in Algebraic Topology" and Tom Diecks "Algebraic Topology"
- Take home midterm and final
- Grading: for graduate students, presumably "you get an A for being alive"

This is a modern class with high tech stuff (he doesn't like Hatcher).

Slogan: algebraic topology is the study of functors from some suitable category[1] of spaces to Abelian groups that 1 take homotopy equivalences to isomorphisms and 2 can often be computed inductively.

We want to solve problems in geometry by turning them into problems into algebra. Broadly, geometry is hard and geometry is easy (true to varying degrees).

Definition 1.1. A category C is a collection of objects $\mathrm{Ob}(C)$, and for each pair $X, Y \in \mathrm{Ob}(C)$ a set[2] $\mathrm{Hom}(X, Y)$, along with an associative map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, and for every $X \in \mathrm{Ob}(C)$ there is an identity map $\mathrm{id}_X \in \mathrm{Hom}(X, Y)$ such that $\mathrm{id}_X \circ \gamma = \gamma \circ \mathrm{id}_X = \gamma$ for all maps γ.

Example.
- *Top*: spaces with continuous maps maps
- *Fin*: finite sets with maps of sets
- *Ab*: abelian groups with group homomorphisms
- *R*: rings with ring maps (likewise, R-modules with R-linear maps)
- (More abstract) $(\mathbf{Z}, \leq)$: objects are $\mathbf{Z}$ and a map $x \to y$ whenever $x \leq y$
- A *monoid* can be thought of as a category with a single object and some morphisms
- A *diagram* represents objects with vertices and morphisms with arrows (usually arrows representing the identity are omitted)

Idea: we can encode everything about an object by its relationship with other objects.

Definition 1.2. In a category C, a map $f : X \to Y$ is an isomorphism if there exists $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Example. Isomorphisms in Top are homeomorphisms, in Fin they are bijections, in Ab they are isomorphisms of groups, etc.

Definition 1.3. For categories C, D, a *functor* is a map $F : \mathrm{ob}(C) \to \mathrm{ob}(D)$ and a map $\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_C(F(X), F(Y))$ that is compatible with composition.

[1] Cartesian closed symmetric monoidal

[2] We will mostly ignore set theory complications. However, they become more relevant when it comes to localization.

Example.

- Forgetful functors Top $\to$ Set, Ab $\to$ Set, Ab $\to$ Gp, etc.
- Functors from a diagram to other categories (i.e., specifying an object for each vertex and a morphism for each arrow)

Returning to our slogan, we want to study functors from Top to Ab that 1 solve problems and 2 can be computed.

Definition 1.4. A functor F is *full* if the map $\mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_D(F(X),F(Y))$ is surjective, and it is *faithful* if that map is injective.

Thus, a fully faithful functor embeds C as a subcategory of D.

Definition 1.5. A functor $F : C \to D$ is an *equivalence* if F is fully faithful and F is essentially surjective (i.e., $\forall d \in \mathrm{ob}(D), \exists c \in \mathrm{ob}(C)$ s.t. $F(c) \cong d$).

An equivalence of categories admits an inverse, but this might require the axiom of choice to construct.

For any category C, for any object $c \in C$ we have functor $\mathrm{Hom}_C(c,-)$ from $C \to$ Set that takes $c' \mapsto \mathrm{Hom}_C(c,c')$ and takes $f : c' \to c''$ to $\varphi \mapsto \varphi \circ f$. We can think of this as a functor $C \to \mathrm{Fun}(C, \mathrm{Set})$.

Likewise, there is a contravariant functor $\mathrm{Hom}(-,c)$, which we can think of as a functor $C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$.

Definition 1.6. The functor category $\mathrm{Fun}(C,D)$ has objects functors $C \to D$ and morphisms natural transformations $F \to G$.

Definition 1.7. A natural transformation between functors $F : C \to D$ and $G : C \to D$ is for each object c a map $F(c) \to G(c)$ such that for every $c \to c'$ the following diagram commutes.

$$\begin{tikzcd} F(c) \arrow[r] \arrow[d] & G(c) \arrow[d] \\ F(c') \arrow[r] & G(c') \end{tikzcd}$$

Lemma 1.8 (Yoneda)**.** *For any category C, the functor $F \to Fun(C^{op}, Set)$ taking $c \mapsto Hom_C(-,c)$ is fully faithful.*

The point is that to understand an object, it suffices to consider all the maps into it. But that's too much! This begs the question, is there a good sub-class of test spaces? Maybe... spheres?

The most basic question about a category is classification: what are the isomorphisms of $\mathrm{ob}(C)$? E.g., for real finite-dimensional vector spaces, the answer is $\{\mathbf{R}^n\}_n$. Then, we might ask: what is the structure of the isomorphism? For real finite-dimensional vector spaces, it is $\mathrm{GL}_n(\mathbf{R})$.

However, for topology both these questions are far too intractable. So we need a weaker notion of equivalence.

Definition 1.9. For $f, g : X \to Y$ in Top, f, g are homotopic if there exists a map $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

That is, a one parameter family of maps γ_x parameterized by $x \in \{0,1\}$. We may think of it un-rigorously as a continuous map $\tilde{H} : I \to \mathrm{Hom}_C(X,Y)$ with $\tilde{H}(0) = f$ and $\tilde{H}(1) = g$, but the codomain is a set not a space.

If we think of Top not just as a category but as an enriched category, then $\mathrm{Map}_T(X,Y)$ is a space and composition is continuous. We endow $\mathrm{Map}_T(X,Y)$ with the compact-open topology: a subbase consists of, for all compact $K \subseteq X$ and open $U \subseteq y$, the set $\{f : f(K) \subseteq U\}$.

In fact, the relationship between $I \xrightarrow{\tilde{H}} \mathrm{Map}_T(X,Y)$ and $H \in \mathrm{Map}_T(X \times I, Y)$ is the concept of an adjunction.

Definition 1.10. $f : X \to Y$ is a homotopy equivalence if $\exists g : Y \to X$ s.t. $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Consequences

1. $\{S^n\}$ as test spaces are basically enough to detect homotopy equivalence
2. There exist nice combinatorial models of spaces up to homotopy

2 Wednesday, September 15

Haynes Miller has good notes on algebraic topology

References for Category Theory

- "Categories in Context" by Emily Riehl
- "Categories for the Working Mathematician" (classic)

We discussed the isomorphism of sets

$$\theta : \mathrm{Map}_T(X \times Y, Z) \cong \mathrm{Map}_T(X, \mathrm{Map}_T(Y, Z))$$

taking f to $x \mapsto f(x, -)$. We claim that θ is natural: that is, compatible with maps $X' \to X$ etc.

Definition 2.1. Functors $F : C \to D$ and $G : D \to C$, are an *adjunction* if there is a natural isomorphism $\mathrm{Map}_D(F(x), y) \cong \mathrm{Map}_C(x, G(y))$. Here, F is the left adjoint and G is the right adjoint.

Since we have a natural isomorphism $\mathrm{Map}(Fx, Fx) \cong \mathrm{Map}(x, GFx)$, there is a natural transformation $\mathrm{Id} \to GF$ called the *unit*. Similarly, there is a natural transformation $FG \to \mathrm{Id}$ called the *counit*.

Example. Let U : groups $\to$ sets be the forgetful functor. Then U is the right adjoint of a functor F : sets $\to$ groups which is the free group functor:

$$\mathrm{Map}_{\mathrm{sets}}(S, UG) \cong \mathrm{Map}_{\mathrm{groups}}(FS, G)$$

Definition 2.2. A *monad* is an endofunctor $M : C \to C$ with natural transformations $MM \to M$ and $\mathrm{id} \to M$.

There is an Endofunctor of sets UF with the unit $\mathrm{id} \to UF$. Then

$$(UF)(UF) \cong U(FU)F \xrightarrow{counit} UF$$

Thus, we have a monad. Let S is an algebra over this monad if $\mu : UFS \to S$ satisfies

$$\begin{tikzcd}
(UF)(UF)S \arrow[r] \arrow[d] & (UF)S \arrow[d] \\
(UF)S \arrow[r] & S
\end{tikzcd}$$

Thus, groups are algebras over the monad UF. Many people say category theory is "to make simple things simple"; instead, we'll say that the goal of category theory is "to make formal things formal".

Now, we will turn to colimits and limits. Suppose X, Y are spaces with $A \subset X$, $A \subset Y$. To glue together X and Y over A, we might take

$$(X \sqcup Y)/[\mathrm{im}(A) \text{ in } X, Y]$$

We want to think about gluing not explicitly but categorically in terms of maps. We will construct $X \sqcup_A Y$ as a colimit, i.e., the following pushout diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr] & \\
Y \arrow[r] \arrow[drr] & X \sqcup_A Y \arrow[dr, dashed] & \\
& & Z
\end{tikzcd}$$

It is unique up to unique isomorphism. A special case is the coproduct is the colimit of the following diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & X \sqcup Y
\end{tikzcd}$$

Another special case is the quotient X/A given by the diagram:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/A \end{array}$$

Limits are dual. Consider the fiber product

$$\begin{array}{ccc} X \times_A Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & A \end{array}$$

This is the product of X and Y with compatibility requirements given by A. If A is $*$, this is just the cartesian product. Note that $\emptyset$ and $*$ play distinguished roles (initial and terminal objects respectively).

Now, we will look at the relationships of colimits and limits with functors (adjunctions).

Proposition 2.3. *Left adjoints preserve colimits and right adjoints preserve limits.*

Given a diagram D whose colimit we want to take, we can think of the colimit as a functor

$$\mathrm{Fun}(D, \mathrm{Top}) \xrightarrow{\mathrm{colim}} \mathrm{Top}$$

Let us return to our adjunction

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

We wish this were an isomorphism in Top, but there are some very bad spaces in Top that could cause problems.

Instead, we work with a "convenient category of spaces". This will be a distinguished subcategory of Top such that the above adjunction is always a homeomorphism.

We want a category of spaces such that $f : X \to Y$ is continuous iff $f|_K$ is continuous for $K \subseteq X$ compact Hausdorff.

Definition 2.4. A space X is *weak Hausdorff* if for any continuous $f : K \to X$ where K is a compact Hausdorff space, $f(K)$ is closed.

Definition 2.5. A subset $A \subset X$ is compactly closed if when $g : K \to X$ with K compact Hausdorff, then $g^{-1}(A)$ is closed.

Theorem 2.6. *The category of weak Hausdorf k-spaces is a convenient category of spaces.*

we call this category U. This is also known as a compactly generated space. There are notes by Strickland which use different terminology.

Question: how do we form limits and colimits in U?

Most familiar spaces are in U — e.g., all locally compact Hausdorff spaces.

We have an inclusion functor $U \to \mathrm{Top}$. There are two functors in the opposite direction: "k-ification", which is friendly, and "weak Hausdorffication", which is bad.

Lemma 2.7. *Let $A \subseteq X$ be a closed inclusion, and $X \to Y$ with $X, Y \in U$. Then the pushout colimit $X \sqcup_A Y$ in Top is also in U. The same is true for $F : \mathbf{N} \to U$.*

Now we return to homotopies. Recall the definition of a homotopy between $f, g : X \to Y$. By our adjunction, this is the same data as a continuous map $I \to \mathrm{Map}(X, Y)$, i.e., a path in $\mathrm{Map}(X, Y)$ from f to g.

A homotopy is also a map $X \to \mathrm{Map}(I, Y)$.

If $f \sim g$ and $g \sim h$, then $f \sim h$, as seen by breaking up I in half and attach the two homotopies. This is the point of departure of the algebraic theory of loop spaces. But for now, we just need to know that homotopy is an equivalence relation on maps.

Homotopies also satisfy $f \sim f' \implies hf \sim hf'$, so we really have a category of homotopies called $\mathrm{Ho}(U)$.

Lemma 2.8. *If X, Y are homotopy equivalent in U, then X, Y are isomorphic in $Ho(U)$.*

If $f : X \to Y$ and $g : Y \to X$ have $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$.

