CS 215 Assignment 3 Report

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Problem 2

Given $y = -\frac{1}{\lambda}\log(x)$ with $\lambda = 5$ and $x \sim U(0,1)$, the pdf $q(y)$ can be analytically derived as follows, Since, $g(x) = -\frac{1}{\lambda}\log(x)$ is a monotonically decreasing function for $x > 0$, the transformation of random variables formula can be applied.

$$\begin{aligned}
y &= -\left(\frac{1}{\lambda}\right)\log(x) \\
x &= \exp(-\lambda y) \\
g^{-1}(y) &= \exp(-\lambda y) \\
\left|\frac{d}{dy}g^{-1}(y)\right| &= \lambda\exp(-\lambda y) \\
\text{Since } x \sim U(0,1), \text{ PDF of } x = p(x) &= \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \\
\therefore \quad q(y) = p\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right| &= \begin{cases} \lambda\exp(-\lambda y) & \text{for } y > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Hence Y is an exponential random variable.

0.1 Posterior Mean

Given data $y_1, y_2, \cdots y_n, n \geq 1$. Since these samples are drawn from a exponential(λ) distribution with unknown λ and $\lambda \sim \text{Gamma}(\alpha, \beta)$ with $\alpha = 5.5$ and $\beta = 1$ (Gamma prior on λ),

- Joint Likelihood:

$$P(y_1, y_2, \cdots y_n|\lambda) = \begin{cases} \lambda^n \exp\{(-\lambda \sum_{i=1}^n y_i)\} & y_i > 0 \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

- Prior:

$$P(\lambda|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}$$

- Posterior:

$$
\begin{aligned}
P(\lambda|y_1, y_2, \cdots y_n) &\propto P(y_1, y_2, \cdots y_n|\lambda) \cdot P(\lambda|\alpha, \beta) \\
&\propto \lambda^n \exp\left\{\left(-\lambda \sum_{i=1}^{n} y_i\right)\right\} \cdot \lambda^{\alpha-1} \exp\{(-\beta\lambda)\} \\
&\propto \lambda^{\alpha+n-1} \exp\left\{\left(-\lambda \left(\sum_{i=1}^{n} y_i + \beta\right)\right)\right\} \\
&\sim \text{Gamma}\left(\alpha + n, \Sigma_{i=1}^{n} y_i + \beta\right)
\end{aligned}
$$

Since the mean of Gamma$(\alpha, \beta) = \frac{\alpha}{\beta}$,

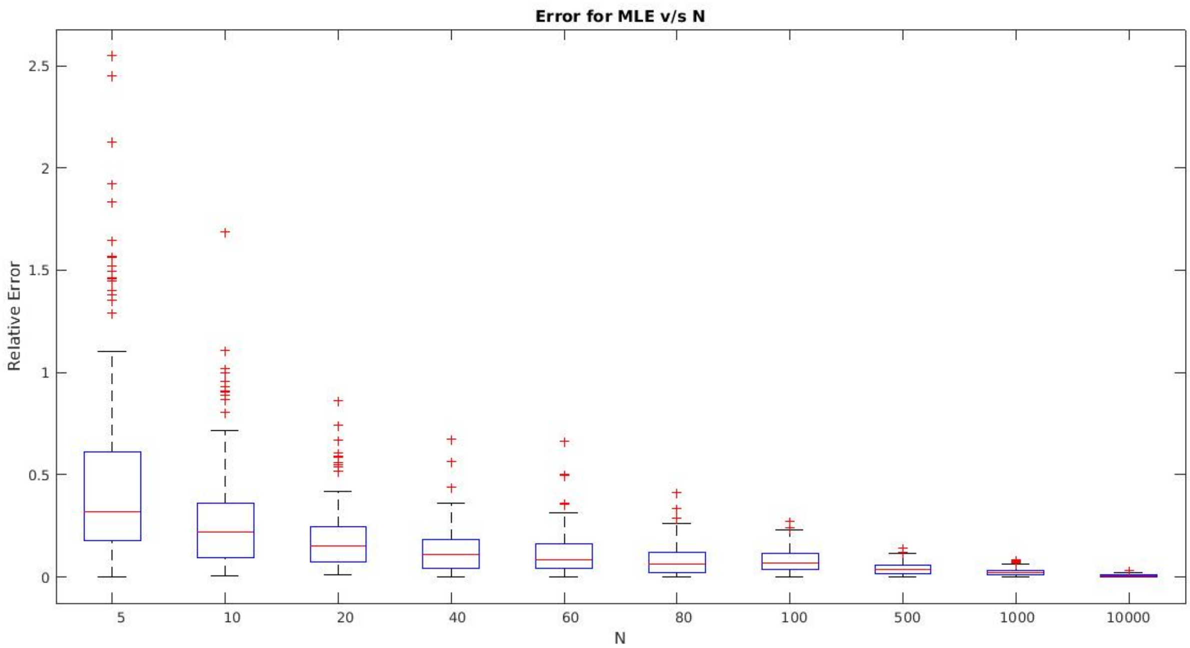
$$
\text{Posterior Mean, } \widehat{\lambda}^{\text{Posterior Mean}} = \frac{\alpha + n}{\sum_{i=1}^{n} y_i + \beta}
$$

