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COMPLEX ANALYSIS

BHM-612

Spiral

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Unit-1

Complex Number System:—

Recall real number system $\mathbb{R}, \mathbb{R}^2, \ldots$
Why to study complex numbers:
Consider quadratic eqⁿ $ax^2+bx+c=0$, $a\neq 0$
with $a, b, c \in \mathbb{R}$.
We know if $b^2-4ac \geq 0$, then solⁿˢ $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$. But if $b^2-4ac<0$, solⁿˢ

Can't be determined. This led to extend the real number system to a bigger system, called the complex number system.

Eg: $x^2+9=0$, solⁿˢ?

$x^2 = -9 = 9\times(-1)$, $x = \pm 3\sqrt{-1}$

We define $\sqrt{-1}$ to be i, called imaginary unit

A Complex Number is defined as $a \pm ib$ where $a, b \in \mathbb{R}$ and $i^2=-1$.
a is called the real part, b is called imaginary part of the complex no. $a+ib$

eg: $3+2i$, $22\pi + i\sqrt{2}$

Notation: z, w etc & $\mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\}$
For $z = a+ib$, $a = \operatorname{Re} z$, $b = \operatorname{Im} z$

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Equality: $Z_1 = a_1 + ib_1$ & $Z_2 = a_2 + ib_2$ are equall iff $a_1 = a_2$ and $b_1 = b_2$

Q. Find a & b such that $2a + i4b$ and $2i$ rep. same Complex number?
$a = 0$, $b = \frac{1}{2}$

Zero Complex no: $Z = a + ib$ is Zero complex no iff $a = 0$, $b = 0$

Order is meaningless: Note that the relations 'greater than' and 'less than' are not defined for complex numbers unlike the Case of real nos. Thus, $i > 0$, $1 - i < 3$ etc are meaningless.

Eg: write the following as Complex nos.

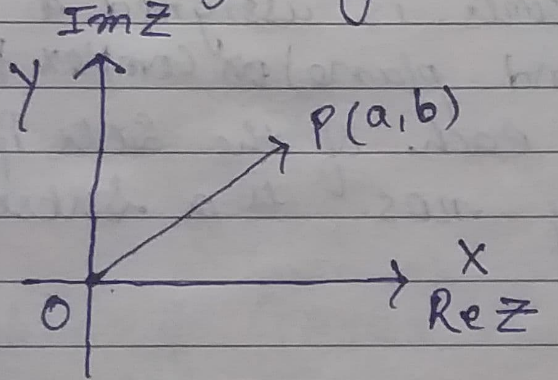
$\sqrt{-19}$, $3 - \sqrt{-5}$

Eg: write the real and imaginary parts of $2 - i\sqrt{2}$ and $\frac{\sqrt{5}i}{7}$

Eg: Find the value of x and y if
i) $4x + i(3x - y) = 3 - 6i$
ii) $(3y - 2) + i(7 - 2x) = 0$

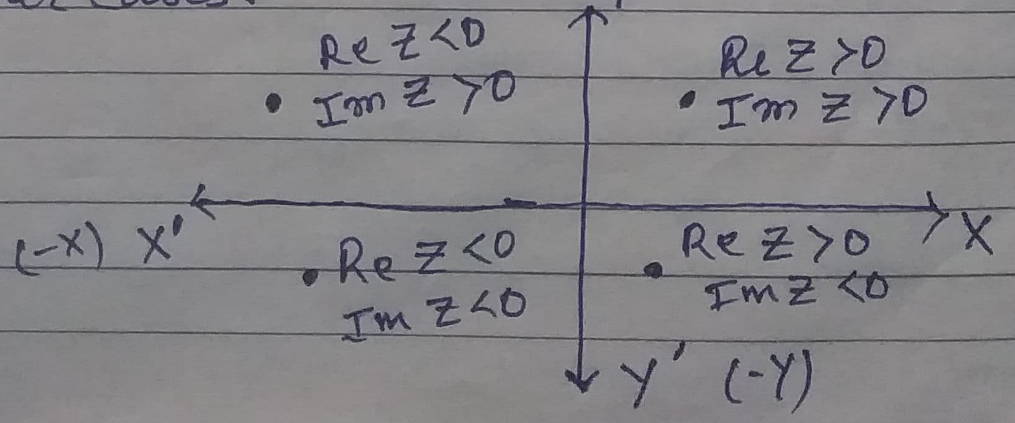
Graphical Representation:

A complex no. $z = a+ib$ can be represented geometrically as a unique point $P(a,b)$ in $\mathbb{R}^2$ with x-coord rep its real part and y-coord rep imaginary part.



(a,0) on the x-axis rep complex no. $a+i0$ and every real no. is represented as a point on the x-axis. This way x-axis is called the real axis. The point (0,b) on the y-axis rep by complex no. $0+ib$ i.e ib an imaginary number. This way y-axis is called imaginary axis.

For other cases:



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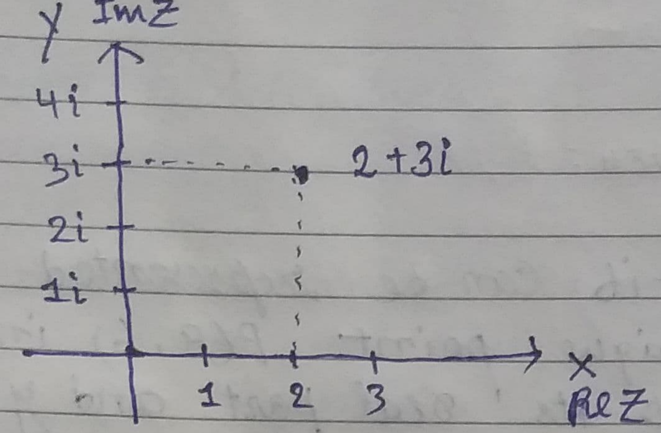
Eg: Represent $2+3i$ by a point in the complex plane.



The plane each of who points is assigned a complex no is called ~~plane~~ (Argand plane) or complex plane. It is obvious that each of the sets of real number and imaginary nos. is a subset of complex numbers.

Modulus: $|Z| = \sqrt{a^2+b^2}$

Conjugate: $z \to \bar{z}$ # Geometrical meaning of $\bar{z}$ ✓

properties: $a = Re(z) = \frac{z+\bar{z}}{2}$

$b = Img(z) = \frac{z-\bar{z}}{2i}$

$z\bar{z} = |z|^2$

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Addition of Complex Nos.

The sum of two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ is defined as the complex no $(a_1 + a_2) + i(b_1 + b_2)$

i.e $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$= (a_1 + a_2) + i(b_1 + b_2)$

Eg. $(2-7i) + (-2+3i) = 2 + (-2) + i(-7+3)$

$= 0 - 4i$

$= -4i$

$(12-4i) + 4i = (12+0) + i(-4+4)$

$= 12 + i \cdot 0 = 12$

Addition of complex numbers observes the following properties:

i) Closure : The sum of two complex no. is is a complex no. Hence the set of complex numbers is closed for addition

ii) Commutativity: $z_1 + z_2 = z_2 + z_1$ How?

Take $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, Then

$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

$= (a_2 + a_1) + i(b_2 + b_1)$

$= (a_2 + ib_2) + (a_1 + ib_1)$

$= z_2 + z_1$

iii) Associativity: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$. Do it

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iv) Additive Identity: Let $(x + iy)$ be the add. id for addition. Then by defn

$(a + ib) + (x + iy) = (a + ib)$

$(a + x) + i(b + y) = a + ib$

by defn of equality of complex nos.

$a + x = a$, $b + y = b$ which gives

$x = 0$, $y = 0$

$\therefore$ The add. id. is the complex no. $0 + i0$, simply denoted by 0.

v) Additive Inverse: Each complex no. $z = a + ib$ possesses its additive inverse. Let $w = c + id$ be its additive inverse. Then by defn $z + w = 0 + i0$

i.e $(a + c) + i(b + d) = 0 + i0$

$\therefore$ $a + c = 0$ and $b + d = 0$

$\therefore$ $c = -a$ and $d = -b$

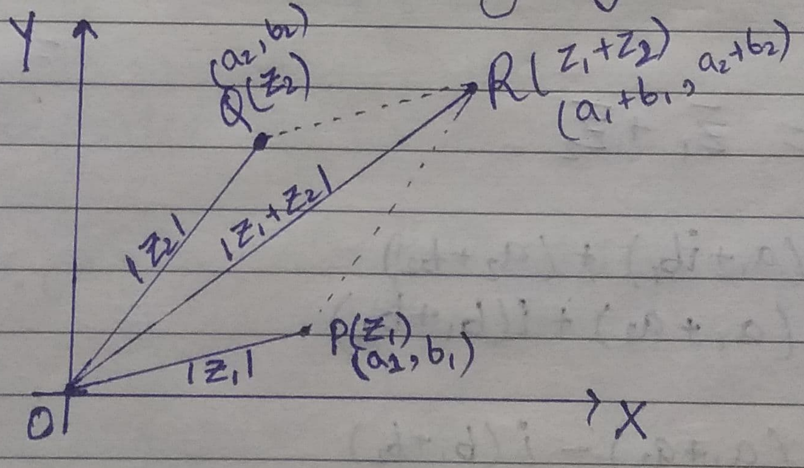
$\therefore$ $w = c + id = -a + i(-b) = -a - ib$ which is the additive inverse of z (denoted by $-z$)

Eg: for $z = -5+7i$, then $-z$ is obtained by changing the sign in z i.e $-z = 5-7i$

geometrically $-z = (-a, -b)$ is the reflection of z through the origin.

Geometrical meaning of Sum: 28 Feb

Let P and Q be two points in Complex plane representing two complex nos. $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ resp. as in the following fig.



Join them to origin. Complete the llgm OPRQ. Then the point R represents the point $R(z_1+z_2)$, and OR is the segment representing the sum of z_1 and z_2 i.e z_1+z_2

$a+ib$ can be written as
$(a+i0) + (0+i)(b+i.0)$
Now think of $a+ib \leftrightarrow a-ib$

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Def: The Conjugate of a Complex no.

$Z = a+ib$ is the Complex no. $a-ib$ and is denoted by $\bar{Z}$.

Eg: $\overline{3+i} = 3-i$

$\overline{1-\frac{\pi}{4}i} = 1+\frac{\pi}{4}i$

Properties: i) $\bar{\bar{Z}} = Z$

$Z = a+ib$, $\bar{Z} = \overline{a+ib} = a-ib$, $\bar{\bar{Z}} = \overline{(\bar{Z})} = \overline{a-ib}$

$= a+ib$

$= Z$

ii) $\overline{Z_1+Z_2} = \bar{Z_1} + \bar{Z_2}$

$Z_1+Z_2 = (a_1+ib_1) + (a_2+b_2)$

$= (a_1+a_2) + i(b_1+b_2)$.