In the beginning, we said we are interested in functors

$$\begin{tikzcd} F : U \arrow[r] \arrow[d] & \mathrm{Ab} \\ \mathrm{Ho}(U) \arrow[ur] & \end{tikzcd}$$

Question: why not just work with $\mathrm{Ho}(U)$? The problem is that some of our constructions don't exist in that category. E.g., the interval is homotopy equivalent to a point, but $D^1 \sqcup_{S^0} D^1 = S^1$ and $* \sqcup_{S^0} * = **$, and a circle is not homotopy equivalent to two points.

3 Monday, October 4th, 2021

In addition to telling us things about math, we should perhaps discuss some other things. He will start recommending restaurants periodically. This time, Barney Greengrass (87th and Amsterdam)

4 CW Complexes

A CW complex is the colimit over n of X_n, where we have a filtration

$$X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_n \longrightarrow \dots$$

And X_n is produced as the pushout of

$$\begin{tikzcd} \bigsqcup S^{n-1} \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ \bigsqcup D^n \arrow[r] & X_n \end{tikzcd}$$

A map of CW complexes $f : X \to Y$ is cellular if $f(X_n \subseteq Y_n)$. This is a filtration preserving map. We will see that any map $f : X \to Y$ where X and Y are CW is homotopic. to a cellular map. This is not a tremendous restriction, but nonetheless it is not true for all maps.

The category of CW complexes we typically work with is

$$\begin{aligned} \text{objects} &= \text{CW complexes} \\ \text{morphisms} &= \text{Cellular maps} \end{aligned}$$

CW Complexes are Hausdorff and are compactly generated weak Hausdorff. This is an element and they live in our convenient category of spaces which we already discussed. One thing we have been silent about is the method of topologization. We can give these structures the "weak topology" where A is open if $A \cap X_n$ is open for each X_n, and we give the X_n filtration the union topology.

The category of complexes is an algebraic model for homotopy theory. Perhaps the slogan has been "we can do homotopy theory in CW complexes, i.e. $\mathrm{Ch}(R)$" We will do an analogue of CW (cellular) theory inside of $\mathrm{Ch}(R)$. In $\mathrm{Ch}(R)$ we need spheres and discs.

Recall, we have cylinders in $\mathrm{Ch}(R)$. It was I with two zero simplices and a 1 simplex such that $\partial(I) = 1 - 0$ (remember, these are the generators of a free complex).

How do we have a cone? $M \otimes I$ is a cylinder. How to we make a cone? $(M \otimes I)/(M \otimes \{1\})$ is the "cone on M." $I(M)_n = M_{n-1} \oplus M_n$ gets (x, y) and $(dx, dy - x)$. (Chain homotopic to 0, of course it is, it's a cone.)

We do we want the cone? A cone is a proxy of a disk.

Now what do we do for spheres? This is R in dimension n for S^n. Natural inclusion $S^n_R \to CS^n_R$ (base of cone). A CW complex in $\mathrm{Ch}(R)$.

$$\begin{tikzcd} \bigoplus S^n_R \arrow[r, "\alpha"] \arrow[d] & X_n \arrow[d] \\ \bigoplus CS^n_R \arrow[r] & X_{n+1} \end{tikzcd}$$

We are making something out of free modules that is "flat" for tensors. It preserves quasi-isomorphisms. We are doing a union of pushouts.

$$Z \otimes M = Z \otimes (\text{colim }_{n\to\infty} M_n) \cong \text{colim}_{n\to\infty}(Z \otimes M_n)$$

If we grade our algebra all the way down $\mathbb{Z}$ for all of $\mathbb{Z}$ then $S^n_R \otimes S^{-n}_R = S^0$ which is the unit. Therefore, in the algebraic sense, the spheres are shifts up and down. Topologically there is not a clear inverse. We see this $Z \otimes S^n \otimes S^{-n} \cong X$.

If we have $X \to CX$ we can take the quotient

$$X \longrightarrow CX \longrightarrow CX/X$$
$$\ldots \longrightarrow H_n(X) \longrightarrow 0 \longrightarrow H_n(X/X) \xrightarrow{\cong} H_{n+1}(X) \longrightarrow 0 \longrightarrow \ldots$$

4.1 A digression on topology and based spaces

Top_* is the category of based spaces, $(X, *)$.

$$\begin{tikzcd} X \arrow[r] & Y \\ *_X \arrow[u] \arrow[r, "\cong"] & *_Y \arrow[u] \end{tikzcd}$$

which takes base points to base points in all morphisms.

We also have a functor from Top to Top_* which takes X to its one point compatification with the one point as the base point and a forgetful map U. This pair is an adjunction

$$((-)_+, U) \text{ is an adjunction}$$

Top_* is the category of algebras over $U(-)_+$ in Top. (Monad)

4.2 Returning to Spaces

Top is a symmetric monoidal category, specifically under the cartesian product. The Cartesian Product is a functor $\times : \text{Top} \times \text{Top} \to \text{Top}$

What about Top_*. $X \wedge Y$ i.e. $X \times Y / X \vee Y$ where $X \vee Y$ is defined as

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd}$$

which essentially collapses $X \times *$ and $* \times Y$. A reason to desire this instead of just naively making the point of both base points as the basepoint is

$$S^n \wedge S^m \cong S^{n+m}$$

$\Sigma X = CX/X \cong S^1 \wedge X$.

If X is CW, $S^n \wedge X$ shifts the cells, and it shifts homology by n. We don't have an inverse here, though, unlike the chain complexes. It will turn out that in some sense there is an inverse. The candidate inverse is what we might write as $\mathcal{N}^n(X) = \text{Map}(S^n, X)$. This is sense $\text{Map}_{\text{Top}_*}(S^n \times X, Y) \cong \text{Map}_*(X, \text{Map}_{\text{Top}_*}(S^n, Y))$

4.3 New CW complexes from old

If A is a subcomplex, X/A is a subcomplex. If X, Y are complexes, then $X \times Y$ is a complex and the n cells are p cells of X and q cells of Y where $p + q = n$¿

Now we will explain some examples of CW complexes. The examples we will do is some basic ones and then we will do $\mathbb{C}\mathrm{P}^{\infty}$, Grassmanians, manifolds, $\mathbb{R}\mathrm{P}^{\infty}$.

1. Any finite set (any discrete set).
2. Any graph (verticles are 0 cells and edges are 1 cells). Clustering is the 1-skeleton of a CW complex over your data.
3. $\mathbb{R}^n$ could have a lattice point structure and integers and fill it in with cells.
4. S^n can smash the boundary of an n-cell down to a 0 cell. There is additionally gluing hemispheres together are their radial belts.

5 Wednesday, October 6th, 2021

The restaurant guidance of the day is suggesting Murray's Sturgeon ("neighbourhood place"), Sadelle's in Soho (eh, it's fancy waiters with bagels and rich people in suits, its a strange vibe), Zabar's ("fun for other reasons"), Russ and Daughters ("other famous New York deli"). This will conclude the opinions on smoked fish. They all get their fish from Acme Fish except, possibly, for Sadelle's.

Today, we will discuss the following topics:

- CW Structures on $\mathbb{RP}^n$ and $G_k(\mathbb{R}^n)$
- classifying some vector bundles
- If we have time, we will talk about the proof of excision.
- Likely on Monday, we could wrap up our discussion of homology and discuss homotopy groups.

5.1 Projective space

$\mathbb{R}\mathrm{P}^n$ is the space of 1-dimensional subspaces of $\mathbb{R}^2$. (generally quotient-ing out by the lines through the origin). Generally it is $S^n/(\mathbb{Z}/2\mathbb{Z})$. It's like the circle upto a sign.

The CW structure on this is determined

$$\begin{array}{ccc} S^n & \longrightarrow & \mathbb{R}\mathrm{P}^n \\ \downarrow & & \downarrow \\ D^{n+1} & \longrightarrow & \mathbb{R}\mathrm{P}^{n+1} \end{array}$$

And therefore $\mathbb{R}\mathrm{P}^n$ has a single cell in dimensions $\leq n$.

There is a natural vector bundle that sits over $\mathbb{R}\mathrm{P}^1$.

$$\begin{array}{c} F_\ell \\ \downarrow \\ E \\ \downarrow \\ \mathbb{R}\mathrm{P}^1 \end{array}$$

$E = \mathbb{R}\mathbf{P}^1 \times \mathbb{R}; \bigcup_{\sigma \in \mathbb{R}\mathrm{P}^1} (\sigma, z)$ where $z \in \sigma$ and $E \to \mathbb{R}P^1$ has $(\ell, z) \to \ell$ $F_\ell = \pi^{-1}(\ell)$.

- F_ℓ is a vector space

- It is locally trivial

if we have a map $X \to \mathbb{R}\mathrm{P}^1$ then the following diagram makes a vector bundle

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & \mathbb{R}\mathrm{P}^1 \end{tikzcd}$$

THe map $P \to X$ is the vector bundle of sorts. There is a space $BO(n)$ such that there is a bijection between $[X, BO(n)]$ and the classes of $\mathbb{R}^n$ bundles on X.

$BO(n)$ is much less crazy than we might expect. We can compute the cohomology and homology. Homology is covariant and cohomology is contravariant so any cohomology class in X gives rise to a cohomology class in X.

There's an idea here, which is a represented functor. The point is that if we have the category Top then we have a functor $\mathsf{Top} \to \mathsf{Set}$. $[-, Z]/\sim$ is homotopy maps of maps to Z. $[-, Z]$ turns out to be a homotopy functor, since this factors through the homotopy category, primarily because we made it by the equivalence quotient.

$$[B/Z, Z] \to [B, Z] \to [A, Z]$$

This might raise a suspicion or a question: is (co)homology represented? The answer will be yes. This will be one way we construct cohomology. The theorem is that $H^n(X, R) \cong (X, K(\mathbb{R}, n))/\sim$ where this is called a Eilenburg Mac Lane space. The cohomology of these spaces will be interesting. Why is the cohomology interesting? Because it's endomorphisms. From the EIlbenberg Steinrod axioms, there is a relation between H^n and H^{n-1}.

$C(X, X)$ is a monoid. The CW complex structure on $G_k(\mathbb{R}^n)$ is the real Grassmanian which is the space of k-planes in $\mathbb{R}^n$. The point is a plane, and the fibre is the plane once we admit a vector bundle. There is a natural map from $G_k(\mathbb{R}^n)$ to $G_k(\mathbb{R}^{n+1})$. It's just adding a dimension (induced by taking x to $(x, 0)$.) Now we can just take the colimit to get $G_k(\mathbb{R}^\infty)$. This is what we want to think of as the classifying space of vector bundles of rank k.

This is a CW-complex. What is that CW structrure on $G_k(\mathbb{R}^n)$?

5.2 Grassmanians

This has a beautiful combination structure where the CW structure comes in the form of Schubert cells. Why is there a CW structure? We have a plane. How do we represent it? Choose a basis for $\mathbb{R}^n$ and then think about the plane projecting onto that basis. Another way of saying that is to say if you think about that plane and you represent it as the row space of a matrix and you perform Gaussian elimination, that matrix will look like a real matrix with jagged leading ones. The numbers are parameterized by the rest of the matrix.

One way of doing this is to produce Schubert symbols. $\mathbb{R}^0 \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \cdots \subseteq \mathbb{R}^n$. V is a k-dimensional subspace. Then, we get $V \cap \mathbb{R}^0 \subseteq \mathbb{R}^1 \cap V \subseteq \cdots \subseteq V \cap \mathbb{R}^n$ starts at dimension 0 and ends at dimension k. Therefore, $\dim(V \cap \mathbb{R}^m) + (0 \text{ or } 1) = \dim(V \cap \mathbb{R}^{m+1})$

We can keep a map of when the dimension changes. We can write the dimension as $v = (0, 1, 1, 2, 3, 4, \ldots k)$, e.g. and then we can write this as $(s_1, s_2, \ldots, s_k))$ as the schubert symbol. $\mathrm{Z}S_i$ is the index of the Ith increment in dimension i.e. $s_1 = v_1 - v_0$, etc.

The Schubert cell is homeomorphic to the interior of D^ℓ for some ℓ. If we take the closure of that we get the disk. These are our cells. We can attach these cells to produce CW decomposition.

V is the row space of some matrix, rank k so it is a $k \times n$ matrix and it will look like

$$\begin{bmatrix} \ldots & \ldots & 1 & 0 & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & 1 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \end{bmatrix}$$

where the dots can be any entry, and the staircase structure formed like this gives us the Schubert cells. Manifolds and nice varieties are CW complexes and in both cases the structures are very interesting. This is the subject of finding triangulations. It literally means a homeomorphism to a simplicial complex.

