Block on a Frictionless Incline Using Lagrange Equations

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Objective

In this assignment, you will:

- Derive the equation of motion for a block of mass m sliding down a frictionless inclined plane of angle α using Lagrange's equations.
- Solve the resulting differential equation analytically.
- Implement the solution in Python using SymPy for symbolic calculations and `matplotlib` for visualization.

1 Problem Statement

Consider a block of mass m placed at the top of a frictionless inclined plane making an angle α with the horizontal. Define a coordinate x along the incline with $x = 0$ at the top (where the block is initially released) and x increasing in the downward direction.

Tasks

1. **Derivation of the Equation of Motion**

   (a) **Kinetic Energy:** The block moves along the plane with speed $\dot{x}$. The kinetic energy is:

   $$T = \frac{1}{2}m\dot{x}^2$$

   (b) **Potential Energy:** When the block has moved a distance x down the plane, its vertical drop is $x \sin\alpha$. Taking the potential energy zero at the top of the incline:

   $$U = -mgx\sin\alpha$$

   (c) **Lagrangian:** Constructing the Lagrangian:

   $$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 + mgx\sin\alpha$$

   (d) **Euler–Lagrange Equation:** Using the Euler–Lagrange equation,

   $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0,$$

   we get:

   $$\frac{d}{dt}(m\dot{x}) - mg\sin\alpha = 0$$

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$$m\ddot{x} = mg\sin\alpha$$

$$\ddot{x} = g\sin\alpha$$

2. **Analytical Solution**

   (a) The equation of motion is a second-order differential equation:

   $$\ddot{x} = g\sin\alpha$$

   Integrating once:

   $$\dot{x} = g\sin\alpha \cdot t + C_1$$

   Given initial condition $\dot{x}(0) = 0$, we get $C_1 = 0$, so:

   $$\dot{x} = g\sin\alpha \cdot t$$

   Integrating again:

   $$x(t) = \frac{1}{2}g\sin\alpha \cdot t^2 + C_2$$

   Applying $x(0) = 0$, we get $C_2 = 0$, so:

   $$x(t) = \frac{1}{2}g\sin\alpha \cdot t^2$$

   (b) The solution shows that the block undergoes uniformly accelerated motion along the incline with acceleration $g\sin\alpha$.

3. **SymPy Implementation and Visualization**

   (a) Using Python and the SymPy library, set up the Lagrangian and derive the equation of motion symbolically.

   (b) Solve the differential equation for $x(t)$ symbolically.

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(c) Plot $x(t)$ over a suitable time interval (e.g., $t \in [0, 5]$ s) using `matplotlib`.

(d) Compare the symbolic solution with the analytical result.

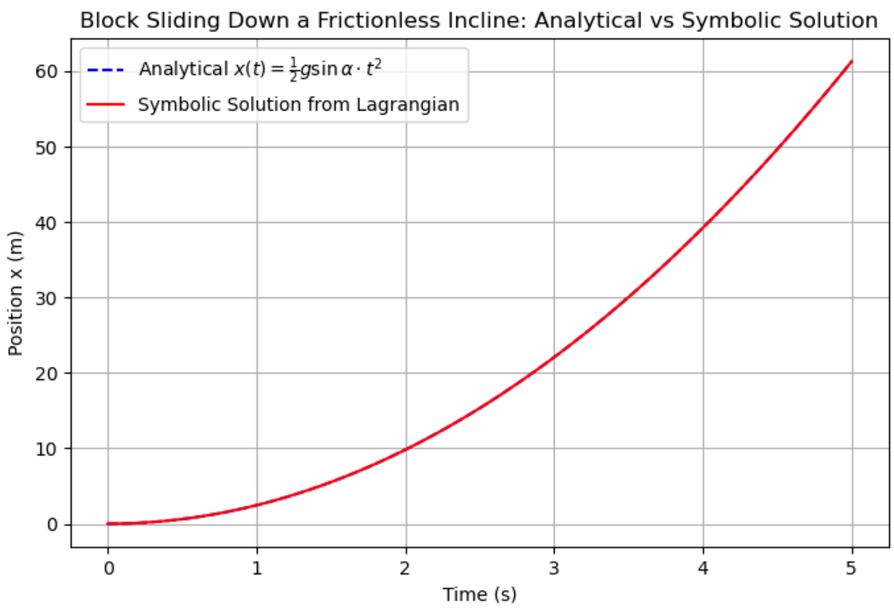

Figure 1: Comparison of the analytical and symbolic solutions for the position $x(t)$ of a block sliding down a frictionless incline. The plot shows the motion over the time interval $t \in [0, 5]$ seconds, illustrating the uniformly accelerated nature of the motion with acceleration $g \sin\alpha$.

2 Python Implementation and Description

The following Python script implements the derivation, solution, and visualization of the motion of a block sliding down a frictionless incline using Lagrangian mechanics. The key steps of the code are as follows:

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- **Symbolic Computation:** The script uses the SymPy library to define the Lagrangian, compute the Euler-Lagrange equation, and solve the resulting second-order differential equation for $x(t)$.
- **Analytical Solution:** The equation of motion is integrated manually to obtain the expected analytical solution:

$$x(t) = \frac{1}{2} g \sin\alpha \cdot t^2$$

- **Numerical Simulation:** The analytical and symbolic solutions are evaluated over the time interval $t \in [0, 5]$ seconds.
- **Visualization:** The script uses Matplotlib to plot both the analytical and symbolic solutions, illustrating that they are identical.

The output of the code includes:

1. The equation of motion derived using the Euler-Lagrange equation.
2. The symbolic solution obtained using SymPy.
3. A numerical comparison showing that the symbolic and analytical solutions match.
4. A plot displaying the motion of the block over time.

3 Submission Guidelines

- Submit a written report (PDF) that includes your complete derivation, analytical solution, and discussion of your results.

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- Include your Python code as a separate file. Ensure the code is well-commented.
- Provide clear plots that illustrate the block's position $x(t)$ over time.

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