-: Differential Geometry :-

UNIT-I

⇒ **Tensor:** Tensor are mathematical objects which can be used to describe physical properties just like scalar & vectors. It has magnitude & two or more direction.

⇒ **Einstein Summation Convention:-**

$$a_1x^1 + a_2x^2 + \cdots + a_nx^n = \sum_{i=1}^{n} a_i x^i = a_i x^i$$

⇒ **Dummy Index (summation Index):-** An Index which is repeated (or occur twice) in a given expression. so that the summation convention applies is called Dummy index.

eg:- $a_i x^i$: $i \to$ Dummy index.

⇒ **Free Index:-** An index appearing once in an expression is called free index.

: $a_{ij} x^j$: $i \to$ free index
j :- Dummy index.

Ex:- If $n=3$, write explicitly the system of equation represented by

$$a_{ij} x^j = b_i \quad , \forall\, j, i = 1, 2, 3.$$

$$a_{i1} x^1 + a_{i2} x^2 + a_{i3} x^3 = b_i$$

for, $i=1$ $\quad a_{11}x^1 + a_{12}x^2 + a_{13}x^3 = b_1$

for, $i=2$ $\quad a_{21}x^1 + a_{22}x^2 + a_{23}x^3 = b_2$

for, $i=3$ $\quad a_{31}x^1 + a_{32}x^2 + a_{33}x^3 = b_3.$

NOTE :- (1) :- Dummy index may be replaced by a suitable symbol not already in use,

$$a_{ii}, a_{jj}, a_{kk}$$

(2) :- No suffix may occur more than twice in an expression.

like $a_{ii}x^i \rightarrow$ not meaningful

$$\left.\begin{aligned} a_{ij}x_j &= a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \\ a_{ii}x_j &= a_{11}x_j + a_{22}x_j + a_{33}x_j \end{aligned}\right\} \text{accepted}$$

(3) :- A free index appears once and only once. which in each additive term in an expression.

$$a_i = \underbrace{\varepsilon_{ijk} b_j C_k + D_{ij} C_j}$$

If here, i is free index, then same at RHS.

$i \rightarrow$ free index

(4) :- The free index may be ~~remanred~~ renamed iff it is renamed in every term.

(5) :- The no. of free indices in a term is equal to the rank of Tensor.

	Notation	rank
Scalar	a	0
vector	a_i	1
Tensor	a_{ij}	2

⇒ kronecker Delta :- (δ^i_j) The term kronecker Delta is defined by

$$\delta^i_j = \begin{cases} 1 & , i=j \\ 0 & , i \neq j \end{cases}$$

(1):- $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 + \dots + \delta^n_n$

$i \to$ Dummy index. $\quad = 1 + 1 + 1 + \dots + 1$

$= n.$

(2):- $\delta^i_k A^k = A^i$

LHS $\quad \delta^i_1 A^1 + \delta^i_2 A^2 + \dots + \delta^i_n A^n$

$i = 1$; $\quad \delta^1_1 A^1 + \delta^1_2 A^2 + \dots + \delta^1_n A^n$

$= \delta^1_1 A^1 = A^1 = A^i$

(3):- If $x^1, x^2, \dots x^n$ are independent variable.

then $\quad \delta^i_j = \dfrac{\partial x^i}{\partial x^j}$

RHS :-

$i = j \Rightarrow \dfrac{\partial x^i}{\partial x^i} = 1 = \delta^i_i$

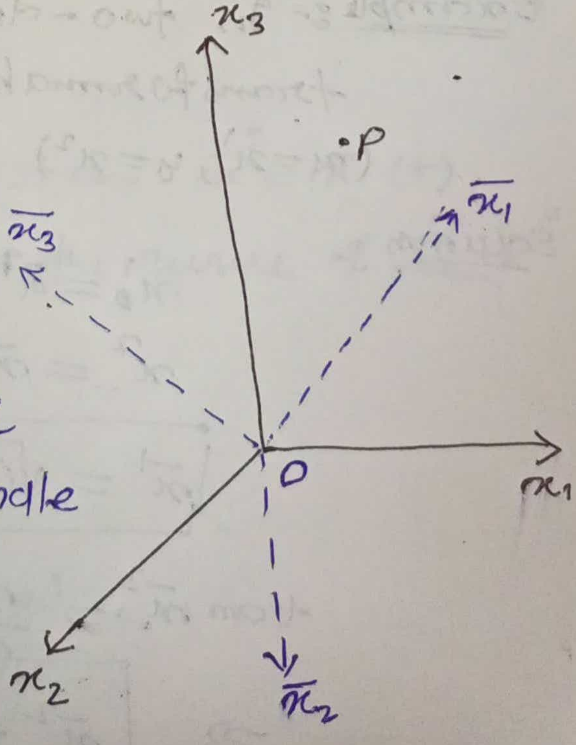
for $i \neq j \Rightarrow \dfrac{\partial x^i}{\partial x^j} = 0$, x^i, x^j are independent.

(4):- $\quad \delta^i_j \delta^j_k = \delta^i_k$ and $\dfrac{\partial x^i}{\partial x^j} \dfrac{\partial x^j}{\partial x^k} = \delta^i_k$.

LHS $\quad \delta^i_j \delta^j_k = \dfrac{\partial x^i}{\partial x^j} \dfrac{\partial x^j}{\partial x^k} = \dfrac{\partial x^i}{\partial x^k} = \delta^i_k$.

$\Rightarrow$ **Coordinate Transformation:-**

consider two set of variables $(x^1, x^2, \dots, x^n)$ & ($\cancel{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n}$) $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ which determine the coordinate of a point in an n-diamensional space in two different coordinate system.



$$\bar{x}^1 = \bar{x}^1 (x^1, x^2, \dots x^n)$$
$$\bar{x}^2 = \bar{x}^2 (x^1, x^2, \dots x^n)$$
$$\vdots$$
$$\bar{x}^n = \bar{x}^n (x^1, x^2, \dots x^n)$$
$$\Rightarrow \quad \bar{x}^i = \bar{x}^i (x^1, x^2, \dots, x^n) \quad \text{--- (1)}$$

differentiating (1) w.r.t x^j, we get

$$\frac{\partial \bar{x}^i}{\partial x^j} = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \frac{\partial \bar{x}^1}{\partial x^3} & \cdots & \frac{\partial \bar{x}^n}{\partial x^n} \\ \vdots & & & & \\ \frac{\partial \bar{x}^n}{\partial x^1} & \frac{\partial \bar{x}^n}{\partial x^2} & \cdots & & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix}_{n \times n}.$$

we shall assume that determinant is non-zero for some range of coordinate x^j so that we can solve eqn (1) for the coordinate x^j.

$$x^i = x^i (\bar{x}^1, \bar{x}^2, \dots \bar{x}^n) \quad \text{--- (2)}$$

eqn (1) & (2) are said to define a coordinate transformation.

Example :- In two-dimensional Euclidian space, the transformation from cartesian coordinate ($x = x^1$, $y = x^2$) to polar coordinate ($r = \bar{x}^1$, $\theta = \bar{x}^2$)

Solution :-

$$x^1 = \bar{x}^1 \cos \bar{x}^2$$
$$x^2 = \bar{x}^1 \sin \bar{x}^2$$

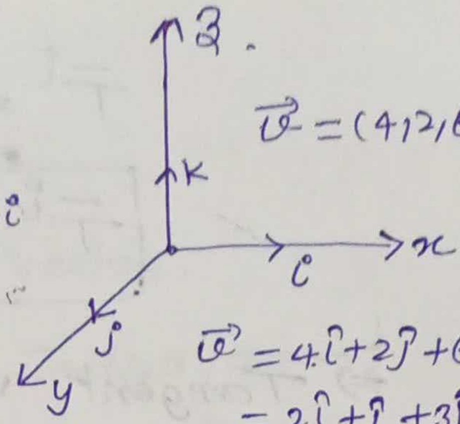
$$\boxed{\bar{x}^1 = \sqrt{(x^1)^2 + (x^2)^2}}$$

$$\tan \bar{x}^2 = \frac{x^2}{x^1}$$

$$\Rightarrow \quad \boxed{\bar{x}^2 = \tan^{-1}\left(\frac{x^2}{x^1}\right)}$$

⇒ Contravariant Tensor :-

Let A^i & $\bar{A}^i$ be the Component of a vector field A in x^i and $\bar{x}^i$ coordinate system respectively. & they are connected by the ~~relat~~ relation.



$\vec{v} = (4,2,6)$

$\vec{v} = 4\hat{i} + 2\hat{j} + 6\hat{k}$
$= 2\hat{i} + \hat{j} + 3\hat{k}$

called contravariant Tensor.

$$\bar{A}^i = A^j \frac{\partial \bar{x}^i}{\partial x^j}$$

Then the vector field A is contravariant tensor of rank 1. (contravariant vector).

Example : show that the tangent vector of a smooth curve transform as a contravariant tensor of rank 1.

Sol^n :- Let γ be a smooth curve Parametrically represented in x^i - coordinate system by

$$x^i = x^i(t)$$
$$= (x^1(t), x^2(t), \ldots, x^n(t))$$

$\gamma(t)$
$= (a\cos t, a\sin t)$
$= (t, t^2, t^3)$

The tangent vector T^i is defined by

$$T^i = \frac{dx^i}{dt}$$

curve in $\bar{x}^i$ coordinate system $\bar{x}^i = \bar{x}^i(t)$

The tangent for vector for the curve γ in $\bar{x}^i$ coordinate system.

$$\bar{T}^i = \frac{d\bar{x}^i}{dt}$$

By chain rule.

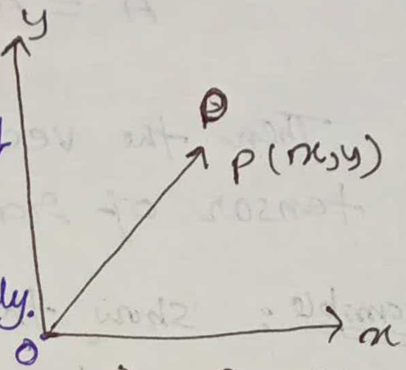
$$\bar{T}^i = \frac{\partial \bar{x}^i}{\partial x^j} \frac{dx^j}{dt}$$

$$\bar{T}^i = \frac{\partial \bar{x}^i}{\partial x^j} T^j$$

$$\boxed{\bar{T}^i = T^j \frac{\partial \bar{x}^i}{\partial x^j}}$$

⇒ Tangent vector transform like Contravarient tensor.

⇒ Covariant Tensor :-



Let A_i & $\bar{A}_i$ be the component of a vector field A in x^i & $\bar{x}^i$ coordinate system respectively, and they are connected by the relation

$$\vec{P} = x\hat{i} + y\hat{j}$$
$$\vec{P}\cdot\hat{i} = x$$
$$\hat{P}\cdot\hat{j} = y.$$

$$\boxed{\bar{A}_i = A_j \frac{\partial x^j}{\partial \bar{x}^i}}$$

then the vector field A is a covariant tensor of rank one (covariant vector).

Ex :- show that the gradient of an arbitrary differentiable function is a covariant vector.

Sol^n :- Let $f(x)$ be a differentiable function in x^i-coordinate system. The gradient of f.

$$\nabla f = \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}, \cdots \frac{\partial f}{\partial x^n}\right) \quad \Bigg| \quad \nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$A_i = \nabla f = \frac{\partial f}{\partial x^i}$$

In $\bar{x}^i$ coordinate system, grad. is defined as

$$\bar{A}_i = \cancel{\partial f}\ \nabla f = \frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i}$$

$$= A_j \frac{\partial x^j}{\partial \bar{x}^i}$$

$$\boxed{\bar{A}_i = A_j \frac{\partial x^j}{\partial \bar{x}^i}}$$

⇒ gradient is a Covariant vector.

Example: If the Component of a Contravariant vector of rank 1 in x-coordinate system are 5 & 7. Find its component in $\bar{x}$-coordinate system if

$$\bar{x}^1 = 7(x^1)^2 \; ; \; \bar{x}^2 = 5(x^1)^2 + 2(x^2)^2$$

Sol^n: given that $A^1 = 5$ & $A^2 = 7$.

$$\bar{A}^i = A^j \frac{\partial \bar{x}^i}{\partial x^j}$$

$$\bar{A}^i = A^1 \frac{\partial \bar{x}^i}{\partial x^1} + A^2 \frac{\partial \bar{x}^i}{\partial x^2}$$

$$\frac{\partial \bar{x}^1}{\partial x^1} = 14x^1$$

$$\frac{\partial \bar{x}^1}{\partial x^2} = 0$$

Now, $i=1$

$$\bar{A}^1 = A^1 \frac{\partial \bar{x}^1}{\partial x^1} + A^2 \frac{\partial \bar{x}^1}{\partial x^2}$$

$$= 5 \times 14x^1 + 7 \times 0$$

$$\bar{A}^1 = 70x^1$$

$$\frac{\partial \bar{x}^2}{\partial x^1} = 10x^1$$

$$\frac{\partial \bar{x}^2}{\partial x^2} = 4x^2$$

again $i=2$

$$\bar{A}^2 = A^1 \frac{\partial \bar{x}^2}{\partial x^1} + A^2 \frac{\partial \bar{x}^2}{\partial x^2}$$

$$\bar{A}^2 = 5 \cdot 10x^1 + 7 \cdot 4x^2$$

$$\bar{A}^2 = 50x^1 + 28x^2$$

Q 8. Find the Component of a vector in polar coordinate System whose component in Cartesian Coordinate system are $\dot{x}, \dot{y}$, and $\ddot{x}, \ddot{y}$.

Sol^n given that $A_1 = \dot{x}$ & $A_2 = \dot{y}$

$(x^1, x^2, \dots x^n)$ $x^1 x^2$ $\quad$ x, y (x, y) (r, θ)

$x^1 = x \;, \; x^2 = y.$

$\bar{x}^1 = r \quad \bar{x}^2 = \theta$

$x^1 = \bar{x}^1 \cos \bar{x}^2 \;, \quad x^2 = \bar{x}^1 \sin \bar{x}^2$

$A_1 = \dot{x} \quad A_2 = \dot{y} \;, \quad \bar{A}_1 = ? = \bar{A}_2 = ?$

$\bar{A}_i = A_j \dfrac{\partial x^j}{\partial \bar{x}^i}$

$\bar{A}_i = A_1 \dfrac{\partial x^1}{\partial \bar{x}^i} + A_2 \dfrac{\partial x^2}{\partial \bar{x}^i}$

$\dfrac{\partial x^1}{\partial \bar{x}^1} = \cos \bar{x}^2$

$\dfrac{\partial x^2}{\partial \bar{x}^1} = \sin \bar{x}^2$

$i = 1 \quad \bar{A}_1 = A_1 \dfrac{\partial x^1}{\partial \bar{x}^1} + A_2 \dfrac{\partial x^2}{\partial \bar{x}^1}$

$\bar{A}_1 = \dot{x} \cos \bar{x}^2 + \dot{y} \sin \bar{x}^2$

$\bar{A}_2 = A_1 \dfrac{\partial x^1}{\partial \bar{x}^2} + A_2 \dfrac{\partial x^2}{\partial \bar{x}^2}$

$\quad = \dot{x}(-\bar{x}^1 \sin \bar{x}^2) + \dot{y}\, \bar{x}^1 \cos \bar{x}^2$

$\dfrac{\partial x^1}{\partial \bar{x}^2} = -\bar{x}^1 \sin \bar{x}^2$

$\dfrac{\partial x^2}{\partial \bar{x}^2} = \bar{x}^1 \cos \bar{x}^2$

$\bar{A}_2 = -\dot{x}\, \bar{x}^1 \sin \bar{x}^2 + \dot{y}\, \bar{x}^1 \cos \bar{x}^2$.

again we have $A_1 = \ddot{x} \quad A_2 = \ddot{y}$

then $\bar{A}_i = A_j \dfrac{\partial x^j}{\partial \bar{x}^i}$

$\bar{A}_i = A_1 \dfrac{\partial x^1}{\partial \bar{x}^i} + A_2 \dfrac{\partial x^2}{\partial \bar{x}^i}$

$\bar{A}_1 = A_1 \dfrac{\partial x^1}{\partial \bar{x}^1} + A_2 \dfrac{\partial x^2}{\partial \bar{x}^1}$

$\bar{A}_1 = \ddot{x} \cos \bar{x}^2 + \ddot{y} \sin \bar{x}^2$.

& $\bar{A}_2 = -\ddot{x}\, \bar{x}^1 \sin \bar{x}^2 + \ddot{y}\, \bar{x}^1 \cos \bar{x}^2$.