5.3 Return to the proof of excision

What is the statement of excision? If we have

$$U \subseteq A \subseteq X$$

and $\overline{U} \subseteq A^\circ$. Then we will say that $H_n(X, A) \cong H_n(X - U, A - U)$ where we just delete U. Part of the point of this condition is that X interacts with A by reaching into the boundary of A, but it can't go past it, and so U is insulated from anything X can see in A.

How are we going to prove this? The proof of this is related to an interesting subject, specifically it is related to the start of geometry measure theory.

Definition 5.1. $\sigma : \Delta^n \to X$ is *small* relative to the excisive triple (U, A, X) if either $\sigma(\Delta^n) \subset A$ or $\sigma(\Delta^n) \cap U = \emptyset$.

1. Small chains computer relative homology.

2. We can always represent a chain by small ones.

The first comment is that we can define $H_n^{\text{Small}}(X)$ and $H_n^{\text{Small}}(X.A)$ by using only the restriction of the singular chains to the small chains to form homology. There is an exact sequence

$$\begin{tikzcd} H_n(A) \arrow[r] \arrow[d] & H_n^{\text{Small}}(X) \arrow[r] \arrow[d] & H_n^{\text{Small}}(X, A) \arrow[d] \\ H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \end{tikzcd}$$

It will be interesting to look at comparisons between H_n and H_N^{Small}

$$\begin{tikzcd} C_n(A-U) \arrow[r] \arrow[d] & C_n(X-U) \arrow[r] \arrow[d] & C_n(X-U, A-U) \arrow[d] \\ C_n(A) \arrow[r] & C_n(X)^{\text{small}} \arrow[r] & C_n(X, A)^{\text{small}} \end{tikzcd}$$

The vertical arrows are inclusions by construction of the definition of a small chain.

Now the point is that the map $C_n(X - U, A - U)$ to $C_n(X, A)$ induces an isomorphism on homology. This factors through $C_n^{\text{small}}(X, A)$

Therefore we get

$$\begin{tikzcd} C_n(X-U, A-U) \arrow[dr, "\cong"] \arrow[dd] & \\ & C_n^{\text{small}}(X, A) \arrow[dl] \\ C_n(X, A) & \end{tikzcd}$$

The key claim is that $C_n^{\text{small}}(X) \to C_n(X)$ induces an isomorphism on homology.

We will introduce what is called a "subdivision" operator to chop up chains. We will take some nice cover of X and localize the chain by cutting up Δ until you have nice piece. Let's discuss subdividing the standard simplex. (This idea has come up before)

Remember, we had the maps of the circle to itself as a simplicial complex. We know that H_1 is isomorphic to $\mathbb{Z}$

This is in some way a proxy for thinking about homotopy and its "size." Let's draw the pictures.

6 Monday, October 11th 2021

Proof of excsision for singular homology. This is the last of the axioms for Singular homology. The rest we've established.

This involves an interesting construction called subdivision where $U \subseteq A \subseteq X$ where $\overline{U} \subseteq A^\circ$, then you have a natural inclusion from the relative homology of $H_n(X-U, A-U) \cong H_n(X, A)$. As a definition, if we have $\sigma : \Delta^n \to X$ is small ("relative to the pair (A,U)") if the image $\sigma(\Delta^n) \subseteq A$ or $X-U$. A chain τ is small if τ is a sum of small simplices.

We are interested in $C_n(X-U, A-U) \to C_n(X, A)$. This factors through the small simplices. Therefore we can get

$$\begin{tikzcd} C_n(X-U, A-U) \arrow[rr] \arrow[dr, "(1)"] & & C_n(X, A) \\ & C_n^{\text{small}}(X, A) \arrow[ur, "(2)"] & \end{tikzcd}$$

The idea is to repeatedly create smaller simplices inside of Δ^n. Each of these subdivisions live in a smaller part of X than the original. Each time you cut it gets smaller, eventually it gets small enough. If we take the chain of the smaller simplices that sum to the large simplex, then they are the same thing but now it's a chain of small simplices.

Theorem: Any map rom $|X| \to |Y|$ where X and Y are simplicial complexes ($|X|$ and $|Y|$ are actual triangle gluing, X and Y are abstract data of triangle gluing). The geometric realization of $\tilde{f}$ is homotopic to f. This says two things of note. We asserted that simplicial complexes are a model for spaces. That means two things. If X is a space, there exists a complex Y such that $|Y| \simeq X$. Additionally, homotopy classes of maps from X to Y can be represented by simplicial maps from $sd_n : X \to Y$.

$$\begin{tikzcd} C_n(A-U) \arrow[r] \arrow[d] & C_n(X-U) \arrow[r] \arrow[d] & C_n(X-U, A-U) \arrow[d, "(1)"] \\ C_n(A) \arrow[r] \arrow[d, "\cong"] & C_n^{\text{small}}(X) \arrow[r] \arrow[d] & C_n^{\text{small}}(X, A) \arrow[d, "(2)"] \\ C_n(A) \arrow[r] & C_n(X) \arrow[r] & C_n(X, A) \end{tikzcd}$$

Diagram commutes, rows are short exact sequences. To deduce excision it suffices to check that $C_n^{\text{small}}(X) \to C_n(X)$ is an equivalence.

Now construct an operator $s.d.C_n(X) \to C_n(X)$, with the two following properties:

1. Simplices of $s.d.(X)$ are $\leq \left(\dfrac{n}{n+1}\right)$ |simplices of X|

2. s.d. is chain homotopic to the identity.

The Lebesgue number relative to a metric space X and an open cover U_α of X is ϵ such that all $B_\epsilon(x) \subset U_\alpha$ for some α. Any compact metric space has a nonzero Lebesgue number. In particular, Δ^n is a compact metric space, and so for any open cover it has a nonzero Lebesgue number.

If we let W_α cover A and B_α cover $X-U$. If we look at $\sigma : \Delta^n \to X$ and $\{\sigma^{-1}(W_\alpha)\} \cup \{\sigma(V_\alpha)\}$ This open cover has a Lebesgue number so that all balls smaller than ϵ are contained in an element of that cover. If we look at $s.d._N$ where N is chosen large enough that that the max dimensino of a simplex is $s.d._N(\sigma) < \epsilon$. Notice a shrewd thing happening here. The singular homology is probing our space with test maps from a nice test space. We are saying we can understand X by probing it with placing triangles into it. This argument is fine. This is buying you this argument though. X is an awful space. It might not be metrizable. It could be an abominable point set construction, but we can still calculate these things that deal primarily with compact metric spaces through this set up.

How do we subdivide a chain? We want a subdivision $s.d. : C_n(X) \to C_n(X)$ but we also want the subdivision to be "natural" in the following manner. For a map $f : X \to Y$ we want

$$\begin{tikzcd} C_n(X) \arrow[r, "s.d."] \arrow[d, "f"] & C_n(X) \arrow[d, "f"] \\ C_n(Y) \arrow[r, "s.d."] & C_n(Y) \end{tikzcd}$$

Inductively we will divide a point into a point, and then place a point at the barycenter, and then connect everything by whatever boundary is required. E.g. Barycenter uf $\frac{1}{n+1}\sum_{i=0}^{n} e_i$.

s.f. of a n-simplex σ by Σ of simplices formed by connecting the barycenter $b(\sigma)$ to $s.d.(\partial\sigma)$. How do we produce the chain homotopy? It's the same as the argument for the product. You define all the 0-simplices. We can extend by computing what the boundary would be and using the fact that $H_i(\Delta^n) = 0$ to lift.

It is of course patently obvious that $|s.d._n\Delta^n| \simeq |\Delta^n|$. We should have the picture in mind here

In the coming weeks we will be covering

- Cofirbrations
- fibrations
- homotopy groups
- exact sequences of cofiber, fiber

Property $f : A \to X$ is a cofibration then $C :\simeq X/A$ (called $fA \rightarrowtail X$)

$f : X \to Y$ Mf mapping cylinder $X \to (X \times I) \cup_f Y)$

This i s always a cofibration. Any map $f : X \to Y$ can be factored

$$\begin{tikzcd} X \arrow[r, tail] \arrow[rr, bend right, "f"'] & Mf \arrow[r, "\cong"] & Y \end{tikzcd}$$

Now suppose X is a space and $x \in X$ then $x : * \to X$ is a nondegenerating at (missed this part)

Can always arrange for X to be nondegenerated based by drawing out a whisker from the space.

Definition 6.1. A map $f : X \to Y$ is a cofibration of f if it satisfies the homotopy extension property (HEP) i.e.

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}\times\{0\}"] \arrow[dd, "f"] & & A \times I \arrow[dl, "h"] \arrow[dd, "f\times\mathrm{id}"] \\ & Z & \\ X \arrow[ur, "\ell"] \arrow[rr, "\mathrm{id}\times\{0\}"] & & Y \end{tikzcd}$$

And we can take a pushout $\tilde{h}$ for any z so that it fits such a diagram. i.e. we can extend the homotopy h to $\tilde{h}$ and it suffices to check on $Z = Mf$ (The universal property so that $Mf \to Z$)

There's also the diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dd] & & A \times I \arrow[dl, "h"] \arrow[dd] \\ & Mf & \\ X \arrow[ur] \arrow[rr] & & X \times I \end{tikzcd}$$

For example, check that $X \to Mf$ is a cofibration $(X \times Y) \cup_f Y$. This is a slightly stronger notion than being a closedf inclusion. Next time we will give a criterion for detecting cofibrations.

Where are we going with this? We want to prove these statements we keep asserting about the homotopy invariance of the quotients. We will think of the sequence $X \to Y$ (cofibration) to Y/X as being a short exact sequence of spaces.

$$[-,X] \longrightarrow [-,Y] \longrightarrow [-,Y/X]$$

$$[Y/X,-] \longrightarrow [Y,-] \longrightarrow [X,-]$$

7 Wednesday October 13th, 2021

Culinary indoctrination: At various points in the history of New York City, the percentage of immigranrts in NYC has exceeded 50 percent. We've recently been covering the immigrant waves of the early 1900s and late 1800s. Notes on New York Pizza:

Do not buy pizza that costs less than \$3. It will not be made with real cheese. Even if you are a disparate grad student, don't do this.

There are a handful of pizza restaurants that are very old with coal ovens. Many of these places aren't actually good. The ones worth going to are Di Fara (Midwood / Borough Park) and Totonno's which is in Coney Island. The Di Fara pizza guy is old so you should go before he dies. There are a bunch of places in Greenwich Village. All of them are terrible. There are places in closer brooklyn that are fine. This is explicitly for the table-set sit down sort of things.

New Jersey has a distinctive pizza style which used to all be in Trenton. They are now in a shopping center and the two worthy ones are Delorenzo's and one other one. They're both in the Robinsville shopping center.

7.1 Cofibrations

Remember we have a map $f : A \to X$ and a map $g : X \to Y$ then the data you are given is

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}\times 0"] \arrow[dd, "f"'] & & A \times I \arrow[dl, "h"] \\ & Y & \\ X \arrow[ur, "g"] & & \end{tikzcd}$$

is given then you can extend this homotopy from A to the rest of X, in the following manner.

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}\times 0"] \arrow[dd, "f"'] & & A \times I \arrow[dl, "h"] \arrow[dd, "f\times\mathrm{id}"] \\ & Y & \\ X \arrow[ur, "g"] \arrow[rr, "\mathrm{id}\times 0"'] & & X \times I \arrow[ul, "\tilde{h}"] \end{tikzcd}$$

Note, $\tilde{h}$ is not unique

As examples, we can take $\emptyset \to X$ and $X \to Y$ homeomorphism.

As another example, if we have a smooth manifold with boundary, $\partial M \rightarrowtail M$ is a cofibration (collars).

