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Optimal Sequencing Policies for Recovery of Physical Infrastructure After Disasters

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Abstract— **In this paper, we consider a disaster scenario where multiple physical infrastructure components suffer damage. After the disaster, the health of these components continue to deteriorate over time, unless they are being repaired. Given this setting, we consider the problem of finding the optimal sequence to repair the different infrastructure components in order to maximize the number of components that are eventually returned to full health. We show that the optimal sequence depends on the relationship between the rate of improvement (when being repaired) and the rate of deterioration (when not being repaired). We explicitly characterize the optimal repair policy as a function of the health states of the different components under certain conditions on the rates of improvement and deterioration.**

I. Introduction

A disaster is a non-routine event that has the potential for catastrophic impacts on physical, natural and social systems. For instance, Hurricane Sandy caused 147 direct casualties along its path and brought damage in excess of $50 billion for the United States [1]. The recovery of infrastructure after a disaster is a massive task which, in turn, also affects the return patterns of displaced communities. It is therefore imperative for emergency-response agencies to determine an optimal sequence to repair the damaged components of physical infrastructure in order to maximize recovery.

There are various studies that involve sequencing of recovery decisions in post-disaster scenarios. Some studies develop network flow-based models for physical infrastructure networks where maximizing flow is considered equivalent to maximizing recovery [2]. Studies in operations research have focused on finding an optimal sequence of targeting different roads in order to remove debris after disasters [3]. However, these studies mainly focus on recovery of individual elements belonging to a certain type of physical infrastructure (e.g., roads in a transportation network) rather than focusing on sequencing decisions for repairing portions of physical infrastructure with varying damage/health levels. Furthermore, these studies do not model degradation of physical infrastructure over time if a component is not being repaired. Such deterioration is commonly observed in real-world post-disaster scenarios; for example road infrastructure deteriorates over time in flooded areas after hurricanes [4].

Our paper considers the problem of finding an optimal sequence (or control policy) that a recovery agency/entity should follow in order to maximize the number of physical infrastructure components that are returned to full health. At a high level, problems of a similar flavor can be found in optimal control of robotic systems [5] that persistently monitor changing environments; there, the goal is to keep the level of uncertainty about some dynamic phenomena below a certain threshold, with the uncertainty growing over time whenever the phenomenon is not being observed. Our problem also has similarities to the problem of allocating resources (e.g., time slots) at a base station to many time-varying competing flows/queues [6]. However, these studies do not consider permanent failure of components or flows being serviced, and instead focus on either bounding the long-term state of the system, or maximizing long-term throughput or stability. Job scheduling problems with degrading processing times [7] as a function of job starting times also have analogies to our problem. Most of these studies either do not consider preemptive schedules (i.e., where one can jump from one job to another), or characterize optimal policies with equal and predefined deadlines for all jobs. In contrast, the deadline (or the time of permanent failure) of an infrastructure component in our problem is a function of its health and the chosen repair sequence. Scheduling analysis of real-time systems [8] also has similarities to our problem but such studies typically focus on real-time tasks that are released for processing at different times; in contrast, all the infrastructure components in our problem are available for repair starting at the same time.

Control problems for switched systems [9], [10] also have similarities to our problem. These studies characterize scheduling control policies so that states (e.g., temperature) of the components (e.g., room) in the system always stay in a given interval. The paper [9] characterizes a lazy scheduling policy where switching decisions are only made when the state of a component reaches either of the interval thresholds. This policy is not optimal as states of some components may violate thresholds if the states of too many components simultaneously reach a threshold. The paper [10] characterizes an optimal policy in which the components that are closest to the threshold values are switched, given some constraints on the number of components that can be switched at a time. In these studies, the system becomes unstable (equivalent to the notion of permanent failure in our problem) if the state of a component violates any of the two interval thresholds. In contrast, our problem has one desirable threshold and one undesirable/failure threshold. The desirable and undesirable thresholds represent the health value of an infrastructure component when it is permanently repaired and permanently