$\Rightarrow$ **Tensor of rank 2:** If quantities A^{ij} in x^i-coordinate system are related to quantities $\bar{A}^{pq}$ in $\bar{x}^i$-coordinate system by the

$$\boxed{\bar{A}^{pq} = A^{ij} \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j}}$$

Then the quantities A^{ij} are called component of contravariant tensor of rank 2.

similarly

$$\boxed{\bar{A}_{pq} = A_{ij} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q}}$$

$A_{ij} \rightarrow$ Component of Covariant Tensor of rank 2.

Next,

$$\boxed{\bar{A}^p_q = A^i_j \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q}}$$

A^i_j are called component of mixed tensor of rank 2.

Example:

$$\bar{A}^{abc}_{d} = A^{ijk}_{l} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^c}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$\bar{A}^{pq}_{rs} = A^{ij}_{kl} \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial x^k}{\partial \bar{x}^r} \frac{\partial x^l}{\partial \bar{x}^s} \quad \left(\begin{array}{c}\text{mixed Tensor} \\ \text{rank 4.}\end{array}\right)$$

$$\bar{A}^{ijk} = A^{pqr} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \quad \text{(contravariant)}$$

(rank 3)

⇒ **Symmetric Tensor :-**

$$A_{ij} = A_{ji} \rightarrow \text{symmetric}$$

$$A_{ij} = -A_{ji} \rightarrow \text{skew-symmetric}$$

Defⁿ :- If two indices of a Tensor either covariant or contravariant are interchanged & the resulting tensor remains the same. The Tensor is said to be symmetric w.r.t indices.

$$A_{ij} = A_{ji} \quad \text{or} \quad A^{ij} = A^{ji}$$

$$\rightarrow A^{ij}_{k} = A^{kj}_{i} \quad (\text{symm. w.r.t } i \,\&\, k).$$

Thm :- The symmetry property of a tensor is independent of the coordinate system.

Proof :- Let a Tensor A^{ijk}_{l} is symmetric w.r.t i, j.

$$\Rightarrow A^{ijk}_{l} = A^{jik}_{l} \quad \text{--- (1)}$$

$\because A^{ijk}_{l}$ transform in a new coordinate system as

$$\bar{A}^{abc}_{d} = A^{ijk}_{l} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^c}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$= A^{jik}_{l} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^c}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^d} \quad [\text{by (1)}]$$

$$= A^{jik}_{l} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^c}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$= \bar{A}^{bac}_{d}. \quad (\Rightarrow \text{given Tensor is symmetric in new coordinate system.})$$

⇒ **Thm :-** The symmetric properties of a Tensor is defined only when the indices are of the same type (either Covariant or Contravariant).

Proof:- Consider the Tensor A^{ijk}_{l} which is symmetric w.r.t i & l, then

$$A^{ijk}_{l} = A^{ljk}_{i} \quad \text{--- (1)}$$

$\because$ A^{ijk}_{l} Transform in a new coordinate system as

$$\bar{A}^{abc}_{d} = A^{ijk}_{l} \frac{\partial \bar{x}^{a}}{\partial x^{i}} \frac{\partial \bar{x}^{b}}{\partial x^{j}} \frac{\partial \bar{x}^{c}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{d}}$$

$$= A^{ljk}_{i} \frac{\partial \bar{x}^{a}}{\partial x^{i}} \frac{\partial \bar{x}^{b}}{\partial x^{j}} \frac{\partial \bar{x}^{c}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{d}} \quad \text{[by (1)]}.$$

(doubt).

$$= A^{ljk}_{i} \frac{\partial \bar{x}^{a}}{\partial x^{l}} \frac{\partial \bar{x}^{b}}{\partial x^{j}} \frac{\partial \bar{x}^{c}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{d}} \quad \text{--- (2)}$$

But according to transformation law of Tensor

$$\bar{A}^{dbc}_{a} = A^{ijk}_{l} \frac{\partial \bar{x}^{d}}{\partial x^{i}} \frac{\partial \bar{x}^{b}}{\partial x^{j}} \frac{\partial \bar{x}^{c}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{a}} \quad \text{--- (3)}$$

from (2) & (3)

$$\boxed{\bar{A}^{abc}_{d} \neq \bar{A}^{dbc}_{a}}$$

⇒ Algebra of Tensor:-

(1.) :- Addition & Substraction :- A linear Combination of Tensors of same type & same rank is a Tensor of same type & same rank.

→ If A_{ij} and B_{ij} are second rank Covariant Tensor and α, β are scalars then

$$C_{ij} = \alpha A_{ij} + \beta B_{ij}$$

is also, a Covariant tensor of rank 2.

$$\bar{C}_{ij} = \alpha \bar{A}_{ij} + \beta \bar{B}_{ij}$$

$$= \alpha A_{kl} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} + \beta B_{kl} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j}$$

$$= (\alpha A_{kl} + \beta B_{kl}) \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j}$$

$$\bar{C}_{ij} = C_{kl} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} .$$

⇒ **Theorem:-** Every Covarient Tensor of rank 2 can be expressed as sum of symmetric and skew-symmetric Tensor.

Proof:-

$A_{ij} \rightarrow$ covariant Tensor of rank 2.

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$$

$$= P_{ij} + Q_{ij}$$

$$P_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$$

$$= \frac{1}{2}(A_{ji} + A_{ij})$$

$$= P_{ji}$$

& $Q_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$

$$= -\frac{1}{2}(A_{ji} - A_{ij})$$

$$= -Q_{ji}$$

⇒ Contraction:- The process of summing over a Covariant & Contravariant index of a Tensor, To get a Tensor s.t the rank of this new Tensor is lowered by 2 (two) is called Contraction.

Example:- Consider a mixed Tensor A^i_{jkl}.

$$\bar{A}^a_{bcd} = A^i_{jkl} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial x^l}{\partial \bar{x}^d}$$

for $a=b$.

$$\bar{A}^a_{acd} = A^i_{jkl} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^a} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$= A^i_{jkl} \frac{\partial x^j}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$= A^i_{jkl} \delta^j_i \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$\bar{A}^a_{acd} = A^i_{ikl} \delta^i_i \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$= A^i_{ikl} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$\bar{B}_{cd} = B_{kl} \frac{\partial x^k}{\partial \bar{x}^c} \frac{\partial x^l}{\partial \bar{x}^d}$$

$$\boxed{\delta^j_i = \frac{\partial x^j}{\partial x^i} = \begin{cases} 1 & i=j \\ 0, & i \neq j \end{cases}}$$

which shows that A^i_{ikl} is a Tensor of rank 2.

⇒ Outer Product:- The Product of Two Tensor is a Tensor whose rank is the sum of the ranks of the given Tensors. This product is called outer product.

Example:- Let A^{ij}_{lmn} & B^k_{pq} are two mixed Tensor

$$\bar{A}^{ab}_{cde} \cdot \bar{B}^{f}_{gh} = \left(A^{ij}_{lmn} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial x^l}{\partial \bar{x}^c} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \right)$$

$$\cdot \left(B^{k}_{pq} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h} \right)$$

$$= A^{ij}_{lmn} B^{k}_{pq} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial x^l}{\partial \bar{x}^c} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h}$$

$$= C^{ijk}_{lmnpq} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial x^l}{\partial \bar{x}^c} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h}$$

$$\bar{C}^{abf}_{cdegh} = C^{ijk}_{lmnpq} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^c} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h}.$$

we can easily see that The rank of Tensor is $8 = (3+5)$.

⇒ **Inner Product:-** Two Tensor may be Combined by first forming their outer product and then Contracting it w.r.t an index of one Tensor and an index of opposite character of the other. This process is called Inner multiplication of Two Tensor and the result is called an Inner Product of Two Tensor.

Ex:-

$$\bar{C}^{abf}_{cdegh} = C^{ijk}_{lmnpq} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^c} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h}$$

Let $a = c$

$$\bar{C}^{abf}_{adegh} = C^{ijk}_{lmnpq} \frac{\cancel{\partial \bar{x}^a}}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^l}{\cancel{\partial \bar{x}^a}} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h}$$

$$= C^{ijk}_{lmnpq} \frac{\partial x^l}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h}$$

$$= C^{ijk}_{lmnpq} \, \delta^l_i \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h}.$$

$$\bar{C}^{abf}_{adegh} = C^{ijk}_{imnpq} \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^f}{\partial x^k} \frac{\partial x^m}{\partial \bar{x}^d} \frac{\partial x^n}{\partial \bar{x}^e} \frac{\partial x^p}{\partial \bar{x}^g} \frac{\partial x^q}{\partial \bar{x}^h}.$$

Example: If A^i and B^j are Contravariant vectors and C_{ij} is a Covariant Tensor (~~vector~~). Find the nature of $A^i B^j C_{ij}$.

Solⁿ:

$$\bar{A}^a \bar{B}^b \bar{C}_{ab} = \left(A^i \frac{\partial \bar{x}^a}{\partial x^i}\right)\left(B^j \frac{\partial \bar{x}^b}{\partial x^j}\right)\left(C_{ij} \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b}\right)$$

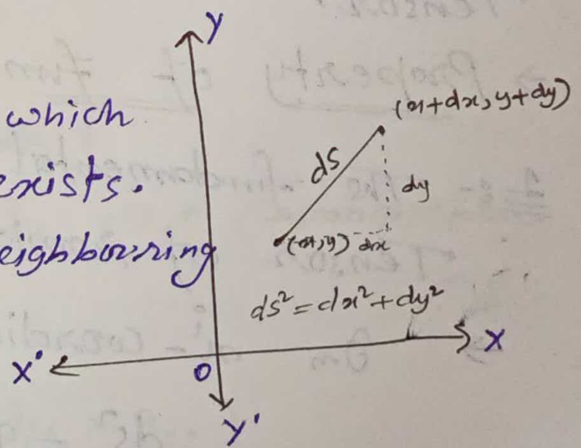
$$= A^i B^j C_{ij} \cancel{\frac{\partial \bar{x}^a}{\partial x^i}} \cancel{\frac{\partial \bar{x}^b}{\partial x^j}} \cancel{\frac{\partial x^i}{\partial \bar{x}^a}} \cancel{\frac{\partial x^j}{\partial \bar{x}^b}}$$

$= A^i B^j C_{ij}$ is a 0 rank Tensor.

$\Rightarrow A^i B^j C_{ij}$ is a scalar.

⇒ Metric Tensor:

Consider a Euclidean plane in which rectangular Coordinate system exists. if (x,y) & $(x+dx, y+dy)$ are neighbouring points in this plane then the distance ds b/w two point.



$$ds^2 = dx^2 + dy^2$$

in 3-Dimensional Euclidean space

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$= 1 \cdot dx^1 dx^1 + 1 \cdot dx^2 dx^2 + 1 \cdot dx^3 dx^3$$

$$= 1 \cdot dx^1 dx^1 + 0 \cdot dx^1 dx^2 + 0 \cdot dx^1 dx^3$$
$$+ 0 \cdot dx^2 dx^1 + 1 \cdot dx^2 dx^2 + 0 \cdot dx^2 dx^3$$
$$+ 0 \cdot dx^3 dx^1 + 0 \cdot dx^3 dx^2 + 1 \cdot dx^3 dx^3$$

$\Rightarrow ds^2 = g_{11}\, dx^1 dx^1 + g_{12}\, dx^1 dx^2 + g_{13}\, dx^1 dx^3$

$+ g_{21}\, dx^2 dx^1 + g_{22}\, dx^2 dx^2 + g_{23}\, dx^2 dx^3$

$+ g_{31}\, dx^3 dx^1 + g_{32}\, dx^3 dx^2 + g_{33}\, dx^3 dx^3$

where $g_{11} = g_{22} = g_{33} = 1.$

& $g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = g_{32} = 0$

$\Rightarrow ds^2 = g_{ij}\, dx^i dx^j \quad , i,j = 1,2,3.$

In n-dimensional space with coordinate $(x^1, x^2, \dots, x^n)$

$$ds^2 = g_{ij}\, dx^i dx^j \quad ; i,j = 1,2,\dots,n$$

The space which satisfies above equation is called Riemannian space and g_{ij} is called fundamental Tensor.

→ Property of fundamental Tensor :-

1 :- The fundamental Tensor g_{ij} is a Covariant Tensor of rank 2.

→ In x^i- coordinate system

$$ds^2 = g_{ij}\, dx^i dx^j \quad \text{—(1)}$$

in $\bar{x}^i$ - coordinate system.

$$ds^2 = \bar{g}_{ij}\, d\bar{x}^i d\bar{x}^j \quad \text{—(2)}$$

from (1) & (2)

$$g_{ij}\, dx^i dx^j = \bar{g}_{ij}\, d\bar{x}^i d\bar{x}^j$$

$$\bar{x}^i = \bar{x}^i (x^1, x^2, x^3, \dots, x^n).$$

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^1} dx^1 + \frac{\partial \bar{x}^i}{\partial x^2} dx^2 + \dots + \frac{\partial \bar{x}^i}{\partial x^n} dx^n.$$

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^\alpha} dx^\alpha \quad , \quad d\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x^\beta} dx^\beta$$

$$g_{ij}\, dx^i dx^j = \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^\alpha} dx^\alpha \frac{\partial \bar{x}^j}{\partial x^\beta} dx^\beta$$

$$= \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial \bar{x}^j}{\partial x^\beta} dx^\alpha dx^\beta$$

i,j are dummy suffix, so, we replace it by α, β respectively.

$$g_{\alpha\beta}\, dx^\alpha dx^\beta = \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial \bar{x}^j}{\partial x^\beta} dx^\alpha dx^\beta$$

$$\Rightarrow g_{\alpha\beta} = \bar{g}_{ij} \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial \bar{x}^j}{\partial x^\beta}$$

$$\boxed{\bar{g}_{ij} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\beta}{\partial \bar{x}^j}}$$

g_{ij} is a covariant Tensor of rank 2.

2:- The fundamental Tensor g_{ij} is a symmetric Tensor.

$\rightarrow$ $$g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) + \frac{1}{2}(g_{ij} - g_{ji})$$

$$g_{ij} = S_{ij} + A_{ij}$$

where $S_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$

& $A_{ij} = \frac{1}{2}(g_{ij} - g_{ji})$

$$ds^2 = g_{ij}\, dx^i dx^j = (S_{ij} + A_{ij})\, dx^i dx^j$$

where S_{ij} is a symmetric Tensor & A_{ij} is a skew sym. Tensor.

$$\Rightarrow g_{ij}\,dx^i dx^j = S_{ij}\,dx^i dx^j + A_{ij}\,dx^i dx^j$$

$$(g_{ij} - S_{ij})\,dx^i dx^j = A_{ij}\,dx^i dx^j \quad \text{—} \circledast$$

Now,

$$A_{ij}\,dx^i dx^j = A_{ji}\,dx^j dx^i$$

on interchanging the dummy suffices in RHS

$$A_{ij}\,dx^i dx^j = -\cancel{A_{ji}} - A_{ij}\,dx^j dx^i$$

$$A_{ij}\,dx^i dx^j + A_{ij}\,dx^i dx^j = 0$$

$$2A_{ij}\,dx^i dx^j = 0$$

$$\Rightarrow A_{ij}\,dx^i dx^j = 0$$

using this in eqn $\circledast$, we get

$$(g_{ij} - S_{ij})\,dx^i dx^j = 0$$

$$\Rightarrow \boxed{g_{ij} = S_{ij}} \Rightarrow \text{sym. Tensor.}$$

3:- $g_{ij}\,dx^i dx^j$ is an invariant (scalar).

$$\bar{g}_{\alpha\beta}\,d\bar{x}^\alpha d\bar{x}^\beta = \left(g_{ij}\frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}\right) d\bar{x}^\alpha d\bar{x}^\beta$$

$$= g_{ij}\left(\frac{\partial x^i}{\partial \bar{x}^\alpha}d\bar{x}^\alpha\right)\left(\frac{\partial x^j}{\partial \bar{x}^\beta}d\bar{x}^\beta\right)$$

$$\bar{g}_{\alpha\beta}\,d\bar{x}^\alpha d\bar{x}^\beta = g_{ij}\,dx^i dx^j$$

$\therefore g_{ij}\,dx^i dx^j$ is an invariant (scalar).

[as $x^i = x^i(\bar{x}^1, \bar{x}^2, \dots \bar{x}^n)$

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^1}d\bar{x}^1 + \frac{\partial x^i}{\partial \bar{x}^2}d\bar{x}^2 + \dots + \frac{\partial x^i}{\partial \bar{x}^n}d\bar{x}^n$$

$$= \frac{\partial x^i}{\partial \bar{x}^\alpha}d\bar{x}^\alpha.$$

and $dx^j = \dfrac{\partial x^j}{\partial \bar{x}^\beta}d\bar{x}^\beta.$]

⇒ Conjugate Tensor:-

$$g^{ij} = \frac{\text{Cofactor of } g_{ij} \text{ in } \det(g_{ij})}{g}.$$

where $g = \det(g_{ij})$

→ Conjugate Tensor is a Contravariant Tensor of rank 2. which is symmetric.