Theorem 7.1. *Cofibrations are stable under cobase change.* $\begin{tikzcd} A \arrow[r] \arrow[d, tail] & B \arrow[d, tail] \\ X \arrow[r] & X \sqcup_A B \end{tikzcd}$ *with cofibration f and g as a map then $B \to X \sqcup_A B$ is a cofibration.*

Similarly, we can use the pushout to construct this map with the following diagram

$$
\begin{tikzcd}
A \arrow[rrrr] \arrow[dr] \arrow[dddd] & & & & A \times I \arrow[dl] \arrow[dddd] \\
& B \arrow[rr] \arrow[dd] & & B \times I \arrow[dl] \arrow[dd] & \\
& & Z & & \\
& X \sqcup_A B \arrow[ur] \arrow[rr] & & X \sqcup_A B \times I \arrow[ul, "\tilde{h}_2"] & \\
X \arrow[ur] \arrow[rrrr] & & & & X \times I \arrow[uull, dashed, bend right, "\tilde{h}_1"']
\end{tikzcd}
$$

Where the existence of $\tilde{h}_2$ is a pushout construction.

A consequence of this is that if we have the pushout

$$
\begin{tikzcd}
\coprod S^n \arrow[r] \arrow[d, tail] & X \arrow[d, tail] \\
\coprod D^{n+1} \arrow[r] & X_{n+1}
\end{tikzcd}
$$

This implies that $A \coprod B \rightarrowtail X \coprod Y$ is a cofibration which implies that $\text{colim} X_n \simeq \text{hocolim} X_n$

Then, to test if $f : A \to X$ is a cofibration, it suffices to look at the "universal test object" Mf, which is the mapping cylinder. We have our diagram again

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dd] & & A \times I \arrow[dl] \arrow[dd] \\
& Mf & \\
X \arrow[ur] \arrow[rr] & & X \times I \arrow[ul, dashed, "h"]
\end{tikzcd}
$$

For any other test object Z, the universal property of the pushout which controls the construction of the mapping cylinder give sus a map $Mf \to Z$ and so we can always construct our homotopy map like this. (pushout below Mf).

Which is to say we always have the map

$$\ell : Mf \to X \times I, x \mapsto (x, 0), (a, t) \mapsto (a, t)$$

Then there is the characterization of cofibrations $X \times I \to Mf \to X \times I$ then you get $X \sqcup_f (A \times I)$ so that we get $X \coprod_f (A \times I) \to X \times I$ is h_0 on X and f on $A \times I$. Then $X \times I$ retracts onto Mf.

Then we have a composite

$$
\begin{tikzcd}
Mf \arrow[r, "\ell"] \arrow[rr, bend right, "\mathrm{id}"'] & X \times I \arrow[r, "h"] & Mf
\end{tikzcd}
$$

This exhibits Mf as a retract of $X \times I$. ℓ always exists by pushout construction. The existence of h for the map to be a retract is what determines whether something is a cofibration.

Then, we can find a criterion for finding cofibrations.

Definition 7.2. A pair, (X, A) is an NDR pair if there exists a function $U : X \to [0, 1]$ such that $U^{-1}(0) = A$ and there is a homotopy $h : X \times I \to X$ such that $h(x, 0) = x$, $h(a, t) = a$ and $h(x, 1) \in A$. for x such that $u(x) < 1$. This data is saying we have a manner of parameterizing families of shells of points in X. The homotopy fixes A at every point and starts at X at the end X is inside A.

As a proposition, (X, A) is an NDR pair if and only if $A \rightarrowtail X$ is a cofibration.

Suppose we have a cofibration $A \rightarrowtail X$. If we project $\pi_1 Mf \to X$ and $\pi_2 Mf \to I$, we can specify that $U(x) = \max_t(t - \pi_1(H(x,t))$ and $H(x,t) = \pi_1(H(x,t))$. This is a map from $X \times I \to X$.

We will show that NDR implies cofibration, the other direction will be an exercise. You use the homotopy you were given to build H.

7.2 A digression on Base Points

So far, this has been unbased. We cna just as easily talk about the based variant which is the same extension condition using based spaces. There are a couple of things that are a different in that context and worth mentioning.

Remember, Top_* has the wedge product $\wedge$ as the analogue of the cartesian product. There are natural homeomorphisms $\mathsf{Map}_{\mathsf{Top}_*} \cong \mathsf{Map}(X, \mathsf{Map}(X, Y))$

We can make the mapping cylinder out of $X \wedge I_+$ is the cylinder and $X \wedge I$ is the based cone.

Reminder,

$$X \wedge Y = X \times Y / (X \times *) \cup (* \times Y)$$

Additionally, $X_+ \wedge Y^+ \cong (X \wedge Y)_+$. Now, for cofibration sequences, suppose we have

$$X \xrightarrow{f} Y \longrightarrow Cf \longrightarrow \Sigma f \longrightarrow \Sigma Y \longrightarrow C(\Sigma f) \cong \Sigma Cf \longrightarrow \Sigma^2 X \longrightarrow ...$$

This is a proxy of exact sequences. On the homotopy classes of maps this is an exact sequence.

$$\mathsf{Map}(\Sigma X, Y) \cong \mathsf{Map}(X, F(S^1, Y)) \cong \mathsf{Map}(X, \Omega Y)$$

where ΩY is the loops on Y. Then we also get

$$\pi_0 \mathsf{Map}(X, Y) \cong [X, Y]$$

If X is a based space and $\pi_0 X$ is the path components of X which is the collection of $[S^0, X]$. So what are the path components of $\mathsf{Map}(X, Y)$?. Two maps f, g are in the same component if there is a path in $\mathsf{Map}(X, Y)$ from f to g. But what is a path in $\mathsf{Map}(X, Y)$? A path in $\mathsf{Map}(X, Y)$ from f to g is a map $\gamma : I \to \mathsf{Map}(X, Y)$ such that $\gamma(0) = f$ and $\gamma(1) = g$ and by adjunction that is the same thing as a map $\bar{\gamma} : X \wedge I_+ \to Y$ such that $\bar{\gamma}(x, 0) = f(x)$ and $\bar{\gamma}(x, 1) = g(x)$, i.e. a homotopy. Therefore the path components of $\mathsf{Map}(X, Y)$ are the homotopy classes $[X, Y]$.

Now, if we have a space and break it into its path components, it's a bunch of sets but it's pretty crude. It's a reasonably thing but can we extrapolate? What else is embedded in $\mathsf{Map}(X, Y)$?

So we want to understand algebraic structure on $[\Sigma X, Y]$. We can look at the sequence

$$Ff \to X \to Y \to Cf \to \Sigma X \to \dots$$

We can see that $[\Sigma X, Y]$ is the path components of $\mathsf{Map}(\Sigma X, Y) \cong \mathsf{X}, \mathsf{\Omega Y}$. Then ΩY is a "monoid" upto homotopy.

If we take $\gamma : S^1 \to X$ and $\tau S^1 \to X$ with $I \to X$ in each then $\gamma + \tau(t) = \gamma(32t)$ for $t \in [0, 1/2]$ and $\tau(2t-1)$ with $t \in [1/2, 1]$ is $[\gamma + \tau] \to X$. We we have $\gamma_1, \gamma_2, \gamma_3 : S^1 \to X$ and $(\gamma_1 + \gamma_2) + \gamma_3 = \gamma_1 + (\gamma_2 + \gamma_3)$. On homotopy classes, rthis means that homotopy classes of maps $[X, \Omega Y]$ is a group. Whats the inverse? It goes the other way.

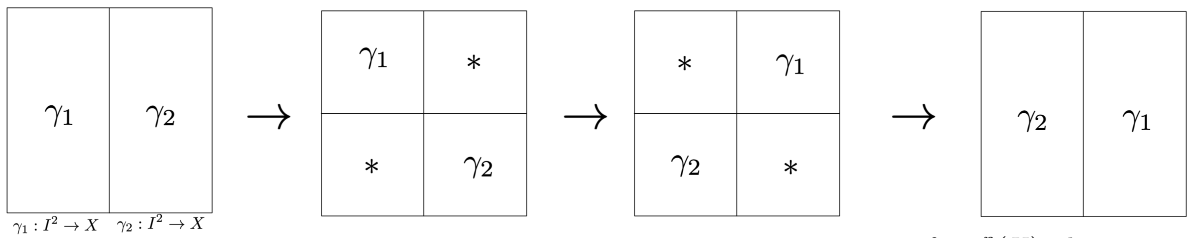
We can extend this to $\mathsf{Map}(X, \Omega Y) \cong \Omega \mathsf{Map}(X, Y)$ with $f, g : X \to \Omega Y$ to have $f * g(x) = f(x)g(x)$.

Moore loops, though, have a problem because of rescaling.

$$\Omega_M = \{\gamma : [0, \ell_\gamma] \to X\}$$

where $\gamma(0) = \gamma(\ell_\gamma)$. This is a monoid. Then for two maps γ, τ we make the concatenation on the domain $[0, \ell_\gamma + \ell_\tau] \to X$ which gives an honest to goodness monoid.

$[X, Y]$ is a set. $[\Sigma X, Y]$ is a group. $[\Sigma^2 X, Y]$ is a group, and it will continue to hold for $[\Sigma^n X, Y]$ for $n \geq 2$. Maps from $\Sigma^2 X \to Y$ is homeomorphic to maps from X to $\Omega^2 Y$. Then $\Omega^2 X \to X$ is the same as the set of all maps form the square to X which takes ∂I^2 to the basepoint. What's the homotopy? It's called the



$\gamma_1 * \gamma_2 \simeq \gamma_2 * \gamma_1$ for $\pi^n(X)$ where $n \geq 2$

Figure 1: A diagram of the homotopy of I^2 that proves that the homotopy groups of dimension 2 and above are Abelian

"spin the wheel homotopy" for $\gamma_1 * \gamma_2$ and $\gamma_2 * \gamma_1$. Therefore homotopy classes of maps on the I^2 commute, so $[\Sigma^2 X, Y]$ is an Abelian group.

There are some unsettling facts here. Going from 0 to 1 loop took us from sets to groups, which is a vast jump (algae has srtarted walking on land) and then 1 to 2 goes to an Abelian group. Then after that we get nothing else. It increases wildly. This is the beginning of the theory of E-N Algebras Culinary indoctrination: B+H Diary, Veselka, Blue + Gold. B+H Dairy is a classic Dairy restaurant, the Blue + Gold is a bar, and Veselka is a 24 hour diner.

Culinary questions: Food from city states or near city states. Any places that are good for Taiwanese or Singaporean food

Long exact sequences of a cofiber sequence

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y/X \end{array}$$

Then Y/X has a universal property with $Y \to Y/X$ in

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y/X \arrow[dl] \\ & Y & \end{tikzcd}$$

Remember, we introduced Cf as a homotopical proxy for Y/X:

Lemma 7.3. *1 If $x \rightarrowtail Y$ is a cofibration then $Y/X \simeq Cf$*

Proof.
$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Cf \end{array}$$

then $Y/X \cong Cf/CX \simeq Cf$ ∎

But what is the universal property of Cf?

$$X \longrightarrow Y \longrightarrow Cf$$

This is null homotopic

$$X \times I \to Cf$$
$$(x,t) \mapsto (*,t)$$

$$\begin{tikzcd} X \arrow[r] & Y \arrow[d] \\ & Z \end{tikzcd} \quad \text{is null homotopic if and only if} \quad \begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d] & Cf \arrow[dl, dashed] \\ & Z & \end{tikzcd}$$

As an immediate consequence, given Z and $X \to Y \to Cf$ has the exact sequence

$$[Cf, Z] \to [Y, Z] \to [X, Z]$$

is an exact sequence of sets.

We can write two sequences

$$X \longrightarrow Y \longrightarrow Cf \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow C(\Sigma f) \longrightarrow \Sigma^2 X \longrightarrow \Sigma^2 Y \longrightarrow \dots$$
$$X \longrightarrow Y \xrightarrow{g_0} Cf \xrightarrow{g_1} Cg_0 \longrightarrow Cg_1 \longrightarrow \dots$$

Each term is the cofiber of the last, and so by this iteration it becomes clear that the second sequence gives rise to a long exact sequence when you apply $[-, Z]$. From the first sequence, we want to show that $[Cg_0, Z] \cong [\Sigma X, Z]$ and we want to show that there exists comparisons between the first sequence and the second which induce the same sequence on $[-, Z]$. Recall from last class that $[\Sigma X, Y]$ is a group and $[\Sigma^2 X, Y]$ is an Abelian group.