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failed, respectively. This model is motivated by the fact after disasters, infrastructure components like roads, water and sewer pipes, etc., deteriorate rapidly in comparison to the deterioration faced during normal times. Thus, it can be assumed that the health of a component does not significantly vary due to normal degradation processes once it is repaired after disasters and thus can be considered to be *permanently repaired*. Also, infrastructure components like pavement sections can deteriorate to such a level after disasters that they become unusable and require full replacement, which is expensive and thus undesirable [11]. At that point, components can be considered to be *permanently failed*. So, the objective of our problem is to maximize the number of components that can be repaired to the desirable threshold value (or permanent recovery) without ever reaching the undesirable threshold (permanent failure). This difference leads us to characterize optimal policies of different types depending on the problem conditions. We show that non-jumping policies, where jumping between different components is not allowed, turn out to be optimal under some conditions. In contrast, the aforementioned switched system studies do not characterize non-jumping policies to be optimal; indeed, jumping is necessary to meet the objectives of those problems.

In the next section, we formally present the problem that we consider in this paper. Then, in the rest of paper, we characterize the optimal control policies for this problem.

II. Problem Statement

There are $N(\geq 2)$ nodes indexed by the set $\mathcal{V} = \{1, 2, \ldots, N\}$, each representing a component of physical infrastructure. For instance, a node could represent a portion of the power grid, the road network in a given area, the communication infrastructure in a region, a group of buildings, etc. There is a recovery agency (also referred to as an *entity*) whose objective is to repair these components after a disaster. We assume that time progresses in discrete time steps, capturing the resolution at which the entity makes decisions about which node to repair. We index the time steps with the variable $t \in \mathbb{N}$. The *health* of each node $j \in \mathcal{V}$ at time step t is denoted by $v_t^j \in [0, 1]$. The initial health of each node j is denoted by $v_0^j \in (0, 1)$. The aggregate state vector for the entire system at each time step $t \in \mathbb{N}$ is given by $v_t = \begin{bmatrix} v_t^1 & v_t^2 & \ldots & v_t^N \end{bmatrix}'$.

Definition 1: We say that node j **permanently fails** at time step t if $v_t^j = 0$ and $v_{t-1}^j > 0$. We say that node j **permanently recovers** (or is fixed) at time step t if $v_t^j = 1$ and $v_{t-1}^j < 1$. If a node permanently fails or permanently recovers, then its health does not change thereafter. □

At each time step t, the entity can target exactly one node to repair during that time step. Thus, the control action taken by the entity at time step t is denoted by $u_t \in \mathcal{V}$. If node j is being repaired by the entity at time step t and it has not permanently failed or recovered, its health increases by a quantity $\Delta_{inc} \in [0, 1]$ (up to a maximum health of 1). If node j is not being repaired by the entity at time step t and it has not permanently failed or recovered, its health decreases by a fixed quantity $\Delta_{dec} \in [0, 1]$ (down to a minimum health of 0). For each node j, the dynamics of the (controlled) recovery process are given by

$$v_{t+1}^j = \begin{cases} 1 & \text{if } v_t^j = 1, \\ 0 & \text{if } v_t^j = 0, \\ \min(1, v_t^j + \Delta_{inc}) & \text{if } u_t = j \text{ and } v_t^j \in (0, 1), \\ \max(0, v_t^j - \Delta_{dec}) & \text{if } u_t \neq j \text{ and } v_t^j \in (0, 1). \end{cases} \tag{1}$$

Definition 2: For any given initial state v_0 and control sequence $\bar{u}_{0:\infty} = \{\bar{u}_0, \bar{u}_1, \ldots\}$, we define the **reward** $J(v_0, \bar{u}_{0:\infty})$ as the number of nodes that become permanently repaired under that sequence. Mathematically, $J(v_0, \bar{u}_{0:\infty}) = |\{j \in \mathcal{V} \mid \exists\, t \geq 0 \text{ s.t. } v_t^j = 1\}|$. □

Based on the dynamics (1) and the reward definition given above, we study the following problem in this paper.

Problem 1 (Optimal Recovery Sequencing): Given a set of N nodes with initial health values v_0, along with repair and degradation rates Δ_{inc} and Δ_{dec}, respectively, find a control sequence $u_{0:\infty}^*$ that maximizes the reward $J(v_0, u_{0:\infty}^*)$.

Before presenting our analysis of the problem, we introduce the concept of a **jump**.