It is known that for exponential(λ) distribution with n samples $y_1, y_2, \cdots y_n$, the ML estimate is

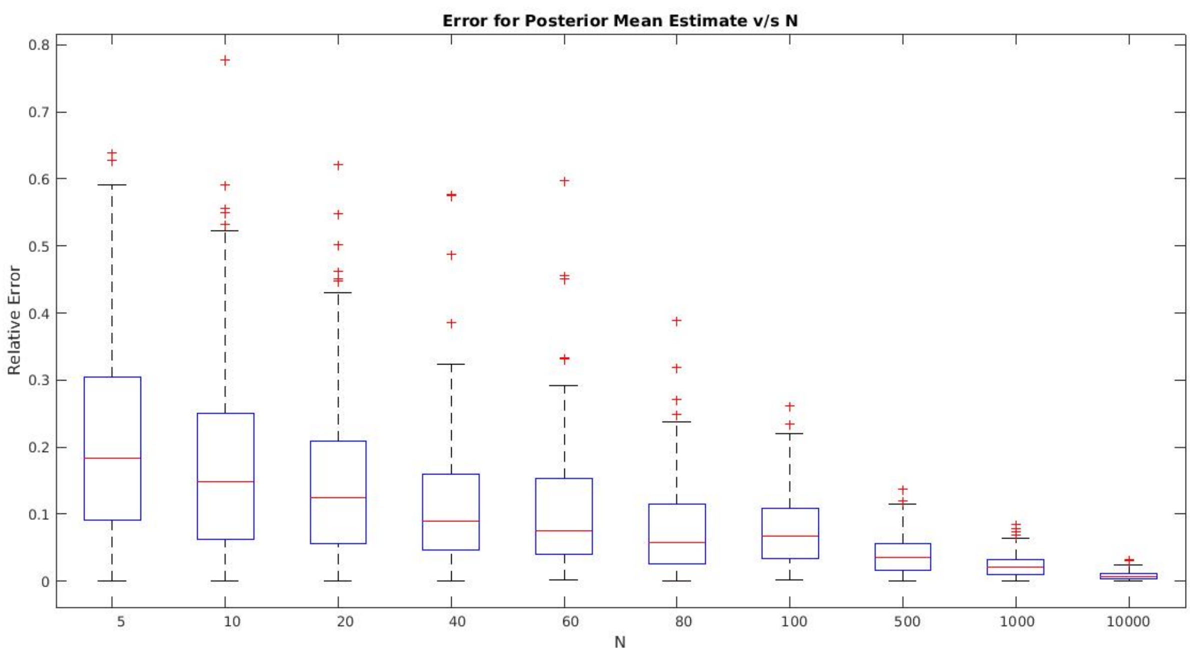
$$
\widehat{\lambda}^{\text{ML}} = \frac{n}{\sum_{i=1}^{n} y_i}
$$

0.2 Boxplots and Interpretation

1. As the value of N increases the error decreases and approaches zero in both the cases
2. The posterior mean estimate is more preferable since since it has a smaller values of relative error than the ML estimate for small values of N. Since, in practice we have only finite data, the posterior mean estimate is a better choice. Also for large values of N, the posterior mean estimate converges to the ML estimate and hence has all the desired asymptotic properties(consistency, asymptotic normality, efficiency) of the ML estimate.



Error for MLE v/s N
Relative Error
2.5
2
1.5
1
0.5
0
5
10
20
40
60
80
100
500
1000
10000
N



Error for Posterior Mean Estimate v/s N
Relative Error
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
5
10
20
40
60
80
100
500
1000
10000
N