Quantum Gravity

Second Technical Exposition
of

Abid Jeem

Submission-1

Washington and Lee University
Department of Physics and Engineering

Professor: Dr. Tom McClain

November 10, 2023

1 Exercise 24.1

Question

Check that this is a solution of Schrödinger's equation:

$$\psi = Ce^{-iEt/\hbar}$$

Solution

To check if the provided solution $\psi = Ce^{-iEt/\hbar}$ satisfies Schrödinger's equation, we need to verify it against the time-dependent Schrödinger equation. The time-dependent Schrödinger equation for a non-relativistic particle in a potential $V(x)$ is given by:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi$$

The provided solution, $\psi = Ce^{-iEt/\hbar}$, is a function of time t only. This means we'll be working with the time derivative part of the Schrödinger equation.

The first step is to differentiate ψ with respect to time:

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}\left(Ce^{-iEt/\hbar}\right) = -\frac{iE}{\hbar}Ce^{-iEt/\hbar}$$

Substituting ψ into the time derivative term of Schrödinger's equation, we get:

$$i\hbar\left(-\frac{iE}{\hbar}Ce^{-iEt/\hbar}\right) = ECe^{-iEt/\hbar}$$

Now, we compare this result with the right-hand side of the Schrödinger equation. Since the provided solution is only time-dependent and does not have a spatial component, the spatial derivative term $\frac{\partial^2 \psi}{\partial x^2}$ is zero. Also, if we assume a constant potential V_0, the equation becomes:

$$ECe^{-iEt/\hbar} = -\frac{\hbar^2}{2m} \cdot 0 + V_0 Ce^{-iEt/\hbar}$$

For this to hold true, E should equal V_0 (the energy of the system in a constant potential), which is consistent with the assumption of the energy being a constant E.

Therefore, the function $\psi = Ce^{-iEt/\hbar}$ is indeed a solution to the time-dependent Schrödinger equation for a particle in a constant potential V_0, where $E = V_0$.

2 Exercise 24.5

Question

Check that the wave operator is the square of a first-order operator defined with the help of these Clifford elements.

Solution

Given the Clifford elements, we express the first-order operator as follows:

$$\gamma_0 \frac{\partial}{\partial t} - \sum_{i=1}^{3} \gamma_i \frac{\partial}{\partial x^i} \tag{1}$$

where γ^0 and γ^i are the Clifford elements.

To demonstrate that the wave operator is the square of this operator, we compute the square of the above expression:

$$\left(\gamma_0 \frac{\partial}{\partial t} - \sum_{i=1}^{3} \gamma_i \frac{\partial}{\partial x^i}\right)^2 = \gamma_0^2 \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \gamma_0 \gamma_i \frac{\partial}{\partial t} \frac{\partial}{\partial x^i} - \sum_{i=1}^{3} \gamma_i \gamma_0 \frac{\partial}{\partial x^i} \frac{\partial}{\partial t} + \sum_{i=1}^{3} \gamma_i \frac{\partial}{\partial x^i} \sum_{j=1}^{3} \gamma_j \frac{\partial}{\partial x^j}$$

$$= \gamma_0^2 \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} (\gamma_0 \gamma_i + \gamma_i \gamma_0) \frac{\partial}{\partial t} \frac{\partial}{\partial x_i} + \sum_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} (\gamma_i \gamma_j + \gamma_j \gamma_i) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \sum_{i=1}^{3} \gamma_i^2 \frac{\partial^2}{\partial (x^i)^2}$$

$$= \gamma_0^2 \frac{\partial^2}{\partial t^2} + \sum_{i=1}^{3} \gamma_i^2 \frac{\partial^2}{\partial (x_i)^2}$$

Using the properties of Clifford algebra, we can simplify the above expression. Specifically, the anticommutative property $\gamma^0 \gamma^i + \gamma^i \gamma^0 = 0$ and $\gamma^i \gamma^i = -1$, the expression simplifies to:

$$\frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial (x_i)^2} \tag{2}$$

which is the standard wave operator. Hence, it is demonstrated that the wave operator is indeed the square of the given first-order operator defined using Clifford elements.

3 Exercise 25.4

Question

Check that the charge values, indicated by the superfixes in the first table, come out right

Solution

The table describes an array of particles made up of combinations of 'up' (u), 'down' (d), and 'strange' (s) quarks.

To solve this exercise, we need to know the charge values of each type of quark:

- Up quark (u) has a charge of $+\frac{2}{3}$ (in proton-charge units).
- Down quark (d) and strange quark (s) each have a charge of $-\frac{1}{3}$.

Given this information, we can calculate the total charge of each particle in the table by summing the charges of its constituent quarks. For example, a particle composed of 'uuu' would have a total charge of $3 \times \frac{2}{3} = 2$, and one composed of 'udd' would have a charge of $\frac{2}{3} + 2 \times \left(-\frac{1}{3}\right) = 0$.

$$\begin{array}{llll} \Delta^{++} & \Delta^{+} & \Delta^{0} & \Delta^{-} \\ \Sigma^{*+} & \Sigma^{*0} & \Sigma^{*-} & \\ \Xi^{*0} & \Xi^{*-} & & \\ \Omega^{-} & & & \end{array}$$

$$\begin{array}{llll} \text{uuu} & \text{uud} & \text{udd} & \text{ddd} \\ \text{uus} & \text{uds} & \text{dds} & \\ \text{uss} & \text{dss} & & \\ \text{sss} & & & \end{array}$$

Calculating the charges for the particles in the given list:

- **uuu**: This particle has three up quarks, so its total charge is $3 \times \frac{2}{3} = 2$.
- **uud**: This particle has two up quarks and one down quark, so its charge is $2 \times \frac{2}{3} + \left(-\frac{1}{3}\right) = 1$.
- **udd**: This particle has one up quark and two down quarks, so its charge is $\frac{2}{3} + 2 \times \left(-\frac{1}{3}\right) = 0$.
- **ddd**: This particle has three down quarks, so its charge is $3 \times \left(-\frac{1}{3}\right) = -1$.
- **uus**: This particle consists of two 'up' quarks and one 'strange' quark. The total charge is calculated as $2 \times \frac{2}{3} + \left(-\frac{1}{3}\right) = \frac{4}{3} - \frac{1}{3} = 1$.

- **uds**: This particle is composed of one 'up' quark (u), one 'down' quark (d), and one 'strange' quark (s). The total charge is $\frac{2}{3} + \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) = 0$.
- **dds**: This particle has two 'down' quarks and one 'strange' quark. The total charge is $2 \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) = -1$.
- **uss**: This particle consists of one 'up' quark and two 'strange' quarks. The total charge is $\frac{2}{3} + 2 \times \left(-\frac{1}{3}\right) = 0$.
- **dss**: This particle has one 'down' quark and two 'strange' quarks. The total charge is $\left(-\frac{1}{3}\right) + 2 \times \left(-\frac{1}{3}\right) = -1$.
- **sss**: This particle is made up of three 'strange' quarks. The total charge is $3 \times \left(-\frac{1}{3}\right) = -1$.