$\overline{Z_1+Z_2} = (a_1+a_2) - i(b_1+b_2)$

$= (a_1-ib_1) + (a_2-ib_2)$

$= \overline{a_1+ib} + \overline{a_2+ib_2}$

$= \bar{Z_1} + \bar{Z_2}$

iii) $\overline{Z_1 Z_2} = \bar{Z_1} \cdot \bar{Z_2}$

$Z_1 \cdot Z_2 = (a_1+ib_1)(a_2+ib_2)$

$= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$

$\overline{Z_1 \cdot Z_2} = (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1)$

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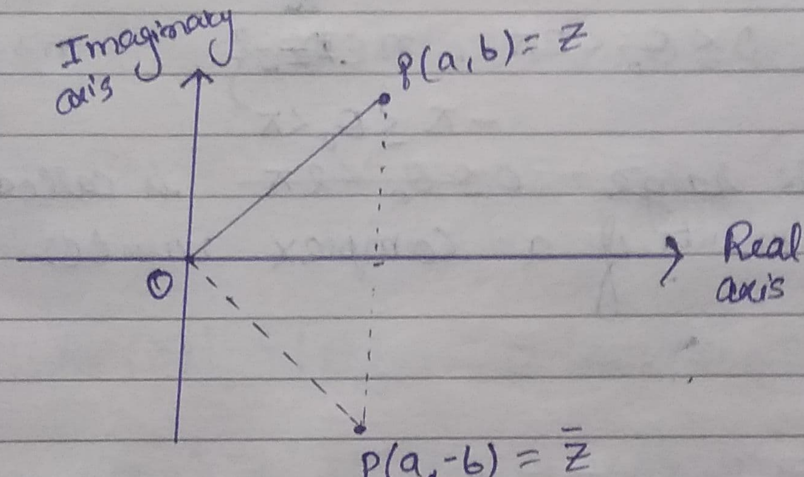
$= (a_1 - ib_1)(a_2 - ib_2)$

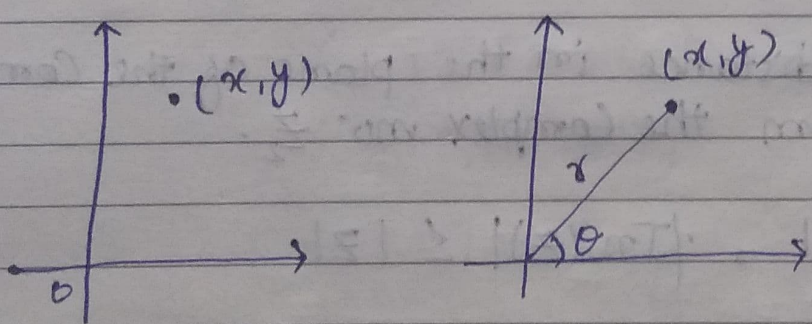
$= \overline{(a_1 + ib_1)} \cdot \overline{(a_2 + ib_2)}$

$= \bar{z}_1 \cdot \bar{z}_2$

iv) $Re(z) = \dfrac{z + \bar{z}}{2}$, $Im(z) = \dfrac{z - \bar{z}}{2i}$



Polar Coordinate:



In polar Co-ordinates, we can represent a point (x, y) as (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.
(θ is not defined for O)

So owing to this, we have a polar rep. of complex no° $x + iy$ as $r(\cos\theta + i\sin\theta)$

r is called the modulus of the complex number

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$x+iy$ denoted by $|Z|$ and θ is called an argument of the complex number

The argument for a complex number $x+iy$ $(\neq 0)$ is infact the set

$$\{\theta_0 + 2n\pi : 0 \le \theta_0 < 2\pi,\ n \in \mathbb{Z}\} \qquad n \to \text{Integer } \mathbb{Z}$$

$$-\pi \le \theta_0 < \pi$$

The angle in the range ~~$0 \le \theta_0 < 2\pi$~~ is called principal argument of a complex number

Properties:

1) $|Z|$ is the distance in the plane of the complex no. Z from 0.

2) $|Z_1 - Z_2|$ is the distance in the plane of the complex number Z_1 from the complex no. Z_2.

3) $|Re(Z)| \le |Z|$, $|Im(Z)| \le |Z|$.

4) $Z\bar{Z} = |Z|^2$

5) Triangle inequality:

$$|Z_1 + Z_2| \le |Z_1| + |Z_2|$$

and equality holds iff Z_1 & Z_2 lie on the same half ray through the origin in the complex plane.

$\overline{z_1 \bar{z}_2} = \bar{z}_1 z_2$

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Proof: $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 + |z_2|^2$

$= |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2$

$\because \dfrac{z+\bar{z}}{2} = Re(z) \Rightarrow z+\bar{z} = 2Re(z)$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$

$\because Re(z)| \leq |z|$

$\Rightarrow |z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$

$\because |z| = |\bar{z}|$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\leq (|z_1| + |z_2|)^2$

$\Rightarrow |z_1+z_2| \leq |z_1| + |z_2|$

The generalised form of Triangular Inequality is

$|z_1+z_2+\dots+z_n| \leq |z_1|+|z_2|+\dots+|z_n|$

It can be proved through mathematical Induction.

In the following we derive another imp. inequality for differences

To prove: $|z_1-z_2| \geq ||z_1|-|z_2||$

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Proof: $|z_1| = |z_1 - z_2 + z_2|$

$\leq |z_1 - z_2| + |z_2|$

$\Rightarrow |z_1| - |z_2| \leq |z_1 - z_2| \quad -(i)$

or $|z_1 - z_2| \geq |z_1| - |z_2|$

Also $|z_2| = |(z_2 - z_1) + z_1|$

$\leq |z_2 - z_1| + |z_1|$

$|z_2| - |z_1| \leq |z_2 - z_1|$

or $|z_2| - |z_1| \leq |z_1 - z_2|$

or $-(|z_1| - |z_2|) \leq |z_1 - z_2|$

or

$|z_1| - |z_2| \geq -|z_1 - z_2| \quad -(ii)$

By (i) and (ii)

$|z_1| - |z_2| \leq |z_1 - z_2|$ ✓

and $-|z_1 - z_2| \leq |z_1| - |z_2|$

now $\because$ $|x| \leq a \Leftrightarrow -a \leq x \leq a$

So the above two inequalities combine to give us ~~together~~

$||z_1| - |z_2|| \leq |z_1 - z_2|$

Q: why the above is called triangle inequality?

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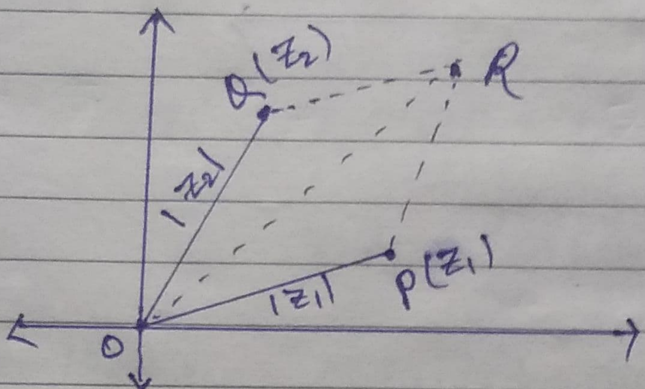
Geometrical Interpretation:

For the $|z_1 + z_2| \leq |z_1| + |z_2|$



then $OP = |z_1|$, $OQ = |z_2| = PR$, $OR = |z_1 + z_2|$

we know that sum of any two sides of a triangle is always greater than the third side,

So in ΔOPR ; we can write

$$OP + PR \geq OR$$

$$\Rightarrow |z_1| + |z_2| \geq |z_1 + z_2|$$

or $$|z_1 + z_2| \leq |z_1| + |z_2|$$

Thus it satisfies the triangle law and hence called as the triangle inequality

✓ For $|z_1 - z_2| \geq |z_1| - |z_2|$

Locate the points z_1, z_2 and $-z_2$ in the complex plane.

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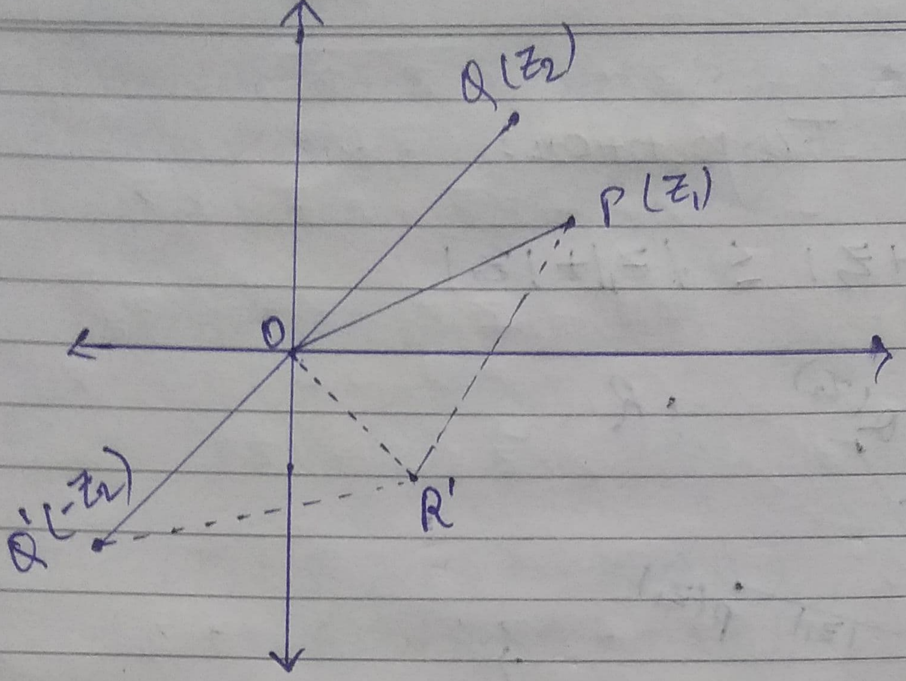


now Complete the $\|$gm with sides OP and OQ'. Then $OP = |z_1|$ and $OQ' = |-z_2|$ (which is same as $|z_2|$).
The point R' represents the complex no $z_1 - z_2$ ($\because$ OP is same as Q'R'). so OR' represents the diagonal of $\|$gm and so $OR' = |z_1 - z_2|$

Consider $\triangle OR'Q'$, we know that the absolute difference of any two sides of a triangle is less than the third side, so

$$Q'R' - OQ' \le OR'$$

or $|z_1| - |z_2| \le |z_1 - z_2|$

Or $|z_1 - z_2| \ge |z_1| - |z_2|$

Again as it satisfies the triangle law, hence its name.

Date 1 March

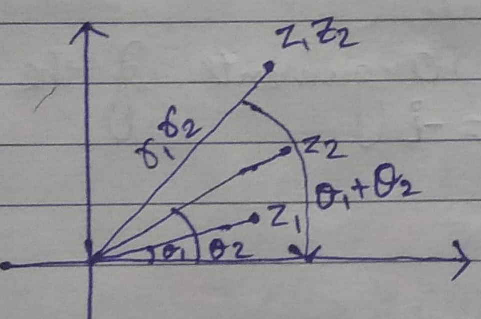
Again we turn our attention to the multiplication of complex numbers. This time we shall use polar forms. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be the two complex nos. ~~[illegible]~~

Then

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

Thus the no. obtained on multiplying has modulus $r_1 r_2$ and argument $\theta_1 + \theta_2$



i.e $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$.