When considering iterated versions of this comparison, we can look at maps in $\Sigma^2 X$ and $\Sigma^2 f$ and these make a permutation

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Cf \arrow[r] \arrow[d, equal] & \Sigma X \arrow[r] & \Sigma Y \arrow[r] & C(\Sigma f) \arrow[r] \arrow[d] & \Sigma^2 X \arrow[r] \arrow[d, "\tau"] & \Sigma^2 Y \arrow[r] \arrow[d] & C(\Sigma^2 f) \arrow[r] & \dots \\ X \arrow[r] & Y \arrow[r] & Cf \arrow[r] & \Sigma X \arrow[r] & \Sigma Y \arrow[r] & C(\Sigma f) \arrow[r] & \Sigma^2 X \arrow[r] & \Sigma^2 Y \arrow[r] & C(\Sigma^2 f) \arrow[r] & \dots \end{tikzcd}$$

This sequence on homotopy gives rise to the LES of a pair (A, X)

$$A \xrightarrow{f} X \longrightarrow Cf \longrightarrow \Sigma A \longrightarrow \Sigma X \longrightarrow \dots$$

Then we can get that

$$H_*(Cf) \cong H_*(Cf, CA) \cong H_*(X \times I, A \times I) \cong H_*(X, A)$$

with the middle isomorphism by excision.

If $A \rightarrowtail X$ is a cofibration then $H_n(X, A) \cong H_n(X/A) \cong H_n(Cf)$.

As an aside, when you work in this context you get a lot of power that is homotopy invariant and we can use homotopy theory to solve our problems. On the other hand, it takes what you said and turns it into what the machinery thinks you wanted to say, but perhaps it doesn't say what you want it to say. From the perspective of this class homotopy is what we're studying, but often there's more important refined data that you want to keep. If you study manifolds or varities it is removing a lot of data to just studying it upto homotopy type.

We have been doing this in the context of the functor $[-, Z]$ since this is the relevant functor for the long exact sequence in homology. What about functors $[Z, -]$? Given a map $X \to Y$ can we form some kind of long exact sequence? We are now starting to follow up on our long intended

Definition 7.4. If X is a based space then

$$\pi_n(X) = [S^n, X]$$

Then we find that $\pi_0(X)$ is the path components, $\pi_1(X)$ is the fundamental group, and $\pi_n(X)$ are Abelian groups.

The Abelian group thing is very helpful, as there's a lot more known structure to Abelian groups. Homotopy groups are hard. Homology is pretty easy. We don't know what the homotopy groups of spheres are.

Definition 7.5. $f : X \to Y$ is a weak equivalence if the induced map $\pi_n f : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n.

We will eventually prove the following theorem

Theorem 7.6. *If $f : X \to Y$ is a weak equivalence and X, Y are CW complexes, then f is a homotopy equivalence.*

As an aside, we can construct $\mathsf{Ho(Top)}$ by looking at CW complexes, category D and formally inverting to weak equivalences. This gives you a universal property through formal inversion. Functors out of CW complexes are functors out of $\mathsf{Ho(Top)}$:

$$\begin{tikzcd} \mathsf{Top} \arrow[rr] \arrow[dr] & & \mathsf{D} \arrow[dl] \\ & \mathsf{Ho(Top)} & \end{tikzcd}$$

The data of a weak equivalence involves the map f. It is not an abstract isomorphism of f. You have to have the map that induces the isomorphism of homotopy groups. There are examples of spaces with equal homotopy groups that are not homotopy equivalents.

As a question, is π_* a homology theory, i.e. does it satisfy the Eilenberg-Steinrod axioms? Unfortunately the answer is no.

Suppose we relax the axiom that says that the value on a point is $\mathbb{Z}$ or R. (i.e. we generalize homology theories, other examples are topological k-theory or cobordism.) The value of any kind of theory like this is heavily constrained on spheres because of the suspension consequence of the axioms. That's just not true for homotopy groups. $\pi_1(S^1) = \mathbb{Z}$ and that's good but in general it is not the case that $\pi_n(S^k)$ is compatible with suspension. We know this for trivial reasons. Let's think about some things we know about homotopy groups.

Lemma 7.7. *1 $\pi_k(S^n) = 0$ for $k < n$. This is in some sense a result of differential topology. It will miss something and you can pull out a point and contract the resulting plane.*

We don't actually know $\pi_n(S^k)$ for most pairs (n, k). We will spend some time talking about $\pi_3(S^2)$ which is not compatible with any kind of Eilenberg Steinrod type axioms.

Can we turn $\{\pi_*(-)]\}$ into a homotopy theory formally? If we had LES associated to a map $f : X \to Y$ in $[Z, -]$ then we might hope to get something like a LES in π_*.

$[\Sigma X, Z]$ and $[Z, \Omega^n X]$ has $\pi_0(\Omega^n X) \cong \pi_n(X)$. This is a reminder of the origin of the algebraic structure on $\pi_n(X)$. Now we will dualize the story of cofibrations, i.e. fibrations.

Cofibrations were answering the question "how should we take the quotient?" Now we will think about the dual, where X and Y are based and we have $f^{-1}(*) \to X \to Y$ fiber.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & Y \end{tikzcd}$$

We get a fiber, which is the same question that we always ask. Is it the case that F is homotopy invariant? When is it? The theory of fibrations will be when the fiber is a homotopy invariant.

$$\begin{tikzcd} * \arrow[r] & Y & X \arrow[l] \\ * \arrow[r] \arrow[u, equal] & Y' \arrow[u, "cong"] & X \arrow[l] \arrow[u, "\cong"] \end{tikzcd}$$

8 October 20th

Culinary indoctrination: Today, we will discuss Sushi. These two places fall into the category of good neighborhood places that are not too expensive and have very good fish. These places are Sushi Yasaka and Tomoe, There's another place called Sushi Zo (branch of an LA place but it's more expensive). Second avenue and 11th street. He says that there's another place for Sushi Yasuda is another recommendation. He said its cheap but to my knowledge it's pretty expensive.

8.1 Return to the cofiber sequences

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Cf \arrow[r] \arrow[dr, "\alpha"] & \Sigma X \arrow[r] & \Sigma Y \arrow[r] & \dots \\ & & & C_g \arrow[u] \arrow[ur, "\beta"] & & \end{tikzcd}$$

We just need to make sure that this vertical component of the diagram commutes. We can get C_g from collapsing the cone CY in Cf. Now we need a map $C_g \wedge T_* \to \Sigma X$ What is it on CX and what is it on CY? $h(x,s,t) = (f(x), t - st)$ What is it on CY? $h(y,s,t) = (y, s+t-st)$. It crushes down the part of Y outside of X. Ultimately, we're trying to say that if we think about the sequence

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g_0"] & Cf \arrow[r] \arrow[d, equal] & \Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma Cf \arrow[r] & \dots \\ X \arrow[r, "f"] & Y \arrow[r, "g_0"] & Cf \arrow[r, "g_1"] & Cg_0 \arrow[r] \arrow[u] & Cg_1 \arrow[r] & \dots \end{tikzcd}$$

This shows that we get commuting in the diagram here from the above diagram

Now we will resume talking about fibrations. A cofiber is a map such that the cofiber and the homotopy cofiber are the same. The slogan is that the fiber and the homotopy fiber are the same, where the fiber of a map from X to Y is specified by the pullback

$$\begin{tikzcd} & * \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \text{which gives} \quad \begin{tikzcd} f^{-1}(*) \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \text{as the pullback}$$

And additionally

$$\begin{tikzcd} Ff \arrow[r] \arrow[d] & py \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where py is $\mathsf{Map}(I_*, Y)$

Definition 8.1. $f : E \to B$ is a fibration if there exists a lift by

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ Y \times I \arrow[r] & B \end{tikzcd} \quad \text{with the same data as} \quad \begin{tikzcd} E \arrow[ddd] & & E^I \arrow[ll] \arrow[ddd] \\ & Y \arrow[ul] \arrow[ur, dashed] \arrow[ddr] & \\ & & \\ B & & B^I \arrow[ll] \end{tikzcd}$$

(F, G) is an adjoint pair

$$\begin{tikzcd} FX \arrow[r] \arrow[d] & Y \arrow[d] \\ FX' \arrow[r] & Y' \end{tikzcd} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & GX \arrow[d] \\ X' \arrow[r] & GY' \end{tikzcd}$$

8.2 Dualizing

If

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "q"] & E \arrow[d, "p"] \\ B' \arrow[r] & B' \end{tikzcd}$$

Then if p is a fibration, q is a fibration.

Recall, $f : X \to Y$ factor $X \rightarrowtail Mf \to Y$ is the universe test symbol for cofibrations. Then similarly for in fibrations, we get $X \to P_f \twoheadrightarrow Y$ as the universal test object for fibrations.

Therefore we can find that

$$\begin{tikzcd} X \times_f PY \arrow[r] \arrow[d] & PY \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$ then we have that $X \times_f PY$ is a mapping path space of f.

$f : E \to B$ is a fibration, then there exists a path lifting function $\ell : E \times_f B^I \to E^I$ and $\theta : E^I \to E \times_f B^I$ where $\theta\ell = \text{id}$. Homotopy fiber is the actual fiber of the map from $E \times_f B^I \to B$.

As a proposition if $f : E \to B$ is a fibration, then the fiber and the homotopy fiber are equivalent. The proof of that is the duel of the proof we gave for the mapping cone and the honest quotient. Given a fibration we can form the sequence

$$\Omega Ff \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow Ff \longrightarrow X \xrightarrow{f} Y$$

Then $\Omega Ff \cong F(\Omega f)$ Then by the same argument as at the beginning, if we have a space Z and we have a long exact sequence $[Z, \Omega X] \to [Z, \Omega Y] \to [Z, Ff] \to [Z, X] \to [Z, Y]$

If we enter S^n as Z this gives rise to a long exact sequence germane to homotopy theory

$\mathsf{Map}(\Sigma X, \Sigma X) \cong \mathsf{Map}(X, \Omega\Sigma X)$

And we can take the adjoint of the unit from $\mathsf{Map}(\Sigma X, \Sigma X)$ in the other group, and this can be helpful. The same thing can be done in the opposite adjoint direction. This gives us

$$\begin{tikzcd} & \Sigma\Omega Ff \arrow[r] \arrow[d] & \Sigma\Omega X \arrow[r] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma\Omega Ff \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] & \\ \ldots \arrow[r] & \Omega Y \arrow[r] \arrow[d, equal] & Ff \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Cf \arrow[r] \arrow[d] & \Sigma X \arrow[r] & \ldots \\ & \Omega Y \arrow[r] & \Omega Cf \arrow[r] & \Omega\Sigma X \arrow[r] & \Omega\Sigma Y \arrow[r] & \Omega\Sigma Cf & \end{tikzcd}$$

Σ shifts by 1 and Ω shifts by -1 and they are adjoint. Additionally, $X \to \Omega\Sigma X$ is not usually an equivalence and $\Sigma\Omega X \to X$ is not usually an equivalence. "The comparison between based fiber and based cofiber sequences"

In Top_*, cofiber sequences and fiber sequences are very different. In $\mathsf{Ch}(R)$ cofiber and fiber sequences coincide up to a sign.

8.3 Homotopy Groups

Now we want to talk about Homotopy Groups. In particular we want to prove the following theorem:

Theorem 8.2. *If $f : X \to Y$ is a weak equivalence (isomorphism on homotopy groups) and X and Y are CW complexes, then X is homotopy equivalent to Y.*

What we will really prove is the following: if Z is a CW complex and $X \to Y$ is a weak equivalence then $[Z, X] \cong [Z, Y]$ is an isomorphism. It preserves weak equivalences in the target variable so it is like a quasi-isomorphism. We will give a fairly modern proof of this.

Let $f : X \to Y$ be a fibration and a weak equivalence (aka an "acyclic" fibration)

And suppose we have the commuting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Then the dashed arrow exists induced by the other arrows (if A and B are CW complexes).

We can then make the following two diagrams

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \text{ is surjective and } \begin{tikzcd} Z \times \partial I \arrow[r] \arrow[d] & XY \arrow[d] \\ Z \times I \arrow[r] \arrow[ur] & Y \end{tikzcd} \text{ is injective}$$

$X \to P_f \twoheadrightarrow Y$

9 October 25th

Culinary Indoctrination: Discussion of doughnuts.

