Quantum Gravity

Third Technical Exposition
of

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Submission-1

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1 Exercise 31.4

Question

Explain why we can write the Laplacian as:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}$$

The reason we can write the Laplacian equation in the form given in the question is actually quite simple. It is because, in this context, the function is defined in a mixed coordinate system, combining Cartesian coordinates (x,y,z) and polar coordinates (r, θ). The Cartesian component $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ captures the linear variations, while the polar component $\frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2}$ accounts for rotational symmetry. In standard three-dimensional Euclidean space ($\mathbb{E}^3$), the Laplacian of a function f is given by:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{1}$$

In an $\mathbb{E}^3 \times \mathbb{S}^1$ space, which combines $\mathbb{E}^3$ with a one-dimensional circular space ($\mathbb{S}^1$), this Laplacian can be extended by including an additional term for the $\mathbb{S}^1$ component:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} \tag{2}$$

This extension accounts for the additional dimension in the $\mathbb{E}^3 \times \mathbb{S}^1$ space, incorporating the contribution from the circular dimension $\mathbb{S}^1$. This extension is possible because the $\mathbb{E}^3 \times \mathbb{S}^1$ space, despite being four-dimensional, is still flat. The additional term $\frac{\partial^2 f}{\partial (\rho\theta)^2}$ represents the contribution from the circular dimension $\mathbb{S}^1$, where ρ is the radius and θ is the angular coordinate.

2 Exercise 31.5

Question

Why can we make the replacement below in the Laplacian discussed in 31.4 for an nth-order $\mathbb{S}^1$*:*

$$\tfrac{1}{\rho^2}\tfrac{\partial^2}{\partial\theta^2} \to -\tfrac{n^2}{\rho^2}$$

Solution

The replacement in the Laplacian for an nth-order $\mathbb{S}^1$ mode can be understood through eigenfunction analysis. Consider a function on $\mathbb{E}^3 \times \mathbb{S}^1$, particularly in the $\mathbb{S}^1$ part, which is circular. A eigenfunction like $e^{in\theta}$, where mode n is an integer, emerges as a solution to the differential equation involving $\frac{\partial^2}{\partial\theta^2}$.When the Laplacian acts on this eigenfunction, it effectively multiplies it by $-n^2$. Specifically, it satisfies the equation:

$$\frac{\partial^2}{\partial\theta^2}e^{in\theta} = -n^2 e^{in\theta} \tag{3}$$

This leads to the replacement in the Laplacian, transforming the term $\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$ into $-\frac{n^2}{\rho^2}$ for such solutions. Therefore, we can understand why this replacement is valid because of the nature of eigenfunctions.

3 Exercise 22.10

Question

Show that any eigenvalue of a Hermitian operator Q is indeed a real number.

Solution

We start by considering a complex number in the form $\langle\phi|L|\psi\rangle$ where ϕ and ψ are states in a Hilbert space, and L is a linear operator. The complex conjugate of this number is given by $\langle\psi|L^*|\phi\rangle$, where L^* is the conjugate transpose (Hermitian adjoint) of L.

For any states ϕ, ψ, and operator L, we have:

$$\langle\phi|L|\psi\rangle = \langle\psi|L^*|\phi\rangle$$

Now, let's take $\phi = \psi$, and consider L to be a Hermitian operator Q. By definition, for a Hermitian operator, $Q = Q^*$. Therefore, we can write:

$$\langle \psi | Q | \psi \rangle = \langle \psi | Q^* | \psi \rangle$$

This implies:

$$\langle \psi | Q | \psi \rangle \in \mathbb{R}$$

indicating that the inner product of a state with itself under a Hermitian operator is a real number.

Now, if ψ is an eigenfunction of Q with eigenvalue λ, then $Q\psi = \lambda\psi$. Substituting this into our inner product, we get:

$$\langle \psi | Q | \psi \rangle = \langle \psi | \lambda \psi \rangle = \lambda \langle \psi | \psi \rangle$$

Since $\langle \psi | \psi \rangle$ is the inner product of a state with itself, it is equal to $||\psi||^2$, which is always a real and positive number. Therefore, we can write:

$$\lambda ||\psi||^2 = \lambda \langle \psi | \psi \rangle \in \mathbb{R}$$

Since $||\psi||^2$ is real and positive, for the entire expression to be real, the eigenvalue λ must also be real. Therefore, we prove that any eigenvalue of a Hermitian operator Q is indeed a real number.