Definition 3: The entity is said to have jumped at some time step t if it starts targeting a different node before fully fixing the node it targeted in the last time step. That is, if $u_{t-1} = j$, $v_t^j < 1$ and $u_t \neq j$ then the entity is said to have jumped at time step t. A control sequence that does not contain any jumps is said to be a **non-jumping sequence**. □

We will split our analysis of the optimal control sequence for Problem 1 into two parts: one for the case where $\Delta_{dec} \geq \Delta_{inc}$, and one for the case where $\Delta_{dec} < \Delta_{inc}$.

III. Optimal Sequences for $\Delta_{dec} \geq \Delta_{inc}$

We first show that non-jumping policies are optimal when $\Delta_{dec} \geq \Delta_{inc}$. Then, we characterize the optimal non-jumping policy and thereby find the globally optimal policy.

A. Sequences without any restriction on jumps

First, we analyze properties of sequences containing at most one jump and later generalize to sequences containing an arbitrary number of jumps. We start with the following result.

Lemma 1: Suppose $\Delta_{dec} \geq \Delta_{inc}$. Consider the two control sequences A and B targeting N nodes shown in Figures 1 and 2, respectively. Suppose sequence A fixes all nodes and contains exactly one jump, where the entity partially fixes node i_1 before moving to node i_2 at time step $\bar{t}_1^A$. Sequence A then fully repairs nodes $i_2, i_3, \ldots, i_k$, before returning to node i_1 and fully repairing it. Sequence B is a non-jumping sequence that targets nodes in the order $\{i_2, i_3, \ldots, i_k, i_1, i_{k+1}, \ldots, i_N\}$. Let t_j^A (resp. t_j^B) be the number of time steps taken to fix node i_j in sequence A (resp. sequence B). Then, sequence B also fixes all nodes, and furthermore, the following holds true:

$$t_j^A \geq t_j^B + \left(2^{j-2}\right)\bar{t}_1^A \quad \forall j \in \{2,\ldots,k\}, \tag{2}$$

$$t_1^A \geq t_1^B + (2^{k-1}-2)\bar{t}_1^A, \tag{3}$$

$$t_j^A \geq t_j^B + \left(2^{j-1}-2^{j-k}\right)\bar{t}_1^A \quad \forall j \in \{k+1,\ldots,N\}. \tag{4}$$

□

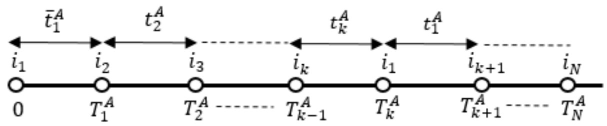

Fig. 1: Sequence A with a single jump.

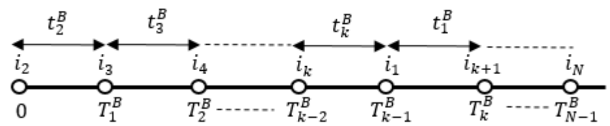

Fig. 2: Non-jumping sequence B.

Proof: Let $T_1^A, T_2^A, \ldots, T_N^A$ be the time steps at which sequence A starts targeting a new node, as shown in Fig. 1. Similarly, let $T_1^B, T_2^B, \ldots, T_{N-1}^B$ be the time steps at which sequence B starts targeting a new node, as shown in Fig. 2.

We start by first proving condition (2). We prove by mathematical induction on the index of nodes in the sequence. Consider $j = 2$. At time step T_1^A in sequence A, the health of node i_2 is given by

$$v_{T_1^A}^{i_2} = v_0^{i_2} - \Delta_{dec}\bar{t}_1^A.$$

We now calculate t_2^A as

$$t_2^A = \left\lceil \frac{1 - v_{T_1^A}^{i_2}}{\Delta_{inc}} \right\rceil = \left\lceil \frac{1 - v_0^{i_2} + \Delta_{dec}\bar{t}_1^A}{\Delta_{inc}} \right\rceil \geq \left\lceil \frac{1 - v_0^{i_2}}{\Delta_{inc}} \right\rceil + \bar{t}_1^A = t_2^B + \bar{t}_1^A.$$

Suppose that condition (2) holds for r nodes where $r < k$. If sequence A fully repairs nodes $i_2, \ldots, i_r$, then so too does sequence B (as each node is reached at an earlier time step in sequence B than in sequence A, by the above inductive assumption). We now compute $v_{T_r^A}^{i_{r+1}}$:

$$v_{T_r^A}^{i_{r+1}} = v_0^{i_{r+1}} - \Delta_{dec}\left(\bar{t}_1^A + t_2^A + \ldots + t_r^A\right).$$

