Deriving OLS

$$Y_i = (\beta_1 + \beta_2 X_i + u_i)$$

$$Y_i = (\hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i)$$

u_i

$$\hat{u}_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i$$

$$\hat{u}_i = Y_i - \hat{Y}_i$$

$$Y_i = \hat{Y}_i + \hat{u}_i$$

$$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i$$

$$\hat{Y}_i = \beta_1 + \hat{\beta}_2 X_i$$

$$\sum_{i=1}^{n} \hat{u}_i^2 = \sum_{i=1}^{n} (Y_i - \hat{\beta}_1 - \beta_2 X_i)^2 = RSS$$

OLS's objective

$$\min_{\hat\beta_1,\hat\beta_2} RSS = \sum_{i=1} (Y_i - \hat\beta_1 - \hat\beta_2 X_i)^2$$

$$(Y_i - \hat\beta_1 - \hat\beta_2 X_i)(Y_i - \hat\beta_1 - \hat\beta_2 X_i)$$

(4) $$\min_{\hat\beta_1,\hat\beta_2} RSS = \sum \Big(\cancelto{0}{Y_i^2} - 2Y_i \cancelto{1}{\hat\beta_1} - 2Y_i \cancelto{1}{\hat\beta_2} X_i + 2\hat\beta_1^{\cancel{2}\,1} + 2\cancelto{1}{\hat\beta_1} \hat\beta_2 X_i + \hat\beta_2^2 \cancelto{0}{X_i^2} \Big)$$

$$f(x) = x^2 \qquad \frac{df}{dx} = 2x^{(2-1)} = 2x$$

First Order Conditions (FOC):

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = \sum\left(-2Y_i + 2\hat{\beta}_1 + 2\hat{\beta}_2 X_i\right) = 0$$

$$\sum_{i=1}^{n} = \left(\cancelto{0}{Y_i^2} - \cancelto{0}{2Y_i\hat{\beta}_1} X_i - 2Y_i \cancelto{1}{\hat{\beta}_2} X_i + \cancelto{0}{\hat{\beta}_1^2} + 2\hat{\beta}_1 \cancelto{1}{\hat{\beta}_2} X_i + \hat{\beta}_2^{\cancelto{1}{2}} X_i^2\right)$$

$$\frac{\partial RSS}{\partial \hat{\beta}_2} = \sum\left(-2Y_iX_i + 2\hat{\beta}_1 X_i + 2\hat{\beta}_2 X_i^2\right) = 0$$

$$\sum_{i=1}^{n} \left(-2Y_i + \underline{2\hat{\beta}_1} + 2\hat{\beta}_2 X_i \right) = 0$$

$$\sum -2Y_i + \sum 2\hat{\beta}_1 + \sum 2\beta_2 X_i = 0$$

$$\cancel{2} \sum \hat{\beta}_1 = \cancel{2} \sum Y_i - \cancel{2} \sum \beta_2 X_i$$

$$\cancel{n} \cdot \hat{\beta}_1 = \cancel{n} \bar{Y} - \hat{\beta}_2 \cancel{n} \bar{X} \qquad (9)$$

$$\sum_{i=1}^{n} Y_i = n \cdot \frac{1}{n} \sum_{i=1}^{n} Y_i = n \cdot \bar{Y}$$

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} \quad 2$$

Solve for $\hat{\beta}_2$

$$-\cancel{2}\sum_{i=1}^{n} Y_i X_i + \cancel{2}\sum_{i=1}^{n} \hat{\beta}_1 X_i + \cancel{2}\sum_{i=1}^{n} \hat{\beta}_2 X_i^2 = 0$$

$$-\sum Y_i X_i + (\bar{Y} - \hat{\beta}_2 \bar{X}) \sum_{i=1}^{n} X_i + \hat{\beta}_2 \sum_{i=1}^{n} X_i^2 = 0$$

$$-\sum Y_i X_i + \bar{Y} \sum X_i - \left[\hat{\beta}_2 \bar{X} \sum_{i=1}^{n} X_i + \hat{\beta}_2 \sum_{i=1}^{n} X_i^2\right] = 0$$

$$\hat{\beta}_2 \left(\sum_{i=1}^{n} X_i^2 - \bar{X} \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} Y_i X_i - \bar{Y} \sum_{i=1}^{n} X_i$$

Divide both sides by: $\sum X_i^2 - \bar{X} \sum X_i$

$$\boxed{\hat{\beta}_2 = \frac{\sum_{i=1}^{n} Y_i X_i - \bar{Y} \sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i^2 - \bar{X} \sum_{i=1}^{n} X_i}}$$

$$\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})}_{\text{is zero}} = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\bar{x}$$

$$= \sum_{i=1}^{n} x_i - n\bar{x}$$

$$= \sum_{i=1}^{n} x_i - n\frac{1}{n}\sum_{i=1}^{n} x_i$$

$$= 0$$

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n} Y_i X_i - \bar{Y} \sum_{i=1}^{n} X_i - \bar{X} \sum (Y_i - \bar{Y})}{\sum_{i=1}^{n} X_i^2 - \bar{X} \sum_{i=1}^{n} X_i - \bar{X} \sum (X_i - \bar{X})}$$

$$= \frac{\sum Y_i X_i - \sum \bar{Y} X_i - \sum \bar{X} Y_i + \sum \bar{Y} \bar{X}}{\sum X_i^2 - \sum \bar{X} X_i - \sum \bar{X} X_i + \sum \bar{X}^2}$$

$$\hat{\beta}_2 = \frac{\hat{\sum_{i=1}} (Y_i - \bar{Y})(X_i - \bar{X})}{\hat{\sum_{i=1}} (X_i - \bar{X})(X_i - \bar{X})}$$

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$$