$ds^2 = g_{ij}\, dx^i dx^j$

g_{ij} → metric Tensor ≈ fundamental Tensor.

$$g = \det(g_{ij}) = \begin{bmatrix} g_{11} & g_{12}, & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}$$

⇒ $g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & ; j=k \\ 0 & ; j \neq k \end{cases}$

Proof:- for $j=k$.

$g_{ij} A^{ij} = d\ (\det)$

$g_{ij} \frac{A^{ij}}{d} = 1$

$g_{ij}\, g^{ij} = 1$

$g_{ij} A^{ik} = 0 \cdot (j \neq k).$

Example:- find fundamental metric Tensor (f.m.t) & Conjugate metric Tensor (c.m.t) in
(i) spherical Polar Coordinate.
(ii) Cylindrical Coordinate System
corresponding to the metric

$$ds^2 = dx^2 + dy^2 + dz^2$$

Sol^n:- given $ds^2 = dx^2 + dy^2 + dz^2$

(i) $ds^2 = g_{ij}\, dx^i dx^j$

$\bar{x}(r, \theta, \phi)$:- ∴ $x^1 = x$, $x^2 = y$ $x^3 = z$.

$\bar{x}^1 = r$, $\bar{x}^2 = \theta$ $\bar{x}^3 = \phi$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$.

now $\quad ds^2 = dx^2 + dy^2 + dz^2$

$$g_{11} = 1 = g_{22} = g_{33} \quad , \; g_{ij} = 0 \; , \; i \neq j \; , \; i,j = 1,2,3.$$

Transformation law of g_{ij}

$$\bar{g}_{ij} = g_{ab} \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial x^b}{\partial \bar{x}^j}$$

$$\bar{g}_{11} = g_{ab} \frac{\partial x^a}{\partial \bar{x}^1} \frac{\partial x^b}{\partial \bar{x}^1}$$

$$\bar{g}_{11} = g_{aa} \frac{\partial x^a}{\partial \bar{x}^1} \frac{\partial x^a}{\partial \bar{x}^1} \qquad \left[\begin{array}{ll} \because g_{ab} = 0 & a \neq b \\ g_{aa} = 1 & a = b \end{array} \right]$$

$$\bar{g}_{11} = g_{11} \frac{\partial x^1}{\partial \bar{x}^1} \frac{\partial x^1}{\partial \bar{x}^1} + g_{22} \frac{\partial x^2}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^1} + g_{33} \frac{\partial x^3}{\partial \bar{x}^1} \frac{\partial x^3}{\partial \bar{x}^1}$$

$$= g_{11} \left(\frac{\partial x^1}{\partial \bar{x}^1} \right)^2 + g_{22} \left(\frac{\partial x^2}{\partial \bar{x}^1} \right)^2 + g_{33} \left(\frac{\partial x^3}{\partial \bar{x}^1} \right)^2$$

$$= \left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2$$

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta.$$

$$= \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta = 1.$$

similarly

$$\bar{g}_{22} = g_{ab} \frac{\partial x^a}{\partial \bar{x}^2} \frac{\partial x^b}{\partial \bar{x}^2}.$$

$$= g_{aa} \frac{\partial x^a}{\partial \bar{x}^2} \frac{\partial x^a}{\partial \bar{x}^2} = \left(\frac{\partial x^a}{\partial \bar{x}^2} \right)^2$$

$$\left[\because \underset{a=b}{g_{aa} = 1} \text{ ad } \underset{a \neq b}{g_{ab} = 0} \right]$$

$$= \left(\frac{\partial x^1}{\partial \bar{x}^2}\right)^2 + \left(\frac{\partial x^2}{\partial \bar{x}^2}\right)^2 + \left(\frac{\partial x^3}{\partial \bar{x}^2}\right)^2$$

$$= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + (-r\sin\theta)^2$$

$$= r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$\bar{g}_{22} = r^2.$$

again. $\bar{g}_{33} = g_{ab}\dfrac{\partial x^a}{\partial \bar{x}^3}\dfrac{\partial x^b}{\partial \bar{x}^3}$

$$= g_{aa}\frac{\partial x^a}{\partial \bar{x}^3}\frac{\partial x^a}{\partial \bar{x}^3} = g_{aa}\left(\frac{\partial x^a}{\partial \bar{x}^3}\right)^2$$

$$\bar{g}_{33} = \underbrace{g_{11}}_{1}\left(\frac{\partial x^1}{\partial \bar{x}^3}\right)^2 + \underbrace{g_{22}}_{1}\left(\frac{\partial x^2}{\partial \bar{x}^3}\right)^2 + \underbrace{g_{33}}_{1}\left(\frac{\partial x^3}{\partial \bar{x}^3}\right)^2$$

$$= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$= (-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0$$

$$= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi$$

$$= r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)$$

$$\bar{g}_{33} = r^2\sin^2\theta$$

$$\bar{g}_{ij} = \begin{bmatrix} \bar{g}_{11} & \bar{g}_{12} & \bar{g}_{13} \\ \bar{g}_{21} & \bar{g}_{22} & \bar{g}_{23} \\ \bar{g}_{31} & \bar{g}_{32} & \bar{g}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$ → fundamental metric Tensor.

$$ds^2 = \bar{g}_{11}\,d\bar{x}^1 d\bar{x}^1 + \bar{g}_{22}\,d\bar{x}^2 d\bar{x}^2 + \bar{g}_{33}\,d\bar{x}^3 d\bar{x}^3$$

$$= \bar{g}_{11}(d\bar{x}^1)^2 + \bar{g}_{22}(d\bar{x}^2)^2 + \bar{g}_{33}(d\bar{x}^3)^2$$

$$ds^2 = 1\cdot dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

$$g = |\bar{g}_{ij}| = r^4 \sin^2\theta.$$

$$\bar{g}^{ij} = \frac{\text{cofactor of } \bar{g}_{ij} \text{ in det } \bar{g}_{ij}}{g = \det(\bar{g}_{ij})}$$

$$\bar{g}^{11} = \frac{\bar{A}^{11}}{g} = \frac{r^4 \sin^2\theta}{r^4 \sin^2\theta} = 1.$$

$$\bar{g}^{22} = \frac{\bar{A}^{22}}{g} = \frac{r^2 \sin^2\theta}{r^4 \sin^2\theta} = \frac{1}{r^2}$$

$$\bar{g}^{33} = \frac{\bar{A}^{33}}{g} = \frac{r^2}{r^4 \sin^2\theta} = \frac{1}{r^2 \sin^2\theta}$$

$$\therefore \quad \bar{g}^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \rightarrow \text{Conjugate metric Tensor.}$$

(ii) for cylindrical coordinate system. $\bar{x}(r, \theta, z)$.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

$$x^1 = x, \quad x^2 = y, \quad x^3 = z.$$

$$\bar{x}^1 = r, \quad \bar{x}^2 = \theta, \quad \bar{x}^3 = z.$$

given metric $ds^2 = dx^2 + dy^2 + dz^2$

$$\therefore \quad g_{11} = g_{22} = g_{33} = 1 \;\&\; g_{ij} = 0 \;\forall\; i \neq j$$

$i, j = 1, 2, 3.$

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

Transformation law of g_{ij}

$$\bar{g}_{ij} = g_{ab} \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial x^b}{\partial \bar{x}^j}$$

$$\bar{g}_{11} = g_{ab} \frac{\partial x^a}{\partial \bar{x}^1} \frac{\partial x^b}{\partial \bar{x}^1}$$

$$\bar{g}_{11} = g_{aa} \frac{\partial x^a}{\partial \bar{x}^1} \frac{\partial x^a}{\partial \bar{x}^1} = g_{aa} \left(\frac{\partial x^a}{\partial \bar{x}^1} \right)^2$$

$$\bar{g}_{11} = g_{11} \left(\frac{\partial x^1}{\partial \bar{x}^1} \right)^2 + g_{22} \left(\frac{\partial x^2}{\partial \bar{x}^1} \right)^2 + g_{33} \left(\frac{\partial x^3}{\partial \bar{x}^1} \right)^2$$

$$= \left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2$$

$$= \cos^2\theta + \sin^2\theta + 0 = 1.$$

again $\bar{g}_{22} = g_{aa} \left(\frac{\partial x^a}{\partial \bar{x}^2} \right)^2$ $\left[\because g_{ab} = 0 \text{ for } a \neq b \atop g_{aa} = 1 \text{ for } a = b \right]$

$$\bar{g}_{22} = g_{11} \left(\frac{\partial x^1}{\partial \bar{x}^2} \right)^2 + g_{22} \left(\frac{\partial x^2}{\partial \bar{x}^2} \right)^2 + g_{33} \left(\frac{\partial x^3}{\partial \bar{x}^2} \right)^2$$

$$= \left(\frac{\partial x}{\partial \theta} \right)^2 + \left(\frac{\partial y}{\partial \theta} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2$$

$$= r^2 \sin^2\theta + r^2 \cos^2\theta + 0 = r^2$$

again $\bar{g}_{33} = g_{aa} \left(\frac{\partial x^a}{\partial \bar{x}^3} \right)^2$ $[\because g_{ab} = 0 \ \forall \ a \neq b.]$

$$\bar{g}_{33} = g_{11} \left(\frac{\partial x^1}{\partial \bar{x}^3} \right)^2 + g_{22} \left(\frac{\partial x^2}{\partial \bar{x}^3} \right)^2 + g_{33} \left(\frac{\partial x^3}{\partial \bar{x}^3} \right)^2$$

$$= \left(\frac{\partial x}{\partial z} \right)^2 + \left(\frac{\partial y}{\partial z} \right)^2 + \left(\frac{\partial z}{\partial z} \right)^2$$

$$= 0 + 0 + 1 = 1.$$

$$\bar{g}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{f.m.t (fundamental metric Tensor)}$$

$$ds^2 = \bar{g}_{11} \, d\bar{x}^1 d\bar{x}^1 + \bar{g}_{22} \, d\bar{x}^2 d\bar{x}^2 + \bar{g}_{33} \, d\bar{x}^3 d\bar{x}^3$$

$$= \bar{g}_{11} (d\bar{x}^1)^2 + \bar{g}_{22} (d\bar{x}^2)^2 + \bar{g}_{33} (d\bar{x}^3)^2$$

$ds^2 = 1 \cdot dr^2 + r^2 d\theta^2 + 1 \cdot dz^2$. (cylindrical coordinates)

Now, $g = |\bar{g}_{ij}| = 1(r^2) = r^2.$

$$\bar{g}^{ij} = \frac{\text{cofactor of } \bar{g}_{ij} \text{ in det } \bar{g}_{ij}}{g}$$

$$\bar{g}^{11} = \frac{\bar{A}^{11}}{g} = \frac{r^2}{r^2} = 1$$

$$\bar{g}^{22} = \frac{\bar{A}^{22}}{g} = \frac{1}{r^2} = \frac{1}{r^2}$$

$$\bar{g}^{33} = \frac{\bar{A}^{33}}{g} = \frac{r^2}{r^2} = 1$$

$$\therefore \bar{g}^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\bar{g}^{ij} \rightarrow$ conjugate metric Tensor.

Example :- Find fundamental metric Tensor & Conjugate metric Tensor of the following -

1. $ds^2 = (dx^1)^2 - 2(dx^2)^2 + 3(dx^3)^3 - 8\, dx^2 dx^3$
2. $ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6\, dx^1 dx^2 + 4\, dx^2 dx^3$

Solution :- 1. $ds^2 = (dx^1)^2 - 2(dx^2)^2 + 3(dx^3)^2 - 8\, dx^2 dx^3$

$$\because ds^2 = g_{ij}\, dx^i dx^j$$

$$\Rightarrow g_{11} = 1,\ g_{22} = -2,\ g_{33} = 3,\ g_{23} = -4 = g_{32}$$

So, now, (f.m.t) $\rightarrow$ fundamental metric Tensor,

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -4 \\ 0 & -4 & 3 \end{bmatrix}$$

$$g^{ij} = (-1)^{i+j} \frac{A^{ij}}{g}$$

$A^{ij} \rightarrow$ cofactor

$g = |g_{ij}|$.

$$g = |g_{ij}| = 1(-6 - \cancel{18}\,{-16}) = \cancel{-44}\ -22.$$

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in det } g_{ij}}{g}$$

$$g^{11} = \frac{A^{11}}{g} = \frac{-22}{-22} = 1$$

$$g^{22} = \frac{A^{22}}{g} = \frac{-3}{+22}, \quad g^{33} = \frac{A^{33}}{g} = \frac{-2}{-22} = \frac{1}{11}$$

$$g^{23} = \frac{(-1)^{3+2}\,-4}{-22} = \frac{+2}{11}, \quad g^{32} = \frac{-4}{-22} = \frac{-2}{11}$$

also due to symmetricity of $g^{ij} \Rightarrow g^{23} = g^{32}$

so, we have Conjugate metric Tensor.

c.m.t

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/+22 & -2/11 \\ 0 & -2/11 & 1/11 \end{bmatrix}$$

2: $ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6\,dx^1dx^2 + 4\,dx^2dx^3$

$\because ds^2 = g_{ij}\,dx^i dx^j$

$\Rightarrow g_{11} = 5, \; g_{22} = 3, \; g_{33} = 4, \; g_{12} = -3 = g_{21}$

$g_{23} = 2 = g_{32}.$

f.m.t

$$g_{ij} = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$g = |g_{ij}| = 5(12-4) + 3(-12-0)$

$= 40 - 36 = 4.$

$$g^{ij} = \frac{\text{Cofactor of } g_{ij} \text{ in det } g_{ij}}{g}$$

$g^{11} = \frac{A^{11}}{g} = \frac{8}{4} = 3, \; g^{12} = +3, \; g^{13} = -\frac{6}{4} = \frac{-3}{2}$

$g^{21} = +3, \; g^{22} = \frac{20}{4} = 5, \; g^{23} = \frac{10}{4} = \frac{-5}{2}$

$g^{31} = -\frac{3}{2} \quad g^{32} = \frac{-5}{2} \quad g^{33} = \frac{6}{4} = \frac{3}{2}.$

c.m.t

$$g^{ij} = \begin{bmatrix} 3 & +3 & -3/2 \\ +3 & 5 & -5/2 \\ -3/2 & -5/2 & 3/2 \end{bmatrix}$$

11-04-22

⇒ **Christoffel symbols:-**

1:- **Christoffel symbols of I^{st} kind:-**

It is denoted by $\Gamma_{ij,k}$ or $[ij,k]$ and defined as

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

⊙ It is symmetric in i, j

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k}\right)$$

$$\Gamma_{ij,k} = \Gamma_{ji,k}$$

2:- **Christoffel symbols of 2^{nd} kind:-**

It is denoted by Γ^i_{jk} or $\left\{ {i \atop jk} \right\}$ and defined as

$$\Gamma^i_{jk} = g^{im}\,\Gamma_{jk,m}$$

$$\boxed{\Gamma^i_{jk} = \frac{1}{2}\, g^{im}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right)}$$

→ It is symmetric in j, k.

$$\rightarrow \Gamma_{ij,k} + \Gamma_{jk,i} = \frac{\partial g_{ik}}{\partial x^j} \quad —— ①$$

Proof :-

sym. to g_{ji}

$$\Gamma_{ij,k} + \Gamma_{jk,i} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \cancel{\frac{\partial g_{jk}}{\partial x^i}} - \cancel{\frac{\partial g_{ij}}{\partial x^k}} \right)$$

$$+ \frac{1}{2}\left(\cancel{\frac{\partial g_{ji}}{\partial x^k}} + \frac{\partial g_{ki}}{\partial x^j} - \cancel{\frac{\partial g_{jk}}{\partial x^i}} \right)$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right) \frac{\partial g_{ik}}{\partial x^j} = \frac{\partial g_{ik}}{\partial x^j}.$$

Example :- prove that $\frac{\partial g^{pq}}{\partial x^m} = -g^{pj}\Gamma^q_{jm} - g^{qj}\Gamma^p_{jm}$

Proof :- $\because$

$$g_{jk}\, g^{kl} = \delta^l_j = \begin{cases} 1 & l = j \\ 0 & l \neq j \end{cases}$$

differentiating "x^m"

$$g^{kl} \frac{\partial g_{jk}}{\partial x^m} + g_{jk} \frac{\partial g^{kl}}{\partial x^m} = 0$$

$$g_{jk} \frac{\partial g^{kl}}{\partial x^m} = -g^{kl} \frac{\partial g_{jk}}{\partial x^m}$$

$$= -g^{kl}\left[\Gamma_{jm,k} + \Gamma_{km,j}\right] \text{ from ①}$$

multiplying both side by g^{jp}.

$$\underbrace{g^{jp} g_{jk}} \frac{\partial g^{kl}}{\partial x^m} = -g^{jp} g^{kl}\, \Gamma_{jm,k} - g^{jp} g^{kl}\, \Gamma_{km,j}$$

$$\delta^p_k \frac{\partial g^{kl}}{\partial x^m} = -g^{jp}\Gamma^l_{jm} - g^{kl}\Gamma^p_{km}$$

$k = p$

$$\frac{\partial g^{pl}}{\partial x^m} = -g^{jp}\Gamma^l_{jm} - g^{kl}\Gamma^p_{km}$$

replace $k \to j$ & $l \to q$.

$$\frac{\partial g^{pq}}{\partial x^m} = -g^{jp}\Gamma^q_{jm} - g^{jq}\Gamma^p_{jm}.$$

$$\boxed{\frac{\partial g^{pq}}{\partial x^m} = -g^{pj}\Gamma^q_{jm} - g^{qj}\Gamma^p_{jm}}$$

<u>Example:</u> Evaluate

(a) $\Gamma_{11,2}$ and $\Gamma_{12,2}$

(b) Γ^1_{22}, and Γ^2_{21} (

in cylindrical coordinate system (r, θ, z).