In the following, we compute powers of complex numbers.
Let $z = r(\cos\theta + i\sin\theta)$, then using above multiplication

$$z^2 = z \cdot z = r^2(\cos 2\theta + i\sin 2\theta).$$

Using mathematical Induction, we can prove for a ~~[illegible]~~ +ve integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Proof 2 March

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Q. Represent the complex no. $z = 1 + i\sqrt{3}$ in polar form.

Solⁿ: we write $x + iy = 1 + i\sqrt{3}$

$r = \sqrt{x^2+y^2} = \sqrt{1^2+3} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$\therefore$ Polar coordinates are $(r, \theta) = \left(2, \frac{\pi}{3}\right)$

Polar form : $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ Ans

Q. Find the moduli and arguments of the complex numbers $z_1 = 1$, $z_2 = -i$

Solⁿ: $z_1 = 1 = 1 + 0 \cdot i$

$\therefore |z_1| = \sqrt{1^2+0^2} = 1$

& $\theta = \arg z_1$
$= 0$

$\because$ 1 lies on the +ve x-axis

general value of argument is $0 + 2\pi K$, $K \in \mathbb{Z}$
$= 2\pi K$

For $z_2 = -i = 0 + (-1)i$

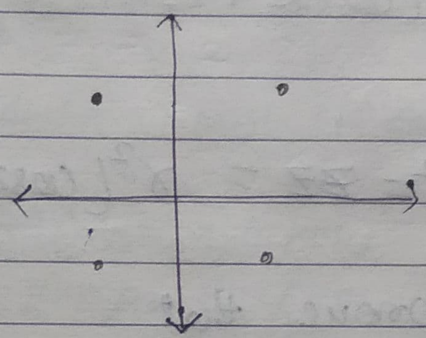
$|z_2| = \sqrt{0^2+(-1)^2} = 1$

$\theta = \arg z_2$, $\because$ $-i$ is purely imaginary, lies on the neg y-axis

So $\theta = -\frac{\pi}{2}$

general value of the argument is $\theta + 2\pi k, k \in \mathbb{Z}$
$= -\frac{\pi}{2} + 2\pi k$

Argument for different quadrants



formulae for the argument of a Complex no.
$z = x + iy$ $\quad (\alpha = \tan^{-1}(y/x))$

Quad.	Sign of x & y	Arg z
I	$x > 0, y > 0$	α
II	$x < 0, y > 0$	$\pi - \alpha$
III	$x < 0, y < 0$	$-(\pi - \alpha)$
IV	$x > 0, y < 0$	$-\alpha$

When $z = x + iy$ lies on the axes

Quad.	type of no.	Sign of x & y	Arg z
IV / I	real & +ve	$x > 0, y = 0$	0
I / II	pure Imag. with $Im(z) > 0$	$x = 0, y > 0$	$\frac{\pi}{2}$
II / III	real & -ve	$x < 0, y = 0$	π
III / IV	pure Imag. with $Im(z) < 0$	$x = 0, y < 0$	$-\frac{\pi}{2}$
Origin	zero	$x = 0, y = 0$	not defined.

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In the following we compute powers of complex numbers (and hence Stablise the De Moivre's Theorem)

De Moivre's Thm : If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

let $z = r(\cos\theta + i\sin\theta)$

By defn of Product, $z^2 = zz = r^2(\cos 2\theta + i\sin 2\theta)$

using M.I, we shall prove that

$z^n = r^n(\cos n\theta + i\sin n\theta)$

for positive integer n

For $n=1$, As $z = r(\cos\theta + i\sin\theta)$

~~$z^1 = r^1(\cos\theta + i\sin\theta)$~~

$z^1 = r^1(\cos 1\cdot\theta + i\sin 1\cdot\theta)$

Suppose it holds for k, then

$z^k = r^k(\cos k\theta + i\sin k\theta)$

$z^{k+1} = z^k \cdot z = r^k(\cos k\theta + i\sin k\theta)\cdot r(\cos\theta + i\sin\theta)$

$= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= r^{k+1}(\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

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So by P.M.I , we conclude

$z^n = r^n(\cos n\theta + i\sin n\theta)$ where we assumed n is a +ve integer

Also for $n=0$

$z^0 = 1$, Now it can be written as $1 = (\cos 0 + i\sin 0)$

$z^0 = r^0(\cos 0 + i\sin 0)$

So it holds for the integer 0.

Also , Next we prove that ~~the~~ it holds for negative integers too.

for $n=-1$

$$z^{-1} = \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{\cos\theta - i\sin\theta}{r(1)}$$

$$= r^{-1}(\cos(-\theta) + i\sin(-\theta))$$

Claim:- $z^{-n} = r^{-n}(\cos(-n\theta) + i\sin(-n\theta))$

using what has been proved, we can show that

$z^{-n} = (z^n)^{-1} = \frac{1}{z^n}$, but we know

$z^n = r^n(\cos n\theta + i\sin n\theta)$

$$z^{-n} = \frac{1}{r^n(\cos n\theta + i\sin n\theta)}$$

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$= \dfrac{\cos n\theta - i\sin n\theta}{r^n (1)}$

$= r^{-n} \left(\cos(-n\theta) + i\sin(-n\theta) \right)$

Thus we conclude that

$z^n = r^n (\cos n\theta + i\sin n\theta)$, for all integer n.

This is known as De Moivre's Theorem / formula

The De Moivre's formula can be used in computing the n^{th} roots of complex numbers.

Remark: The De Moivre's formula can be extended to rationals also.

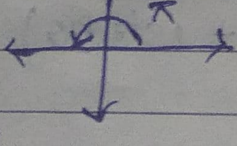
Q. Find the fourth roots of $z = -3$

Soln. To apply De Moivre's we write $z = -3$ in polar form

$z = -3 = -3 + i \cdot 0$

$\therefore r = |z| = \sqrt{(-3)^2 + 0^2} = 3$

As $-3 \equiv (-3, 0)$ lies on -ve x-axis, so argument is π



we use generalised argument, which will be $\pi + 2n\pi$, $n \in \mathbb{Z}$

$\therefore$ $Z = 3(\cos(\pi+2n\pi) + i\sin(\pi+2n\pi))$

Fourth roots of Z are given by $Z^{1/4}$.

Applying the De Moivre's thm (for rationals), we can write

$$Z^{1/4} = 3^{1/4}\left(\cos\frac{(\pi+2n\pi)}{4} + i\sin\frac{(\pi+2n\pi)}{4}\right)$$

where n will be 0, 1, 2, or 3.

then putting these values, we get all the roots

$$w_1 = 3^{1/4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= 3^{1/4}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$w_2 = 3^{1/4}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 3^{1/4}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \quad \left\{\because \frac{3\times \cancel{180}^{45}}{\cancel{4}} = 135^\circ = 90^\circ + 45^\circ\right.$$

$$w_3 = 3^{1/4}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$= 3^{1/4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \quad \left\{\because 5\times45^\circ = 225^\circ = 180 + 45^\circ\right.$$

$$w_4 = 3^{1/4}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$= 3^{1/4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \quad \left\{\because 7\times45^\circ = 315^\circ = 360 - 45^\circ\right.$$

All the roots are $3^{1/4}\left(\pm\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right)$

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Q Find cube roots of unity $z = 1$

Solⁿ $z = 1 = 1(\cos 0 + i \sin 0)$

using generalised argument,

$z = 1(\cos(0 + 2n\pi) + i\sin(0 + 2n\pi))$ where $n \in \mathbb{Z}$

By De Moivre's theorem,

$$z^{1/3} = 1^{1/3}\left(\cos\left(\frac{0+2n\pi}{3}\right) + i\sin\left(\frac{0+2n\pi}{3}\right)\right)$$

$$= \cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right)$$

where $n = 0, 1, 1, 2$

we obtain

$\omega_1 = \cos 0 + i \sin 0 = 1 + i \cdot 0 = 1$

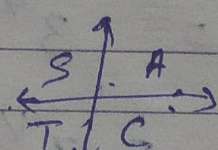
$\omega_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ $\qquad \frac{2\pi}{3} = 120 = 90 + 30^\circ$

$= -\sin 30^\circ + i\cos 30^\circ$

$= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$\omega_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ $\qquad 240 = 180 + 60$

↳ 3rd quad.



$= -\cos 60^\circ - i\sin 60$

$= \frac{-1}{2} - \frac{i\sqrt{3}}{2}$

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if we denote the 2^{nd} root by ω, then the third roots is ω^2:

$$\omega^2 = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

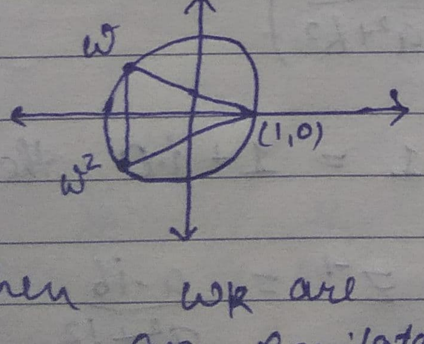
we note that $1+\omega+\omega^2=0$ and $\omega^3=1$

Geometry: $1 \to (1,0)$

$$-\frac{1}{2} + \frac{i\sqrt{3}}{2} \to \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$-\frac{1}{2} - \frac{i\sqrt{3}}{2} \to \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

they lie on x-axis, in 2^{nd} and 3^{rd} quad. respectively. Since the complex no: 1; $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2}-\frac{i\sqrt{3}}{2}$ are of unit modulus, it follows that $1, \omega, \omega^2$ lie on a unit circle with centre at 0.



when we are joined through st. lines, they form an equilateral triangle.

Note: $\{1, \omega, \omega^2\}$ is a group, infact cyclic.

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In the following we ~~[illegible]~~ delve into the multiplication of complex nos (written as $a+ib$).

The notable properties of multiplication of complex no are the following:

1) Closure: The product of two complex numbers is again a complex no. Hence the set of complex no is closed under multiplication.

2) Commutative law: $z_1 z_2 = z_2 \cdot z_1$

3) Associative law: $(z_1 \cdot z_2) z_3 = z_1 (z_2 \cdot z_3)$

4) Multiplicative ~~[illegible]~~ Id: $1 = 1 + 0i$

5) Inverse $z = a + ib (\neq 0)$; the multiplicative inv. of z is given by $z^{-1} = \frac{a-ib}{a^2+b^2}$

because $zz^{-1} = (a+ib) \cdot \left(\frac{a-ib}{a^2+b^2}\right)$

$= \frac{a^2+b^2}{a^2+b^2} = 1 = 1 + i \cdot 0$, the multiplicative identity. So $z^{-1} = \frac{a-ib}{a^2+b^2}$

or $\frac{\bar{z}}{|z|^2}$

Eg: If $z = 4+3i$, then what is the multiplicative inverse of z,

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Solⁿ: $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4-3i}{16+9} = \frac{4}{25} - \frac{3}{25}i$

6) Distributive law: distributes over addition i.e

$z_1(z_2+z_3) = z z_2 + z_1 z_3$

$(z_1+z_2)z_3 = z_1 z_3 + z_2 z_3$

we conclude that the distributive law holds for Complex numbers

$(\mathbb{C}, +, \cdot)$ forms a field.