Thus,

$$\begin{aligned} t_{r+1}^A &= \left\lceil \frac{1 - v_0^{i_{r+1}} + \Delta_{dec}\left(\bar{t}_1^A + t_2^A + \ldots + t_r^A\right)}{\Delta_{inc}} \right\rceil \\ &\geq \left\lceil \frac{1 - v_0^{i_{r+1}} + \Delta_{dec}\left(t_2^B + \ldots + t_r^B\right)}{\Delta_{inc}} + \right. \\ &\qquad \left. \frac{\Delta_{dec}\left(\bar{t}_1^A + \bar{t}_1^A + 2\bar{t}_1^A + \ldots + 2^{r-2}\bar{t}_1^A\right)}{\Delta_{inc}} \right\rceil \\ &\geq t_{r+1}^B + 2^{r-1}\bar{t}_1^A. \end{aligned}$$

So, we have shown condition (2) by induction. We now prove condition (3). Node i_1 is targeted again in sequence A at time step T_k^A, at which point its health is

$$v_{T_k^A}^{i_1} = v_0^{i_1} + \bar{t}_1^A\Delta_{inc} - \left(t_2^A + \ldots + t_k^A\right)\Delta_{dec}. \tag{5}$$

Thus, the number of time steps taken to repair node i_1 in sequence A (the second time it is targeted in the sequence) is

$$t_1^A = \left\lceil \frac{1 - v_0^{i_1} - \bar{t}_1^A\Delta_{inc} + \left(t_2^A + \ldots + t_k^A\right)\Delta_{dec}}{\Delta_{inc}} \right\rceil. \tag{6}$$

Note that

$$t_2^A + \ldots + t_k^A \geq t_2^B + \ldots + t_k^B + \left(2^{k-1}-1\right)\bar{t}_1^A, \tag{7}$$

by condition (2). Furthermore, in sequence B, the health of node i_1 at the time it is targeted is given by

$$v_{T_{k-1}^B}^{i_1} = v_0^{i_1} - \Delta_{dec}(t_2^B + t_3^B + \cdots + t_k^B).$$

Comparing this to the health of node i_1 in sequence A at the time it is targeted (given by (5)), and using (7), we see that since i_1 is assumed to not have failed in sequence A, it will not have failed in sequence B as well. Thus, the number of time steps required to repair i_1 in sequence B is given by

$$t_1^B = \left\lceil \frac{1 - v_0^{i_1} + \Delta_{dec}(t_2^B + t_3^B + \cdots + t_k^B)}{\Delta_{inc}} \right\rceil. \tag{8}$$

Thus, using (6), (7) and (8), we have

$$t_1^A \geq t_1^B + \left(2^{k-1}-2\right)\bar{t}_1^A,$$

proving condition (3).

We now prove condition (4) via mathematical induction. Consider node i_{k+1}. At the time step when this node is targeted in sequence A, its health is

$$v_{T_{k+1}^A}^{i_{k+1}} = v_0^{i_{k+1}} - \Delta_{dec}\left(\bar{t}_1^A + t_2^A + \ldots + t_k^A + t_1^A\right).$$

If node i_{k+1} has not failed at this point in sequence A, it has also not failed when it is reached in sequence B (as all nodes prior to i_{k+1} are fixed faster in sequence B than in sequence A, as shown above). Thus, using (2) and (3),

$$t_{k+1}^A \geq t_{k+1}^B + \left(2^k-2\right)\bar{t}_1^A.$$

Suppose condition (4) holds for $j \in \{k+1,\ldots,r\}, r < N$ nodes. Consider node i_{r+1}. Then, a similar inductive argument can be used to show that

$$t^A_{r+1} \geq t^B_{r+1} + \left(2^r - 2^{r+1-k}\right)\bar{t}^A_1.$$

This proves the third claim. ■

The above result considered sequences containing exactly one jump. This leads us to the following key result pertaining to the optimal control policy when $\Delta_{dec} \geq \Delta_{inc}$.