<u>Sol^n:</u> $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$

$ds^2 = g_{ij}\, dx^i dx^j$

$g_{11} = 1 \quad g_{22} = r^2, \quad g_{33} = 1.$ & $g_{ij} = 0 \;\forall\, i \neq j.$

$$\Gamma_{ij,k} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right] \qquad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma_{11,2} = \frac{1}{2}\left[\frac{\partial g_{12}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right]$$

$$= \frac{1}{2}\left[0 + 0 - 0\right] \qquad \left[\because \frac{\partial 1}{\partial x^2} = 0\right]$$

$[x^2 = \theta]$
$x^3 = z$
$x^1 = r$

$$= 0$$

$$\Gamma_{12,2} = \frac{1}{2}\left[\cancel{\frac{\partial g_{12}}{\partial x^2}} + \frac{\partial g_{22}}{\partial x^1} - \cancel{\frac{\partial g_{12}}{\partial x^2}}\right]$$

$$= \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = \frac{1}{2}\frac{\partial (r^2)}{\partial r} = \frac{1}{2}2r = r \qquad [\because x^1 = r]$$

$\Gamma_{12,2} = r.$

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = r^2.$$

$$g^{11} = \frac{A^{11}}{g} = 1, \quad g^{22} = \frac{A^{22}}{g} = \frac{1}{r^2}, \quad g^{33} = \frac{A^{33}}{g} = 1.$$

$\&\ g^{ij} = 0, \forall i \neq j$

$$\therefore g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\rightarrow \Gamma^i_{jk} = g^{im} \Gamma_{jk,m}$

$$\Gamma^1_{22} = g^{1m} \Gamma_{22,m}$$

$$= g^{11} \Gamma_{22,1} + \underbrace{g^{12}}_{=0} \Gamma_{22,2} + \underbrace{g^{13}}_{=0} \Gamma_{22,3}$$

$$= 1 \cdot \Gamma_{22,1} + 0 + 0$$

$$= \frac{1}{2} \left(\underbrace{\frac{\partial g_{21}}{\partial x^2}}_{=0} + \underbrace{\frac{\partial g_{21}}{\partial x^2}}_{=0} - \frac{\partial g_{22}}{\partial x^1} \right)$$

$$= \frac{1}{2} \left(0 + 0 - \frac{\partial (r^2)}{\partial r} \right) = \frac{1}{2}(-2r) = -r.$$

$$\Gamma^2_{21} = g^{2m} \Gamma_{21,m}$$

$$= \underbrace{g^{21}}_{=0} \Gamma_{21,1} + g^{22} \Gamma_{21,2} + \underbrace{g^{23}}_{=0} \Gamma_{21,3}.$$

$$= g^{22} \Gamma_{21,2} = g^{22} \cdot \frac{1}{2} \left[\frac{\partial g_{22}}{\partial x^1} + \overbrace{\frac{\partial g_{12}}{\partial x^2}}^{=0} - \overbrace{\frac{\partial g_{21}}{\partial x^2}}^{=0} \right]$$

$$= \frac{1}{r^2} \cdot \frac{1}{2} \left[\frac{\partial g_{22}}{\partial x^1} \right] = \frac{1}{2r^2} \left[\frac{\partial (r^2)}{\partial r} \right] = \frac{1}{2r^2}(2r) = \frac{1}{r}.$$

$$\Gamma^2_{21} = \frac{1}{r}.$$

⇒ Covariant derivative :- The covarient derivative of a covariant Tensor A_i w.r.t x^k is denoted by $A_{i;k}$ and defined by.

$$\boxed{A_{i;k} = \frac{\partial A_i}{\partial x^k} - A_j \Gamma^j_{ik}}$$

Similarly, The covariant derivative of a contravariant Tensor A^i w.r.t x^k is denoted by $A^i_{;k}$ and defined by

$$\boxed{A^i_{;k} = \frac{\partial A^i}{\partial x^k} + A^j \Gamma^i_{jk}.}$$

Similarly, the covariant derivative of a higher rank Tensor is defined as

$$A_{ij;k} = \frac{\partial A_{ij}}{\partial x^k} - A_{lj} \Gamma^l_{ik} - A_{im} \Gamma^m_{jk}.$$

& $$A^{ij}_{;k} = \frac{\partial A^{ij}}{\partial x^k} + A^{lj} \Gamma^i_{lk} + A^{im} \Gamma^j_{mk}.$$

& $$A^i_{j;k} = \frac{\partial A^i_j}{\partial x^k} - A^i_m \Gamma^m_{jk} + A^m_j \Gamma^i_{mk}.$$

Example :- Show that :-

(1):- $g_{ij;p} = 0$ (2): $g^{ij}_{;p} = 0.$

Solⁿ :-

1:- $$g_{ij;p} = \frac{\partial g_{ij}}{\partial x^p} - g_{lj} \Gamma^l_{ip} - g_{im} \Gamma^m_{jp} \quad \text{—(1)}$$

$$\rightarrow \frac{\partial g_{ij}}{\partial x^p} = \Gamma_{ip,j} + \Gamma_{jp,i}$$

(done)

$$\frac{\partial g^{ij}}{\partial x^p} = -g^{ih}\Gamma^{j}_{hp} - g^{jh}\Gamma^{i}_{hp}$$

$$\rightarrow \Gamma^{l}_{ip} = g^{lk}\Gamma_{ip,k}.$$

$$\Gamma^{m}_{jp} = g^{mk}\Gamma_{jp,k}.$$

from ①, we have.

$$= \Gamma_{ip,j} + \Gamma_{jp,i} - g_{lj}\, g^{lk}\Gamma_{ip,k} - g_{im}\, g^{mk}\Gamma_{jp,k}.$$

$$= \Gamma_{ip,j} + \Gamma_{jp,i} - \delta^{k}_{j}\Gamma_{ip,k} - \delta^{k}_{i}\Gamma_{jp,k}.$$

$$= \Gamma_{ip,j} + \Gamma_{jp,i} - \Gamma_{ip,j} - \Gamma_{jp,i} = 0.$$

$$\Rightarrow \quad g_{ij;p} = 0$$

2:- $g^{ij}_{;p} = 0$

$$g^{ij}_{;p} = \frac{\partial g^{ij}}{\partial x^p} + g^{mj}\Gamma^{i}_{mp} + g^{im}\Gamma^{j}_{mp}.$$

$$= -g^{ih}\Gamma^{j}_{hp} - g^{jh}\Gamma^{i}_{hp} + g^{mj}\Gamma^{i}_{mp} + g^{im}\Gamma^{j}_{mp}$$

$$= -g^{ih}\Gamma^{j}_{hp} - g^{jh}\Gamma^{i}_{hp} + g^{hj}\Gamma^{i}_{hp} + g^{ih}\Gamma^{j}_{hp}$$

due to symmetricity of conjugate metric Tensor.

$$= -g^{ih}\Gamma^{j}_{hp} - g^{jh}\Gamma^{i}_{hp} + g^{jh}\Gamma^{i}_{hp} + g^{ih}\Gamma^{j}_{hp} = 0$$

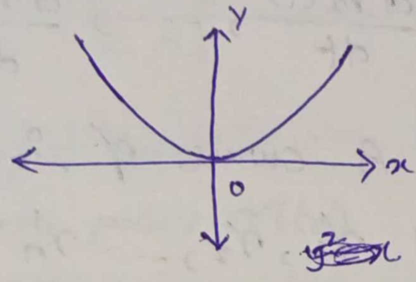
$$\Rightarrow \quad g^{ij}_{;p} = 0.$$

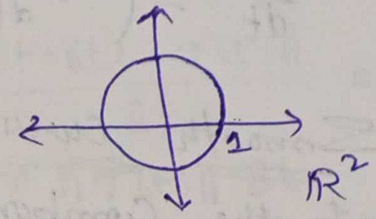
□ 1st unit end.

UNIT-II

Differential Geometry

⇒ **Level curve :-**





$$C = \{ (x,y) : f(x,y) = c \ (\text{constant}) \}$$

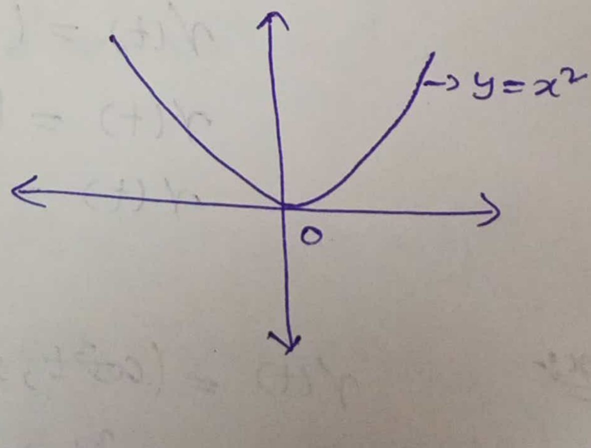
⇒ **Curve :-** A Parametrized curve in $\mathbb{R}^n$ is a map

$$\gamma : (\alpha, \beta) \to \mathbb{R}^n$$

for some α, β with $-\infty \leq \alpha < \beta \leq \infty$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots \gamma_n(t)).$$

Ex :- Parametrization of $\gamma(t)$ of parabola.
$y = x^2$ is given as



$$\gamma(t) = (t, t^2)$$
$$\gamma(t) = (2t, 4t^2)$$
$$\gamma(t) = (t^2, t^4)$$
$$\gamma(t) = (t^3, t^6).$$

⇒ **Tangent vector :-** If $\gamma(t)$ is a parametrized curve, its first derivative $\frac{d\gamma}{dt}$ is called tangent vector of γ at the point $\gamma(t)$.

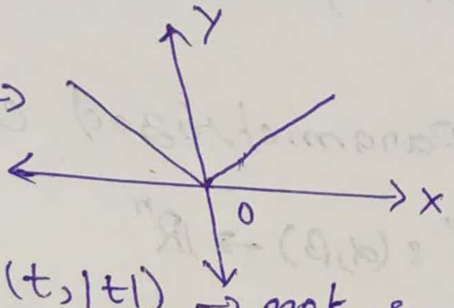
$$\gamma : (\alpha, \beta) \to \mathbb{R}^n$$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \ldots, \gamma_n(t))$$

$$\frac{d\gamma}{dt} = \left(\frac{d\gamma_1(t)}{dt}, \frac{d\gamma_2(t)}{dt}, \ldots, \frac{d\gamma_n(t)}{dt} \right)$$

$\Rightarrow$ Smooth curve :- A curve γ is smoth if each of the Components $\gamma_1, \gamma_2, \ldots \gamma_n$ is smooth. i.e if all the derivatives $\frac{d\gamma_i}{dt}, \frac{d^2\gamma_i}{dt^2}, \frac{d^3\gamma_i}{dt^3}, \ldots$ exist $\forall\ i = 1, 2, \ldots n$.

$y = |x| \longrightarrow$



here $\gamma(t) = (t, |t|) \to$ not smoth as $|t|$ derivative does not exist

Example :- $x^2 + y^2 = 1$ circle.

we have

$$\gamma(t) = (t, \sqrt{1-t^2})$$

$$\gamma(t) = (t, -\sqrt{1-t^2})$$

$$\gamma(t) = (\cos t, \sin t) \checkmark$$

Ex:-

$$\gamma(t) = (\cos^3 t, \sin^3 t) \to \text{Astroid.}$$

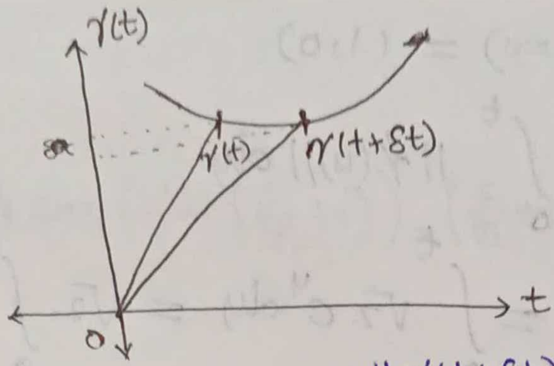
$$\cos^2 t + \sin^2 t = 1$$

$$x = \cos^3 t, \quad y = \sin^3 t$$

$$\cos^2 t = x^{2/3} \quad \sin^2 t = y^{2/3}$$

$$\Rightarrow \boxed{x^{2/3} + y^{2/3} = 1}$$

⇒ Arc length :-



$$\frac{\|\gamma(t+\delta t) - \gamma(t)\|}{\delta t} \quad \delta t$$

$$\Rightarrow \|\dot{\gamma}(t)\| \, \delta t$$

The arc length of a curve γ starting at the point $\gamma(t_0)$ is a function $s(t)$ given by

$$\boxed{s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| \, dt}$$

$\gamma : (\alpha, \beta) \to \mathbb{R}^n$

$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$

Example :- $\gamma(t) = (e^t \cos t, e^t \sin t)$, to lagarithmic spiral. Find the arc length.

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| \, dt$$

we have

$$\gamma(t) = (e^t \cos t, e^t \sin t)$$

$$\Rightarrow \dot{\gamma}(t) = (-e^t \sin t + e^t \cos t, e^t \cos t + e^t \sin t)$$

Now $\|\dot{\gamma}(t)\|^2 = \dot{\gamma}(t) \cdot \dot{\gamma}(t)$

$$= (-e^t \sin t + e^t \cos t) + [e^t(\cos t + \sin t)]^2$$

$$= e^{2t}(\sin^2 t + \cos^2 t - 2\sin t \cos t)$$
$$+ e^{2t}(\cos^2 t + \sin^2 t + 2\sin t \cos t)$$

$$= e^{2t}(2\sin^2 t + 2\cos^2 t) = e^{2t} \cdot 2(\sin^2 t + \cos^2 t)$$

$$\|\dot{\gamma}\|^2 = 2e^{2t} = (\sqrt{2}\, e^t)^2$$

$$\|\dot{\gamma}(t)\| = \sqrt{2}\, e^t$$

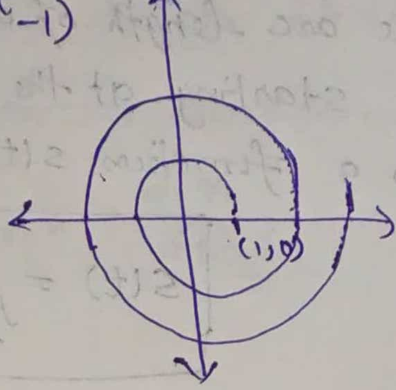
Hence, Arc length of γ starting at $t_0 = 0$.

$\gamma(t_0 = 0) = (1, 0)$.

$$s(t) = \int_0^t \|\dot{\gamma}(u)\| \, du$$

$$= \int_0^t \sqrt{2} e^u \, du = \sqrt{2} \int_0^t e^u \, du = \sqrt{2} \left[e^u \right]_0^t$$

$$s(t) = \sqrt{2} (e^t - e^0) = \sqrt{2} (e^t - 1)$$



⇒ Definition :- If $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is a parametrized curve, its speed at the point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$. and γ is said to be unit speed curve if $\dot{\gamma}(t)$ is unit vector $\forall \; t \in (\alpha, \beta)$ i.e $\|\dot{\gamma}(t)\| = 1$.

$$s(t) = \int_{t_0}^t \|\dot{\gamma}(t)\| \, dt$$

$$\Rightarrow \frac{ds}{dt} = \|\dot{\gamma}(t)\|$$

rate of change of distance along the curve.

⇒ Definition :- A point $\gamma(t)$ of a parametrized curve γ is called regular point if $\dot{\gamma}(t) \neq 0$.

A curve is regular if all its points are regular.