Division: for $z_1 = a+ib$, and $z_2 = c+id$, we have (for $z_2 \neq 0$)

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$= \frac{ac+b}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$

Eg: Divide the Complex no. $z_1 = 3+i$ by $z_2 = 1+i$

Solⁿ $\frac{z_1}{z_2} = \frac{3+i}{1+i} = \frac{(3+i)(1-i)}{(1+i)(1-i)}$

$$= \frac{3-3i+i+1}{1+1} = \frac{4-2i}{2}$$

$$= 2-i \text{ Ans}$$

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Conjugate of quotient :—

for $Z_2 \neq 0$, $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$

Proof: $Z_1 = a+ib$, $Z_2 = c+id$

$$\frac{Z_1}{Z_2} = \frac{ac+bd}{c^2+d^2} - \frac{i(ad-bc)}{c^2+d^2}$$

$$\therefore \quad \overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{ac+bd}{c^2+d^2} + \frac{i(ad-bc)}{c^2+d^2}$$

and Now we find $\frac{\overline{Z_1}}{\overline{Z_2}} = \frac{a-ib}{c-id}$

$$= \frac{(a-ib)(c+id)}{(c-id)(c+id)}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{i(ad-bc)}{c^2+d^2}$$

so, $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$

Absolute value of quotient:

for $Z_2 = 0$, $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$

Proof: We can write $Z_1 = \left(\frac{Z_1}{Z_2}\right) Z_2$

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$$\therefore |z_1| = \left|\left(\frac{z_1}{z_2}\right)\cdot z_2\right|$$

$$= \left|\frac{z_1}{z_2}\right| \cdot |z_2|$$

$$\frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right| \quad \text{or} \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

3 March:

1

<u>de Moivre's formula</u>: If we put $z = r(\cos\theta + i\sin\theta)$, $|z| (= r) = 1$ in $z^n = r^n(\cos n\theta + i\sin n\theta)$, we obtain

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \cdots (2)$$

The result in (2) is known as the de Moivre's formula. This is useful in deriving certain trigonometric identities involving $\cos n\theta$ and $\sin n\theta$.

For example, if we put $n=2$ in (2), we get trig. id. for $\cos 2\theta$ and $\sin 2\theta$:

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

$$\cos^2\theta + i^2\sin^2\theta + 2i\cos\theta\sin\theta = \cos 2\theta + i\sin 2\theta$$

$$\cos^2\theta - \sin^2\theta + i(2\sin\theta\cos\theta) = \cos 2\theta + i\sin 2\theta$$

which gives $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$. ✓

de Moivre's formula in (2) also holds for rationals, which can be seen in the following. Consider the rationals of the type $\frac{1}{b}$, $b \neq 0$, $b \in \mathbb{N}$.

e.g. take $b=2$. We note that

$$(\cos\theta + i\sin\theta)^{1/2}(\cos\theta + i\sin\theta)^{1/2} = (\cos\theta + i\sin\theta)^1 \text{ which is } \cos\theta + i\sin\theta.$$

This can be written as (*)

$$(\cos A + i\sin A)(\cos A + i\sin A) = \cos\theta + i\sin\theta$$

which is $(\cos A + i\sin A)^2 = \cos\theta + i\sin\theta$

which is $(\cos 2A + i\sin 2A) = \cos\theta + i\sin\theta$ (by de Moivre's thm) for +ve integers

which gives $2A = \theta$, which gives $A = \frac{\theta}{2}$

Then by ⊛, we get $(\cos\theta + i\sin\theta)^{1/2} = (\cos A + i\sin A)$ which is $\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$.

That is, we obtain

$$(\cos\theta + i\sin\theta)^{1/2} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} = \cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta,$$

which proves de Moivre's formula for the rational $\frac{1}{2}$.

Take a general rational of the type $\frac{1}{b}$, $b \neq 0$, $b \in \mathbb{N}$ and consider $(\cos\theta + i\sin\theta)^{1/b}$. Then one can write

$$\underbrace{(\cos\theta + i\sin\theta)^{1/b}(\cos\theta + i\sin\theta)^{1/b}\cdots(\cos\theta + i\sin\theta)^{1/b}}_{b \text{ times}} = (\cos\theta + i\sin\theta)^{1} \text{ or just } \cos\theta + i\sin\theta$$

$(**)$

which can be written as

$$(\cos B + i\sin B)(\cos B + i\sin B)\cdots(\cos B + i\sin B) = \cos\theta + i\sin\theta$$

which is written as

$$(\cos B + i\sin B)^{b} = \cos\theta + i\sin\theta$$

which is

$$\cos bB + i\sin bB = \cos\theta + i\sin\theta \quad \text{(by de Moivre's thm for +ve integers)}$$

which gives $bB = \theta$, which gives $B = \frac{\theta}{b}$. Then by $(**)$, we get

$$(\cos\theta + i\sin\theta)^{1/b} = \cos\frac{\theta}{b} + i\sin\frac{\theta}{b} = \cos\frac{1}{b}\theta + i\sin\frac{1}{b}\theta$$

which proves de Moivre's result for rationals of the type $\frac{1}{b}$, $b > 2$.

Next consider rationals of the type $\frac{a}{b}$, where $a \neq 1$ and $b \in \mathbb{N}$, (which is always possible, e.g. $-\frac{10}{3}$ can be written as $\frac{-10}{3}$ etc.)

Consider $(\cos\theta + i\sin\theta)^{a/b}$, which can be written as $\left[(\cos\theta + i\sin\theta)^{1/b}\right]^{a}$, which can be written as $\left[\cos\frac{1}{b}\theta + i\sin\frac{1}{b}\theta\right]^{a}$ (proved above), which will be $\cos\left(a\cdot\frac{1}{b}\theta\right)+i\sin\left(a\cdot\frac{1}{b}\theta\right)$ $(\because a\in\mathbb{Z})$, which is just $\cos\frac{a}{b}\theta+i\sin\frac{a}{b}\theta$, which proves de Moivre's formula for all rationals.

Roots of Complex numbers: We know from Algebra that -2 and 2 are said to be square roots of the number 4 because $(-2)^2=4$ and $(2)^2=4$. Said differently, two square roots of 4 are solutions (distinct) of the algebraic eqn $w^2=4$. In general, we say that a number w is an n^{th} root of a non-zero complex number z if $w^n=z$. We will show that there are exactly n solutions of the eqn $w^n=z$.

Let $z=r(\cos\theta+i\sin\theta)$ and $w=\rho(\cos\phi+i\sin\phi)$ be the polar forms. Then the eqn $w^n=z$ is $\rho^n(\cos n\phi+i\sin n\phi)=r(\cos\theta+i\sin\theta)$, from where we can write $\rho^n=r$ and $\cos n\phi+i\sin n\phi=\cos\theta+i\sin\theta$ ...(1)

From first part, we define $\rho=\sqrt[n]{r}$ to be the unique positive n^{th} root of the positive real number r, and from second part, by equality of two complex numbers, we get $\cos n\phi=\cos\theta$ and $\sin n\phi=\sin\theta$. These equalities, in turn, indicate that the arguments θ and ϕ are related as $n\phi=\theta+2k\pi$, where $k\in\mathbb{Z}$, which gives $\phi=\frac{\theta+2k\pi}{n}$, $k\in\mathbb{Z}$. Note that as k takes on the values $k=0,1,2,\cdots,n-1$, we obtain n distinct n^{th} roots of z. These roots have the same modulus $\sqrt[n]{r}$ but different arguments. For $k\geq n$, we find that things get repeated, because the circular functions are 2π-periodic. To see it concretely, take $k=n+m$, where $m=0,1,2,\cdots$. Then we compute the value of ϕ as

$$\phi = \frac{\theta + 2(n+m)\pi}{n} = \frac{\theta + 2n\pi + 2m\pi}{n} = \frac{\theta + 2m\pi}{n} + 2\pi$$

and that $\sin\phi = \sin\left(\frac{\theta + 2m\pi}{n} + 2\pi\right) = \sin\left(\frac{\theta + 2m\pi}{n}\right)$

and $\cos\phi = \cos\left(\frac{\theta + 2m\pi}{n} + 2\pi\right) = \cos\left(\frac{\theta + 2m\pi}{n}\right)$.

To summarize, the n nth roots of a non-zero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$

where $k = 0, 1, 2, \cdots, n-1$. $\quad\cdots(2)$

Q1. Find cube roots of unity.

Sol^n: Basically we have to solve the eqn $w^3 = 1$.

Obtain polar form. It is given by $1(\cos 0 + i\sin 0)$. By eqn (2), with $n=3$, we obtain

$$w_k = \sqrt[3]{1}\left[\cos\left(\frac{0 + 2k\pi}{3}\right) + i\sin\left(\frac{0 + 2k\pi}{3}\right)\right], \quad k = 0, 1, 2.$$

Hence, the three roots are

$k=0$, $w_1 = \cos 0 + i\sin 0 = 1$

$k=1$, $w_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

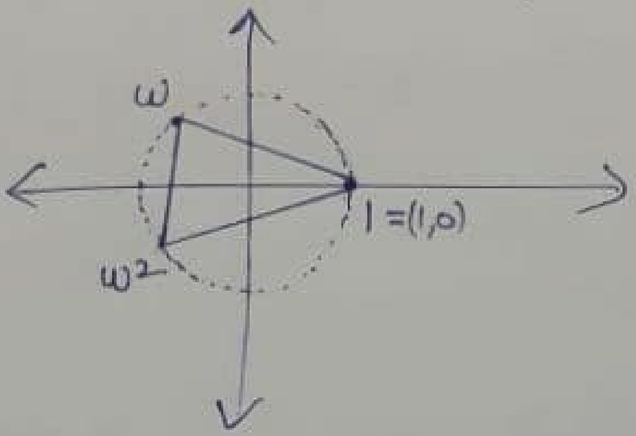
$k=2$, $w_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

If we denote the second root by ω, then the third root is seen to be ω^2:

$$\omega^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Note that $1+\omega+\omega^2=0$ and $\omega^3=1$.

Geometry: 1, ω and ω^2 lie on a circle of radius 1, centered at 0, forming sides of an equilateral triangle.



$\{1, \omega, \omega^2\}$ forms a finite cyclic group.

- The procedure for solving $w^n=1$ is easy to generalize in solving $z^n=c$, for any non-zero complex number c, ~~If $c=\rho(\cos\phi+i\sin\phi)$ and $z=r(\cos\theta+i\sin\theta)$, then $z^n=c$ iff~~ as seen before.

In case of $z^n=c$, where $z=r(\cos\theta+i\sin\theta)=re^{i\theta}$ and $c=\rho(\cos\phi+i\sin\phi)=\rho e^{i\phi}$, the n n^{th} roots of c are equally spaced points lying on the circle $C_{\rho^{1/n}}(0)=\{z : |z|=\rho^{1/n}\}$ and form vertices of a regular polygon with n sides.

Also, it is to note that if ξ is any particular solution to the eqn $z^n=c$, then all solns can be generated by multiplying ξ by the various n^{th} roots of unity. That is, the solutions are given by $\xi, \xi\omega, \xi\omega^2, \ldots, \xi\omega^{n-1}$, where ω is a root of unity.

