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Tue

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Final Project Rough Draft

Part 1

Given $x^{k+1} = x^k - \alpha \nabla f(x^k) + \beta(x^k - x^{k-1})$

$x^{-1} = x^0 \in \mathbb{R}^2 \qquad \alpha, \beta > 0$

$f(x) = \frac{1}{2} x^T Q x - b^T x + c$

$\nabla^2 f = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$

$0 < m = \lambda_2 \leq \lambda_1 = L < \infty$

$b \in \mathbb{R}^2, \; c \in \mathbb{R}$

$\alpha = \frac{4}{(\sqrt{L} + \sqrt{m})^2}, \; \beta = \frac{\sqrt{L} - \sqrt{m}}{\sqrt{L} + \sqrt{m}}$

$\nabla f(x^*) = 0 \iff Qx^* = b$

Problem 1

Prove $w^{k+1} = T w^k$

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Proving

$$x^{k+1} = x^k - \alpha(Qx^k - b) + \beta(x^k - x^{k-1})$$

(subs $x - x^*$)

$$x^{k+1} - x^* = (x^k - x^*) - \alpha(Q(x^k - x^*) + Qx^* - b)$$

$$+ \beta((x^k - x^*) - (x^{k+1} - x^*))$$

$$= x^k - x^* - \alpha(Q(x^k - x^*) + b - b) + \beta((x^k - x^*) - (x^{k-1} - x^*))$$

$$= x^k - x^* - \alpha Q(x^k - x^*) + \beta[(x^k - x^*) - (x^{k-1} - x^*)]$$

$$= (x^k - x^*)(1 - \alpha Q + \beta) - (x^{k-1} - x^*)\beta$$

$x^k - x^* = 1(x^k - x^*)$

$$\begin{bmatrix} x^{k+1} - x^* \\ x^k - x^* \end{bmatrix} = \begin{bmatrix} I - \alpha Q + \beta & -\beta \\ I & 0 \end{bmatrix} \begin{bmatrix} x^k - x^* \\ x^{k-1} - x^* \end{bmatrix} \therefore$$

Problem 2

Prove

$T = \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix}$, T_1 depends on λ_1 of Q
T_2 depends on λ_2 of Q

$$T_i = \begin{bmatrix} 1 - \alpha\lambda_i + \beta & -\beta \\ 1 & 0 \end{bmatrix}, \quad i = 1,2$$

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Proving $U^T Q U = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$,

$$\begin{bmatrix} U^T & 0 \\ 0 & U^T \end{bmatrix} T \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} I\alpha\overset{Q}{\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}} + \beta I & -\beta I \\ I & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c|c} 1-\alpha\lambda_1+\beta & 0 & -\beta & 0 \\ 0 & 1-\alpha\lambda_2+\beta & 0 & -\beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{c|c|c|c} 1-\alpha\lambda_1+\beta & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1-\alpha\lambda_2+\beta & -\beta \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$= \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix},$$

$$T_i = \begin{bmatrix} 1-\alpha\lambda_i+\beta & -\beta \\ 1 & 0 \end{bmatrix}$$

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Problem 3

Find $\mu_{i,1}$ $\mu_{i,2}$ of T_i as function of λ_i, α, β

Finding $\det(rI - T_i) = \det\begin{bmatrix} (r-(1-\alpha\lambda_i+\beta) & -(-\beta) \\ -(1) & r \end{bmatrix}$

$$= r(r-(1-\alpha\lambda_i+\beta) + (-(1))(-(-\beta))$$

$$= \overset{(a)}{r^2} - \overset{(b)}{r(1-\alpha\lambda_i+\beta)} + \overset{(c)}{\beta} = 0,$$

$$\mu_{i\frac{1}{2}} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-(1-\alpha\lambda_i+\beta)) \pm \sqrt{(1-\alpha\lambda_i+\beta)^2-4(1)(\beta)}}{2(1)}$$

$$= \frac{(1-\alpha\lambda_i+\beta) \pm \sqrt{(1-\alpha\lambda_i+\beta)^2-4\beta}}{2} \quad \therefore$$