Example :-

$$\gamma(t) = (e^t \cos t, e^t \sin t)$$

as we have done earlier

$\|\dot{\gamma}(t)\| = \sqrt{2} e^t \to$ regular curve.

$$s(t) = \sqrt{2} (e^t - 1)$$

$$\frac{s}{\sqrt{2}}+1=e^t$$

$$\gamma(s)=\left(\left(\frac{s}{\sqrt{2}}+1\right)\cos\left(-\ln\left(\frac{s}{\sqrt{2}}+1\right)\right),\left(\frac{s}{\sqrt{2}}+1\right)\sin\left(-\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\right)$$

$\|\dot{\gamma}(s)\|=?$

$$\dot{\gamma}(s)=\left(\frac{1}{\sqrt{2}}\cos\left(-\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)-\cancel{\left(\frac{s}{\sqrt{2}}+1\right)}\sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\cdot\frac{1}{\cancel{\frac{s}{\sqrt{2}}+1}}\cdot\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\right.$$

$$\left.+\cancel{\left(\frac{s}{\sqrt{2}}+1\right)}\cos\left(-\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\cdot\frac{1}{\cancel{\frac{s}{\sqrt{2}}+1}}\cdot\frac{1}{2}\right)$$

$$=\left(\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta,\ \frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right)$$

where $\theta=\ln\left(\frac{s}{\sqrt{2}}+1\right)$

$$\dot{\gamma}(s)=\left(\frac{1}{\sqrt{2}}(\cos\theta-\sin\theta),\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)\right)$$

$$\|\dot{\gamma}(s)\|^2=\frac{1}{2}(\cos\theta-\sin\theta)^2+\frac{1}{2}(\cos\theta+\sin\theta)^2$$

$$=\frac{1}{2}\left[2(\sin^2\theta+\cos^2\theta)\right]=\frac{1}{2}\cdot 2=1.$$

$$\|\dot{\gamma}(s)\|=1.$$

$\Rightarrow$ $\gamma:(\alpha,\beta)\rightarrow\mathbb{R}^3$, curvature, Torsion.

$\Rightarrow$ <u>Curvature</u>:- If γ is a unit speed curve with parameter 's' its curvature $k(s)$ at the point $\gamma(s)$ is defined to be.

$\|\ddot{\gamma}(s)\|$.

<u>Example</u>:- $\gamma(t)=(x_0+R\cos t,\ y_0+R\sin t)$

Centre $=(x_0,y_0)$ & Radius $=R$.

$\gamma(t) = (x_0 + R\cos t, y_0 + R\sin t)$

$\dot{\gamma}(t) = (-R\sin t, R\cos t)$

$\|\dot{\gamma}(t)\|^2 = (-R\sin t)^2 + (R\cos t)^2$

$= R^2(\sin^2 t + \cos^2 t)$

$= R^2$

$\|\dot{\gamma}(t)\| = R = \frac{ds}{dt}$

Now $s = \int_0^t \|\dot{\gamma}(u)\| \, du.$

$s = \int_0^t R \, du = Rt.$

$\Rightarrow t = \frac{s}{R}.$

$\tilde{\gamma}(s) = (x_0 + R\cos(\frac{s}{R}), y_0 + R\sin(\frac{s}{R}))$

$\dot{\tilde{\gamma}}(s) = (-\cancel{R}\sin(\frac{s}{R}) \cdot \frac{1}{\cancel{R}}, \cancel{R}\cos(\frac{s}{R}) \cdot \frac{1}{\cancel{R}})$

$= (-\sin(\frac{s}{R}), \cos(\frac{s}{R}))$

$\|\dot{\tilde{\gamma}}(s)\|^2 = \sin^2(\frac{s}{R}) + \cos^2(\frac{s}{R}) = 1$

$\|\dot{\tilde{\gamma}}(s)\| = 1.$ ~~&~~ unit speed curve. $\tilde{\gamma}(s)$.

curvature $k(s) = \|\ddot{\tilde{\gamma}}(s)\|.$

$\ddot{\tilde{\gamma}}(s) = (-\cos(\frac{s}{R}) \cdot \frac{1}{R}, -\sin(\frac{s}{R}) \cdot \frac{1}{R})$

$\|\ddot{\tilde{\gamma}}(s)\|^2 = [-\frac{1}{R}\cos(\frac{s}{R})]^2 + [-\frac{1}{R}\sin(\frac{s}{R})]^2$

$= \frac{1}{R^2}[\cos^2(\frac{s}{R}) + \sin^2(\frac{s}{R})] = \frac{1}{R^2}$

$\|\ddot{\tilde{\gamma}}(s)\| = \frac{1}{R} = k(s)$

if $R \to \infty$ then $k(s) \to 0$

i.e curvature 0.

$\Rightarrow$ Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Then its curvature is

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}.$$

Proof :- Let $\tilde{\gamma}$ be a unit speed parametrization of γ.

$$\tilde{\gamma}(s) = \gamma(t)$$

differentiating w.r.t t

$$\tilde{\gamma}'(s) \frac{ds}{dt} = \dot{\gamma}(t)$$

as we know

$$\kappa = \|\tilde{\gamma}''(s)\|$$

$$= \left\| \frac{d}{ds}\left(\frac{d\tilde{\gamma}}{ds}\right) \right\|$$

$$= \left\| \frac{d}{ds}\left(\frac{\dot{\gamma}(t)}{ds/dt}\right) \right\|$$

$$= \left\| \frac{\frac{d}{dt}\left(\frac{\dot{\gamma}}{ds/dt}\right)}{ds/dt} \right\|$$

$$= \left\| \frac{\frac{\frac{ds}{dt}\ddot{\gamma} - \dot{\gamma}\frac{d^2s}{dt^2}}{(ds/dt)^2}}{ds/dt} \right\| \quad \text{— (1)}$$

$\because \frac{ds}{dt} = \|\dot{\gamma}(t)\|.$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = \|\dot{\gamma}(t)\|^2 = \dot{\gamma}\cdot\dot{\gamma} \quad \text{— (a)}$$

diff. w.r.t 't'

$$2\frac{ds}{dt}\cdot\frac{d^2s}{dt^2} = \dot{\gamma}\cdot\ddot{\gamma} + \ddot{\gamma} + \dot{\gamma}$$

$$2\frac{ds}{dt}\frac{d^2s}{dt^2} = 2\dot{\gamma}\cdot\ddot{\gamma}$$

$$\Rightarrow \frac{ds}{dt}\frac{d^2s}{dt^2} = \dot{\gamma}\cdot\ddot{\gamma} \quad \text{— (2)}$$

Using (2) in (1), we have

$$= \left\| \frac{\frac{ds}{dt}\ddot{\gamma} - \dot{\gamma}\frac{(\dot{\gamma}\cdot\ddot{\gamma})}{ds/dt}}{\left(\frac{ds}{dt}\right)^3} \right\|$$

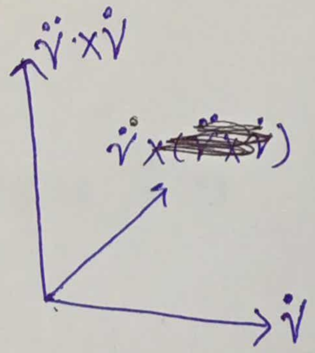
$$= \left\| \frac{\left(\frac{ds}{dt}\right)^2\ddot{\gamma} - \dot{\gamma}(\dot{\gamma}\cdot\ddot{\gamma})}{\left(\frac{ds}{dt}\right)^4} \right\|$$

$a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$

$$= \left\| \frac{(\dot{\gamma}\cdot\dot{\gamma})\cdot\ddot{\gamma} - \dot{\gamma}(\dot{\gamma}\cdot\ddot{\gamma})}{\left(\frac{ds}{dt}\right)^4} \right\|$$

$\because \dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma}) = (\dot{\gamma}\cdot\dot{\gamma})\ddot{\gamma} - (\dot{\gamma}\cdot\ddot{\gamma})\dot{\gamma}$

$$\Rightarrow \quad \kappa = \frac{\|\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})\|}{\|\dot{\gamma}\|^4}$$



$\dot{\gamma}$ and $\ddot{\gamma}\times\dot{\gamma}$ are orthogonal

$\|a\times b\| = \|a\|\|b\|$

$$\Rightarrow \quad \kappa = \frac{\|\dot{\gamma}\|\,\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^4}$$

$$\boxed{\kappa = \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}}$$

Example :- A circular helix with the axis as z-axis is a curve of the form

$$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$$

$-\infty < \theta < \infty$, $a, b \rightarrow$ constant.

Sol^n.

$$k = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

Now,

$$\dot{\gamma}(\theta) = (-a\sin\theta, a\cos\theta, b)$$

$$\ddot{\gamma}(\theta) = (-a\cos\theta, -a\sin\theta, 0)$$

$$\ddot{\gamma} \times \dot{\gamma} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\cos\theta & -a\sin\theta & 0 \\ -a\sin\theta & a\cos\theta & b \end{vmatrix}$$

$$= \hat{i}(-ab\sin\theta) - \hat{j}(-ab\cos\theta) + \hat{k}(-a^2\cos^2\theta - a^2\sin^2\theta)$$

$$= (-ab\sin\theta, ab\cos\theta, -a^2)$$

$$\|\ddot{\gamma} \times \dot{\gamma}\|^2 = a^2b^2\sin^2\theta + a^2b^2\cos^2\theta + a^4$$

$$= a^2b^2 + a^4 = a^2(a^2+b^2)$$

$$\|\ddot{\gamma} \times \dot{\gamma}\| = a\sqrt{a^2+b^2}$$

$$\|\dot{\gamma}\|^2 = a^2\cos^2\theta + a^2\sin^2\theta + b^2 = a^2+b^2$$

$$\|\dot{\gamma}\| = \sqrt{a^2+b^2}$$

curvature

$$k = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{a\sqrt{a^2+b^2}}{(\sqrt{a^2+b^2})^3} = \frac{|a|}{a^2+b^2}$$

$$k = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

For line $\gamma(t) = t + b \Rightarrow \dot{\gamma}(t) = 1. \Rightarrow \ddot{\gamma}(t) = 0$

$\|\dot{\gamma}\| = 1. \quad \Rightarrow k = \|\ddot{\gamma}\| = 0.$

Example :- Find the Curvature ?

1. $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$

2. $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$

Sol :- given curve γ

$$\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, t/\sqrt{2}\right)$$

$$\dot{\gamma}(t) = \left(\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right)$$

$$\ddot{\gamma}(t) = \left(\frac{1}{2}\cdot\frac{1}{2}(1+t)^{-\frac{1}{2}}, \frac{1}{2}\cdot\frac{1}{2}(1-t)^{-1/2}, 0\right)$$

$$\ddot{\gamma}\times\dot{\gamma} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{1}{4}(1+t)^{-\frac{1}{2}} & \frac{1}{4}(1-t)^{-1/2} & 0 \\ \frac{1}{2}(1+t)^{\frac{1}{2}} & -\frac{1}{2}(1-t)^{\frac{1}{2}}, & \frac{1}{\sqrt{2}} \end{vmatrix} = \hat{i}\left(\frac{1}{4\sqrt{2}}(1-t)^{-1/2}\right) - \hat{j}\left(\frac{1}{4\sqrt{2}}(1+t)^{-1/2}\right) + \hat{k}\left(-\frac{1}{8}-\frac{1}{8}\right)$$

$$\ddot{\gamma}\times\dot{\gamma} = \left(\frac{1}{4\sqrt{2}}(1-t)^{-1/2}, -\frac{1}{4\sqrt{2}}(1+t)^{-1/2}, -\frac{1}{4}\right)$$

$$\|\ddot{\gamma}\times\dot{\gamma}\|^2 = \frac{1}{32}(1-t)^{-1} + \frac{1}{32}(1+t)^{-1} + \frac{1}{16}$$

$$= \frac{1}{32}\left[\frac{1}{1-t} + \frac{1}{1+t} + 2\right]$$

$$= \frac{1}{32}\left[\frac{1+\cancel{t} + 1-\cancel{t} + 2(1-t^2)}{1-t^2}\right]$$

$$= \frac{1}{32}\left[\frac{2+2(1-t^2)}{1-t^2}\right]$$

$$= \frac{\cancel{2}}{32}\left[\frac{2-t^2}{1-t^2}\right] = \frac{1}{16}\left(\frac{2-t^2}{1-t^2}\right)$$

$$\|\ddot{\gamma}\times\dot{\gamma}\| = \frac{1}{4}\sqrt{\frac{2-t^2}{1-t^2}}.$$

$$\dot{\gamma} = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}} \right)$$

$$\|\dot{\gamma}\|^2 = \frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}$$

$$= \frac{1}{4}\left[1+\cancel{t}+1-\cancel{t}+2\right] = \frac{1}{4}(4) = 1$$

$\|\dot{\gamma}\| = 1. \Rightarrow$ It is unit speed curve the $k = \|\ddot{\gamma}\|$.

$$k = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{1}{4}\sqrt{\frac{2-t^2}{1-t^2}}.$$

2. $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t \right)$

$$\dot{\gamma}(t) = \left(-\frac{4}{5}\sin t, -\cos t, +\frac{3}{5}\sin t \right)$$

$$\ddot{\gamma}(t) = \left(-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t \right)$$

$$\ddot{\gamma} \times \dot{\gamma} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\frac{4}{5}\cos t & \sin t & \frac{3}{5}\cos t \\ -\frac{4}{5}\sin t & -\cos t & \frac{3}{5}\sin t \end{vmatrix}$$

$$= \hat{i}\left(\frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t \right) - \hat{j}\left(-\frac{12}{25}\sin t\cos t + \frac{12}{25}\sin t\cos t \right)$$

$$+ \hat{k}\left(\frac{4}{5}\cos^2 t + \frac{4}{5}\sin^2 t \right)$$

$$\ddot{\gamma} \times \dot{\gamma} = \left(\frac{3}{5}, 0, \frac{4}{5} \right)$$

$$\|\ddot{\gamma} \times \dot{\gamma}\|^2 = \frac{9}{25} + \frac{16}{25} = 1.$$

$$\Rightarrow \|\ddot{\gamma} \times \dot{\gamma}\| = 1.$$

$$\|\dot{\gamma}\|^2 = \frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t$$

$$= \sin^2 t + \cos^2 t = 1.$$

$$k = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{1}{1^3} = 1. \quad \boxed{k(s) = 1}$$

⇒ Space curve :- let $\gamma(s)$ be a unit speed curve in $\mathbb{R}^3$ and let $\vec{t} = \dot{\gamma}$ be its unit tangent vector. if the curvature is non-zero. we define principal normal of γ at the point $\gamma(s)$ to be a vector

$$n(s) = \frac{1}{\kappa(s)} t'(s).$$

$\kappa = \|\ddot{\gamma}\|$
$t = \dot{\gamma}$
$\dot{t} = \ddot{\gamma}$

$$\|t\| = t \cdot t = 1.$$

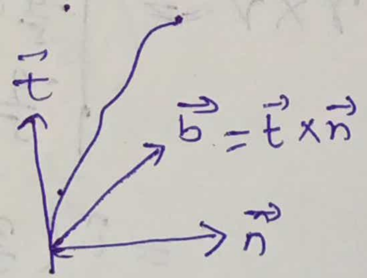
diff. w.r.t s.

$$\dot{t} \cdot t + t \cdot \dot{t} = 0$$

$$t \cdot \dot{t} = 0$$

→ t & n are unit orthogonal vector then $b = t \times n$ is an unit vector ⊥ to both t & n.