$\because \xi^n=c$ $\therefore$ for any $j=0,1,2,\ldots,n-1$, $(\xi\omega^j)^n=\xi^n(\omega^j)^n=\xi^n(\omega^n)^j$
$=\xi^n(1)=\xi^n=c$.

Primitive n^{th} root: The root corresponding to $k=1$ is known as the primitive n^{th} root.

For example: $\omega_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ is the primitive cube root of unity.

Another example: consider the eqn $z^8=1$. It's eight eighth roots of unity will be given by

$$w_k = \sqrt[8]{1}\left[\cos\left(\frac{0+2k\pi}{8}\right) + i\sin\left(\frac{0+2k\pi}{8}\right)\right]$$

$$k=0,1,2,3,\cdots,7.$$

The primitive eighth root will be obtained as

$$w_1 = \sqrt[8]{1}\left[\cos\left(\frac{0+2\pi}{8}\right) + i\sin\left(\frac{0+2\pi}{8}\right)\right]$$

$$= 1\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}.$$

Ex. Find cube roots of $8i$.

They are given by

$$w_k = \sqrt[3]{r}\left[\cos\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right)\right]$$

$$k=0,1,2$$

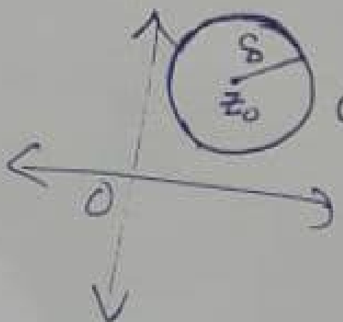
$k=0$, $w_1 = \sqrt[3]{8}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = 2\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = \sqrt{3}+i$

$k=1$, $w_2 = \sqrt[3]{8}\left[\cos\left(\frac{\frac{\pi}{2}+2\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+2\pi}{3}\right)\right] = -\sqrt{3}+i$

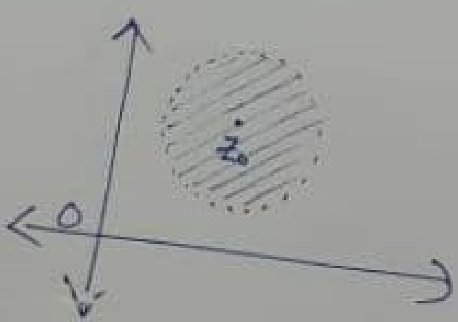
$k=2$, $w_3 = \sqrt[3]{8}\left[\cos\left(\frac{\frac{\pi}{2}+4\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+4\pi}{3}\right)\right] = -2i.$

In the following we define certain sets in the complex plane $\mathbb{C}$.

Circle : let z_0 be a fixed point in $\mathbb{C}$ and $\delta>0$. Then the set of points z such that $|z-z_0|=\delta$ is called a circle with center z_0 and radius δ.

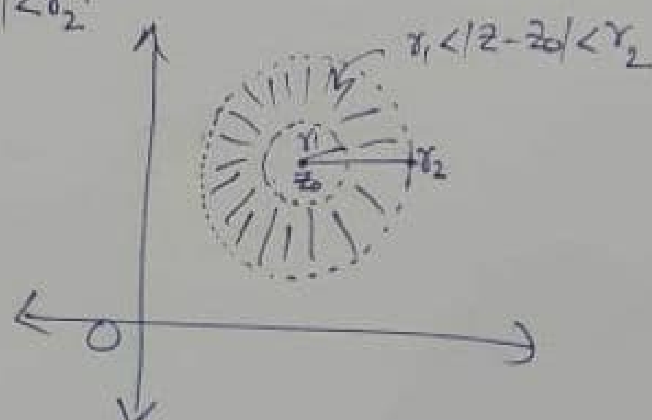


Neighbourhood (nbd): Let $z_0 \in \mathbb{C}$ be fixed and $\delta>0$ is given number. The δ-nbd of z_0 is the set of all points z s.that $|z-z_0|<\delta$. The δ-nbd of z_0 is denoted by $N(z_0;\delta)$ or by $D_\delta(z_0)$ or by $D(z_0;\delta)$.

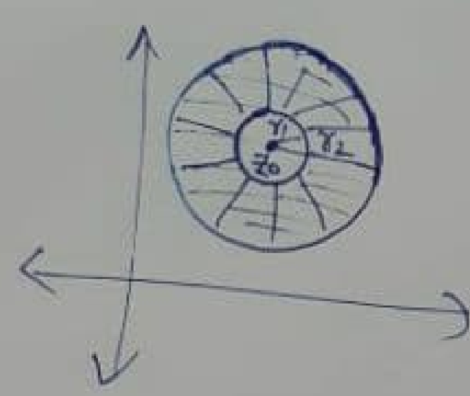


If the point z_0 itself is removed, it is called the deleted nbd of z_0, denoted by $D_\delta^*(z_0)$: $0<|z-z_0|<\delta$.
$D_\delta(z_0)$ is known as open disk with center z_0 and radius δ.
If boundary points are also included, it is called a closed disk, denoted by $\overline{D_\delta(z_0)}$: $|z-z_0|\leq\delta$.

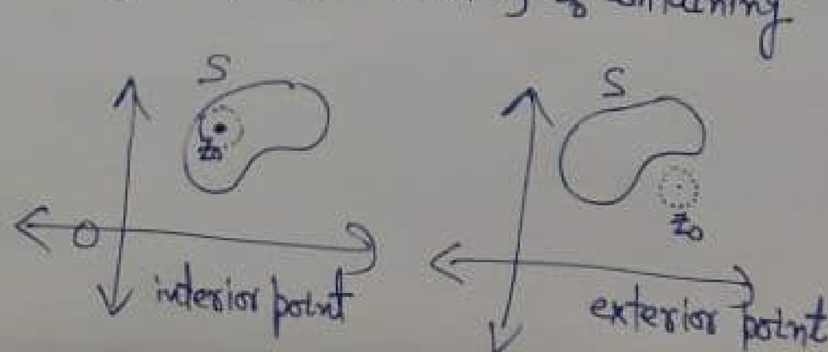
<u>open annulus</u>: Let $z_0 \in \mathbb{C}$ fixed and $r_1, r_2 > 0$. The open annulus with center at z_0 is defined as the set of all points z such that $r_1 < |z - z_0| < r_2$.



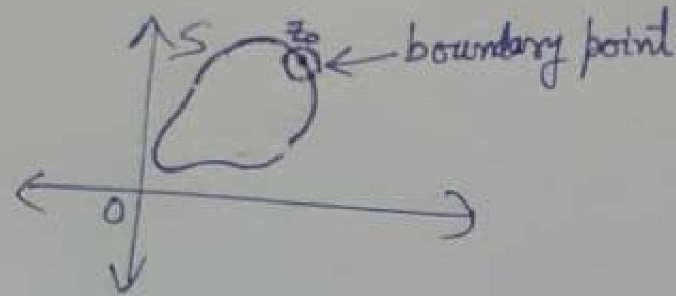
<u>Closed annulus</u>: The above is called closed annulus if the boundary points are also included, written as $r_1 \leq |z - z_0| \leq r_2$.



<u>Interior and exterior points</u>: Let $\phi \neq S \subseteq \mathbb{C}$. A point z_0 is said to be an interior point of the set S if S is a nbd of z_0. The point z_0 is exterior point of S if $\exists$ a δ-nbd of z_0 containing no points of S.

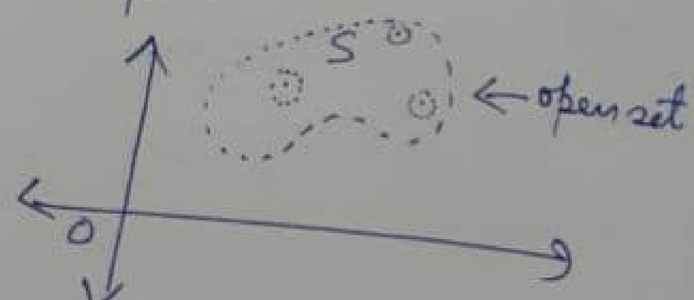


Boundary point: Let $z_0 \in \mathbb{C}$ and $\phi \neq S \subseteq \mathbb{C}$, subset. The point z_0 is said to be a boundary point of the set S if every nbd of z_0 contains points of S and points not in S.



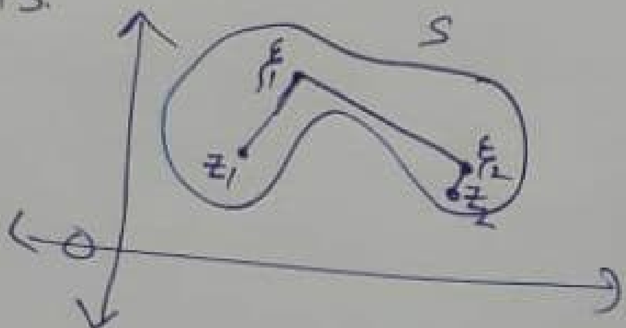
Limit point (l.p.): A point z_0 is said to be a l.p. of a set $S \subseteq \mathbb{C}$, if every nbd of z_0 contains at least one point of S other than z_0.

Open set: A set $S \subseteq \mathbb{C}$ is said to be open if every point of S is an interior point.



Closed set: A set $S \subseteq \mathbb{C}$ is said to be closed if its compliment is an open set. (or equivalently, if S contains all its limit points).

Connected Set: A set $S \subseteq \mathbb{C}$ is said to be connected if each pair of points of S can be joined by a polygonal line consisting of a finite no. of line segments joined end-to-end and lying entirely in S.



Domain: A subset $D \subseteq \mathbb{C}$ is called a domain if D is open and connected.

Region: A domain together with all, some or none of its boundary points is called a region.

To determine regions in the complex plane:

1. Determine the region $\{z \in \mathbb{C} : |z+1-2i| = 3\}$.

Consider $|z+1-2i| = 3$, which can be written as

$|(x+1)+i(y-2)| = 3$, which is

$$\sqrt{(x+1)^2+(y-2)^2} = 3$$

$$(x+1)^2+(y-2)^2 = 9$$

$$x^2+1+2x+y^2+4-4y = 9$$

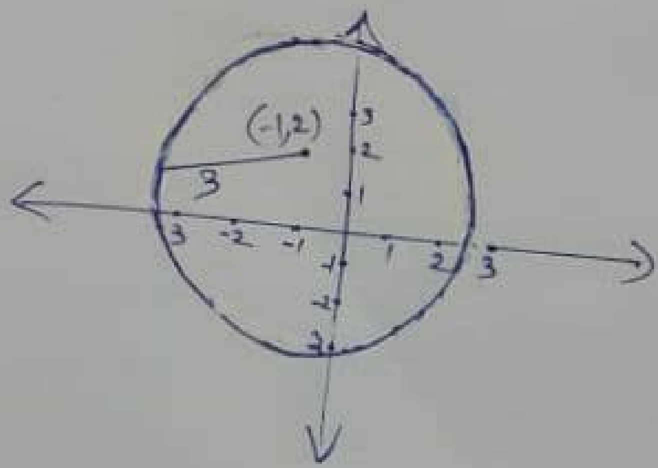
$$x^2+y^2+2x-4y = 4$$

Compare it with $x^2+y^2+2gx+2fy+c=0$

We get $2g = 2$, $2f = -4$, $c = -4$

i.e. $g = 1$, $f = -2$, $c = -4$

It is a circle with center $= (-g, -f) = (-1, 2)$ and radius $= \sqrt{g^2+f^2-c} = \sqrt{1+4+4} = \sqrt{9} = 3$. Thus the region $\{z : |z+1-2i| = 3\}$ represents a circle, centered $(-1, 2)$, rad. $= 3$.



