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Due Tuesday August 8, 2018

Math 314 Homework 3

Name: ____________________

Written Problems

Instructions. (319 points) After reading the sections and completing the MyMathLab assignments, write out the solutions to the following problems (or if you prefer, you can use LaTex to type your solutions). This assignment covers sections 5.1-6.8 of the text. Work should be shown to support your answers and to receive partial credit. Please review the written assignment requirements, and ask questions about problems before submitting your work.

(3^{pts}) **1.** Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$? Why or why not?

(3^{pts}) **2.** Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? Why or why not?

(8^{pts}) **3.** Find a basis for the eigenspace corresponding to the given eigenvalue.

(a) (3 pts) $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$ with $\lambda = 7$

(b) (5 pts) $A = \begin{bmatrix} 5 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & -2 & 4 \end{bmatrix}$ with $\lambda = 4$

(4^{pts}) **4.** Find the eigenvalues of the matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$.

(3^{pts}) **5.** Construct an example of a 2×2 matrix with only one distinct eigenvalue.

(16^{pts}) **6.** Find the characteristic polynomial of the given matrix and list its real eigenvalues.

(a) (3 pts) $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$

(b) (3 pts) $\begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$

(c) (5 pts) $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$

(d) (5 pts) $\begin{bmatrix} 4 & 0 & -1 \\ -1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix}$

(4^{pts}) **7.** List the real eigenvalues for the matrix $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 3 & 3 & -5 \end{bmatrix}$. Include any multiplicities.

(4pts) **8.** Let $A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ Find h such that the eigenspace for $\lambda = 4$ is two-dimensional.

(6pts) **9.** Given $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ Let $A = PDP^{-1}$, use this information to find A^4.

(6pts) **10.** Given $\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ Use the factorization $A = PDP^{-1}$ to find A^{4k}, where k is an arbitrary positive integer.

(6pts) **11.** Given $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix}$ Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

(9pts) **12.** Diagonalize each matrix, if possible.

(a) (3 pts) $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$

(b) (6 pts) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{bmatrix}$ *(you will need to find the real eigenvalues first).*

(3pts) **13.** A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why or why not?

(18pts) **14.** Let $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$

(a) (2 pts) Find $\mathbf{w}\cdot\mathbf{w}$

(b) (2 pts) Find $\mathbf{x}\cdot\mathbf{w}$

(c) (3 pts) Find $\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$

(d) (3 pts) Find $\dfrac{1}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$

(e) (4 pts) Find $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$

(f) (4 pts) Find $\left(\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x}$

(6pts) **15.** Find a unit vector in the direction of the given vector.

(a) (3 pts) $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

(b) (3 pts) $\begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix}$

(3pts) **16.** Find the distance between $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $\mathbf{z} \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

(3pts) **17.** Determine if $\mathbf{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{z} \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$ are orthogonal vectors.

(13^{pts}) **18.** Let W be a subspace of $\mathbb{R}^n$, and let $W^\perp$ be the set of all vectors orthogonal to W.

(a) (4 pts) Take $\mathbf{z} \in W^\perp$, and let $\mathbf{u}$ represent any element of W. Then $\mathbf{z}\cdot\mathbf{u} = 0$. Take any scalar c and show that $c\mathbf{z}$ is orthogonal to $\mathbf{u}$.

(b) (4 pts) Take $\mathbf{z}_1, \mathbf{z}_2 \in W^\perp$, and let $\mathbf{u}$ represent any element of W. Show that $\mathbf{z}_1 + \mathbf{z}_2$ is orthogonal to $\mathbf{u}$.

(c) (2 pts) What can you conclude about $\mathbf{z}_1 + \mathbf{z}_2$?

(d) (3 pts) Prove that $W^\perp$ is a subspace of $\mathbb{R}^n$.

(4^{pts}) **19.** Determine if the set of vectors $\left\{ \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \right\}$ are orthogonal.

(11^{pts}) **20.** Given the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

(a) (3 pts) Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$.

(b) (5 pts) Determine if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

(c) (3 pts) Express $\mathbf{x}$ as a linear combination fo the $\mathbf{u}'s$.

(4^{pts}) **21.** Compute the orthogonal projection of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto the line through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin.

(4^{pts}) **22.** Let $\mathbf{a} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{a}$ as the sum of a vector in $Span\,\mathbf{b}$ and a vector orthogonal to $\mathbf{b}$.

(4^{pts}) **23.** Determine if the set of vectors $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$ are orthonormal. If the set is only orthogonal, normalize the vectors to produce an orthonormal set.

