Introduction to Lagrangian Mechanics: A Step-by-Step Worksheet

Contents

1 Introduction

In classical mechanics, the Lagrangian formulation provides a powerful and elegant way to derive the equations of motion of a system. Instead of analyzing individual forces, we focus on the system's *kinetic energy* T and *potential energy* V, combined in the form

$$L = T - V.$$

From this quantity, we apply the **Euler–Lagrange equations** to obtain the equations of motion (EOM):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

where q_i are the *generalized coordinates* of the system, and $\dot{q}_i$ are the corresponding velocities.

In this worksheet, we will explore:

1. The **Simple Harmonic Oscillator (SHO)**
2. The **Pendulum (Single Pendulum)**
3. The **Double Pendulum** (demonstrating chaotic dynamics)

Each example demonstrates how to set up the Lagrangian, find the equations of motion, and (when possible) solve or at least discuss the solution.

2 Simple Harmonic Oscillator (SHO)

2.1 Physical Setup

The SHO consists of a **mass** m attached to a **spring** with spring constant k. We assume:

- The mass can move along a single axis (e.g., the x-axis).
- The spring obeys Hooke's law with potential energy $\frac{1}{2}kx^2$.
- No friction or damping is present.

2.2 Lagrangian Setup

1. **Coordinate:** Let x be the displacement of the mass from equilibrium.
2. **Kinetic Energy:** $T = \frac{1}{2}m\dot{x}^2$.
3. **Potential Energy:** $V = \frac{1}{2}kx^2$.
4. **Lagrangian:** $L = T - V = \frac{1}{2}m\,\dot{x}^2 - \frac{1}{2}k\,x^2$.

2.3 Euler–Lagrange Equation

We have a single generalized coordinate $q_1 = x$. Hence,

$$\frac{\partial L}{\partial x} = -k\,x, \quad \frac{\partial L}{\partial \dot{x}} = m\,\dot{x}.$$

Taking the time derivative,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\,\dot{x}) = m\,\ddot{x}.$$

The Euler–Lagrange equation becomes:

$$m\,\ddot{x} - \big(-k\,x\big) = 0 \quad \Longrightarrow \quad m\,\ddot{x} + k\,x = 0.$$

2.4 Solving the ODE

This well-known ODE is:

$$\ddot{x} + \omega^2\,x = 0, \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}.$$

The general solution is:

$$x(t) = A\cos(\omega t) + B\sin(\omega t),$$

with constants A, B determined by initial conditions.

3 Pendulum (Single Pendulum)

3.1 Physical Setup

Consider a **mass** m at the end of a massless **rod** or **string** of length L, free to swing in a plane under gravity g. Let θ be the angular displacement from the vertical.

3.2 Kinetic and Potential Energy

1. **Coordinate:** The generalized coordinate is θ.
2. **Position:** If the pivot is at the origin, we have $x = L\sin\theta, \quad y = -L\cos\theta$.
3. **Velocity:** $\dot{x} = L\dot{\theta}\cos\theta, \quad \dot{y} = L\dot{\theta}\sin\theta$. Hence $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\big(L\dot{\theta}\big)^2$.
4. **Potential Energy:** A common choice is $V = m\,g\,L(1 - \cos\theta)$. This differs by a constant from $-m\,g\,y$ but yields the same physics.

3.3 Lagrangian and Euler–Lagrange Equation

$$L = T - V = \tfrac{1}{2} m L^2 \dot{\theta}^2 - m\,g\,L(1 - \cos\theta).$$

Then:

$$\frac{\partial L}{\partial \theta} = m\,g\,L\,\sin\theta, \quad \frac{\partial L}{\partial \dot{\theta}} = m\,L^2\,\dot{\theta}.$$

Taking the time derivative,

$$\frac{d}{dt}\big(mL^2\dot{\theta}\big) = mL^2\ddot{\theta}.$$

The Euler–Lagrange equation:

$$mL^2\ddot{\theta} - m\,g\,L\,\sin\theta = 0 \quad \Longrightarrow \quad \ddot{\theta} + \frac{g}{L}\sin\theta = 0.$$

For small angles ($\sin\theta \approx \theta$), this reduces to $\ddot{\theta} + (g/L)\theta = 0$, a simple harmonic oscillator.

4 Double Pendulum

4.1 Physical Setup

A **double pendulum** has two masses:

- m_1, attached to a pivot by a rod/string of length L_1.
- m_2, attached to m_1 by a rod/string of length L_2.

Define θ_1 and θ_2 as the angles each pendulum makes with the vertical.

4.2 Coordinates and Velocities

Let us position the origin at the pivot for the first mass. Then:

$$\begin{aligned} x_1 &= L_1 \sin(\theta_1), & y_1 &= -L_1 \cos(\theta_1), \\ x_2 &= x_1 + L_2 \sin(\theta_2), & y_2 &= y_1 - L_2 \cos(\theta_2). \end{aligned}$$

Differentiate to get velocities:

$$\begin{aligned} \dot{x}_1 &= L_1 \cos(\theta_1)\,\dot{\theta}_1, & \dot{y}_1 &= L_1 \sin(\theta_1)\,\dot{\theta}_1, \\ \dot{x}_2 &= \dot{x}_1 + L_2 \cos(\theta_2)\,\dot{\theta}_2, & \dot{y}_2 &= \dot{y}_1 + L_2 \sin(\theta_2)\,\dot{\theta}_2. \end{aligned}$$

4.3 Kinetic and Potential Energy

$$T = \frac{1}{2} m_1 \big(\dot{x}_1^2 + \dot{y}_1^2\big) + \frac{1}{2} m_2 \big(\dot{x}_2^2 + \dot{y}_2^2\big).$$

$$V = -m_1 g\, y_1 - m_2 g\, y_2.$$

Hence the *Lagrangian* is

$$L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = T - V.$$

4.4 Equations of Motion

We treat θ_1 and θ_2 as the two generalized coordinates. The Euler–Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0.$$

After a fair bit of trigonometric simplification, these become **two coupled, nonlinear ODEs** usually represented as:

$$(m_1 + m_2)L_1\ddot{\theta}_1 + m_2L_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2L_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + (m_1 + m_2)g\sin(\theta_1) = 0,$$
$$m_2L_2\ddot{\theta}_2 + m_2L_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) + m_2L_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + m_2g\sin(\theta_2) = 0.$$

These equations cannot be solved with simple elementary functions in the general case. Instead, one typically **numerically integrates** them (e.g., using Python or another computational tool) and observes the *chaotic* dynamics.

4.5 Key Remarks on the Double Pendulum

- For small oscillations, one can approximate θ_1 and θ_2 around a stable equilibrium, but the full nonlinear system exhibits chaotic motion.
- Energy is conserved in the ideal (no damping) double pendulum.
- Small changes in initial conditions can lead to dramatically different trajectories (**sensitive dependence on initial conditions**).