Problem 3: Dantzig-Wolfe Decomposition

In this problem, you are asked to implement the Dantzig-Wolfe decomposition for the following linear optimization problem:

$$
\begin{aligned}
(MP_{\boldsymbol{x}}) \quad \min \quad & \boldsymbol{c}^\top \boldsymbol{x} && (5)\\
\text{s.t.} \quad & \boldsymbol{D}\boldsymbol{x} = \boldsymbol{b}_0 && (6)\\
& \boldsymbol{F}\boldsymbol{x} = \boldsymbol{b} && (7)\\
& \boldsymbol{x} \geq \boldsymbol{0}. && (8)
\end{aligned}
$$

Here (6) is the coupling constraint. (7)-(8) define a polyhedral P. Denote the extreme points of P as $\{\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^N\}$. Each point $\boldsymbol{x} \in P$ can be written as a convex combination of these extreme points as

$$
\begin{aligned}
& \boldsymbol{x} = \sum_{i=1}^{N} \lambda_i \boldsymbol{x}^i \\
& \sum_{i=1}^{N} \lambda_i = 1 \\
& \lambda_i \geq 0 \qquad \forall i = 1, \ldots, N.
\end{aligned}
$$

Substitute this extreme point representation to $(MP_{\boldsymbol{x}})$, we have an equivalent problem $(MP_{\boldsymbol{\lambda}})$ in the $\boldsymbol{\lambda}$ variable as

$$
\begin{aligned}
(MP_{\boldsymbol{\lambda}}) \quad \min \quad & \sum_{i=1}^{N} \lambda_i \left(\boldsymbol{c}^\top \boldsymbol{x}^i\right) \\
\text{s.t.} \quad & \sum_{i=1}^{N} \lambda_i \left(\boldsymbol{D}\boldsymbol{x}^i\right) = \boldsymbol{b}_0 \\
& \sum_{i=1}^{N} \lambda_i = 1 \\
& \lambda_i \geq 0 \qquad \forall i = 1, \ldots, N.
\end{aligned}
$$

Now we can apply column generation to $(MP_{\boldsymbol{\lambda}})$ as discussed in class.

1. Start from a subset I of extreme points of the polyhedron P. Solve the following **restricted master problem**:

$$
\begin{aligned}
(RMP) \quad \min \quad & \sum_{i \in I} \lambda_i \left(\boldsymbol{c}^\top \boldsymbol{x}^i\right) \\
\text{s.t.} \quad & \sum_{i \in I} \lambda_i \left(\boldsymbol{D}\boldsymbol{x}^i\right) = \boldsymbol{b}_0 && (9)\\
& \sum_{i \in I} \lambda_i = 1 && (10)\\
& \lambda_i \geq 0 \qquad \forall i \in I.
\end{aligned}
$$

   Let $\hat{\boldsymbol{y}}$ be the optimal dual variable associated with the coupling constraint (9), and let $\hat{r}$ be the optimal dual variable associated with the convexity constraint (10).

2. Solve the **pricing problem**:

$$\begin{aligned} \hat{Z} = \min \quad & \left(\boldsymbol{c}^\top - \hat{\boldsymbol{y}}^\top \boldsymbol{D}\right) \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{F}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Denote the optimal solution of the above pricing problem as $\hat{\boldsymbol{x}}$. There are two possibilities:

(a) If $\hat{Z} - \hat{r} \geq 0$, then all the reduced costs are nonnegative. Terminate the column generation algorithm, and report the optimal solution.

(b) Otherwise, we add $\hat{\boldsymbol{x}}$ as a newly generated extreme point to I, and go to 1.

Question:

1. Implement the Dantzig-Wolfe decomposition code using the data in the following Question 2(a). Submitted the complete mos file both online and in the physical copy.

2. Do the following two experiments with your code:

   (a) $\boldsymbol{c} = [-4, -1, -6, -3, -5], \boldsymbol{D} = [3, 2, 4, 3, 5], \boldsymbol{b}_0 = 25$, the polyhedron P is the five dimensional cube $1 \leq x_i \leq 2$ for $i = 1, ..., 5$. Start with two initial extreme points $(1, 2, 1, 1, 1)$ and $(2, 2, 2, 1, 1)$.

   (b) $\boldsymbol{c} = [-4, -1, -6, -3, -5], \boldsymbol{D} = [3, 2, 4, 3, 5], \boldsymbol{b}_0 = 25$, the polyhedron P is the five dimensional cube $1 \leq x_i \leq 3$ for $i = 1, ..., 5$. Start with two initial extreme points $(1, 1, 1, 1, 1)$ and $(3, 3, 3, 3, 3)$.

   For each experiment, print out the optimal solution of the (RMP) and the associated dual solution, also print out the minimum reduced cost and the new extreme points generated in each intermediate iteration of the algorithm. Of course, you also need to print out the final optimal solutions in $\boldsymbol{x}$ variable and $\boldsymbol{\lambda}$ variable, respectively.