This vector $b(s)$ is called the Binormal vector of γ at the point $\gamma(s)$. Thus $\{t, n, b\}$ is an orthonormal Basis of $\mathbb{R}^3$.



$$t = n \times b, \quad n = b \times t, \quad b = t \times n$$

$\because b = t \times n$

$$\dot{b} = \dot{t} \times n + t \times \dot{n}$$

$$= \kappa\, n \times n + t \times \dot{n}$$

$[\because n = \frac{1}{\kappa}\dot{t}]$

$$= 0 + t \times \dot{n}$$

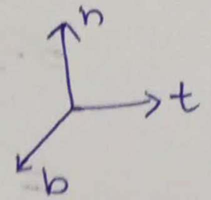
$$\dot{b} = t \times \dot{n}$$

$$\|b\| = b \cdot b = 1$$

diff. $\dot{b} \cdot b + b \cdot \dot{b} = 0$

$$\Rightarrow b \cdot \dot{b} = 0$$

$\Rightarrow \quad b' \perp b \,, \; b' \perp t$



$\therefore$ b' is $\perp$ to t & b, so, b' must be $\parallel$ to n

$$\therefore \quad b' = -\tau n.$$

where τ is called Torsion & (-ve sign is a convension)

$$b' \cdot b' = (-\tau n) \cdot (-\tau n)$$

$$\|b'\|^2 = \tau^2 \, n \cdot n$$

$$\boxed{\|b'\| = \tau}$$

$\Rightarrow$ **Proposition:-** Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing Curvature then its Torsion is given by

$$\tau = \frac{(\dot\gamma \times \ddot\gamma) \cdot \dddot\gamma}{\|\dot\gamma \times \ddot\gamma\|^2}$$

Proof:- Let γ is a unit speed curve

$$b' = -\tau n.$$

$$b' \cdot n = -\tau n \cdot n \qquad \left[\because n = \frac{t'}{k}\right]$$

$$b' \cdot n = -\tau$$

$$\Rightarrow \tau = -b' \cdot n.$$

$$= -n \cdot (\dot{\overline{t \times n}})$$

$$= -\frac{t'}{k} (t' \times n + t \times n')$$

$t = \dot\gamma$

$n = \frac{1}{k} t'$

$n = \frac{1}{k} \ddot\gamma$

$t' = \ddot\gamma$

$$= -\frac{1}{k} t' (k n \times n + t \times n')$$

$$= -\frac{1}{k} t' (t \times n')$$

$$= -\frac{1}{k} \ddot\gamma \left[\dot\gamma \times \frac{d}{dt}\left(\frac{1}{k} \ddot\gamma\right)\right]$$

$$= -\frac{1}{k} \ddot\gamma \left[\dot\gamma \times \left(\frac{k \dddot\gamma - \ddot\gamma k'}{k^2}\right)\right]$$

$$= -\frac{1}{k}\ddot{\gamma}\left[\dot{\gamma}\times\left(\frac{1}{k}\dddot{\gamma} - \frac{1}{k^2}\dot{k}\ddot{\gamma}\right)\right]$$

$$= -\frac{1}{k^2}\ddot{\gamma}\cdot(\dot{\gamma}\times\dddot{\gamma}) + \frac{1}{k^3}\ddot{\gamma}\underbrace{(\dot{\gamma}\times\ddot{\gamma})}_{=0}$$

$$= \frac{1}{k^2}\dddot{\gamma}(\dot{\gamma}\times\ddot{\gamma}) \qquad [\because a.(b\times c) = -c.(b\times a)]$$

$\because$ $\dot{\gamma}$ and $\ddot{\gamma}$ are $\perp$

$$\|\dot{\gamma}\times\ddot{\gamma}\| = \|\dot{\gamma}\|\|\ddot{\gamma}\|$$

$$\|\dot{\gamma}\times\ddot{\gamma}\| = k.$$

$$\tau = \frac{\dddot{\gamma}(\dot{\gamma}\times\ddot{\gamma})}{\|\dot{\gamma}\times\ddot{\gamma}\|^2}$$

$k = \|\ddot{\gamma}\|$

$\|\dot{\gamma}\| = 1.$

$(\dot{\gamma}\cdot\dot{\gamma}) = 1$

$\ddot{\gamma}\cdot\dot{\gamma} + \dot{\gamma}\cdot\ddot{\gamma} = 0$

$\dot{\gamma}\cdot\ddot{\gamma} = 0$

$\Rightarrow$ Proof in general

$$\gamma(t) = \gamma(s)$$

$$\dot{\gamma} = \gamma'\frac{ds}{dt} \qquad \left[\gamma' = \frac{d\gamma}{ds}\right]$$

$$\ddot{\gamma} = \gamma'\frac{d^2s}{dt^2} + \gamma''\left(\frac{ds}{dt}\right)^2$$

$$\dddot{\gamma} = \gamma'\frac{d^3s}{dt^3} + \frac{d^2s}{dt^2}\gamma''\frac{ds}{dt} + \gamma'' 2\frac{ds}{dt}\cdot\frac{d^2s}{dt^2} + \gamma'''\left(\frac{ds}{dt}\right)^3$$

$$\dddot{\gamma} = \gamma'''\left(\frac{ds}{dt}\right)^3 + 3\frac{ds}{dt}\frac{d^2s}{dt^2}\gamma'' + \gamma'\frac{d^3s}{dt^3}$$

$$\dot{\gamma}\times\ddot{\gamma} = \left(\gamma'\frac{ds}{dt}\right)\times\left(\gamma'\frac{d^2s}{dt^2} + \gamma''\left(\frac{ds}{dt}\right)^2\right)$$

$$= \frac{ds}{dt}\frac{d^2s}{dt^2}\gamma'\times\gamma' + \left(\frac{ds}{dt}\right)^3\gamma'\times\gamma''$$

$$\dot{\gamma}\times\ddot{\gamma} = 0 + \left(\frac{ds}{dt}\right)^3\gamma'\times\gamma''$$

$$\dddot{r}(\dot{r}\times\ddot{r}) = \left(\frac{ds}{dt}\right)^6 r'''(r'\times r'')$$

$$\|\dot{r}\times\ddot{r}\|^2 = \left(\frac{ds}{dt}\right)^6 \|r'\times r''\|^2$$

$$\dddot{r}(\dot{r}\times\ddot{r}) = \frac{\|\dot{r}\times\ddot{r}\|^2}{\|r'\times r''\|^2}\, r'''(r'\times r'')$$

$$\Rightarrow \quad \tau = \frac{\dddot{r}(\dot{r}\times\ddot{r})}{\|\dot{r}\times\ddot{r}\|^2} = \frac{r'''(r'\times r'')}{\|r'\times r''\|^2}$$

Q Find the Torsion of the circular helix.

$$r(\theta) = (a\cos\theta, a\sin\theta, b\theta) \qquad \tau = \frac{(\dot{r}\times\ddot{r})\cdot\dddot{r}}{\|\dot{r}\times\ddot{r}\|^2}$$

Sol^n :-

$$r(\theta) = (a\cos\theta, a\sin\theta, b\theta)$$

$$\dot{r}(\theta) = (-a\sin\theta, a\cos\theta, b)$$

$$\ddot{r}(\theta) = (-a\cos\theta, -a\sin\theta, 0)$$

$$\dddot{r}(\theta) = (a\sin\theta, -a\cos\theta, 0).$$

Now,
$$\dot{r}\times\ddot{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & a\cos\theta & b \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix} = \hat{i}(-ab\sin\theta) - \hat{j}(+ab\cos\theta) + \hat{k}(a^2).$$

$$\dot{r}\times\ddot{r} = (+ab\sin\theta, -ab\cos\theta, a^2)$$

$$\|\dot{r}\times\ddot{r}\|^2 = (ab)^2(\sin^2\theta+\cos^2\theta) + a^4$$

$$= a^2b^2 + a^4 = a^2(b^2+a^2)$$

$$\|\dot{r}\times\ddot{r}\| = a\sqrt{a^2+b^2}$$

again
$$(\dot{r}\times\ddot{r})\cdot\dddot{r} = (+ab\sin\theta, -ab\cos\theta, a^2)\cdot(a\sin\theta, -a\cos\theta, 0)$$

$$= +a^2b\sin^2\theta + a^2b\cos^2\theta + 0$$

$$= +a^2b = |+a^2b| = a^2b$$

Now, $\tau = \dfrac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$

$$\tau = \frac{a^2 b}{a^2 \sqrt{(a^2+b^2)}} = \frac{|b|}{a^2+b^2}$$

$$\boxed{\kappa = \frac{|a|}{a^2+b^2}} \qquad \boxed{\tau = \frac{|b|}{a^2+b^2}}$$

$\Rightarrow$ $\gamma \rightarrow$ be unit-speed curve:-

$$t = \dot{\gamma}$$

& $n = \frac{1}{\kappa}\dot{t} \Rightarrow \dot{t} = \kappa n.$

$$\dot{b} = -\tau n.$$

$\rightarrow$ $n = b \times t$

diff. $\dot{n} = \dot{b} \times t + b \times \dot{t}$

$$= -\tau n \times t + b \times \kappa n$$

$$= -\tau (n \times t) + \kappa (b \times n)$$

$$= -\tau(-b) + \kappa(-t)$$

$$\boxed{\dot{n} = \tau b - \kappa t}$$

serret-frenel formula for the curve γ.

Theorem:- let γ be a unit speed curve in $\mathbb{R}^3$ with non-zero curvature then

$\dot{t} = \kappa n$

$\dot{b} = -\tau n$

$\dot{n} = -\kappa t + \tau b$

$$\begin{bmatrix} \dot{t} \\ \dot{n} \\ \dot{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

above equation is called "serret-frenet" equation.

$(t, n, b, \kappa, \tau) \rightarrow$ serret-frenet apparatus.

Example :- Find the serret-frenet apprratus for the following curve.

1. $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}} \right)$

$t = \dot{\gamma}$, $n = \frac{1}{k}\dot{t}$, $b = t \times n$, $k = \|\ddot{\gamma}\|$, $\tau = \|\dot{b}\|$

Now,

$$\dot{\gamma}(t) = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\|\dot{\gamma}\|^2 = \frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}$$

$$= \frac{1}{4}\left[1+\cancel{t}+1-\cancel{t}+2\right] = 1.$$

$\|\dot{\gamma}\| = 1.$ unit speed curve.

$\rightarrow t = \dot{\gamma}(t) = \left(\frac{1}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}} \right)$

$$\dot{t} = \ddot{\gamma}(t) = \left(\frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0 \right)$$

$$k = \|\ddot{\gamma}\| = \sqrt{\frac{1}{16}\frac{1}{1+t} + \frac{1}{16}\frac{1}{1-t}}$$

$$= \sqrt{\frac{1}{16}\left(\frac{1}{t+1} + \frac{1}{1-t}\right)}$$

$$= \sqrt{\frac{1}{16}\left(\frac{1-\cancel{t}+1+\cancel{t}}{1-t^2}\right)}$$

$$\rightarrow k = \sqrt{\frac{1}{8(1-t^2)}} = \frac{1}{\sqrt{8(1-t^2)}}$$

Now

$$n = \frac{1}{k}\dot{t}$$

$$= \sqrt{8(1-t^2)}\left(\frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0 \right)$$

$$\rightarrow n = \left(\frac{1}{\sqrt{2}}(1-t)^{1/2}, \frac{1}{\sqrt{2}}(1+t)^{1/2}, 0 \right)$$

$$\hat{b} = \hat{t} \times \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2}(1+t)^{1/2} & -\frac{1}{2}(1-t)^{1/2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(1+t)^{1/2} & \frac{1}{\sqrt{2}}(1+t)^{1/2} & 0 \end{vmatrix}$$

$$b = \hat{i}\left(-\frac{1}{2}(1+t)^{1/2}\right) - \hat{j}\left(-\frac{1}{2}(1+t)^{1/2}\right) + \hat{k}\left(\frac{1}{2\sqrt{2}}(1+t) + \frac{1}{2\sqrt{2}}(1+t)\right)$$

$$\rightarrow b = \left(-\frac{1}{2}(1+t)^{1/2}, \frac{1}{2}(1-t)^{1/2}, \frac{1}{2\sqrt{2}}\right)$$

$$\dot{b} = -\tau n$$

$$\dot{b}\cdot\dot{b} = (-\tau n)\cdot(-\tau n) = \tau^2 (n\cdot n)$$

$$\|\dot{b}\|^2 = \tau^2$$

$$\|\dot{b}\| = \tau.$$

$$\dot{b} = \left(-\frac{1}{4}(1+t)^{-1/2}, -\frac{1}{4}(1-t)^{-1/2}, 0\right)$$

$$\tau = \|\dot{b}\| = \sqrt{\frac{1}{16}\frac{1}{1+t} + \frac{1}{16}\frac{1}{1-t}}$$

$$= \sqrt{\frac{1}{16}\left(\frac{1-\cancel{t}+1+\cancel{t}}{1-t^2}\right)}$$

$$\rightarrow \tau = \frac{1}{\sqrt{8(1-t^2)}} = k.$$

2:- $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$

Solⁿ:- $\dot{\gamma}(t) = \left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right)$

$$\|\dot{\gamma}\|^2 = \frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t = 1$$

$\|\dot{\gamma}\| = 1.$ unit speed curve.

$$\rightarrow t = \dot{\gamma}(t) = \left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right)$$

Now $\dot{t} = \ddot{\gamma}(t) = \left(-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t\right)$

$$\Rightarrow k = \|\ddot{\gamma}\| = \sqrt{\frac{16}{25}\cos^2 t + \sin^2 t + \frac{9}{25}\cos^2 t} = 1.$$

$$\rightarrow k = 1 = \|\ddot{\gamma}\|$$

$$n = \frac{1}{k} t' = \frac{1}{1}\left(\frac{-4}{5}\cos t, \sin t, \frac{3}{5}\cos t\right)$$

$$\rightarrow n = \left(-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t\right)$$

$$b = t \times n = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{4}{5}\sin t & -\cos t & \frac{3}{5}\sin t \\ -\frac{4}{5}\cos t & \sin t & \frac{3}{5}\cos t \end{vmatrix}$$

$$= \hat{i}\left\{-\frac{3}{5}(\cos^2 t + \sin^2 t)\right\} - \hat{j}\left\{\frac{-12}{25}\sin t\cos t + \frac{12}{25}\sin t\cos t\right\}$$

$$+ \hat{k}\left\{-\frac{4}{5}(\sin^2 t + \cos^2 t)\right\}.$$

$$= \hat{i}\left(-\frac{3}{5}\right) + \hat{j}\cdot 0 + \hat{k}\left(-\frac{4}{5}\right)$$

$$\rightarrow b = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right)$$

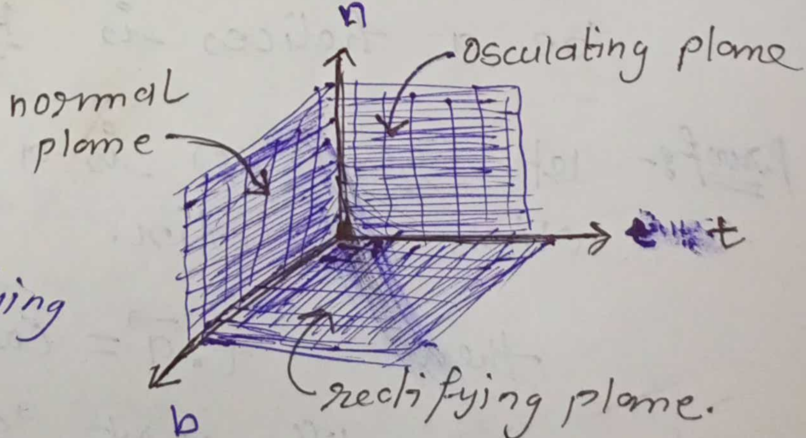
$$b' = (0, 0, 0) \quad \rightarrow b' = -\tau n.$$

$$\|b'\| = 0$$

$$\Rightarrow \tau = \|b'\| = 0$$

$$\rightarrow \tau = 0.$$

$\Rightarrow$ Three planes :-



1. Rectifying plane:-
plane spanned by $\{t, b\}$ is called rectifying plane.

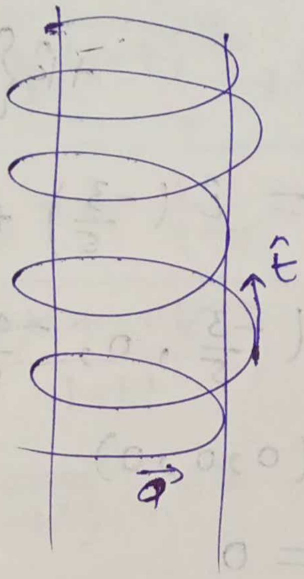
2. osculating plane:-
plane spanned by $\{t, n\}$ is called osculating plane.

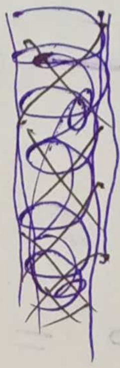
3. Normal plane:-

plane spanned by $\{n, b\}$ is called normal plane.

⇒ **Helices** :- A cylindrical Helices or helices is a curve lying on the surface of a cylinder s.t unit tangent vector 't' makes a constant angle 'θ' with a fixed vector '$\vec{a}$', i.e.

$$t \cdot \vec{a} = \cos\theta = \text{constant}$$





⇒ **Theorem** :- A necessary & sufficient condition for a unit speed curve $\vec{r} = \vec{r}(s)$ to be a helices is $\frac{\kappa}{\tau}$ = constant.

Proof:- Let $\vec{r} = \vec{r}(s)$ is a unit speed curve which is helix.

then $t \cdot \vec{a} = \cos\theta$.

diff. w.r.t 's'

$$t' \cdot a + t \cdot \overbrace{\frac{da}{ds}}^{\to = 0} = 0 \quad \text{[a fixed vector]}$$

$\because n = \frac{1}{\kappa} t' \Rightarrow t' = \kappa n$.