2. What is the region $\{z \in \mathbb{C} : |z-2| = |z-4|\}$

Work out $|z-2| = |z-4|$

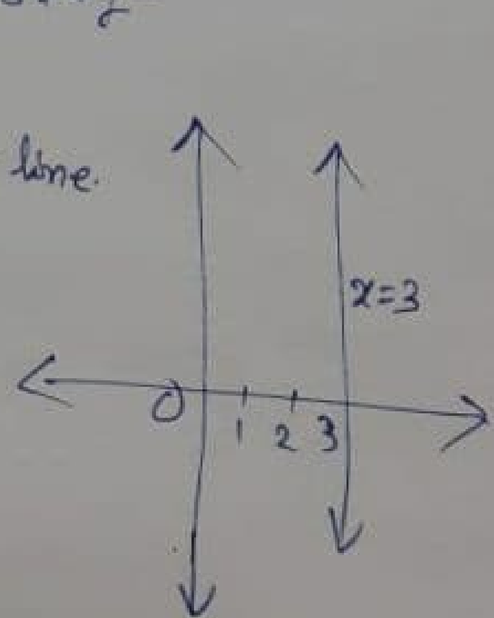
$$|(x-2)+iy| = |(x-4)+iy|$$

$$(x-2)^2+y^2 = (x-4)^2+y^2$$

$$x^2+4-4x+y^2 = x^2+16-8x+y^2$$

$$4x = 12$$

$x = 3$, a straight line.



3. Determine $\{z \in \mathbb{C} : |z+4| = 2|z-i|\}$.

Work on $|z+4| = 2|z-i|$

$$|(x+4)+iy| = 2|x+(y-1)i|$$

$$(x+4)^2+y^2 = 4[x^2+(y-1)^2]$$

which simplifies to $x^2+y^2-\frac{8}{3}x-\frac{8}{3}y-4=0$

which can be written as

$$x^2+y^2-\frac{8}{3}x-\frac{8}{3}y+\left(-\frac{4}{3}\right)^2+\left(-\frac{4}{3}\right)^2 = 2\times\left(-\frac{4}{3}\right)^2+0,$$

(with -4 written above the y term)

which is $\left(x-\frac{4}{3}\right)^2+\left(y-\frac{4}{3}\right)^2 = 4+\frac{32}{9} = \frac{68}{9}$.

Compare it with $(x-h)^2+(y-k)^2 = r^2$, it is a circle with center $=(h,k)=\left(\frac{4}{3},\frac{4}{3}\right)$, radius $= r = \frac{\sqrt{68}}{3}$. ✓

4. Obtain the region $\{z \in \mathbb{C} : |z-1|+|z+1| \le 3\}$.

Work with $|z-1|+|z+1| \le 3$

$$|(x-1)+iy|+|(x+1)+iy| \le 3$$

$$\sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2} \le 3$$

Let's write it as

$$\sqrt{(x-1)^2+y^2} \le 3-\sqrt{(x+1)^2+y^2}$$

which on squaring gives,

$$x^2+1-2x+y^2 \le 9+x^2+1+2x+y^2-6\sqrt{(x+1)^2+y^2}$$

which is $9+4x \ge 6\sqrt{(x+1)^2+y^2}$

which on further squaring gives

$$81+16x^2+72x \ge 36(x^2+1+2x+y^2)$$

which is as

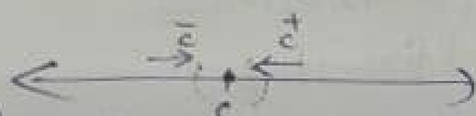
$$20x^2+36y^2 \le 45$$

which is $\frac{20}{45}x^2 + \frac{36}{45}y^2 \leq 1$

or, $\frac{x^2}{(9/4)} + \frac{y^2}{(5/4)} \leq 1$,

which represents a closed elipse including its interior.

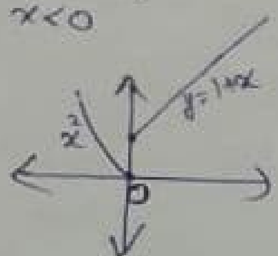
Limit (real case): We say $f(x)$ has limit L as x approaches a point c, written as $\lim_{x\to c} f(x) = L$, iff $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ both exist and are equal, where $\lim_{x\to c^-} f(x)$ is determined in a small nbd of c to its left and $\lim_{x\to c^+} f(x)$ is determined in a small nbd of c to its right.



Check the existence of limit at $x=0$ for the function

Example 1: $f(x) = \begin{cases} 1+x, & x \geq 0 \\ x^2, & x < 0 \end{cases}$.

Sol. Graph of $f(x)$.



$LHL = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h),\ h>0$

$= \lim_{h\to 0} (0-h)^2 = \lim_{h\to 0} h^2 = 0^2 = 0.$

$RHL = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} [1+(0+h)] = \lim_{h\to 0} (1+h) = 1+0 = 1.$

$LHL \neq RHL$

$\lim_{x\to 0} f(x)$ does not exist.

ε-δ definition: We say $f(x)$ approaches the limit L as x approaches the point c, if for each $\varepsilon>0$ $\exists$ a $\delta>0$ s.that $0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$

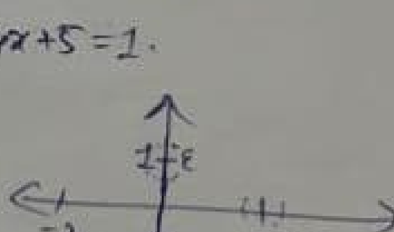
or

$|f(x)-L|<\varepsilon$ whenever $0<|x-c|<\delta$.

Example 1: Using ε-δ definition, prove that $\lim_{x\to 2} x^2-4x+5 = 1$.

Pf: Let $\varepsilon>0$, then seek a $\delta>0$ s.that

$|f(x)-1|<\varepsilon$ whenever $0<|x-2|<\delta$.



Work with $|f(x)-1|<\varepsilon$

$\Leftrightarrow |(x^2-4x+5)-1|<\varepsilon$

$\Leftrightarrow |x^2+4-4x|<\varepsilon$

$\Leftrightarrow |(x-2)^2|<\varepsilon$

$\Leftrightarrow |x-2|<\sqrt{\varepsilon}$

On choosing $\delta=\sqrt{\varepsilon}$, we are done, verify?

whenever $|x-2| < \delta = \sqrt{\varepsilon}$

then $|(x-2)^2| < \varepsilon$

then $|x^2+4-4x| < \varepsilon$

then $|(x^2-4x+5)-1| < \varepsilon$

then $|f(x)-1| < \varepsilon$

Thus, whenever $0 < |x-2| < \delta$ then $|f(x)-1| < \varepsilon$,

i.e. $|f(x)-1| < \varepsilon$ whenever $0 < |x-2| < \delta$. $\therefore \lim_{x\to 2} f(x) = 1$.

Limit (of two variable function).

Let $u = u(x,y)$, real valued function of two real variables x and y.
We say the function $u(x,y)$ has the limit u_0 as (x,y) approaches the point (x_0,y_0)
if for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.
$|u(x,y) - u_0| < \varepsilon$ whenever

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

u_0 is real or complex? It is a real number.

Ex. Show that if $u(x,y) = \dfrac{2x^3}{x^2+y^2}$, then $\lim\limits_{(x,y)\to(0,0)} u(x,y) = 0$.

Pf: $u_0 = 0$, let $\varepsilon > 0$ be any given number such that $|u(x,y) - 0| < \varepsilon$.
Then seek a $\delta > 0$ s.t. $0 < \sqrt{x^2+y^2} < \delta \Rightarrow |u(x,y) - 0| < \varepsilon$.
Work with $|u(x,y) - 0|$
substitute $x = r\cos\theta$, $y = r\sin\theta$

$$= \left|\frac{2x^3}{x^2+y^2} - 0\right| = \left|\frac{2x^3}{x^2+y^2}\right|.$$

$$= \left|\frac{2r^3\cos^3\theta}{r^2(\cos^2\theta + \sin^2\theta)}\right| = |2r\cos^3\theta| = 2|r||\cos^3\theta| \le 2r \cdot 1 = 2r$$

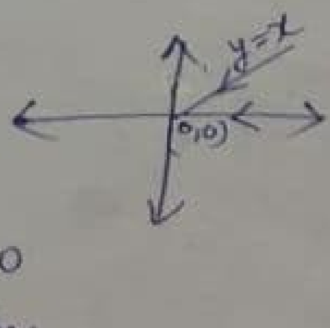
Then $|u(x,y) - 0| < \varepsilon \Rightarrow 2r < \varepsilon \Leftrightarrow r < \frac{\varepsilon}{2} \Leftrightarrow \sqrt{x^2+y^2} < \frac{\varepsilon}{2} \Leftrightarrow \sqrt{(x-0)^2+(y-0)^2} < \frac{\varepsilon}{2}$.
If we choose $\delta = \varepsilon/2$, then we find that
$|u(x,y) - 0| < \varepsilon$ whenever $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

$$\Leftrightarrow \lim_{(x,y)\to(0,0)} \frac{2x^3}{x^2+y^2} = 0. \checkmark$$

Remark: (Negation): If we can find two curves C_1 and C_2 that end at (x_0,y_0) along which $u(x,y)$ approaches to two distinct values, say u_1 and u_2, then $u(x,y)$ does not have a limit as (x,y) approaches (x_0,y_0).

Ex. Show that the function $u(x,y) = \dfrac{xy}{x^2+y^2}$ does not have a limit as (x,y) approaches $(0,0)$.

Pf: If we approach $(0,0)$ along the x-axis,



then $\lim\limits_{(x,y)\to(0,0)} u(x,y) = \lim\limits_{x\to 0} u(x,0)$

$$= \lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = \lim_{x\to 0}\left(\frac{0}{x^2}\right) = \lim_{x\to 0}(0) = 0$$

But if we approach $(0,0)$ along the line $y = x$, then

$$\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{x\to 0} u(x,x) = \lim_{x\to 0}\frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0}\frac{x^2}{2x^2} = \frac{1}{2}.$$

Thus the two limits are different. Hence we conclude that $u(x,y)$ does not have a limit as (x,y) approaches $(0,0)$. $\checkmark$

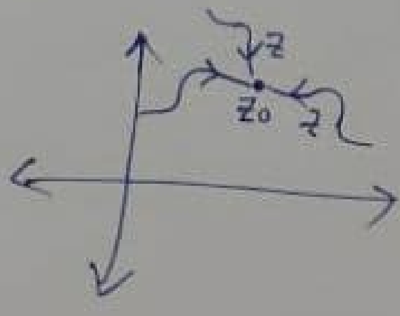
Limit of $f(z)$: Let $f(z)$ be a complex function of the complex variable z that is defined for all values of z in some nbd of z_0 except perhaps at the point z_0. We say that f has the limit w_0 (a complex number, in general) as z approaches z_0, provided the value of $f(z)$ can be made as close as we want to the value w_0 by taking z to be sufficiently close to z_0. When this happens, we write

$$\lim_{z \to z_0} f(z) = w_0$$

In terms of ε-δ: We say $\lim_{z\to z_0} f(z) = w_0$ if for each $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.
Equivalently $f(z) \in D_\varepsilon(w_0)$ whenever $z \in D^*_\delta(z_0)$.

