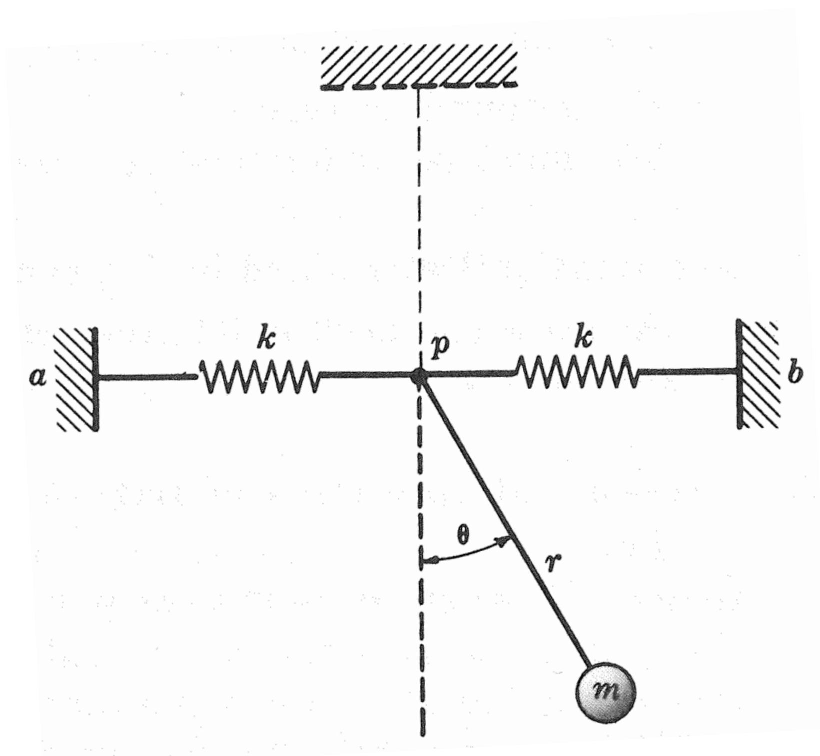
Chapter 7 – Lagrange's Equations

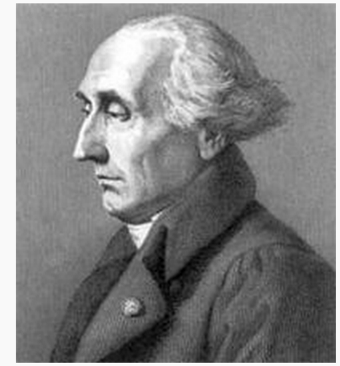
Joseph-Louis Lagrange (born **Giuseppe Lodovico Lagrangia** [1][2][3] (also reported as *Giuseppe Luigi Lagrangia* [4]), 25 January 1736 in Turin, Piedmont; died 10 April 1813 in Paris) was an Italian Enlightenment Era mathematician and astronomer. He made significant contributions to all fields of analysis, number theory, and both classical and celestial mechanics.

In 1766, On the recommendation of Euler and d'Alembert, Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, Prussia, where he stayed for over twenty years, producing volumes of work and winning several prizes of the French Academy of Sciences. Lagrange's treatise on analytical mechanics (*Mécanique Analytique*, 4. ed., 2 vols. Paris: Gauthier-Villars et fils, 1888–89), written in Berlin and first published in 1788, offered the most comprehensive treatment of classical mechanics since Newton and formed a basis for the development of mathematical physics in the nineteenth century.

In 1787, at age 51, he moved from Berlin to Paris and became a member of the French Academy. He remained in France until the end of his life. He was significantly involved in the decimalisation in Revolutionary France, became the first professor of analysis at the École Polytechnique upon its opening in 1794, founding member of the Bureau des Longitudes and Senator in 1799.



Joseph-Louis Lagrange



Joseph-Louis (Giuseppe Luigi), comte de Lagrange

Born	Giuseppe Lodovico Lagrangia 25 January 1736 Turin, Piedmont-Sardinia
Died	10 April 1813 (aged 77) Paris, France
Residence	Piedmont France Prussia
Citizenship	Sardinia-Piedmont French Empire
Fields	Mathematics Mathematical physics
Institutions	École Polytechnique
Academic advisors	Leonhard Euler Giovanni Battista Beccaria
Doctoral students	Joseph Fourier Giovanni Plana Siméon Poisson
Known for	(see list) Analytical mechanics Celestial mechanics Mathematical analysis Number theory

Generalized Momenta and Ignorable Coordinates

$$q_i \quad i = 1, \ldots, n \quad generalized\,coordinates$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = F_i \quad generalized\,forces$$

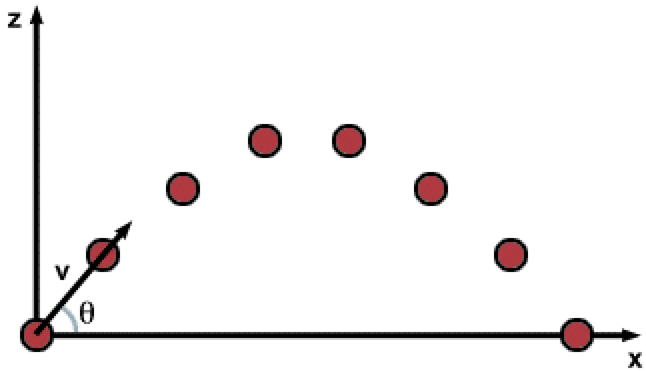
$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \quad generalized\,momenta$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \Leftrightarrow \quad F_i = \frac{dp_i}{dt} \qquad Lagranges'\,Equations$$

