4/2/2023 (1) Adam Cruz

Assignment 3 Final Draft

1)

— Given: $f(x) = \frac{1}{2} x^T Q x - \vec{b}^T x + C$

$Q \succ 0, \quad 0 < m = \lambda_n \leq \lambda_{n-1} \leq \lambda_1 = L < \infty$

$\begin{pmatrix} Q > 0 \\ Q^T = Q \end{pmatrix},$

— (a)

- prove $\nabla f = Qx - \vec{b}$
- $\vec{b} = 0, \; c = \vec{0}, \quad Q = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}, \quad \underset{L}{\underset{\parallel}{\lambda_{11}}} > \underset{m}{\underset{\parallel}{\lambda_{22}}}$

$f(x_1, x_2) = \frac{1}{2} (x_1, x_2) \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= \frac{1}{2} (\lambda_{11} x_1^2 \; \lambda_{22} \cdot x_2^2)$

$\nabla^2 f = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix} = Q$

(2)

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$$\nabla f(x) = \nabla\left(\frac{1}{2}x^TQx - b^Tx + C\right)$$

$$= \nabla\left(\frac{1}{2}x^TQx\right) - b + 0$$

$$= Qx - b$$

needs more proving

(b)

Given

$$\nabla f(x) = Qx - \vec{b}$$

$$f(x) = \frac{1}{2}x^TQx - b^Tx + C$$

Prove: ~~prove~~ $Qx^* - b = 0$ s.t. $\min_x f(x) = f(x^*)$

$$x^{k+1} = x^k - \alpha \nabla f(x^k + \beta(x^k - x^{k-1})) + \beta(x^k - x^{k-1})$$

$$x^{-1} = x^0$$

$$k = \frac{L}{m}$$

~~if $Qx^* - b = 0$, $x^* = Q^{-1}b$~~

~~$\nabla f(x) = Qx - b = Q(x - x^*)$~~

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1)
(2 cont)

$\nabla f(x)$ defined as $\langle \nabla f(x), l \rangle = \lim\limits_{\varepsilon \to 0} \dfrac{f(x+\varepsilon \vec{l}) - f(x)}{\varepsilon}$

$\cancel{\langle \nabla f(x), l \rangle = \frac{1}{2}(x+\varepsilon\vec{l})^T Q (x+\varepsilon\vec{l}) - b^T(x+\varepsilon\vec{l}) + l}$

$\langle \nabla f(x), \vec{l} \rangle$

$$= \lim_{\varepsilon \to 0} \frac{\frac{1}{2}(x+\varepsilon\vec{l})^T Qx(x+\varepsilon\vec{l}) - b^T(x+\varepsilon\vec{l}) + l - \frac{1}{2}x^TQx + b^Tx - c}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\frac{1}{2}(x+\varepsilon\vec{l})^T Q (x+\varepsilon\vec{l}) - \frac{1}{2}x^TQx}{\varepsilon} - \langle b, \vec{l} \rangle$$

$$= \lim_{\varepsilon \to 0} \frac{\left(\frac{1}{2}x^TQx + \frac{1}{2}\varepsilon\vec{l}^TQx + \frac{1}{2}\varepsilon x^TQ\vec{l} + \frac{1}{2}\varepsilon^2\vec{l}^TQ\vec{l} - \frac{1}{2}x^TQx\right)}{\varepsilon}$$

$$- \langle b, \vec{l} \rangle$$

(4)

$$= \frac{1}{2}\langle Qx, \vec{l}\rangle + \frac{1}{2}\vec{l}^{\,T}Qx - \langle b, \vec{l}\rangle$$

$$= \frac{1}{2}\langle Qx, \vec{l}\rangle + \frac{1}{2}\langle Qx, \vec{l}\rangle - \langle b, \vec{l}\rangle$$

$$= \langle Qx - \vec{b}, l\rangle = \langle \nabla f(x), \vec{l}\rangle$$

$$\therefore \quad Qx^* - b = 0 \text{ and } \min_x f(x) = f(x^*)$$

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2)