(5^{pts}) **24.** Let $\{\mathbf{u}_1, \dots \mathbf{u}_4\}$ be an orthogonal basis for $\mathbb{R}^4$. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$. Write $\mathbf{v}$ as the sum of two vectors, one in $Span\{\mathbf{u}_1\}$ and the other in $Span\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

(7^{pts}) **25.** Given $\mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$.

(a) (4 pts) Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set.

(b) (3 pts) Find the orthogonal projection of $\mathbf{y}$ onto $Span\{\mathbf{u}_1, \mathbf{u}_2\}$.

(4^{pts}) **26.** Let $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$ where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$. If $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$, write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

(4^{pts}) **27.** Let $\mathbf{z} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$. Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{u} + c_2\mathbf{v}$.

(4^{pts}) **28.** The set $\left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} \right\}$ is a basis for a subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W.

(4^{pts}) **29.** Find an orthogonal basis for the column space of $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

(6^{pts}) **30.** Let $A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & -6 \end{bmatrix}$, $Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$. Suppose that Q is obtained by applying the Gram-Schmidt process to the columns of A. Find an upper triangular matrix R such that $A = QR$.

(4^{pts}) **31.** Suppose $A = QR$, where R is an invertible matrix. Show that A and Q have the same column space.

(8^{pts}) **32.** Let p and q in $\mathcal{P}_2$, define $< p, q >= p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

(a) (4 pts) Compute $< p, q >$ where $p(t) = 3t - t^2, q(t) = 3 + 2t^2$.

(b) (4 pts) Compute the orthogonal projection of q onto the subspace spanned by p.

(4^{pts}) **33.** Let $\mathcal{P}_3$ have the inner product given by evaluation at -3, -1, 1, and 3. Let $p_0(t) = 1, p_1(t) = t$, and $q(t) = \frac{1}{4}t^2 - \frac{5}{4}$. Find the best approximation to $p(t) = t^3$ by polynomials in $Span\{p_0, p_1, q\}$

(3^{pts}) **34.** Use the inner product axioms to verify $\| \mathbf{u} + \mathbf{v} \|^2 + \| \mathbf{u} - \mathbf{v} \|^2 = 2 \| \mathbf{u} \|^2 + 2 \| \mathbf{v} \|^2$.

(9^{pts}) **35.** Let $V = C[0, 1]$ have the inner product given by the integral. Let $f(x) = 5x - 3$, and $g(x) = x^3 - x^2$.

(a) (3 pts) Compute $< f, g >$.

(b) (3 pts) Compute $\| f \|$.

(c) (3 pts) Compute $\| g \|$.

(4^{pts}) **36.** Suppose 5 out of 25 data points in a weighted least-squares problem have a y-measurement that is less reliable than the others, and they are to be weighted half as much as the other 20 points. One method is to weight the 20 points by a factor of 1 and the other 5 by a factor of $\frac{1}{2}$. A second method is to weight the 20 points by a factor of 2 and the other 5 by a factor of 1. Do the two methods produce different results? Explain.

(8^{pts}) **37.** To make a trend analysis of six evenly spaced data points, one can use orthogonal polynomials with respect to evaluation at the points $t = -5, -3, -1, 1, 3$, and 5.

(a) (5 pts) Show that the first three orthogonal polynomials are $p_0(t) = 1, p_1(t) = t$, and $p_2(t) = \frac{3}{8}t^2 - \frac{35}{8}$.

(b) (3 pts) Fit a quadratic trend function to the data $(-5, 1), (-3, 1), (-1, 4), (1, 4), (3, 6), (5, 8)$.

(4^{pts}) **38.** Let $V = C[0, 2\pi]$ have the inner product given by the integral. Show that $\sin(mt)$ and $\sin(nt)$ are orthogonal for all positive integers m and n.

MatLab Problems

For this part of the assignment, you will need to use Matlab to complete these problems. This assignment covers sections 5.1-6.8 of the text. Work should be shown to support your answers and to receive partial credit. You should attach the printout of your solutions to the written assignment and copies of your MatLab files should be uploaded into Canvas.

(4^{pts}) **39.** Let $A = \begin{bmatrix} 5 & -2 & 2 & -4 \\ 7 & -4 & 2 & -4 \\ 4 & -4 & 2 & 0 \\ 3 & -1 & 1 & -3 \end{bmatrix}$. Find the eigenvalues of the matrix, and use the eigenvalues to produce a basis for each eigenspace.

(6^{pts}) **40.** Let $B = \begin{bmatrix} -23 & 57 & -9 & -15 & -59 \\ -10 & 12 & -10 & 2 & -22 \\ 11 & 5 & -3 & -19 & -15 \\ -27 & 31 & -27 & 25 & -37 \\ -5 & -15 & -5 & 1 & 31 \end{bmatrix}$. Find the eigenvalues of the matrix, and use the eigenvalues to produce a basis for each eigenspace.