(generalized force) = (rate of change of generalized momentum) (7.17)

When the Lagrangian is independent of a coordinate q_i (i.e., invariant when q_i varies with all other q_j held fixed), that coordinate is said to be ***ignorable*** and the corresponding generalized momentum is ***conserved***.

Projectile Motion Revisited



$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$U = mgz$$

Let $\mathcal{L} = T - U$ *be the Lagrangian of the system*

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \qquad \frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

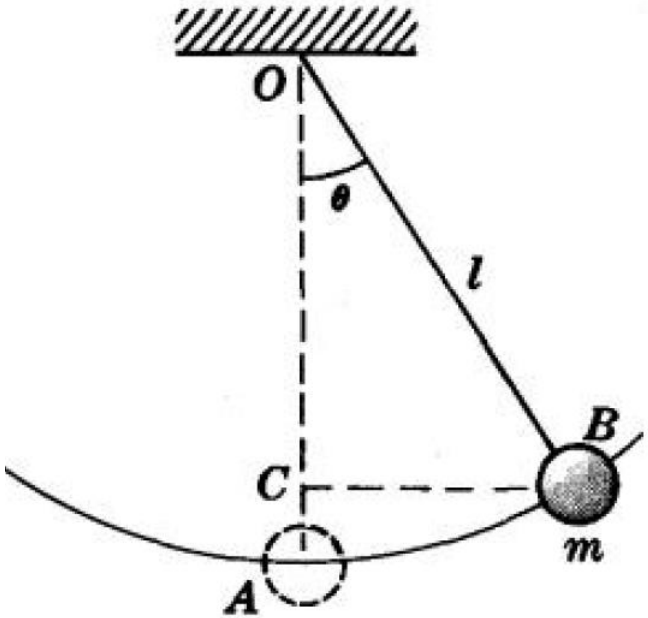
Lagrange's Equations for Unconstrained Motion

$$\mathcal{L} = T - U$$

the Lagrangian of the system

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{x}} & \Leftrightarrow \quad F_x = m\ddot{x} \\ \dfrac{\partial \mathcal{L}}{\partial y} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{y}} & \Leftrightarrow \quad F_y = m\ddot{y} \quad \Leftrightarrow \quad \vec{F} = m\vec{a} \\ \dfrac{\partial \mathcal{L}}{\partial z} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{z}} & \Leftrightarrow \quad F_z = m\ddot{z} \end{cases}$$

The Simple Pendulum



$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(l\dot{\theta})^2 = \tfrac{1}{2}ml^2\dot{\theta}^2$$

$$U = mg(OA - OC) = mg(l - l\cos\theta)$$

$$L = T - U = \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

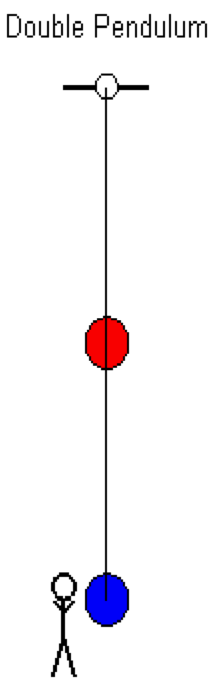
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

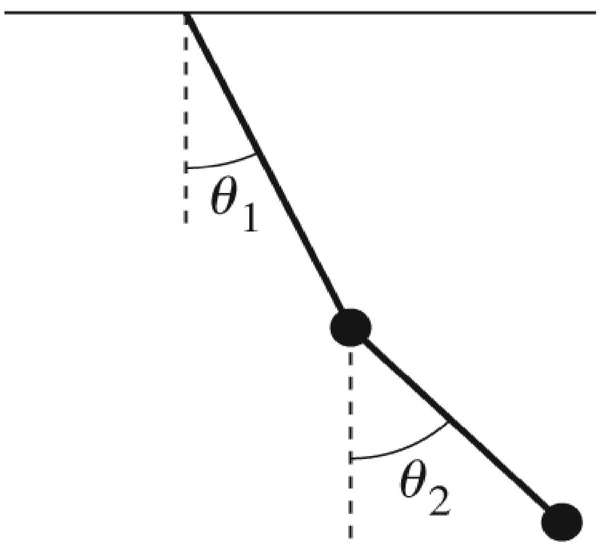
$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta, \qquad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \qquad \text{or} \qquad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

A Double Pendulum

For more on this, see Section 11.4 of Chapter 11: Coupled Oscillators and Normal Modes.