— Given

$$T = \begin{bmatrix} (1+\beta)(I-\alpha Q) & -\beta(I-\alpha Q) \\ I & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$U^T Q U = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \Lambda$$

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_i = \begin{bmatrix} (1+\beta)(1-\alpha\lambda_i) & -\beta(1-\alpha\lambda_i) \\ 1 & 0 \end{bmatrix}, \quad i = 1,2$$

— Prove $\Pi \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}^T T \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} = \operatorname{diag}(T_1, T_2)$

(6)

$$-\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}^T T \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} = I^2 U^T T U$$

$$= U^T T U = \begin{bmatrix} (1+\beta)(1-\alpha\Lambda) & -\beta(I-\alpha\Lambda) \\ I & 0 \end{bmatrix}$$

$$\equiv R$$

$$\Pi R \Pi^T = \left[\begin{array}{c|c|c|c} (1+\beta)(1-\alpha\lambda_1) & 0 & -\beta(1-\alpha\lambda_1) & 0 \\ 0 & (1+\beta)(1-\alpha\lambda_2) & 0 & -\beta(1-\alpha\lambda_2) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

(switch cols 2&3)

$$= \left[\begin{array}{c|c|c|c} (1+\beta)(1-\alpha\lambda_1) & -\beta(1-\alpha\lambda_1) & 0 & 0 \\ 0 & 0 & (1+\beta)(1-\alpha\lambda_2) & -\beta(1-\alpha\lambda_2) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(switch rows 2&3)

$$= \left[\begin{array}{c|c|c|c} (1+\beta)(1-\alpha\lambda_1) & -\beta(1-\alpha\lambda_1) & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & (1+\beta)(1-\alpha\lambda_2) & -\beta(1-\alpha\lambda_2) \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$= \mathrm{diag}(T_1, T_2) \therefore$$

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3)

Given $\rho_k \geq 0, \rho_0 = 0, \rho_k^2 = \dfrac{(1-\rho_k^2)^2}{(1-\rho_{k-1}^2)^2}$

Prove ~~Prove~~ $1-\rho_k^2 \leq \dfrac{2}{k+2}$

Proving $k=0, 1-\rho_0^2 = 1-0 \leq \dfrac{2}{0+2} = 1$ (base)

$1-\rho_k^2 \leq \dfrac{2}{k+2}$, show $1-\rho_{k+1}^2 \leq \dfrac{2}{k+3}$ (maintenance)

Show by contrapositive (Suppose $1-\rho_{k+1}^2 > \frac{2}{k+3}$)

$1-\rho_{k+1}^2 > \dfrac{2}{k+3}; \rho_{k+1}^2 < 1 - \dfrac{2}{k+3} = \dfrac{k+1}{k+3} \Longleftrightarrow \dfrac{1}{\rho_{k+1}^2} > \dfrac{k+3}{k+1}$

Iterative Relation

$\rho_{k+1}^2 = \dfrac{(1-\rho_{k+1}^2)^2}{(1-\rho_k^2)^2}$

Square both sides

$\underbrace{\dfrac{4}{(k+2)^2} \geq (1-\rho_k^2)^2}_{\text{(substitute)}} = \dfrac{(1-\rho_{k+1}^2)^2}{\rho_{k+1}^2} = (1-\rho_{k+1}^2)^2 \cdot \left(\dfrac{1}{\rho_{k+1}^2}\right)^2$

$> \left(\dfrac{2}{k+3}\right)^2 \left(\dfrac{k+3}{k+1}\right) = \dfrac{4(k+3)}{(k+1)(k+3)^2}$

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$$= \frac{4}{(k+1)(k+3)},$$

$$\frac{4}{(k+2)^2} > \frac{4}{(k+1)(k+3)}, \quad (k+1)(k+3) > (k+2)^2,$$

$$k^2 + 4k + 3 > k^2 + 4k + 4, \quad 0 > 1$$

$$\Rightarrow 1 - \rho_{k+1}^2 \not> \frac{2}{k+3}$$