Theorem 1: Let there be $N(\geq 2)$ nodes and $\Delta_{dec} \geq \Delta_{inc}$. If there is a sequence U with one or more jumps that fixes $x(\leq N)$ nodes, then there exists a non-jumping sequence that fixes x nodes in less time. Thus, non-jumping sequences are optimal when $\Delta_{dec} \geq \Delta_{inc}$. □

Proof: We prove that given a sequence with an arbitrary number of jumps that fixes $x(\leq N)$ nodes, one can come up with a sequence that fixes x nodes but has one less jump than the given sequence (and fixes in less time than the given sequence). One can iteratively apply this result on the obtained sequences to eventually yield a non-jumping sequence that fixes x nodes in less time as compared to the given sequence.

Consider the given sequence U that fixes x nodes and suppose it contains one or more jumps. Denote the set of x nodes that are fixed by sequence U as $\mathcal{Z}$. Remove all the nodes in U that are not fully fixed. This gives a new sequence V that only targets nodes in the set $\mathcal{Z}$. Consider the *last* jump in the sequence, and suppose it occurs at time step T. Denote the portion of the sequence V from time step $T-1$ onwards by A, and denote the portion of the sequence V from time step 0 to time step $T-2$ by A'. Now, note that sequence A contains exactly one jump. Thus, by Lemma 1, we can replace sequence A with another sequence B that contains no jumps and fully fixes all nodes that are fully fixed in A in less time. Create a new sequence V' by concatenating the sequence A' and the sequence B. Thus, V' is a sequence with one fewer jump than U, and fixes all the x nodes in set $\mathcal{Z}$ and does so in less time. The first part of the result thus follows. The fact that non-jumping policies are optimal is then immediately obtained by considering U to be any optimal policy. ■

B. Non-jumping sequences

Theorem 1 showed that non-jumping policies are optimal for Problem 1 when $\Delta_{dec} \geq \Delta_{inc}$. We now characterize the optimal non-jumping policy in the set of all non-jumping policies. The following lemma provides a useful result for a later proposition.

Lemma 2: Let there be $N(\geq 2)$ nodes and consider a non-jumping sequence that fixes all the nodes. Under that sequence, suppose the order in which the nodes are targeted is $(i_1,\ldots,i_N)$ and that t_j is the number of time steps the entity takes to fix node i_j. Define $A_1 = v_0^{i_1}$ and $A_k = v_0^{i_k} - \Delta_{dec}\sum_{j=2}^{k}\left\lceil\frac{1-A_{j-1}}{\Delta_{inc}}\right\rceil$ for $k \in \{2,\ldots,N\}$. Then, the following holds true:

$$\sum_{p=1}^{N-1} t_p = \sum_{j=2}^{N}\left\lceil\frac{1-A_{j-1}}{\Delta_{inc}}\right\rceil. \tag{9}$$

□

The claim follows immediately from mathematical induction by noting that A_j is the health of node i_j when it is reached in the sequence, and thus $t_j = \left\lceil\frac{1-A_j}{\Delta_{inc}}\right\rceil, \forall j \in \{1,\ldots,N\}$.

The next proposition presents the necessary and sufficient conditions for a non-jumping sequence to fix all nodes.

Proposition 1: Let there be $N(\geq 2)$ nodes and consider a non-jumping sequence. Suppose the order in which the nodes are targeted in the sequence is $(i_1,\ldots,i_N)$. Define $A_1 = v_0^{i_1}$ and $A_k = v_0^{i_k} - \Delta_{dec}\sum_{j=2}^{k}\left\lceil\frac{1-A_{j-1}}{\Delta_{inc}}\right\rceil$ for $k \in \{2,\ldots,N\}$. Then the following conditions are necessary and sufficient for all the nodes to eventually get fixed:

$$A_k > 0 \quad \forall k \in \{1,\ldots,N\}. \tag{10}$$

□

The proof of this preposition follows trivially from the definition of A_k, namely that A_k is the health of node i_k at the time step when all nodes before i_k in the sequence under consideration are repaired fully and the entity starts repairing node i_k.

Based on Proposition 1, we now provide a result to determine a sequence that fixes the maximum number of nodes, under certain conditions on the initial health values and rates of recovery and degradation.