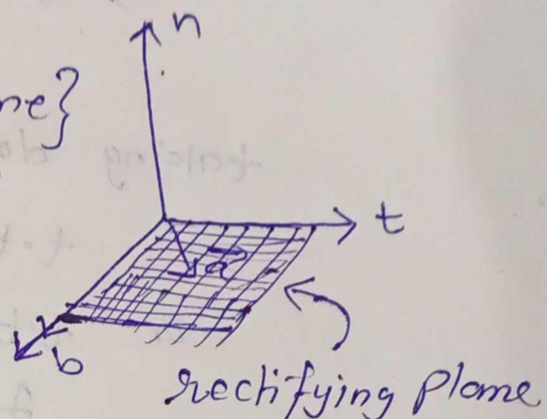
$$\Rightarrow \kappa\, n \cdot a = 0$$

$\Rightarrow \; n \cdot a = 0 \quad (\because k \neq 0)$

$\Rightarrow \; a \perp n.$

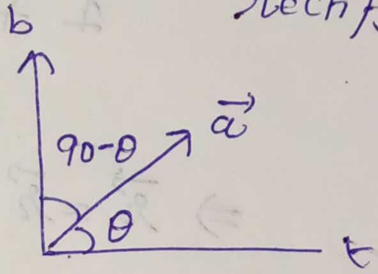
$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

$\Rightarrow \vec{a}$ will lie in the plane of t & b {rectifying plane} making θ angle with t and $(90-\theta)$ with b.



$\Rightarrow \vec{a} = t\cos\theta + b\cos(90-\theta)$

$\vec{a} = t\cos\theta + b\sin\theta$



diff. w.r.t 's'

$$\frac{d\vec{a}}{ds} = t'\cos\theta + b'\sin\theta$$

$$0 = kn\cos\theta - \tau n\sin\theta$$

$$\hat{n}\cdot 0 = \hat{n}\cdot(k\cos\theta - \tau\sin\theta)$$

$$\Rightarrow k\cos\theta = \tau\sin\theta \;, \; n \neq 0$$

$$\frac{k}{\tau} = \tan\theta = \text{constant}.$$

Conversely: let $\frac{k}{\tau} = \text{constant} = c$

$$\Rightarrow k = \tau c. \quad \text{—(1)}$$

$$\because \; t' = kn \Rightarrow k = \frac{t'}{n}.$$

from (1) $\quad t' = c\tau n. \qquad [\because b' = -\tau n]$

$$= -c(-\tau n)$$

$$t' = -cb'$$

$$\Rightarrow t' + cb' = 0$$

$$\frac{dt}{ds} + c\frac{db}{ds} = 0$$

$$\frac{d}{ds}(t + cb) = 0$$

$\Rightarrow t + cb = a$ (on integrating)

$a \to$ constant.

taking dot product with t.

$$t \cdot t + c b \cdot t = a \cdot t$$

or $t \cdot t + c\, t \cdot b = t \cdot a$

$$1 + c \cdot 0 = t \cdot a$$

$$\boxed{\hat{t} \cdot \vec{a} = 1} \text{ (constant)}$$

$\Rightarrow \vec{r} = \vec{r}(s)$ is a helix.

□ unit 2 end.

<u>Reminding:</u>

curve

1: $k = 0 \Rightarrow$ (straight line)

2: $\tau = 0 \Rightarrow$ (plane curve)

3: $\tau = 0$, $k =$ constant $\Rightarrow$ (part of circle)

4: $\tau =$ constant, $k =$ constant $\Rightarrow$ (helix)

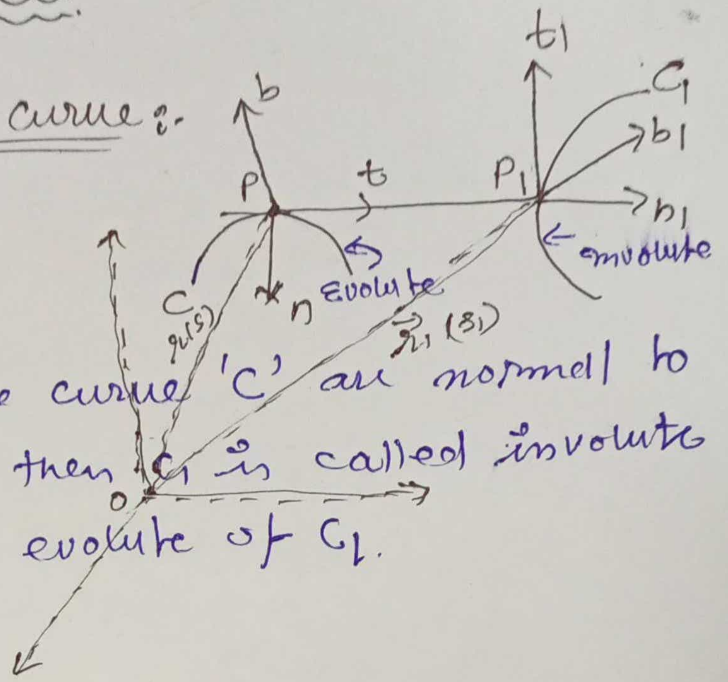
5: Radius of curvature $= \frac{1}{k}$.

6: Radius of Torsion $= \frac{1}{\tau}$.

26/04/2022

UNIT-III

⇒ Involute and Evolute curve :-



If the tangents to the curve 'C' are normal to another curve 'C_1', then C_1 is called involute of the C and C is evolute of C_1.

⇒ Equation of involute :- Let C_1 be an involute of C and the Equation of C be $r = \vec{r}(s)$ then them any point on C_1 is given by

$$\vec{r}_1(s_1) = \vec{r}(s) + \lambda \vec{t} \qquad \lambda(s).$$

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

diff. w.r.t 's_1'

$$\frac{d\vec{r}_1}{ds_1} = \left(\frac{dr}{ds} + \lambda' t + \lambda t' \right) \frac{ds}{ds_1}$$

$$\vec{t}_1 = (t + \lambda' t + \lambda k n) \frac{ds}{ds_1}$$

$\because t \perp t_1$

dot product with 't'

$$t_1 \cdot t = (t \cdot t + \lambda' t \cdot t + \lambda k\, n \cdot t) \frac{ds}{ds_1}$$

$$0 = (1 + \lambda' + 0) \frac{ds}{ds_1}$$

$$0 = (1 + \lambda') \frac{ds}{ds_1}$$

$$\frac{ds}{ds_1} \neq 0$$

$$\Rightarrow \lambda' = -1 = \frac{d\lambda}{ds}$$

$$\lambda = -s + c$$
$$\text{or } \lambda = c - s.$$

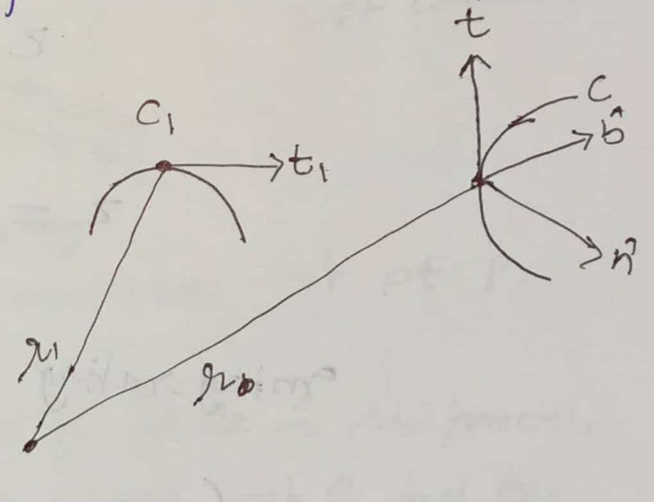
$$\therefore \boxed{\vec{r}_1 = \vec{r} + (c-s)\hat{t}}$$

Remark :- 1 :- for different value of c, we get different involute of the curve C

2 :- The d/s b/w corresponding points of two involute is Constant.

$$r_1 - r = (c-s)\,t$$
$$\|r_1 - r\| = \|c-s\| = |c-s|$$

$\Rightarrow$ **Equation of Evolute :-** If the tangent of the curve C_1 is orthogonal to the tangent of the curve C. then C_1 is called evolute of C and C is involute of curve C_1.



$\because \; t_1 \perp t$

t_1 lies in normal plane (plane spanned by $\hat{n}$ & $\hat{b}$)

$$\vec{r}_1 = \vec{r} + \lambda\hat{n} + \mu\hat{b}$$

diff. w.r.t 's_1'

$$\bar{t}_1 = (t + \lambda' n + \lambda n' + \mu' b + \mu b')\frac{ds}{ds_1}$$

$$t_1 = [t + \lambda' n + \lambda(-kt + \tau b) + \mu' b + \mu(-\tau n)]\frac{ds}{ds_1}$$

Taking inner prod. with t:

$$0 = (1 - k\lambda)\frac{ds}{ds_1} \Rightarrow \lambda = \frac{1}{k} = \rho$$

$t_1 = [(\lambda' - \mu\tau) n + (\lambda\tau + \mu') b] \frac{ds}{ds_1}$

$t_1 \parallel \lambda\vec{n} + \mu\vec{b}$

$t_1 = \delta(\lambda\vec{n} + \mu\vec{b})$

$[(\lambda' - \mu\tau)n + (\lambda\tau + \mu')b] \frac{ds}{ds_1} = \delta(\lambda n + \mu b)$

$$\begin{cases} \dfrac{\lambda' - \mu\tau}{\lambda} = \dfrac{\delta}{ds/ds_1} \\ \dfrac{\lambda\tau + \mu'}{\mu} = \dfrac{\delta}{ds/ds_1} \end{cases}$$

$\Rightarrow \dfrac{\lambda' - \mu\tau}{\lambda} = \dfrac{\lambda\tau + \mu'}{\mu}$

$\lambda'\mu - \mu^2\tau = \lambda^2\tau + \lambda\mu'$

$\tau = \dfrac{\lambda'\mu - \lambda\mu'}{\lambda^2 + \mu^2}$

$\tau = \dfrac{d}{ds}\left[\tan^{-1}\left(\dfrac{\lambda}{\mu}\right)\right]$

integrating b/s.

$a + \int \tau\, ds = \tan^{-1}\left(\dfrac{\lambda}{\mu}\right) = \cot^{-1}\left(\dfrac{\mu}{\lambda}\right)$

$a \to$ integrating constant.

$\mu = \lambda \cot(a + \int \tau\, ds)$ $\quad \because \lambda = \frac{1}{k} = \rho$

$= \rho \cot(a + \int \tau\, ds)$

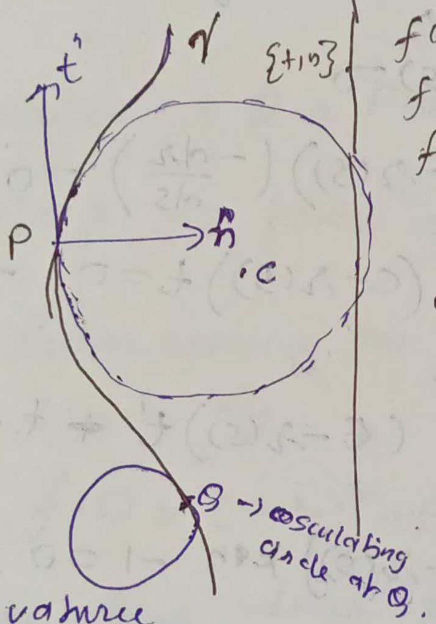
$r_1 = r + \lambda n + \mu b$

$$\boxed{r_1 = r + \rho n + \rho \cot\left(a + \int \tau\, ds\right) \cdot b}$$

This is the req. Equation of Evolute.

⇒ **Osculating Circle:-**

Let γ be given space curve & P be any point on it. The circle having three (3) point of contact with the given space curve at pt P. is called osculating circle at P.



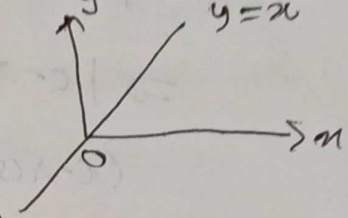
$y = x$ at $x = 0$

$f(x) = x$

$f(0) = 0$

$f'(x) = 1 \neq 0$

(one pt contact or simple contact.



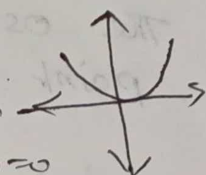
• $y = x^2$

$\{+ive\}$ $f(x) = x^2$ $f(0) = 0$

$f'(x) = 2x$, $f'(0) = 0$

$f''(x) = 2 \neq 0$ $f''(0) = 2$.

Two point contact



$y = x^3$, $f(x) = x^3$.

$f'''(x) \neq 0$

3-pt contact.

→ The radius of osculating circle is called radius of curvature at point P. & centre of osculating circle is called centre of curvature at pt. P.

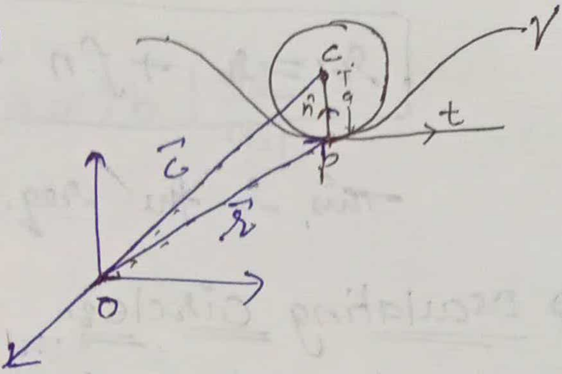
⇒ **Theorem:** The radius of osculating circle is reciprocal of the curvature of the curve at P and the position vector of the centre of osculating circle is $C = r + \rho n$ where $\rho = 1/\kappa$.

Proof :- Let

C → Position vector of centre of osculating circle

$|C - r| = a$ (Eqⁿ of circle).

$r = r(s)$ (eqⁿ of curve)



at common point

$$|c - r(s)| = a$$

$$\Rightarrow |c - r(s)|^2 = a^2$$

$$(c - r(s))(c - r(s)) = a^2$$

$$f(s) = (c - r(s))^2 - a^2$$

The osculating circle has three point of contact with given curve.

$f'(s) = 0$
$f''(s) = 0$
$f'''(s) \neq 0$

$\therefore f'(s) = 0$

$$2(c - r(s))\left(-\frac{dr}{ds}\right) = 0$$

$$(c - r(s))\, t = 0 \quad \text{— (i)}$$

$\because f''(s) = 0$

$$(c - r(s))\, t' + t \cdot t = 0$$

$$[c - r(s)]\, kn - 1 = 0$$

$$[c - r(s)]\, n = \frac{1}{k} = \rho \quad \text{— (ii)}$$

~~from~~ by (i) & (ii)

$$c - r(s) = 0 \cdot t + \rho \cdot n$$

$$\Rightarrow c - r(s) = \rho n$$

$$\Rightarrow \boxed{c = r + \rho n}$$

⇒ <u>osculating</u> <u>sphere</u> :- A sphere having four pt of contact with the curve at pt P is called osculating sphere at P on the curve.

→ The centre of osculating sphere is called centre of spherical curvature and its radius is called the spherical curvature.

Thm :- If $\bar{r} = \bar{r}(s)$ is the given curve then the centre C and ~~co~~ radius R of spherical curvature at a pt P are given by.

$$\boxed{C = r + \rho n + \sigma \rho' b} \quad \& \quad R = \sqrt{\rho^2 + \sigma^2 \rho'^2}$$

where $\rho = \frac{1}{k}$ & ~~σ~~ $\sigma = \frac{1}{\tau}$

Proof :-

Let C → centre & R → Radius of osculating sphere.

$$|c - r| = R \Rightarrow |c - r|^2 = R^2$$

The points intersecting the curve and the sphere are given by

$$f(s) = (c - r)^2 - R^2 = 0$$

∵ osculating sphere has four point contact with the given curve at point P.

$f'(s) = 0$, $f''(s) = 0$, $f'''(s) = 0$

$f'(s) = 0 \Rightarrow \quad (c - r(s))\, t = 0 \quad$ —(I)

$f''(s) = 0 \Rightarrow \quad (c - r(s))\, t' - t \cdot t = 0 \quad$ —(II)

$$(c - r(s))\cdot n = \frac{1}{k} = \rho$$

$f'''(s) = 0 \Rightarrow \quad (c - r(s))\, t'' + t' \cdot t - t \cdot t' - t' \cdot t = 0$

or $\quad -t \cdot n + (c - r(s)) \cdot n' = \frac{-k'}{k^2}$

$$-k\underbrace{(c-r(s))\cdot t}_{=0 \text{ from } (1)} + (c-r(s))\cdot \tau b = \frac{-k'}{k^2}$$

$$(c-r(s))\cdot b = \frac{-k'}{k^2}\cdot\frac{1}{\tau}$$

$$(c-r(s))\cdot b = \rho'\sigma \quad \text{—(IV)}$$

from (I), (II) & (III)

$$c-r(s) = 0\cdot t + \rho\cdot n + \rho'\sigma\cdot b$$

$$\therefore \boxed{c = r + \rho n + \rho'\sigma b}$$

Now,

$$c-r = \rho n + \rho'\sigma b$$

$$|(c-r)|^2 = (\rho n + \rho'\sigma b)\cdot(\rho n + \rho'\sigma b)$$

$$R^2 = \rho^2 + \rho'^2\sigma^2$$

$$\boxed{R = \sqrt{\rho^2 + \rho'^2\sigma^2}}$$

□

⇒ Theorem: Necessary & sufficient condition for a curve to lie on a sphere is that

$$\boxed{\rho/\sigma + \frac{d}{ds}(\sigma\rho') = 0}$$

at every point on the curve.