Q1. Show that if $f(z) = \bar{z}$, then $\lim_{z\to z_0} f(z) = \bar{z}_0$, where z_0 is any complex number.

Solⁿ: (ε-δ approach)
Let $\varepsilon > 0$, seek $\delta > 0$.
Claim $\delta = \varepsilon$ works, how?



Take any $z \in D_\delta(z_0)$. Then we have $|z - z_0| < \delta$, which will mean $|z - z_0| < \varepsilon$, which is same as $|\overline{(z - z_0)}| < \varepsilon$, which is written as $|\bar{z} - \bar{z}_0| < \varepsilon$, which on applying defⁿ of f, becomes

$\left|f(z)-\overline{z}_0\right|<\varepsilon$, which is same as $f(z)\in D_{\varepsilon}(\overline{z}_0)$.

Thus we have found that $f(z)\in D_{\varepsilon}(\overline{z}_0)$ whenever $z\in D_{\delta}(z_0)$, which also means that $f(z)\in D_{\varepsilon}(\overline{z}_0)$ whenever $z\in D^{*}_{\delta}(z_0)$, which proves the result.

Using "ε-δ" definition, 42

Q2 Show that $\lim_{z\to 1+i} \frac{z^2-2i}{z^2-2z+2} = 1-i$

Proof: According to definition of limit, $\lim_{z\to 1+i} \frac{z^2-2i}{z^2-2z+2} = 1-i$, if for each $\varepsilon>0$, there exists a $\delta>0$ such that $\left|\frac{z^2-2i}{z^2-2z+2}-(1-i)\right|<\varepsilon$

whenever $0<|z-(1-i)|<\delta$.

Proving that the limit exists requires that we find an appropriate value of δ for a given value of ε. In other words, for a given value of ε, we must find a positive number δ with the property that when $0<|z-(1-i)|<\delta$, then $\left|\frac{z^2-2i}{z^2-2z+2}-(1-i)\right|<\varepsilon$.

$\therefore$ For a given $\varepsilon>0$, how to find $\delta>0$?

One way of finding δ is to "work backward". The idea is to start with the inequality $\left|\frac{z^2-2i}{z^2-2z+2}-(1-i)\right|<\varepsilon$, and then use properties of complex numbers and the modulus to manipulate this inequality until it involves the expression $|z-(1-i)|$. Thus, a natural first step is to factor N^r and D^r of 1^{st} term in $\left|\frac{z^2-2i}{z^2-2z+2}-(1-i)\right|<\varepsilon$

$$\Rightarrow \left|\frac{(z-1-i)(z+1+i)}{(z-1-i)(z-1+i)}-(1-i)\right|<\varepsilon \Rightarrow \left|\frac{z+1+i}{z-1+i}-(1-i)\right|<\varepsilon$$

$$\Rightarrow \left|\frac{(z+1+i)-(1-i)(z-1+i)}{z-1+i}\right|<\varepsilon \Rightarrow \left|\frac{1+iz-i}{z-1+i}\right|<\varepsilon$$, take i common from N^r.

$$\Rightarrow \left|\frac{(z-1)-i}{(z-1)+i}\right|<\varepsilon \Rightarrow |z-1-i|<\varepsilon\,|(z-1)+i|$$

$$\Rightarrow |z-(1-i)|<\varepsilon\,|(x-1)+i(y+1)| \Rightarrow |z-(1-i)|<\varepsilon\sqrt{(x-1)^2+(y+1)^2} \quad \text{---(1)}$$

Thus (1) indicates that we should take $\delta=\varepsilon\sqrt{(x-1)^2+(y+1)^2}$, (note that $\delta>0$).

(Keep in mind that the choice of δ is not unique)

Having found δ, we now present the formal proof:

Given $\varepsilon>0$, let $\delta=\varepsilon\sqrt{(x-1)^2+(y+1)^2}$. Then if $0<|z-(1+i)|<\delta$, then we have

$|z-1+i|<\varepsilon\sqrt{(x-1)^2+(y+1)^2}$

ie $\frac{|z-(1+i)|}{\sqrt{(x-1)^2+(y+1)^2}}<\varepsilon$ ie $\frac{|z-(1+i)|}{|(z-1)+i|}<\varepsilon$ ie $\left|\frac{1+iz-i}{z-1+i}\right|<\varepsilon$ ie

$$\left|\frac{z+1+i}{z-1+i}-(1-i)\right|<\varepsilon$$

i.e $$\left|\frac{z^2-2i}{z^2-2z+2}-(1-i)\right|<\varepsilon$$

i.e $|f(z)-l|<\varepsilon$

proving that $\lim_{z\to 1+i}\frac{z^2-2i}{z^2-2z+2}=1-i$.

~~[illegible]~~

3 (Redone): Show that if $f(z)=\bar{z}$, then $\lim_{z\to z_0} f(z)=\bar{z}_0$, where z_0 is any complex number.

<u>Proof:</u> Given $\varepsilon>0$, seek a $\delta>0$ such that

$|f(z)-l|<\varepsilon$ whenever $0<|z-z_0|<\delta$. and come up with expression involving $|z-z_0|$.

Start with $|f(z)-l|<\varepsilon$, $\cancel{|\bar{z}-\bar{z}_0|<\varepsilon}$, $\cancel{\text{then } |\bar{z}-\bar{z}_0|<\varepsilon}$, $|\overline{z-z_0}|<\varepsilon$

So Now by $|f(z)-l|<\varepsilon$, we can write $|\bar{z}-\bar{z}_0|<\varepsilon$ then

then $|z-z_0|<\varepsilon \quad \cdots (1)$

Eqn (1) indicates that we should take $\delta=\varepsilon$.

Having found δ, we now write the formal proof:

Given $\varepsilon>0$, let $\delta=\varepsilon$. Then if $0<|z-z_0|<\delta$, then we have

$0<|z-z_0|<\varepsilon$, (or) $|z-z_0|<\varepsilon$, then we have $|\overline{z-z_0}|<\varepsilon$,

then we have, $|\bar{z}-\bar{z}_0|<\varepsilon$, then we have $|f(z)-l|<\varepsilon$,

proving that $\lim_{z\to z_0} f(z)=\bar{z}_0$.

limit of polynomial :

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ be a polynomial.

For any $z_0 \in \mathbb{C}$, what is $\lim_{z \to z_0} P(z)$?

$$\lim_{z\to z_0} P(z) = \lim_{z\to z_0} a_0 + \lim_{z\to z_0} a_1 z + \lim_{z\to z_0} a_2 z^2 + \dots + \lim_{z\to z_0} a_n z^n \quad \dots\dots (1)$$

$\because a_0$ is constant $\therefore \lim_{z\to z_0} a_0 = a_0$.

$\lim_{z\to z_0} a_1 z = ?$

Let $f(z) = a_1 z$, assume $a_1 \neq 0$ (if 0 then limit will be 0).

$\lim_{z\to z_0} f(z) = ?$

~~Choose $\varepsilon>0$ and seek $\delta>0$~~ claim is that the limit will be $a_1 z_0$. How?

choose $\varepsilon > 0$ s.that $|f(z) - a_1 z_0| < \varepsilon$ and seek $\delta > 0$ s.that $|z - z_0| < \delta$. or $(0 < |z - z_0| < \delta)$.

By $|f(z) - a_1 z_0| < \varepsilon$, we get $|a_1 z - a_1 z_0| < \varepsilon$, which is $|a_1||z - z_0| < \varepsilon$, which gives $|z - z_0| < \frac{\varepsilon}{|a_1|}$.

which also means that $0 < |z - z_0| < \frac{\varepsilon}{|a_1|}$.

Take $\delta = \frac{\varepsilon}{|a_1|}$, and we are done.

Thus we get that $\lim_{z\to z_0} a_1 z = a_1 z_0$.

Similarly other limits are computed, and we get by (1)

$$\lim_{z\to z_0} = a_0 + a_1 z_0 + a_2 z_0^2 + \dots + a_n z_0^n, \text{ which is } P(z_0).$$

Thus limit of polynomial exists for any $z_0 \in \mathbb{C}$.

Limit of Quotient: If $f(z) = \frac{P(z)}{Q(z)}$, then $\lim_{z \to z_0} f(z) = ?$

If $Q(z_0) \neq 0$, then it is given by $\frac{P(z_0)}{Q(z_0)}$.

If $P(z_0) = 0 = Q(z_0)$, then $P(z)$ and $Q(z)$ both can be factored as
$$P(z) = (z - z_0) P_1(z)$$
$$Q(z) = (z - z_0) Q_1(z).$$

If $Q_1(z_0) \neq 0$, then $\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \lim_{z \to z_0} \frac{(z-z_0) P_1(z)}{(z-z_0) Q_1(z)}$.

$\because z \to z_0 \therefore z - z_0$ is non-zero $\therefore$ it can be cancelled out, to get $= \lim_{z \to z_0} \frac{P_1(z)}{Q_1(z)}$, which is given by $\frac{P_1(z_0)}{Q_1(z_0)}$. ✓

For example, $\lim_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} = ?$
(Redone)

$$z_0 = 1+i, \quad P(z_0) = 1 - 1 + 2i - 2i = 0$$
$$Q(z_0) = 1 - 1 + 2i - 2 - 2i + 2 = 0$$

$\therefore (z - 1 - i)$ is a factor of $P(z)$ and $Q(z)$ both.
They can be written as
$$P(z) = (z - 1 - i)(z + 1 + i)$$
$$Q(z) = (z - 1 - i)(z - 1 + i)$$

Then the limit is obtained by

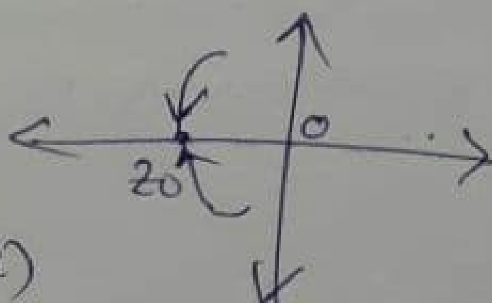
$$\lim_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} = \lim_{z \to 1+i} \frac{z + 1 + i}{z - 1 + i},$$ which can be computed as
$$\frac{1 + i + 1 + i}{1 + i - 1 + i} = \frac{2 + 2i}{2i} = 1 - i,$$

Ans ✓

Ex 1 Show that $\lim_{z \to -1} Arg(z)$ does not exist where $(-\pi < Arg(z) \leq \pi)$

Sol^n: Approach -1 through two different paths and show that the limit $\lim_{z \to -1} Arg(z)$ does not exist.