Problem 4

Prove $(1-\alpha\lambda_i+\beta)^2 - 4\beta < 0$

Proving $(1-\alpha\lambda_i+\beta)^2 < 4\beta$

$$\Leftrightarrow -2\sqrt{\beta} < 1-\alpha\lambda_i+\beta < 2\sqrt{\beta}$$

$$\alpha = \frac{4}{(\sqrt{L}+\sqrt{m})^2}, \quad \beta = \frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}} = \frac{\sqrt{L-m}}{\sqrt{L}+\sqrt{m}},$$

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$$1-\alpha\lambda_i+\beta = 1-\left(\frac{4}{(\sqrt{L}+\sqrt{m})^2}\right)\lambda_i + \frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}}$$

$$= \frac{(\sqrt{L}+\sqrt{m})^2+(\sqrt{L}+\sqrt{m})(\sqrt{L}-\sqrt{m})-4\lambda_i}{(\sqrt{L}+\sqrt{m})^2}$$

$$= \frac{L+m+\sqrt{Lm}+L-m-4\lambda_i}{(\sqrt{L}+\sqrt{m})^2}$$

$$= \frac{2L+\sqrt{Lm}-4\lambda_i}{(\sqrt{L}+\sqrt{m})^2}$$

need to prove $$-2\left(\frac{\sqrt{L-m}}{\sqrt{L}+\sqrt{m}}\right) < \frac{2L+\sqrt{Lm}-4\lambda_i}{(\sqrt{L}+\sqrt{m})^2} < 2\left(\frac{\sqrt{L-m}}{\sqrt{L}+\sqrt{m}}\right)$$

$$\iff -\cancel{2}\sqrt{L-m}(\sqrt{L}+\sqrt{m}) < \cancel{2}L+\sqrt{Lm}-\overset{2}{\cancel{4}}\lambda_i < \cancel{2}\sqrt{L-m}(\sqrt{L}+\sqrt{m})$$

$$\iff (-L-\sqrt{Lm}-\sqrt{L-m})(\sqrt{L}+\sqrt{m}) < -2\lambda_i < (-L-\sqrt{Lm}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

$$\iff (L+\sqrt{Lm}-\sqrt{L-m})(\sqrt{L}+\sqrt{m}) < 2\lambda_i < (L+\sqrt{Lm}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

$$\iff (\sqrt{L(L+m)}-\sqrt{L-m})(\sqrt{L}+\sqrt{m}) < 2\lambda_i < (\sqrt{L(L+m)}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

$$\iff (\sqrt{L}-\sqrt{L-m})(\sqrt{L}+\sqrt{m}) < 2\lambda_i < (\sqrt{L}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

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$Z_1 = m$

$L \geq m$

$\frac{a^2-b^2}{a+b} = a-b$

$$(\sqrt{L}-\sqrt{L-m})(\sqrt{L}+\sqrt{m}) < 2m < (\sqrt{L}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

$$\left(\frac{(\cancel{L})-(\cancel{L}-m)}{\sqrt{L}+\sqrt{L+m}}\right)(\sqrt{L}+\sqrt{m}) < 2m < (\sqrt{L}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

$$\frac{m}{\sqrt{L}+\sqrt{L-m}}(\sqrt{L}+\sqrt{m}) < 2m < (\sqrt{L}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

(L) $$\frac{\cancel{m}}{\sqrt{L}+\sqrt{L-m}}(\sqrt{L}+\sqrt{m}) < 2\cancel{m}$$

$$\underbrace{\sqrt{L}+\sqrt{m}}_{\sqrt{L}>\sqrt{m}} < 2(\sqrt{L}+\sqrt{L-m})$$

$$\sqrt{m} < \sqrt{L} + 2\overbrace{\sqrt{L-m}}^{>0};$$

(R) $$2m < (\sqrt{L}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

$$2m < \left(\frac{\cancel{L}-(\cancel{L}-m)}{\sqrt{L}-\sqrt{L-m}}\right)(\sqrt{L}+\sqrt{m}) \qquad \frac{a^2-b^2}{a-b} = a+b$$

$$2\cancel{m} < \frac{\cancel{m}}{\sqrt{L}-\sqrt{L-m}}(\sqrt{L}+\sqrt{m})$$

$$2 < \frac{\sqrt{L}+\sqrt{m}}{\sqrt{L}-\sqrt{L-m}}, \quad \left(\cdot\frac{1}{\sqrt{m}}\right) \left(\cdot\frac{1}{\sqrt{m}}\right)$$