Proof :- Necessary Condition :- Let the curve lie on a sphere, i.e. the sphere is osculating sphere and its radius is given by

$$R^2 = \rho^2 + \sigma^2\rho'^2$$

diff. w.r.t 's'.

$\Rightarrow 0 = 2\rho\rho' + 2\sigma\sigma'\rho'^2 + 2\sigma^2\rho'\rho''$

= dividing by $2\sigma\rho'$

$\Rightarrow \; = \frac{\rho}{\sigma} + \sigma'\rho' + \sigma\rho''$

$\Rightarrow \boxed{0 = \frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')}$ - (a)

→ sufficient condition :- Let $\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0$

$\Rightarrow \frac{\rho}{\sigma} + \sigma'\rho' + \sigma\rho'' = 0$

$\Rightarrow 2\rho\rho' + 2\sigma\sigma'\rho'^2 + 2\sigma^2\rho\rho'' = 0$

$\Rightarrow \frac{d}{ds}(\rho^2 + \sigma^2\rho'^2) = 0$ - (b)

$\rho^2 + \sigma^2\rho'^2 = \text{constant} = R^2$.

also the centre of osculating sphere is given by

$c = r + \rho n + \sigma\rho' b$.

diff. w.r.t. 's'

$\frac{dc}{ds} = t + \rho' n + \rho n' + \sigma'\rho' b + \sigma\rho'' b$
$+ \sigma\rho' b'$

$= \cancel{t} + \cancel{\rho' n} + \rho(\cancel{-\kappa t} + \tau b) + \sigma'\rho' b + \sigma\rho'' b$
$+ \sigma\cancel{\rho'}(\cancel{-\tau n})$.

$\rho = \frac{1}{\kappa}$

$\sigma = \frac{1}{\tau}$

~~$= t + \rho' n + \rho t$~~

$$=(\rho\tau+\sigma'\rho'+\sigma\rho'')\cdot b.$$

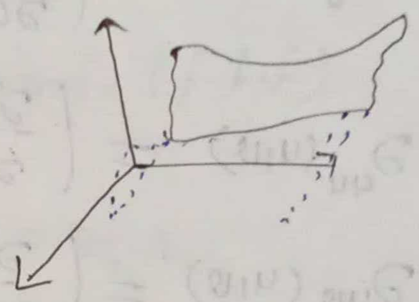
$$=\left(\rho\cdot\frac{1}{\sigma}+\sigma'\rho'+\sigma\rho''\right)b.$$

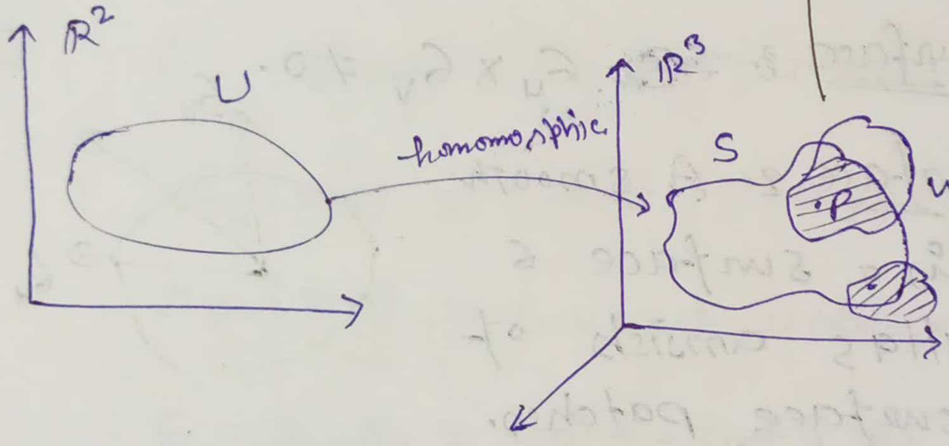
$$=\left(\frac{\rho}{\sigma}+\frac{d}{ds}(\sigma\rho')\right)b$$

$$=\quad 0b=0$$

UNIT-4

⇒ **surface** :- A ~~surface~~ subset S of $\mathbb{R}^3$, is a surface if for every point $P \in S$, there is an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing P s.t $S \cap W$ is homeomorphic to U





Thus a surface is a collection of homeomorphism

$$\sigma : U \subseteq \mathbb{R}^2 \longrightarrow S \cap W \subseteq \mathbb{R}^3$$

which we call surface patches or reparametrization,

⇒ **Higher order derivative and smooth ~~curve~~ surface** :-

$$\sigma : U \subset \mathbb{R}^2 \longrightarrow S \cap W \subset \mathbb{R}^3$$

$$\sigma(u,v) = \left(f_1(u,v), f_2(u,v), f_3(u,v) \right)$$

$$\sigma_u(u,v) = \left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}, \frac{\partial f_3}{\partial u}\right)$$

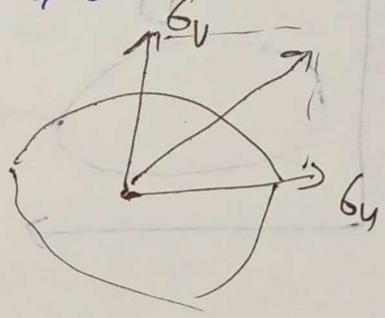
$$\sigma_v(u,v) = \left(\frac{\partial f_1}{\partial v}, \frac{\partial f_2}{\partial v}, \frac{\partial f_3}{\partial v}\right)$$

$$\partial_{uu}(u,v) = \left(\frac{\partial^2 f_1}{\partial u^2}, \frac{\partial^2 f_2}{\partial u^2}, \frac{\partial^2 f_3}{\partial u^2}\right)$$

$$\partial_{uv}(u,v) = \left(\frac{\partial^2 f_1}{\partial u \partial v}, \frac{\partial^2 f_2}{\partial u \partial v}, \frac{\partial^2 f_3}{\partial u \partial v}\right)$$

$$\sigma_{vv}(u,v) = \left(\frac{\partial^2 f_1}{\partial v^2}, \frac{\partial^2 f_2}{\partial v^2}, \frac{\partial^2 f_3}{\partial v^2}\right)$$

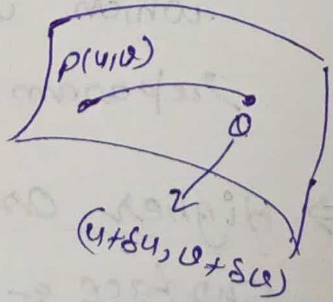
⇒ **Regular surface** :- $\sigma_u \times \sigma_v \neq 0$.



⇒ **Smooth Surface** :- A smooth surface is a surface σ whose atlas consists of regular surface patches.

⇒ **First fundamental form** :- (d/s b/w two pts)

Let $\vec{r} = \vec{r}(u,v)$ be a regular surface



$$dr = \frac{\partial r}{\partial u}.du + \frac{\partial r}{\partial v}dv$$

$$= r_u du + r_v dv$$

$$ds^2 = |dr|^2$$

$$= dr.dr$$

$$= (r_u du + r_v dv).(r_u du + r_v dv)$$

$$= r_u.r_u du^2 + (r_u.r_v) du dv + (r_v.r_u) dv du + (r_v.r_v) dv^2$$

$$= (r_u \cdot r_u)\, du^2 + 2(r_u \cdot r_v)\, du\, dv + (r_v \cdot r_v)\, dv^2$$

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2$$

called first fundamental form. (f.f.f)

$$E = r_u \cdot r_u \,,\; F = r_u \cdot r_v \quad G = r_v \cdot r_v .$$

Example: find f.f.f of helicoidal surface

$$x = u\cos v \quad y = u \sin v \;,\; z = cv.$$

sol^n $\therefore$ $r(u,v) = (u\cos v, u\sin v, cv)$

$$r_u(u,v) = (\cos v, \sin v, 0)$$

$$r_v(u,v) = (-u\sin v, u\cos v, c)$$

$$E = r_u \cdot r_u = \cos^2 v + \sin^2 v = 1.$$

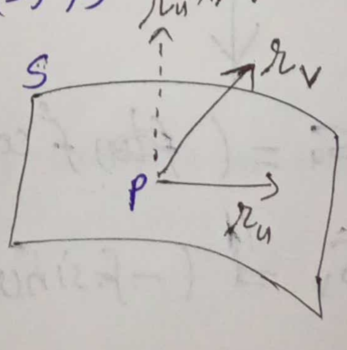
$$F = r_u \cdot r_v = 0$$

$$G = r_v \cdot r_v = u^2 + c^2$$

$$ds^2 = du^2 + (u^2+c^2)\, dv^2.$$

$\Rightarrow$ second fundamental form :- (s.f.f)

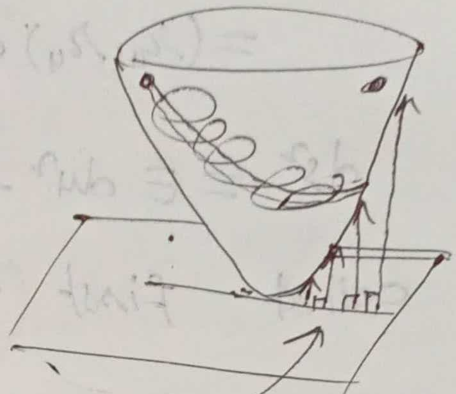
$r_u, r_v \in \vec{T_p}\, s$ $\rightarrow$ tangent at P on S.



$$\nu = \frac{r_u \times r_v}{\| r_u \times r_v \|}$$

$\hookrightarrow$ unit surface normal.

$$\begin{cases} L = r_{uu} \cdot U \\ M = r_{uv} \cdot U \\ N = r_{vv} \cdot U \end{cases}$$



→ called the coefficient of s.f.f. (second fundamental form).

s.f.f measure $\perp$ d/s.

$$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

↳ s.f.f.

<u>Example</u> :- find second fundamental form of surface of revolution.

$$\sigma(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u)).$$

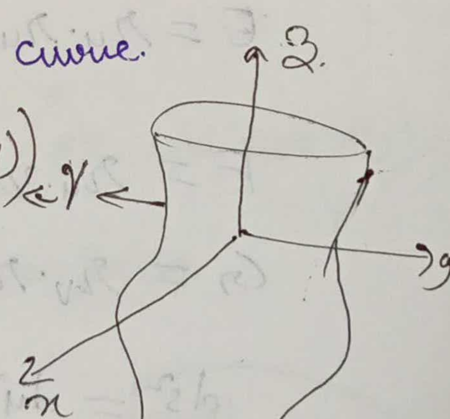
<u>Sol</u>^{n}

→ unit speed curve.

$\gamma = (f(u), 0, g(u)) \leftarrow \gamma$

$\|\dot{\gamma}\| = 1.$

$\dot{f}^2 + \dot{g}^2 = 1.$ —(9)



$$\sigma_u = (\cancel{f(u)}\ \dot{f}\cos v,\ \dot{f}\sin v,\ \dot{g})$$

$$\sigma_v = (-f\sin v,\ f\cos v,\ 0)$$

$$\sigma_u \times \sigma_v = (-f\dot{g}\cos v,\ -f\dot{g}\sin v,\ \dot{f}f)$$

$$\|G_u \times G_v\| = \sqrt{f^2\dot{g}^2 + f^2\dot{f}^2} = \sqrt{f^2(\dot{f}^2+\dot{g}^2)}$$

$$= f \quad \text{by (a)}$$

$$U = \frac{G_u \times G_v}{\|G_u \times G_v\|} = \frac{1}{f}(-\dot{g} f\cos v, -\dot{g} f\sin v, f\dot{f})$$

$$= (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f}).$$

Now $G_{uu} = (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}) =$

$G_{uv} = (-\dot{f}\sin v, \dot{f}\cos v, 0)$

$G_{vv} = (-f\cos v, -f\sin v, 0)$

$L = G_{uu} \cdot U = (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}) \cdot (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f})$

$= -\dot{g}\ddot{f} + \ddot{g}\dot{f}$

$M = G_{uv} \cdot U = (-\dot{f}\sin v, \dot{f}\cos v, 0) \cdot (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f})$

$= 0$

$N = G_{vv} \cdot U = f\dot{g}$

So, second fundamental form.

$$\boxed{\mathrm{II} = (\ddot{g}\dot{f} - \dot{g}\ddot{f})\,du^2 + (f\dot{g})\,dv^2}$$

If the surface ~~sphere~~ is unit sphere, we can take.

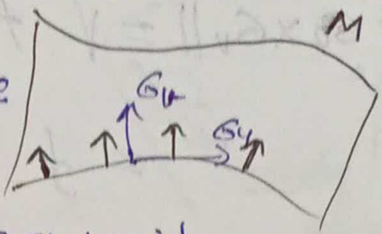
$f(u) = \cos u, \quad g(u) = \sin u$

then

$\mathrm{II} = (+\sin^2 u + \cos^2 u)\,du^2 + \cos^2 u\,dv^2$

$= du^2 + \cos^2 u\,dv^2$, $\square$ S.F.F for sphere.

⇒ <u>shape</u> <u>operator</u> : $G: U \subset \mathbb{R}^2 \to M \cap W \subseteq \mathbb{R}^3$



Let $M \subset \mathbb{R}^3$ be regular surface and G_u be a tangent vector at a point $p \in M$ and U be a unit surface normal. Then the shape operator denoted by $S_p(G_u)$ where $G(u,v)$ is a surface patch of the surface is a linear map

$$S : T_pM \to T_pM$$

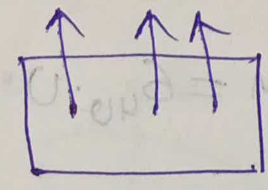
defined by

$$S(G_u) = -\nabla_{G_u} U$$

$$S(G_v) = -\nabla_{G_v} U$$

<u>Ex</u>: plane

for a plane U = constant



$$S(G_u) = 0.$$

<u>Ex</u>:- sphere

$$G(\theta, \phi) = (a\sin\theta\cos\phi,\ a\sin\theta\sin\phi,\ a\cos\theta)$$

$$G_\theta(\theta,\phi) = (a\cos\theta\cos\phi,\ a\cos\theta\sin\theta,\ -a\sin\theta)$$

$$G_\phi(\theta,\phi) = (-a\sin\theta\sin\phi,\ a\sin\theta\cos\phi,\ 0)$$

$$G_\theta \times G_\phi = (a^2\sin^2\theta\cos\phi,\ a^2\sin^2\theta\sin\phi,\ a^2\sin\theta\cos\theta)$$

$$\|G_\theta \times G_\phi\| = a^2 \sin\theta$$

$$U = \frac{G_\theta \times G_\phi}{\|G_\theta \times G_\phi\|}$$

$$= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Now the shape operator.

$$S(G_\theta) = -\nabla_{G_\theta} U$$

$$= \frac{-\partial}{\partial\theta}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$= -(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$= \frac{-1}{a} G_\theta + 0\cdot G_\phi$$

& $S(G_\phi) = -\nabla_{G_\phi} U$

$$= \frac{-\partial}{\partial\phi}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$= -(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$$

$$= 0\cdot G_\theta - \frac{1}{a} G_\phi$$

$$[S] = \begin{bmatrix} \frac{-1}{a} & 0 \\ 0 & \frac{-1}{a} \end{bmatrix}$$

⇒ Normal & Principal Curvature :-

Defⁿ :- If $\gamma(t) = \sigma(u(t), v(t))$ is a unit speed curve on a ~~sphere~~ ~~surphase~~ surface patch σ, its normal curvature is given by

$$k_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$$

Defⁿ :- The principal curvature of a ~~sphere~~ surface patch are the roots of the equation.

$$\begin{vmatrix} L-kE & M-kF \\ M-kF & N-kG \end{vmatrix} = 0$$

Ex : Sphere

$E=1$, $F=0$, $G=\cos^2\theta$

$L=1$, $M=0$ $N=\cos^2\theta$

$$\begin{vmatrix} 1-k & 0 \\ 0 & \cos^2\theta - k\cos^2\theta \end{vmatrix} = 0$$

$$(1-k)(\cos^2)(1-k) = 0$$

$$(1-k)^2 = 0$$

$k=1$ (Repeated root).

Ex :- Cylinder

$$\sigma(u,v) = (\cos u, \sin v, u)$$

$E = 1 \qquad F = 0 \qquad G = 1$

$L = 0 \qquad M = 0 \qquad N = \cancel{0}1.$

$$\begin{vmatrix} 0-k & 0 \\ 0 & 1-k \end{vmatrix} = 0 \Rightarrow k(1-k) = 0$$

$$\Rightarrow k = 0, 1.$$