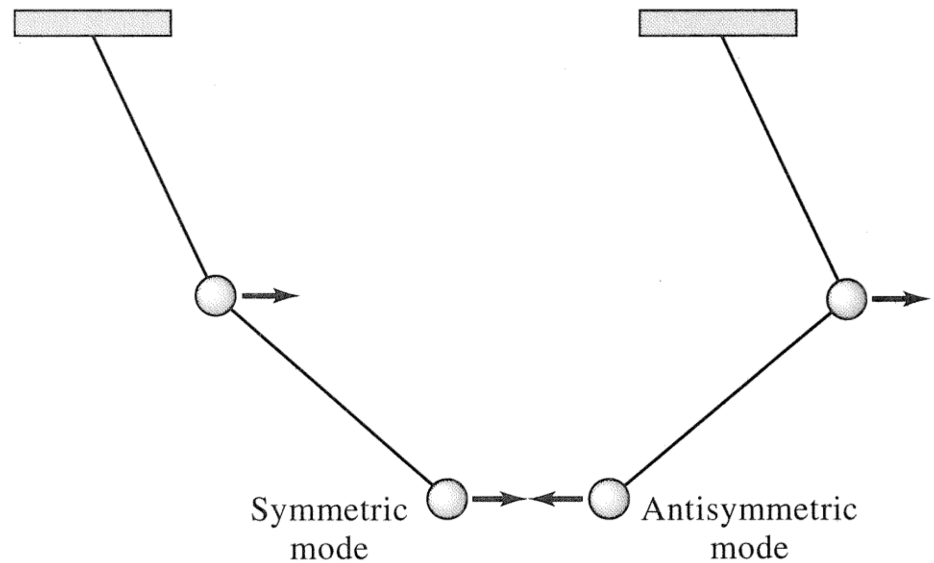
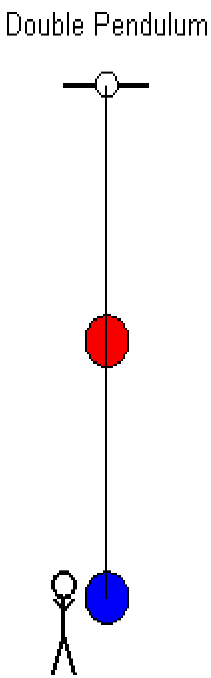
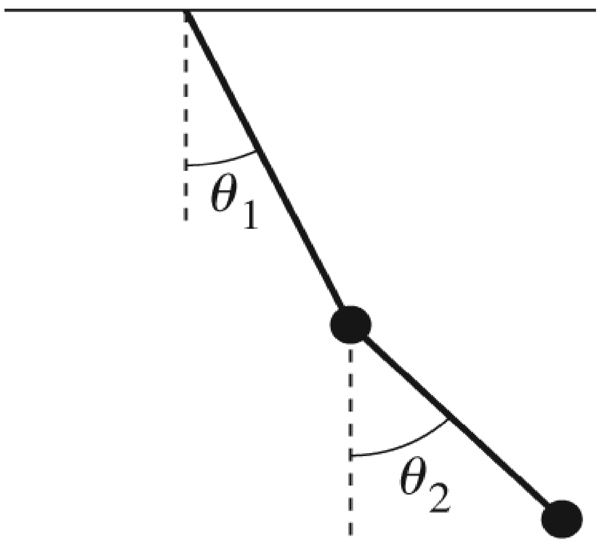


Figure 6.7

A Double Pendulum





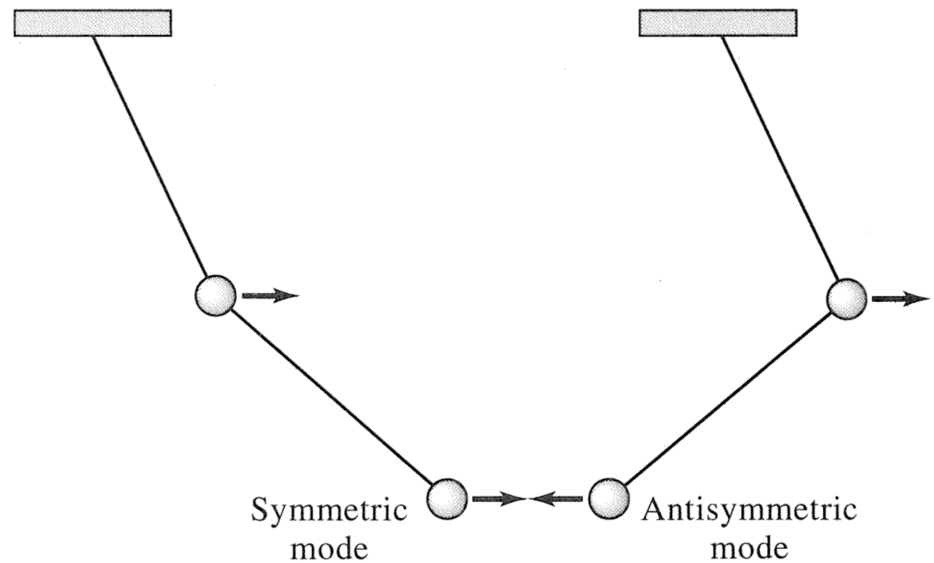


Figure 6.7