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$$2 < \frac{\sqrt{\frac{L}{m}}+1}{\sqrt{\frac{L}{m}}-\sqrt{\frac{L}{m}-1}} = \frac{\sqrt{k}+1}{\sqrt{k}-\sqrt{k-1}} \quad \left(k=\frac{L}{m}>1\right)$$

$$2\sqrt{k} - 2\sqrt{k-1} < \sqrt{k}+1$$

$$\sqrt{k} - 2\sqrt{k-1} < 1$$

$$\sqrt{k} - \sqrt{k-1} < 1 + \sqrt{k-1};$$ don't get this, just trust it

$$1 < \underbrace{(1+\sqrt{k-1})}_{>1}\underbrace{(\sqrt{k}-\sqrt{k-1})}_{\geq 1};$$

$\lambda_2 = L$ $$(\sqrt{L}-\sqrt{L-m})(\sqrt{L}+\sqrt{m}) < 2L < (\sqrt{L}+\sqrt{L-m})(\sqrt{L}+\sqrt{m})$$

$$(L)\ (\sqrt{L}-\sqrt{L-m})(\sqrt{L}+\sqrt{m}) < 2L$$

$$L + \sqrt{mL} - \sqrt{L^2-mL} < 2L,$$

$$L > \sqrt{mL} - \sqrt{L^2-mL}$$

(I give up, since this part is optional)

$$L > \frac{mL-(L^2-mL)}{\sqrt{mL}+\sqrt{L^2-mL}} = \frac{L(2m-L)}{\sqrt{mL}+\sqrt{L^2-mL}}$$

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Problem 5

Prove $\rho(T) \equiv \max_{i=1,2} (|\mu_{i,1}|, |\mu_{i,2}|) = \sqrt{\beta} \approx 1 - k^{-1/2}$

$|\mu_{i,1}|, |\mu_{i,2}| = \sqrt{\beta}, \ i = 1,2$

where $k = \dfrac{L}{m} >> 1$

from problem 3

Proving $\mu_{i\,1/2} = \dfrac{(1-\alpha\lambda_i\beta) \pm \sqrt{\overbrace{(1-\alpha\lambda_i+\beta)^2 - 4\beta}^{<0}}}{2}$

$$= \frac{(1-\alpha\lambda_i\beta) \pm i\sqrt{4\beta - (1-\alpha\lambda_i\beta)}}{2}$$

$$|\mu_{i\,1/2}|^2 = \frac{\cancel{(1-\alpha\lambda_i+\beta)^2} \mp (4\beta - \cancel{(1-\alpha\lambda_i\beta)^2})}{4}$$

$$= \frac{4\beta}{4} = \beta$$

$$|\mu_{i\,1/2}| = \sqrt{\beta} = \sqrt{\frac{(\sqrt{L}-\sqrt{m})(\cdot\frac{1}{\sqrt{m}})}{(\sqrt{L}+\sqrt{m})(\cdot\frac{1}{\sqrt{m}})}} = \sqrt{\frac{\sqrt{\frac{L}{m}}-1}{\sqrt{\frac{L}{m}}+1}}$$

$$= \sqrt{\frac{\sqrt{k}-1 \ \cdot(\frac{1}{\sqrt{k}})}{\sqrt{k}+1 \ \cdot(\frac{1}{\sqrt{k}})}} = \sqrt{\frac{1-1/\sqrt{k}}{1+1/\sqrt{k}}} = \frac{\sqrt{1-\frac{1}{\sqrt{k}}}}{\sqrt{1+1/\sqrt{k}}}$$

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$$= \left(1 - \frac{1}{\sqrt{k}}\right)^{1/2}\left(1 + \frac{1}{\sqrt{k}}\right)^{-1/2}$$

$$\approx \left(1 - \frac{1}{2\sqrt{k}}\right)\left(1 + \frac{1}{2\sqrt{k}}\right) \Leftarrow$$ don't get this just trust it

$$= 1 - \frac{1}{\sqrt{k}} + \frac{1^2}{4k} \approx 1 - \frac{1}{\sqrt{k}}$$

$$\left((\sqrt{k}-1)/(\sqrt{k}+1) = 1 - \frac{1}{\sqrt{k}}\right) = 1 - k^{-1/2} \therefore$$

Missing Problem 2