(8^{pts}) **41.** Construct a random integer-valued 4×4 matrix C.

(a) (2 pts) Do C and C^T have the same characteristic polynomial?

(b) (2 pts) Do C and C^T have the same eigenvectors.

(c) (4 pts) Perform the same analysis on D a 5×5 matrix. Report your conclusions.

(8^{pts}) **42.** Let $A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & c & 25 \end{bmatrix}$. For each value of c in the set $\{32, 31.9, 31.8, 32.1, 32.2\}$, compute the characteristic polynomial of A and the eigenvalues. In each case, create a graph of the characteristic polynomial $p(t) = det(A - tI)$for $0 \leq t \leq 3$. Describe how the graphs reveal the changes in the eigenvalues as c changes.

(8^{pts}) **43.** Diagonalize the matrix $\begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$. Use the eigenvalue command to find the eigenvalues and compute bases for the eigenspaces.

(20^{pts}) **44.** Construct a pair of random vectors $\mathbf{u}, \& \mathbf{v} \in \mathbb{R}^4$. Let $A = \begin{bmatrix} .5 & .5 & .5 & .5 \\ .5 & .5 & -.5 & -.5 \\ .5 & -.5 & .5 & -.5 \\ .5 & -.5 & -.5 & .5 \end{bmatrix}$

(a) (4 pts) Denote the columns of A by $a_1, \ldots, a_4$. Compute the length of each column.

(b) (6 pts) Compute $a_1 \cdot a_2$, $a_1 \cdot a_3$, $a_1 \cdot a_4$, $a_2 \cdot a_3$, $a_2 \cdot a_4$,and $a_3 \cdot a_4$.

(c) (4 pts) Compute and compare the lengths of $\mathbf{u}, \mathbf{v}, A\mathbf{u}$, and $A\mathbf{v}$.

(d) (4 pts) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$. Compare this to the cosine of the angle between $A\mathbf{u}$, and $A\mathbf{v}$.

(e) (2 pts) Repeat this for another pair of vectors and form a conjecture about the effect of A on vectors.

(17pts) **45.** Let $A = \begin{bmatrix} -6 & -3 & 6 & 1 \\ -1 & 2 & 1 & -6 \\ 3 & 6 & 3 & -2 \\ 6 & -3 & 6 & -1 \\ 2 & -1 & 2 & 3 \\ -3 & 6 & 3 & 2 \\ -2 & =1 & 2 & -3 \\ 1 & 2 & 1 & 6 \end{bmatrix}$

(a) (4 pts) Find the matrix U found by normalizing each column of A.

(b) (6 pts) Compute U^TU and UU^T. How do these differ?

(c) (3 pts) Generate a random vector in $\mathbf{y} \in \mathbb{R}^8$ and compute $\mathbf{p} = UU^T\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \mathbf{p}$. Explain why $\mathbf{p}$ is in Col$A$.

(d) (4 pts) Find the closest point to $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in Col$U$. Write the commands used to solve this problem.

(6pts) **46.** Given $A = \begin{bmatrix} -10 & 13 & 7 & -11 \\ 2 & 1 & -5 & 3 \\ -6 & 3 & 13 & -3 \\ 16 & -16 & -2 & 5 \\ 2 & 1 & -5 & -7 \end{bmatrix}$.

(a) (3 pts) Find an orthogonal basis for the column space of A.

(b) (3 pts) Produce a QR factorization of matrix A.

(4pts) **47.** Let $\mathcal{P}_4$ given by evaluation at -2, -1, 0, 1, and 2. Let $p_0(t) = 1, p_1(t) = t$, and $p_2(t) = t^2$.Apply the Gram-Schmidt process to the set$\{p_0, p_1, p_2, t^3, t^4\}$ to create an orthogonal basis for $\mathcal{P}_4$.

(5pts) **48.** Let $V = C[0, 2\pi]$ have the inner product given by the integral. Use the Gram-Schmidt process to create an orthogonal basis for the subspace spanned by $\{1, \cos t, \cos^2 t, \cos^3 t\}$

(10pts) **49.** Let f_4 and f_5 be the fourth-order and fifth-order Fourier approximations in $C[0, 2\pi]$ to the square wave function: $f(t) = 1$ for $0 \le t \le \pi$ and $f(t) = -1$ for $\pi \le t \le 2\pi$.

(a) (3 pts) Produce the graphs of f_4 on the interval $[0, 2\pi]$.

(b) (3 pts) Produce the graphs of f_5 on the interval $[0, 2\pi]$.

(c) (4 pts) Produce the graphs of f_5 on the interval $[-2\pi, 2\pi]$.