Theorem 2: Let there be $N(\geq 2)$ nodes and suppose $\Delta_{dec} = n\Delta_{inc}$, where n is a positive integer. Also, for each node $j \in \{1,\ldots,N\}$, suppose there exists a positive integer m_j such that $1 - v_0^j = m_j\Delta_{inc}$. Then, the non-jumping sequence that targets nodes in decreasing order of their initial health is optimal. □

Proof: Consider any optimal (non-jumping) sequence B, and let x be the number of nodes fixed by that sequence. Denote this set of $x(\leq N)$ nodes as $\mathcal{Z}$. Let $\{i_1,\ldots,i_x\}$ be the order in which the sequence B fixes the x nodes. The conditions $\Delta_{dec} = n\Delta_{inc}$ and $1 - v_0^j = m_j\Delta_{inc}, \ \forall j \in \{1,\ldots,N\}$ ensure that no node gets fixed halfway through a time step. Thus, the necessary and sufficient conditions to fix x nodes if a non-jumping sequence B targets the nodes in the order $(i_1,\ldots,i_x)$ are given by $A_k > 0, \forall k \in \{1,\ldots,x\}$ from Proposition 1, where $A_1 = v^0_{i_1}$ and $A_k = v_0^{i_k} - n\sum_{j=2}^{k}(1 - A_{j-1})$ for $k \in \{2,\ldots,x\}$. Note that the ceiling functions in the definition of A_k in Proposition 1 are dropped due to the conditions on the health values and the rates of repair and degradation.

As $\Delta_{dec} = n\Delta_{inc}$, we can expand these conditions as

$$v_0^{i_k} - n\sum_{j=2}^{k}\left((1 - v_0^{i_{j-1}})(1+n)^{k-j}\right) > 0 \quad \forall k \in \{2,\ldots,x\}. \tag{11}$$

The conditions (11) can be alternatively written as

$$v_0^{i_1}n + v_0^{i_2} > n, \tag{12}$$

$$v_0^{i_1} n(1+n) + v_0^{i_2} n + v_0^{i_3} > n(1+n) + n, \qquad (13)$$

$$\vdots$$

$$v_0^{i_1} n\,(1+n)^{x-2} + v_0^{i_2} n(1+n)^{x-3} + \ldots + v_0^{i_{x-1}} n + v_0^{i_x} > n\,(1+n)^{x-2} + n(1+n)^{x-3} + \ldots + n. \qquad (14)$$

The RHS of the above conditions do not depend on the sequence in which the nodes are fixed. Consider the left-hand side (LHS) of the above conditions. In condition (12), the LHS would be the largest when node i_1 has the largest initial health (as coefficients corresponding to $v_0^{i_1}$ and $v_0^{i_2}$ are n and 1, respectively). In condition (13), the LHS would be the largest when node i_1 has the largest initial health and node i_2 has the second largest initial health (as coefficients corresponding to $v_0^{i_1}, v_0^{i_2}, v_0^{i_3}$ are $n(1+n), n, 1$, respectively). Proceeding in this manner until the last condition (equation (14)), we see that the LHS would be largest when i_1 is the node with largest initial health, i_2 is the node with the second largest initial health and so on. Thus, the non-jumping sequence C that targets the nodes of set $\mathcal{Z}$ in decreasing order of their initial health values would also fix x nodes and hence will be optimal (since it fixes the same number of nodes as the optimal sequence B). Consider another non-jumping sequence D that targets the top x nodes with the largest initial health values from the N nodes. Then, the sequence D would also fix x nodes. That is because each node in sequence D has a higher initial health value (or at least equal) to the corresponding node in sequence C and thus sequence D satisfies the conditions (12)-(14). Thus, the policy of targeting the nodes in decreasing order of their initial health values would also fix x nodes, and hence is optimal. ■

Theorem 1 shows that non-jumping policies are optimal when $\Delta_{dec} \geq \Delta_{inc}$. Furthermore, Theorem 2 shows that under certain conditions on the initial health values and repair/degradation rates, repairing the nodes in decreasing order of their initial health is optimal. **Equivalently, under the conditions given in these theorems, the optimal sequence is a feedback policy that targets the healthiest node at each time step**.

IV. Optimal Sequences for $\Delta_{dec} < \Delta_{inc}$

We first characterize the globally optimal policy. In contrast to the previous section where we showed that the optimal policy is a non-jumping policy when $\Delta_{dec} \geq \Delta_{inc}$, we will show that the optimal policy involves jumps when $\Delta_{dec} < \Delta_{inc}$. We start with the following general result.