1. Doughnut Plant
2. Doughnut Pub (7th Ave. & 14th st., Open 24 hours and worthy of note)
3. Peter Pan (Greenpoint, Polish)

There are two major schools of doughnut production, one of which is a classic American doughnut shop (like Dunkin Donuts but better). Peter Pan is much better then Doughnut pub, both of which are of this type. The other type of place is artisanal hipster doughnuts. There are lots of them around the city. The original location is at Grand near Essex. Easier ones to go to, including one at 23rd st.

Opinions of cake doughnuts vs yeast doughnuts. He has normative views and he regards the difference as a permissible personal variation. In terms of these places, Yeast is better than Cake at Doughnut Pub, both are excellent at Doughnut Plant.

He used to go to Tokyo a great deal for a friend/other researchers. There was a place called Mr. Doughnut and it was a Massachusetts chain. There were lots of strange nostalgia for 1960s Boston in the middle of Japan.

9.1 Interaction between Homotopy Groups and Cohomology.

We will finish proving the whitehead theorem.

Theorem 9.1. *If $X \to Y$ is a weak equivalence and Z is a CW-Complex, then $[Z, X] \to [Z, Y]$ is an isomorphism.*

Last class we reduced this proof to $X \twoheadrightarrow Y$ is a fibration and weak equivalence, then there is a lift $B \to X$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "\cong"] \\ B \arrow[r] & Y \end{tikzcd}$$

We are proving that cofibration (CW inclusions) have the left lifting property. against fibrations. We procede with induction. It suffices to consider the case of

$$\begin{tikzcd} S^{n+1} \arrow[r] \arrow[d] & X \arrow[d, two heads, "\cong"] \\ D^n \arrow[r] & Y \end{tikzcd}$$

Observation: it suffices to replace $D^n \to Y$ by a homotopic map. (use the fact that $X \to Y$ is a fibration).

Consider the map $D^n \to D^n$. We will take D^n and carve out an annulus and restrict the map to that. We will send the center part to a point. Basically just contracting the middle circle to a point and shifting everything else into it. Evidently this is homotopic to the identity. Name this map f. Then we get

$$\begin{tikzcd} S^{n+1} \arrow[rr] \arrow[d] & & X \arrow[d] \\ D^n \arrow[r, "f"] & D^n \arrow[r] & Y \end{tikzcd}$$

Because $X \to Y$ is a fibration, we can solve the lifting already. If we look at the fiber over the image of 0, it must be contractible. Why? because of the weak equivalence, so π_1 must be equal.

Remember, if X and Y are CW complexes, then if $f : X \to Y$ is a weak equivalence. then f is a homotopy equivalence.

Any space X is weakly equivalent to a CW complex. There are various proofs, most of the classical ones are not functorial, so ours will also not be functorial

Proof. $\bigwedge_\alpha S^{n_\alpha}_\alpha \to X$ and is a surjection on each π_n. If we let Z_i be a space resulting from the wedging of spheres. Then, we can kill the kernel. $\pi_n(Z_i)$ surjects onto $\pi_n(X)$ then we have homotopic $\gamma, \gamma' : S^n \to Z_i$ in $\pi_n(X)$. Now attach a cylinder that gives a homotopy between γ and γ' from Z_i to Z_{i+1}. One observation to make is that whitehead implies that this does not screw up any prior work. From a low dimensional sphere there are no nontrivial maps into a higher dimensional sphere. If we do something like this, once we fix π_n and move on, we wont mess it up by fixing π_{n+1}. □

Terminology: X is n-connected if $\pi_i(X) = 0$ for $i < n$. $X \to Y$ is a n-equivalence of if $\pi_i X \cong \pi_i Y$ for $i < n$ surjective at n. We are going to define cohomology by way of defining Eilenberg Maclane spaces. We will give a definition of cohomology as the functor represented by Eilenberg Maclane spaces and then it will turn out that there are other ways to say it.

First, the Hurewicz Map. this will be a map from $\pi_n(X)$ to $H_n(X)$, and this is reduced by killing the extra class in degree 0.

The map will take $\gamma \mapsto \gamma_*(1)$ for $\gamma \in \pi_n(X)$.

Lemma 9.2. *1 If $X = \bigwedge_\alpha S^n$ then this is an isomorphism. Something more general that we will prove but not today is the following, which is that X is n-connected and the map $\pi_n(X) \to H_n(X)$ is an isomorphism if $n \geq 1$ and if $n = 1$ then this is Abelianization.*

Remember: the theory of Abelian groups is easy and the theory of all groups is hard, in particular if we have this kind of we have lots of fun things involving noncomputability. The word problem is undecidable. The argument is a diagonalization. This means we can embed undecidable questions into questions about spaces. This bad behavior happens on π_1. That's the only on that can not be π_1.

Now, one comment is that this theorem $(*)$ depends on something called "homotopy excision". This is an interesting result which says something like the following. For homology, we just have excision, we can glue stuff. One thing that excision amounts to is saying that we have something like Mayer-Vietoris. If we have a cover and a union we can relate all the homologies. Homotopy excision says if we decompose things and the pieces are pretty connected, then we can say something about the connectivity of what happens when we glue them together. This is a much for subtle statement. This is a hard, nontrivial theorem. We will prove this and it is related to something interesting. How do you increase connectivity, here's an easy way to do it. take X and take $\Sigma^n X$ which is the same thing as $S^n \wedge N$. If you make things more connected, we make excision work better (maybe we can't exactly see this). If we push up connectivity, the excision formula looks nicer and nicer. The fact that excision doesn't quite work is the same thing as loops in suspension not quite being inverse. Everything gets solved if we stabilize by passing to the category of spectra.

What is the relationship betwen $\mathsf{Map}(X, Y)$ and $\mathsf{Map}(\Sigma X, \Sigma Y)$. BY adjunction, we're asking for the relationship $[X, Y]$ and $[X, \Omega\Sigma Y]$. This is something called the freudenthal Suspension Theorem. If they're connected enough, it's an equivalence. We should think of these as sorts of approximation results. Now we are going to use the Hurewicz map to establish the uniqueness on CW complexes (which is enough by weak equivalences) of ordinary homology. We will just sketch this. Suppose that we have some theory E that satisfies the EIlenberg Steinrod axioms. Associated to E, we also have cellular theory. $E(X_n, X_{n-1})$ was $C_n^E(X)$ is the boundary of complex coming from LES of pair.

Since $X_{n+1} \rightarrowtail X_n$ of $E(X_n, X_{n+1})) \cong E(X_n/X_{n+1})$ form another cellular theory $\pi_n(X_n/X_{n+1}$. Define for any pair (n, G) with G Abelian group and $n \in \mathbb{N}$ a space $K(G, n)$ such that

- $K(G, n)$ is a CW-complex
- $K(G, n)$ is unique up to homotopy equivalence
- $\pi_1 K(G, n) = 0$ unless $i = n$ when $\pi_n K(G, n) \cong G$.

There's something else to say whcih is that homotopy classes of maps $[K(G_1, n), K(G_2, n)]$ is bijective (isomorphism of sets) to homomorphisms from G_1 to G_2. There's a sort of dichotomy theorem where small complexes tend to have enormous and hard to deal with homotopy groups. This will be a great deal like our CW approximation process. We will produce something n-connected that has the right homotopy group at π_n that has crazy stuff up about. But then we will kill everything about π_n. It's an infinite number since as we glue stuff on to kill stuff.

As observation, we know that $K(G, n) \cong \Omega K(G, n+1)$. Why? Because of the long exact sequence of homotopy groups, i.e. what loops does. For any space X we have the path loop fibration which is $\Omega X \to \text{Paths} X \to X$. $\text{Paths} X$ is contractible, so we get a long exact sequence of π_i which implies that π_i of X is isomorphic to π_{i+1} of ΩX. That's all that's happening here. This says something fascinating. If you weren't so exhausted, you would feel excited. This tells you that this thing has a product coming from the loop structure, which is cool and perhaps unexpected. We can then induct as well where

k times $\Omega(\Omega(\ldots \Omega(G, n+k))) = \Omega(G, n)$.

The non-functorial ad-hoc way to do this is the following. We can make $M(G, n)$, the Moore space. $H_1(M(G, n)) = 0$ where $i = n$ and $H_n(M(G, n)) \cong G$

We know that $\pi_n(M(G, n)) \cong H_n(M(G, n)) \cong G$. For a space like this that is n-connected we can replaced it with something that has no n-cells. It it's homologically n-connected then it's also broadly n-connected

$\bigwedge S_\gamma^{n+1}$ for $\gamma \in \pi_{n+1}(M(G, n))$. We can certainly do that. This has a map to $M(G, n)$. We pick a representative of each homotopy class then we wedge them together. Then we get this map. Now what do we do?

Now take the homotopy cofiber of this map. If we write $M(G < n) = Z_0$ we can call that Z_1. $\pi_n(Z_1) \cong G$. Playing a game like this does not mess anything up at G. $\pi_{n+1}(Z_1) \cong 0$. We're filled all the class of maps by placing a disk on top of them, in which they can be contracted. The next step up is crazy. We've added a potentially infinite number of spheres. Now we repeat infinitely many times and take the limits. This is not functorial in any sense, and it is huge. It is a CW complex. Provided $M(G, n)$ is a CW complex, this is a CW complex. We do know some explicit example. S^1 is a $K(\mathbb{Z}, 1)$. Once we have this, we can define a new invariant. We will call this invariant cohomology. It will be defined as $H^n(X, R) = [X, K(R, n)]$. This is a contravariant functor from spaces to Abelian groups.

10 October 27th, 2021

Culinary Indoctrination for today is Defonte's, a sandwich shop in Red Hook. There are lots of strange hipster spaces in old warehouses, the waterfront is very pretty.

The midterm will probably be around November 10th. It will be a take-home and it will be multiple days long. It will likely be structured that he sends us the midterm on Monday and then we will have 12 hours to work on it, but that timeperiod can be taken any time from Monday to Friday (something along those lines)

10.1 Moore Spaces

This was characterized by the property that a Moore space M has (G, n) so that $H_n(M(G, n)) \cong G$, otherwise $H_m(M(G, n)) = 0$ for $m \neq n$.

We will build this using the presentation of the group G in terms of Free Groups, and then we will model it. In general, we know that G is isomorphic to F/R where F and R are free, and R are the relations, then we have a surjection $f \to G$ and R is the kernel of the map $F \to G$ This is essentially the co-unit of a adjunction.

We need to model this in topology. We will have $M(G,n)$ as a cofiber of $\bigwedge_\alpha S^n_\alpha \to \bigwedge_\beta S^n_\beta$ where the α wedge represents generators of R and the β wedge represents the generators of F. The long exact sequence of homology tells us that we get the right answer.

Now, what is the map $\bigwedge_\alpha S^n_\alpha \to \bigwedge_\beta S^n_\beta$. Pick a generator in R, γ. γ is equal to some sum $\sum_{i=1}^m k_i \beta_i$ where β_i are generators of F and k_i are integers. Now, we have

$$S^n \longrightarrow \bigwedge_m S^n \xrightarrow{\deg k_i} \bigwedge \beta_i$$

This is essentially a linear algebra game.

Now, if we recall, we built $K(G,n)$ by attaching cells to $M(G,n)$ to kill all the higher homotopy groups. We did this in a large process where we went inductively. By Hurewicz theorem we know that $\pi_n(G < n) \cong G$ by the Hurewicz theorem. Eilenberg-Maclane spaces.

There is an elaboration of this process, which produces "Postnikov Sections" or "Postnikov Tower." Homology is computed inductively, but it is a formal decomposition of a space where the instructions for building it are controlled by the algebra. You could ask a similar question about Homotopy groups. That is what the postnikov tower is. The idea is that we can end up with a tower

$$\begin{tikzcd}
 & P^{n+2}X \arrow[d] \\
 & P^{n+1}X \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & P^n X \arrow[d] \\
 & \vdots
\end{tikzcd}$$

$P^n X$ is formed by infinitely many attached cells to X to kill all homotopy groups above n.

$$\dots \longrightarrow k(\pi_{n+1}(X), n+1) \longrightarrow P^{n+1}X \longrightarrow P^n X \longrightarrow k(\pi_n X, n+2) \longrightarrow \dots$$

Then $P^{n+1}X$ is the hofiber of $P^n X \to k(\pi_{n+1}, n{+}2)$. This fiber has another name, which is $H^{n+2}(P^n X; \pi_{n+1}X)$. Whatever else we might have thought abouut cohomology, this is a pretty good reason to think that cohomology is interesting. This is called the k-invariant. This is the gluing data that puts spaces together. If you want to build iup spaces there are two kinds of data that make it nontrivial. There's cellular attaching maps, and alternatively you have to know the cohomological twisting invariant to put it together the other way. This is a step towards algebraicizing spaces.

We might not talk about this explicitly, but another place where this happens is in obstruction theory. The idea is that we have a map on something and we want to extend it. Typically the data we want to extend things live in a cohomolgy group. Obstruction theory is interesting but we may or may not have time to truly talk about it.

Something else we should say that is X is eqiuivalent to the homotopy limit of $P^n X$. You can recover the space from the tower very explicitly from passing to the homotopy limit.

10.2 Cohomology

Remember, we had the definition $H^n(X;R) = [X, k(R,n)]$ What do we now know about this functor? Immediately we now several things. We can conclude that cohomology is a homotopy invariant. $X \simeq Y$ and then $H^n(X) \cong H^n(Y)$. The next thing we can see is $H^n(\Sigma X)$ is $[\Sigma X, k(R,n)]$ and by adjunction this is $[X, \Omega, k(R,n)] \cong [X, k(R; n-1)]$ coming from the path fibration, which is $H^{n-1}(X)$

Then lastly we get that $A \to X \to CF$ has the long exact sequence

$$\ldots \longrightarrow H^n(Cf) \longrightarrow H^n(X) \longrightarrow H^n(A) \longrightarrow H^{n+1}(Cf) \longrightarrow \ldots$$

$H^n(S^n) \cong [S^n, k(R,n)] \cong \pi_n k(R,n) \cong R$ and so we can find that H^n satisfies the Eilenberg Steinrod Axioms. We will get another construction of cohomology that is useful for other things.

Most notably, the way we wrote it with the cofiber used excision.

We have cohomology and it satisfies our axioms and we will do lots of things with it but first a question. What did we need to know about $\{k(R,n)\}$ to do this? We didn't actually care about the homotopy groups of these spaces. We needed the fact that $k(R,n) \cong \Omega k(R, n+1)$ mainly, and then one more thing. Suppose that we have a collection of spaces $E_0, E_1, \ldots$ and we stipulate that $f : E_i \to \Omega E_{i+1}$ has $E_i \cong_f E_{i+1}$. We can define $E^n(X) = [X, E_n]$. What do we get out of that? It's homotopy invariant, it satisfies the suspension axiom, it has the long exact sequence associated to a cofibration, and it clearly has the direct sum property, since that's how we defined it. It does not necessarily have the property that $E^n(S^n) \cong [S^n, E^n]$ is as desired. This is what's called a generalized cohomology theory. You've given up on what's called the dimension axiom. We can still do this and there are many interesting generalized cohomology theories. There are two ways to look at this. On the one hand, examples of such theories include MU, KU, (complex cobordism [what happens if spaces are equivalent if they are upto homotopy the boundary of a manifold.], complex K-theory[Bundles on a space]).

A collection of spaces like this is called a spectrum. In general, you can prove a sort-of theorem. Any spectrum like this gives rise to a general cohomology and any cohomology theory gives rise to a spectrum. This is called Brown Representability. Why is this a representability theorem? The collection of functors is represented by homotopy maps into a space. It's not the case that they give rise to a spectrum directly but it's up to some basic equivalences. This theorem is not a terrible proof, you could learn it relatively quickly, in an afternoon or so.

There's something else. We can form the category of spectra. That's the category of sequences of spaces that have a property like $E_i = \Omega E_{i+1}$. The objects are E_i and F_i arnd you need maps such that the following diagram commutes

$$\begin{tikzcd}
E_i \arrow[r] \arrow[d] & F_i \arrow[d] \\
\Omega E_{i+1} \arrow[r] & \Omega F_{i+1}
\end{tikzcd}$$

We can call this category Sp. This is the formal stabilization of Top_*. This is the original thing that is nontrivial that is stable which you can make out of spaces. There are a couple of things to say about this. One thing to say is that this tells you is that this is another way to think about cohomologies. Another thing to say about this category is that this is the original incarnation of the idea that went into a triangulated category. The derived category in Algebraic Geometry, Vertier wrote down those axioms by in part knowing about the category of spectra. Invariants of geometric spaces should ultimately be controlled by some kind of derived category which will look something like this. One last thing to say about the category of spectra that relates to number theory is that this is a symmetric monoidal category with a tensor product. It's a generalization of the cup product.

If we think about the category of commutative rings, it has an initial object, $\mathbb{Z}$. The initial commutative ring is no longer $\mathbb{Z}$. It's built from spheres. You can construct invariants of that which can control, e.g. the van-diver conjecture is true. Lots of data about p-adic functions for extensions of $\mathbb{Z}$. There's a lot of number theoretic data sitting in this category.

We will now give a "hands on" definition of cohomology. $C^n(X;R) = \mathsf{Hom}_R(C_n(X);R)$ This assembles into a cochain complex.

Remember we have the chain complexes which are the duel of the cochain complexes

$$\text{Chain complex}\quad \begin{array}{c}\vdots\\ \downarrow\\ A_n\\ \downarrow{\scriptstyle\partial}\\ A_{n-1}\\ \downarrow\\ \vdots\end{array}\quad \text{with } \partial^2 = 0 \text{ and cochain complex}\quad \begin{array}{c}\vdots\\ \uparrow\\ A_n\\ {\scriptstyle d}\uparrow\\ A_{n-1}\\ \uparrow\\ \vdots\end{array}\quad \text{with } d^2 = 0.$$

What happens in this cochain complex? $f \in C^n(X; R)$ then $\partial f(z) = (-1)^{n+1} f(\partial z)$ Define H^n as the cohomology of the complex. The points of this definition for our purposes are the following

1. We can check using the Eilenberg Steinrod Axioms that this is isomorphic to the $[-, k(R, n)]$. We can just check this directly.

2. This is an elaboration of the adjoint given in homology implying that $H^n(X; R)$ defined by cochian is equivalent to $[X, K(R, n)]$ If we get a new way to define cohomology, we relaly just need to check that it satifies the axioms and everything will line up to be the same, essentially.

We can define $\Delta : X \to X \times X$ as the diagonal map which gives us $C^*(X \times X; R) \to C^*(X; R) \cong C^*(X; R) \otimes C^*(X; R)$. This is very cool. Every space has a diagonal map. That's something about being alive. This means that every space has $H^*(X)$ becomes a graded commutative ring with $a \vee b = (-1)^{\deg a \cdot \deg b} b \vee a$. Algebras are nice because we grow up with it (we do not grow up co-adding).

For algebras, theres lot of ways to reduce, co-algebras have all of this fail. $H^n(X) \otimes H^m(X) \to H^{m+n}(X)$ gives us

$$[X, k(R, n)] \otimes [X, k(R, m)] \to [X \times X; k(R \otimes R, n + m)] \to_{\Delta_*} [X, K(R \otimes R, n + m)] \to [X, k(R, m + n)]$$

11 November 3rd, 2021

Today's discussion of food will be about Uzbek food. There are various large populations of Uzbek people, including in Forest Hills and Rio. There's also Tam tov at 47th st in the Diamond District. The other restaurant of this kind is Cheburechnaya in queens in Rego Park. Most of the people who showed up in Queens were from Tashkent and so there are lots of interesting aspects of Korean cuisine in Uzbek food especially in Tashkent. He's also looking for Korean Fried Chicken restaurants.

We will discuss the Universal Coefficient Theorem. This tells you what happens when you change groups or rings when you compute cohomology. If you have the space X there is a short exact sequence.

$$0 \longrightarrow \mathrm{Ext}^1(H_{p-1}) \longrightarrow H^p(X; G) \longrightarrow \mathrm{Hom}(H_p(X), G) \longrightarrow 0$$

The last two nonzero terms are the adjoint of the evaluation. This means this is an answer to the question of when is this comparison an isomorphism. What it's saying is that it's not always an isomorphism but it is always surjective, and the isomorphism depends on controlling the Ext group. In general, it's also saying that $H^p(X, G)$ gets information from $H_p(X)$ and $H_{p-1}(X)$. We will say some things about the relevant homological algebra for Ext.

$$0 \longrightarrow G \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$$

where this sequence is short exact.

I_0 and I_1 are injective implies $H^p(X; I) \cong \mathrm{Hom}(H_p(X), I)$

For any A, $\mathrm{Hom}(A, -)$ has an exact sequence.

$$0 \longrightarrow \mathrm{Hom}(A, G) \longrightarrow \mathrm{Hom}(A, I_0) \longrightarrow \mathrm{Hom}(A, I_1) \longrightarrow \mathrm{Ext}(A, G) \longrightarrow 0$$

Apply this functor to a short exact sequence. This exact sequence is not determined by choice of injective structures.

$$\dots \longrightarrow H^{p-1}(X;I_0) \longrightarrow H^{p-1}(X;I_1) \longrightarrow H^p(X;G) \longrightarrow H^p(X;I_0) \longrightarrow H^p(X;I_1) \longrightarrow \dots$$

Using adjunction change the non-$H^p(X;G)$ terms into their Hom equivalents.

So what does it mean to be injective?

Let $A \to B$ be an injection of Abelian groups or more broadly R-modules. I is injective if Hom(B,I)$\to$ Hom(A,I) is surjective,. An alternative characterization is that I is injective if Hom($-$,I) preserves short exact sequences. If R is a field, then all modules are injective We get $V \rightarrowtail V'$ with $\theta(V) \subseteq V'$ take a basis for $\theta(V)$ which we call $\alpha_1, \dots \alpha_k$ and extend to β_i. Note, we did not assume finite dimensionality.

We now want to prove that $H^p(X;I) \to \mathrm{Hom}(H_p(X), I)$ is an isomorphism when I is injective. For this, let's draw a diagram. B_p and Z_p are boundaries and cycles.

$$\begin{array}{ccccc} & B_p & C_{p+1}X & & \\ 0 & Z_p & C_pX & B_{p-1} & 0 \\ & H_pX & C_{p-1} & Z_{p-1} & \\ & 0 & & & \end{array}$$

Now we apply $\mathrm{Hom}(-,I)$ and we can get

$$\begin{tikzcd}
 & \mathrm{Hom}(B_p,I) \arrow[r, tail] & C^{p+1}X & & \\
0 & \mathrm{Hom}(Z_p,I) \arrow[l] \arrow[u] & C^pX \arrow[l, two heads] \arrow[u] & \mathrm{Hom}(B_p,I) \arrow[l, tail] & 0 \arrow[l] \\
 & \mathrm{Hom}(H_pX,I) \arrow[u, tail] & C^{p-1}X \arrow[u] \arrow[r, two heads] & \mathrm{Hom}(Z_{p-1},I) \arrow[u, two heads] & \\
 & 0 \arrow[u] & & &
\end{tikzcd}$$

What's the point We can identified the comparison map $H^p(X;I) \to \mathrm{Hom}(H^pX, I)$ in terms of the diagram. The map from $C^px \twoheadrightarrow \mathrm{Hom}(Z_p, I)$ induces a map $H^p(X,I)$ onto the image of f in $\mathrm{Hom}(Z_p,I)$ And we can invert f since it is an injection. It's clearly a surjection and a diagram chase shows that it must also be an injection. Here we are using a couple of places where I is injective, notably in identifying maps as surjective.

Recall, in conclusion, $H^p(X,I) \cong \mathrm{Hom}(H_p(X), I)$ when I is injective.

Next, we need to get injective resolutions. There is a theorem that says that R-mod "has enough injectives." Professor Blumberg thinks this is from Grothendieck. For any module M there exists an injective i and an injection $0 \to M \to I$. We can make anything sit inside an injective module. We can ake cokernels in the sequence

$0 \longrightarrow M \longrightarrow I_0 7i-1 \longrightarrow I_2 \longrightarrow \dots$ and embed then in the module. then, when R is a PID, this terminates after two terms. after I_1. More explicitly, in $\mathbb{Z}$ we will call a group divisible if for all $g \in G$ there exists and n and Y such that $ny = g$. Divisible groups are injective. This tells us what to do for finitely generated $\mathbb{Z}$ modules, as we get torsion pieces and free pieces. For the free piece we have

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Then we can get the make $M \to \coprod_{\mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$, . The map and the index are injections.