Lemma 3: Let there be $N (\geq 2)$ nodes. Then, the necessary condition for all the nodes to eventually get fixed (regardless of the rates of repair and deterioration) is that there exists a permutation $(i_1, \ldots, i_N)$ such that

$$v_0^{i_j} > (N-j)\Delta_{dec}, \quad \forall j \in \{1, \ldots, N\}. \qquad (15)$$

□

Proof: Suppose there exists a sequence that fixes all the nodes. At each time step t, use $\mathcal{C}^t$ to denote the set of nodes that have not been targeted at least once by the entity prior to t. Note that $\mathcal{C}^0 \supseteq \mathcal{C}^1 \supseteq \ldots \supseteq \mathcal{C}^{N-1}$. At $t = 0$, $|\mathcal{C}^t| = N$ where $|\mathcal{C}^t|$ denotes the cardinality of set $\mathcal{C}^t$. At time $t = 1, |\mathcal{C}^t| = N-1$ as there are $N-1$ nodes that have not been targeted by the entity at least once. Each node k belonging to the set $\mathcal{C}^1$ should have initial health value larger than Δ_{dec} to survive until $t = 1$. At $t = 2, |\mathcal{C}^t| \geq N-2$ as there are at least $N-2$ nodes that have not been targeted by the entity at least once. Each node k belonging to the set $\mathcal{C}^2$ should have initial health value larger than $2\Delta_{dec}$ to survive until $t = 2$. Repeating this argument for the next $N-4$ time steps proves that there should be an permutation $(i_1, \ldots, i_N)$ that should satisfy the conditions (15) for all the nodes to eventually get fixed. Note that (15) represent necessary conditions that need to be satisfied by *any* sequence that fixes all the nodes, regardless of the rates of repair and deterioration. ■

We now provide the following result for the case when the rate of repair is significantly larger than the rate of deterioration.

Lemma 4: Let there be $N (\geq 2)$ nodes, and suppose $\Delta_{inc} > (N-1)\Delta_{dec}$. Suppose there exists a permutation $(i_1, \ldots, i_N)$ such that

$$v_0^{i_j} > (N-j)\Delta_{dec}, \quad \forall j \in \{1, \ldots, N\}. \qquad (16)$$

Then, the sequence that fixes the least healthy node at each time step will fix all the nodes. □

Proof: Suppose the initial health values of the nodes satisfy the order $v_0^1 \geq \ldots \geq v_0^N$. If the sequence that fixes the least healthy node at each time step is followed, then after the completion of the first time step, the health values of the nodes are given by

$$v_1^N = v_0^N + \Delta_{inc} > (N-1)\Delta_{dec},$$
$$v_1^j = v_0^j - \Delta_{dec} > (N-1-j)\Delta_{dec} \;\; \forall j \in \{1, \ldots, N-1\}.$$

Note that the least health of nodes in the sequence after completion of the first time step would be of either node $N-1$ or N. First, suppose $v_1^N < v_1^{N-1}$. Since node N has the least health, $v_1^j > (N-1)\Delta_{dec} \; \forall j \in \{1, \ldots, N-1\}$. Thus, condition (16) is trivially satisfied. Now, suppose $v_1^N > v_1^{N-1}$. Then, the N nodes satisfy the condition (16), but with the indices reordered to correspond to a new permutation of nodes. That is, the lowest health (i.e., that of node $N-1$) is positive, the second lowest node has health larger than Δ_{dec} and so on. Thus, if the health values of the nodes satisfy the conditions in equation (16) at any time step then they would also satisfy the conditions in the next time step. Therefore, no node's health would become zero at any time. Furthermore, if node j is targeted by the entity at a time step and it does not get permanently recovered in less than a complete time step then the average health of all the nodes increases by at least $\frac{\Delta_{inc} - (N-1)\Delta_{dec}}{N}$, $\; j \in \{1, \ldots, N\}$. Note that $\frac{\Delta_{inc} - (N-1)\Delta_{dec}}{N} > 0$ as $\Delta_{inc} > (N-1)\Delta_{dec}$. So, either the increase in average health at each time step is positive or a node gets permanently recovered in a time step or both. Therefore, all the nodes would eventually get fixed. ■

The next result shows that it is optimal to target the least healthy node at each time step, under certain conditions on the rates of repair and degradation. Thus, non-jumping policies are no longer necessarily optimal when $\Delta_{dec} < \Delta_{inc}$.

Theorem 3: Let there be $N(\geq 2)$ nodes and $\Delta_{inc} > (N-1)\Delta_{dec}$. Then, the optimal policy is to target the least healthy node at each time step. □

Proof: Consider a sequence U that fixes $x(\leq N)$ nodes. Denote the set of x nodes as $\mathcal{Z}$. By Lemma 3, there exists a permutation $(i_1, \ldots, i_x)$ of the nodes in the set $\mathcal{Z}$ such that

$$v_0^{i_j} > (x-j)\Delta_{dec}, \quad \forall j \in \{1, \ldots, x\}. \tag{17}$$

Since the nodes in the set $\mathcal{Z}$ satisfy (17), the sequence S that targets the least healthy node at each time step in $\mathcal{Z}$ fixes all the nodes in set $\mathcal{Z}$ by Lemma 4.

Denote the set of nodes $\{1, \ldots, x\}$ that satisfy conditions (17) at time step t as $\mathcal{B}_t$. Then, $\mathcal{B}_0$ is the set $\mathcal{Z}$. Suppose the order of initial health values in the set $\mathcal{Z}$ is $v_0^1 \geq \ldots \geq v_0^x$. We show that there exists a set $\mathcal{B}_t$ at each time step if the least healthy node (among the N nodes) is targeted at each time step. In the first time step, either the node with least health value from the set $\mathcal{B}_0$ is targeted or a node outside the set $\mathcal{B}_0$ is targeted. If a node from the set $\mathcal{B}_0$ is targeted then the nodes from set $\mathcal{B}_0$ satisfy the conditions (17) at the end of first time by Lemma 4. Thus, the set $\mathcal{B}_1$ would be the same as set $\mathcal{B}_0$. Consider the other case in which a node c not belonging to the set $\mathcal{B}_0$ is targeted. Then, the health value of node c after the first step would be greater than $(x-1)\Delta_{dec}$ as $\Delta_{inc} > (N-1)\Delta_{dec} \geq (x-1)\Delta_{dec}$. Also, the $x-1$ nodes with the largest health values in the set $\mathcal{B}_0$ would satisfy the following due to conditions (17):

$$v_1^j = v_0^j - \Delta_{dec} > (x-j-1)\Delta_{dec}, \quad \forall j \in \{1, \ldots, x-1\}.$$

Thus, there exists a permutation of nodes in the set $\mathcal{B}_1$ (consisting of node c and $x-1$ nodes from $\mathcal{B}_0$) that satisfy the conditions (17) after the first time step. We can repeat this argument for all the subsequent time steps, noting that there would always be a set $\mathcal{B}_t$ of size x that would satisfy the conditions (17) and thus x nodes would eventually get fixed. Thus, if there is a sequence U that fixes $x(\leq N)$ nodes then the sequence that targets the least healthy node at each time step also fixes x nodes. The result thus follows. ■

It can be seen that optimal control sequences depend on the relationship between Δ_{dec} and Δ_{inc}. When $\Delta_{dec} \geq \Delta_{inc}$, targeting the healthiest node at each time step is the best policy (under certain conditions on the initial health values), whereas targeting the least healthy node at each time step is the best policy when $\Delta_{inc} > (N-1)\Delta_{dec}$.

V. Summary

In this paper, we characterized optimal sequencing policies of recovery actions in a post-disaster scenario, where multiple physical infrastructure components have been damaged, and an agency wishes to fully repair as many components as possible. We find that the properties of optimal policies depend on the relationship between the repair rate and the deterioration rate. We show that when the deterioration rate is larger than the repair rate, it is optimal to target the healthiest component at each time step (under certain conditions on the initial health values). If the repair rate is sufficiently greater than the deterioration rate, then it is optimal to target the least healthy node at each step.

There are several interesting avenues for future research. Characterizing optimal sequences with non-constant deterioration and repair rates is one such promising direction. Extending the problem to take into account stochasticity in deterioration and repair rates is another future avenue. Also, developing state estimation methods for exact measurement of the health values and the rates of the nodes will be an interesting study. Furthermore, incorporating interdependencies between different infrastructure components into the sequencing decisions also has importance for real-world scenarios.

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