The last fact we needed was uniqueness up to chain homotopy of injective resolutions. Let us sketch this. The real theorem is that if we have

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \dots$$

as an exact sequence of injective resolutions and we have a similar long exact sequence for N_i. Then there exists a chain map $\{I_i\} \to \{N_i\}$ which is unique upto chain homotopy

We can set it up with a diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & N_0 \arrow[r] \arrow[d, dashed, "f_0"] & N_1 \arrow[r] & N_2 \arrow[r] & \dots \\ 0 \arrow[r] & M \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & I_2 \arrow[r] & \dots \end{tikzcd}$$

f_0 exists because I_0 is injective. Then we can get the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & N_0 \arrow[r] \arrow[d, dashed, "f_0"] & \operatorname{im}(N_0 \to N_1) \arrow[r] & N_1 \arrow[r] & \dots \\ 0 \arrow[r] & M \arrow[r] & I_0 \arrow[rr] & & I_1 \end{tikzcd}$$

Construct f_1 as $f_1(dx) = df_0(x)$ on the image We can then get f_1 on N_1 using injectivity. We can perform this on any map $f - \tilde{f}$. This conclusdes the discussion of universal coefficients in Cohomology. There is a dual statement in Homology using Tor. We will explore this in exercises. The slogan: Ext is the derived Hom. Hom is not homotopy invariant, but Ext is. When we see this for the first time, the appearance of tor and ext in these statements looked bizarre and arbitrary, but they are certainly not.

For the next few classes we will talk about Poincare homology, the stable category, and the relations between many of the objects we have already been discussing.

11.1 Poincare Duality

We will be talking about some of the algebraic structures we have on cohomology. First, the cup product, coming from the diagonal. For any space, we have $\Delta : X \to X \times X$. We know that we have an isomorphism $C^*(X \times X; R \otimes R) \cong C^*(X; R) \otimes C^*(X; R)$. This induces a product on H^*

$$H^*(X; R) \otimes H^*(X; R) \to H^*(X; R)$$

and using degrees, we get

$$H^p(X; R) \otimes H^{n-p}(X; R) \to H^n(X; R)$$

We know things about this product. They ultimately boil down to facts about the diagonal map. For instance, we know that if we have X and we have

$$\begin{tikzcd} X \arrow[r, "\delta"] \arrow[d, "\Delta"] & X \times X \arrow[d, "\mathrm{id}\times\Delta"] \\ X \times X \arrow[r, "\mathrm{id}\times\Delta"] & X \times X \times X \end{tikzcd}$$

commutes which tells us that the cup product is associative. We are working with compact manifolds. When M is orientable, we'll have a distinguished class in $H_n(M; R)$ such that the evaluation pairing gives rise to a map $H_n(M; R) \to R$. This is called the fundamental class. The statement of Poincare Duality is

$$H^{n-p}(M; R) \otimes H^p(M; R) \to H^n(M; R) \to R$$

is a "perfect pairing" meaning that there is an induced isomorphism $H^{n-p}(M; R) \cong \operatorname{Hom}(H^{n-p}(M; R), R) \cong H_p(M; R)$ Somehow this is telling us that there's data in our top homology group.

We can put this in terms of the cap product: This is a product $H^p(X; R) \otimes H_n(X; R) \to H_{n-p}(X; R)$ Usually the cup product is written $\cup$ and the cap product is written $\cap$, and $\langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta \cap \gamma \rangle$ it's basically an adjoint to the cup0 product, and $\langle -, - \rangle$ is the evaluation pairing. we will leave the definition of this next time. We will say now some things about what an orientation is.

11.2 Poincare Duality continued

Poincare Duality is very old but the real reason it's true takes us to the beginning of modern homotopy theory. We will say something about what Poincare Duality is about. It's about Local vs. Global considerations. $H_n(X)$ is a global invariant of the shape of X. It's sensitive to everything that happens to X upto homotopies. On the other hand, if we think about manifolds, whether or not something is a manifold is determined by local similarity to Euclidean space (at every point there is a tangent space). For reasons that will become clear, Poincare duality is connected to ideas about orientation. The first somewhat naive statements is that the vector spaces of homology and cohomology line up properly. But there is a more delicate and informative way of understanding it relating to orientations.

One way of encoding local information is understanding the relative homology of $\tilde{H}_n(X, X - \{x\})$. By excision, this is the same as $H_n(U, U - \{x\})$ where $U \subset X$ is open and $x \in U$. There is a natural map $H_n(X) \to H_n(X, X - \{x\})$ which is something liek a quotient map. A lot of the game will be understanding what we can recover from data in the relative homology about the original homology. We will now say something about manifolds and then something more general.

Image that X is a n-dimensional manifold. Then, $H_n(X, X - \{x\})$ is isomorphic to $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Which U do we take? We take a chart. We have some kind of function from the set of X to Ab that assigns x to $\mathbb{Z}$. The underlying set of X has no maps or structures. This knows nothing about the topology of x. That's not what we meant. We know stuff. By continuity and functoriality for H_n, we know that these vary continuously, in some sense. In order to say this precisely we need some language. We will present both languages.

We can make a definition. Define $\prod_1 X$ as the *fundamental groupoid*. What is a groupoid? Well first, a monoid (a category with one object). A group is a monoid where all the maps are isomorphisms, at least in category theory. This is in some sense a nice way to formulate this because we can extend it.

A **Groupoid** is a category where all the maps are isomorphisms. This is some kind of generalized groups. "A group with many objects" [3] There's a very famous paper due to Steve Mitchell which takes the perspective of doing algebra in a ring with many objects.

Now we want to define the **Fundamental Groupoid**. The objects are the points of X and the maps from $x \to y$ is a homotopy class of paths. Maps from x to y are given by homotopy classes of paths $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. It's a groupoid because there's a path out and there's a path back between the points, which again are objects

Formally, $\mathsf{Ob}(\Pi X) = X$ and $\mathsf{Mor}(\Pi X) = \{\gamma : [0,1] \to X\}$

A local system of Abelian groups is a functor from $\Pi_1 X \to \mathsf{Ab}$. An n-manifold X has a local system $x \to H_n(X, X - \{x\})$ (*) (You have to check that this actually works). We can also describe this as a bundle. We saw this in passing before when we talked about how vector bundles were classified. In general, there's no reason for Ab to be Abelian groups. This could be some other kind of algebraic category. It's very useful and enriching to study generalizations of local systems to different algebraic categories than Ab.

We can also think of this in terms of bundles (covering space).

$$\begin{array}{c} \coprod_{x \in X} H_n(X, X - \{x\})(**) \\ \downarrow \\ X \end{array} \quad \text{where we topologize this to make it an honest bundle/covering space.}$$

Equivalently, we will give this a name. $\mathcal{O}_x$ which we will use to refer to both $(*)$ and $(**)$ We are going to be interested in the space of sections $\Gamma(X, O_x)$, which is a map where if you go both up and down that morphism, it is the identity.

We can also consider the sub covering space, which has fiber $H_n(X, X - \{x\})^\times$, i.e. the units. This can be more or less interesting depending on what your ring is. If you're working in $\mathbb{Z}^\times$, you have ± 1. In $\mathbb{F}_2$ there's just one unit. If you ring is something random, the units can be very complicated. We will write $\mathcal{O}_X(R)$ for a version of this with coefficients in the ring R. We're going to say that an orientation of M, (we've switched from X to M here) is a section of this subbundle. of generators.

A fundamental class of M is an element $\alpha \in H_n(M)$ such that $H_n(M) \to H_n(M, M - \{x\})$ hits the elements specified by the orientation.

[3] Additionally, as an aside, we can define something of the sort of a DG-category which is like a Ring with many objects."

The first result towards poincare duality is that there is a map from $H_n(M,R) \cong \Gamma(M, \mathcal{O}_M^R)$ under the assumption that M is a compact n manifold. The map is taking the image of a class M to $M - \{x\}$. The basic idea here is that sections are sort of like cohomology.

How do we prove this? Give an inductive proof. We can call the map $\psi : H_n(M,R) \to \Gamma(M, \mathcal{O}_M^R)$. Every one of our textbooks has a unique way to prove Poincare duality. It may not always be clear, but everyone is actually doing the same thing but with slightly different local approaches. This proof is along the lines of the proof in Miller's notes, which comes mostly from Spanier.

We first have a lemma. If we are going to induct, we need to know something about what kinds of functors we have. We need to study the properties of Γ as a functor. The notation is that we will define $H_n(M \mid A) = H_n(M, M - A)$. These are chains on M which are supported on A.

Lemma 11.1. *1 If ψ is an isomorphism for $H_n(X \mid A)$ and $H_n(M \mid B)$ and $H_n(M \mid A \cap B)$ then it is an isomorphism for $H_n(M \mid A \cup B)$. It's a patching result.*

Proof.

$$\begin{tikzcd}
0 \arrow[r] & H_n(A \mid A \cup B) \arrow[r] \arrow[d, "\cong"] & H_n(M \mid A) \oplus H_n(M \mid B) \arrow[r] \arrow[d, "\cong"] & H_n(M \mid A \cap B) \arrow[d, dashed, "\cong"] \\
0 \arrow[r] & \Gamma(A \cup B) \arrow[r] & \Gamma(A) \oplus \Gamma(B) \arrow[r] & \Gamma(A \cap B)
\end{tikzcd}$$

□

Now, we need another lemma.

Lemma 11.2. *2 $H_n(X \mid -; R)$ and $\Gamma(-, \mathcal{O}_M^R)$ commute with inverse limits. They're both covariant so they take inverse limits to inverse limits.*

Recall the functoriality of our constructions. $H_n(M \mid A) = H_n(M, M - A)$ and $A \to A'$ with $H_n(M, M - A') \to H_n(M, M - A)$ and $\Gamma(A', \mathcal{O}_M^R \to \Gamma(A, \mathcal{O}_M^R)$ which is contravariant and given by restriction. We are going to induct over decomposing M as an intersection of finite unions of compact and convex sets. This style of decomposition occurs frequently. If we have M compact and we fix ℓ and we know that $M = \bigcup_{i \in I_\ell} B(n, i/\ell)$ By compactness, we know that $|I_\ell|$ is finite. For every ℓ, there's a collection of points and balls that covers ℓ. Consider the collection $M_k = \bigcap_{j=1}^k \bigcup_{i \in I} B(m, 1/j)$

If $A \subset M$ is a compact subset of M, a n-manifold, then $H_i(M \mid A) = 0$ for $i > n$ and $H_n(M \mid A; R) = \Gamma(A, \mathcal{O}_M^R)$. This space of sections is nice, it glues, but we don't know what it is yet.

There are stages of building up the proof. First, $M = \mathbb{R}^n$ and A is compact and convex subset. The point now is that A. is a compact and convex subset, so it is homotopy equivalent to a disk. We now know that $H_n(\mathbb{R}^n \mid D^n) \to \Gamma(D^n \mid \mathcal{O}_M^R)$ and these are both equivalent to $H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$.

In the second case, $M = \mathbb{R}^n$ and A is a finite union of compact and convex sets. What happens now? We have it for a single compact and convex set, so if we are a finite union of them, that's why we have Mayer Vietoris. We can just keep inducting up. Now, for the third step, when $M = \mathbb{R}^n$, A is compact, A is an inverse limit, i.e. intersection, of finite unions of compact and convex sets. Since the two invariants in question commute with inverse limits, we decompose the comparison mappings. Once we have that we're done by proceeding work.

We will pause for a second to make a comment or observation. We have cavalierly assumed that we're working with finite CW complexes. Our constructions of cohomology, for example, agree on finite CW complexes but maybe not infinite spaces. Choices we need to make to make them line up might come up here. there will be more detailed information about this in the next lecture. In the most general forms of Poincare duality end up using Cech cohomology. Singular cohomology doesn't work properly for noncompact things in this instance. Our cohomology was originally given as maps to the Eilenberg Maclane spaces.