From upper side,



$$\lim_{z \to z_0} f(z) = \lim_{z \to -1} Arg(z) = \pi \quad \text{---(1)}$$

From lower side

$$\lim_{z \to z_0} f(z) = \lim_{z \to -1} Arg(z) = -\pi \quad \text{---(2)}$$

By (1) and (2), the two values do not agree.

$\therefore$ The limit $\lim_{z \to -1} Arg(z)$ does not exist.

Also, Think of $\log z$.

for continuity of $Arg(0)$?

The limit of $f(z)$ can be determined by the limits of its real and imaginary parts.

<u>Thm 1</u> Let $f(z) = u(x,y) + i\,v(x,y)$ be a complex function that is defined in some nbd of z_0, except perhaps at $z_0 = x_0 + iy_0$. Then

$$\lim_{z \to z_0} = w_0 = u_0 + i v_0 \quad \text{---}(1)$$

iff $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ --- (2)

<u>Pf</u>: Assume that the statement (1) is true and show that the statement (2) is true. $\because$ (1) is true, so we have for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$$f(z) \in D_\varepsilon(w_0) \text{ whenever } z \in D^*_\delta(z_0)$$

ie $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Now, $f(z) - w_0$ can be written as $f(z) - w_0 = u(x,y) - u_0 + i(v(x,y) - v_0)$, nothing that $|x| = |Re(z)| \le |z|$, & $|y| = |Im(z)| \le |z|$, we can write from above

$|u(x,y) - u_0| \le |f(z) - w_0| < \varepsilon$ and $|v(x,y) - v_0| \le |f(z) - w_0| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

ie $|u(x,y) - u_0| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$|v(x,y) - v_0| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$\Rightarrow \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

which establishes (2).

Conversely, suppose (2) is true. Then for each $\varepsilon > 0$ $\exists$ $\delta_1 > 0$ and $\delta_2 > 0$ s.t.

$|u(x,y) - u_0| < \frac{\varepsilon}{2}$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1$

$|v(x,y) - v_0| < \frac{\varepsilon}{2}$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2$

where $u_0 + i v_0 = w_0$.

Let us choose $\delta = \min\{\delta_1, \delta_2\}$, so we can write

$|u(x,y) - u_0| < \frac{\varepsilon}{2}$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$|v(x,y) - v_0| < \frac{\varepsilon}{2}$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

Now using triangle inequality, we can write

$$|f(z) - w_0| \le |u(x,y) - u_0| + |v(x,y) - v_0|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

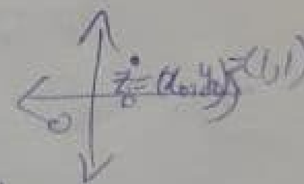
$\Rightarrow |f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

$\Rightarrow \lim_{z \to z_0} f(z) = w_0 = u_0 + i v_0$

which establishes (1).

Hence the theorem is proved.

Q1. Show that $\lim_{z \to 1+i} (z^2 - 2z + 1) = -1$.



Pf; Use Thm1.

We have $f(z) = z^2 - 2z + 1 = (x+iy)^2 - 2(x+iy) + 1$

$= x^2 - y^2 + 2ixy - 2x - 2iy + 1$

$= (x^2 - y^2 - 2x + 1) + i(2xy - 2y) = u(x,y) + i v(x,y)$

$= u + iv$

Compute, $\lim_{(x,y) \to (1,1)} u(x,y) = \lim_{(x,y) \to (1,1)} (x^2 - y^2 - 2x + 1) = 1 - 1 - 2 + 1 = -1 = u_0$

$\lim_{(x,y) \to (1,1)} v(x,y) = \lim_{(x,y) \to (1,1)} (2xy - 2y) = 2 \times 1 \times 1 - 2 \times 1 = 0 = v_0$

Thm1.

$\lim_{z \to 1+i} f(z) = u_0 + i v_0 = -1 + i \cdot 0 = -1.$

Q2. Show that $\lim_{z\to 0} \frac{z}{\bar{z}}$ does not exist.

Solⁿ:

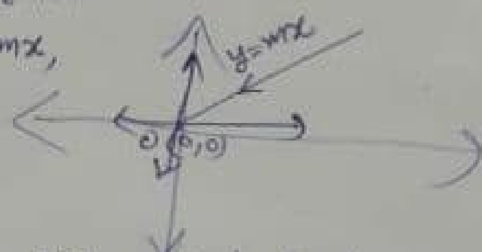
$$f(z) = \frac{z}{\bar{z}} = \frac{x+iy}{x-iy} = \frac{x+iy}{x-iy} \times \frac{x+iy}{x+iy} = \frac{x^2-y^2}{x^2+y^2} + i \cdot \frac{2xy}{x^2+y^2}$$
$$= u(x,y) + i v(x,y)$$

where $u(x,y) = \frac{x^2-y^2}{x^2+y^2}$, $v(x,y) = \frac{2xy}{x^2+y^2}$

Compute $\lim_{(x,y)\to(0,0)} u(x,y)$ and $\lim_{(x,y)\to(0,0)} v(x,y)$.

For the latter case, put $y = mx$,

then

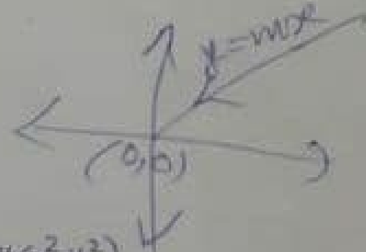


$$\lim_{(x,y)\to(0,0)} v(x,y) = \lim_{x\to 0} \frac{2x \cdot mx}{x^2+m^2x^2}$$
$$= \lim_{x\to 0} \frac{2mx^2}{x^2(1+m^2)} = \lim_{x\to 0} \frac{2m}{1+m^2}, \text{ which depends on } m.$$

$\hookrightarrow \lim_{(x,y)\to(0,0)} v(x,y)$ does not exist.

Hence, in the light of Thm 1, the $\lim_{z\to 0} \frac{z}{\bar{z}}$ does not exist.

Q3: Show that $\lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$ does not exist.



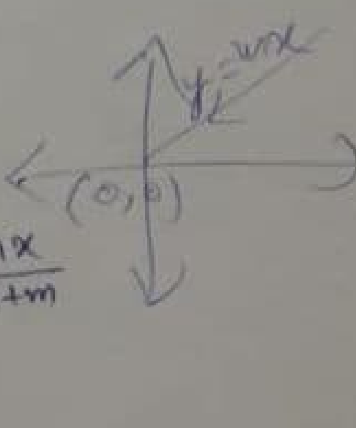
$$f(z) = \left(\frac{z}{\bar{z}}\right)^2 = \left(\frac{x+iy}{x-iy}\right)^2 = \left(\frac{x^2-y^2}{x^2+y^2} + i \cdot \frac{2xy}{x^2+y^2}\right)^2$$
$$= \left(\frac{x^2-y^2}{x^2+y^2}\right)^2 - \left(\frac{2xy}{x^2+y^2}\right)^2 + i\, \frac{4xy(x^2-y^2)}{(x^2+y^2)^2}$$
$$= u(x,y) + i v(x,y).$$

Choose $y = mx$, and compute $\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{x\to 0} \left(\frac{1-m^2}{1+m^2}\right)^2 - \left(\frac{2m}{1+m^2}\right)^2$,

which depends on m, so it does not exist.

$\hookrightarrow$ Thm 1 $\Rightarrow \lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$ does not exist.

Q4: Check $\lim_{z\to 0} \frac{Re(z) \cdot Im(z)}{Re(z) + Im(z)}$.



Solⁿ:

$$f(z) = \frac{Re(z) \cdot Im z}{Re(z) + Im(z)} = \frac{xy}{x+y} = u(x,y) + i v(x,y)$$

$u(x,y) = \frac{xy}{x+y}$, put $y = mx$ $\lim_{x\to 0} \frac{mx^2}{x+mx} = \lim_{x\to 0} \frac{mx}{1+m}$

which does not exist (e.g. for $m = -1$).

Thm 1 $\Rightarrow \lim_{z\to 0} f(z)$ does not exist.

We can redo the following question by using Theorem 1.

We have $f(z) = \bar{z}$, which is $x - iy$.

Then $u(x,y) = x$ and $v(x,y) = -y$.

Take $z = z_0 = x_0 + iy_0$.

Obtain $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = \lim\limits_{x\to x_0} x = x_0 = u_0$ (say)

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = \lim_{y\to y_0} (-y) = -y_0 = v_0 \text{ (say)}$$

Then $\lim\limits_{z\to z_0} f(z)$ will be $u_0 + iv_0 = x_0 + i(-y_0) = x_0 - iy_0$ which is $\bar{z}_0$.

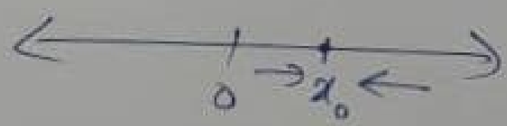
Hence the result. ✓

Continuity of $f(x)$: $f(x)$ is contin. at x_0 if

(i) $\lim\limits_{x\to x_0} f(x)$ exists

(ii) $f(x)\Big|_{x=x_0} = f(x_0)$ exists

(iii) Both are equal i.e. $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

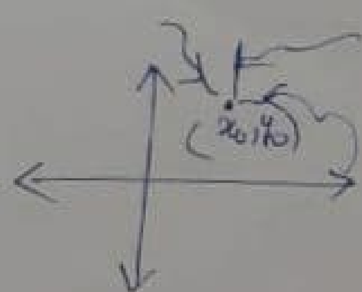


Continuity of $f(x,y)$: At (x_0,y_0).

If (i) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

(ii) $f(x,y)\Big|_{(x,y)=(x_0,y_0)} = f(x_0,y_0)$ exists

(iii) Both are equal.



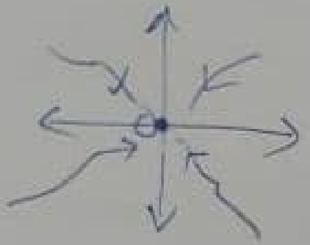
Ex. Check the continuity of the function

$$u(x,y) = \begin{cases} \dfrac{2x^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at the origin.

Sol^n: As seen before, we find that $\lim\limits_{(x,y)\to(0,0)} u(x,y) = 0$.

And from definition, we have



$$u(x,y)\Big|_{(x,y)=(0,0)} = u(0,0) = 0,$$

so that we get $\lim\limits_{(x,y)\to(0,0)} u(x,y) = u(0,0)$,

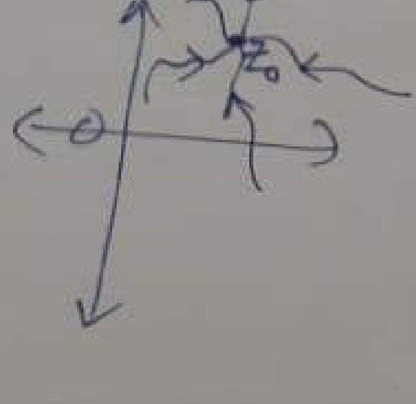
which proves that $u(x,y)$ is contin. at the origin.

Continuity of f(z):

Let $f(z)$ be a complex function of the complex variable z that is defined for all values of z in some nbd of z_0. We say f is contin. at z_0 if three conditions are satisfied:

i) $\lim\limits_{z\to z_0} f(z)$ exists

ii) $f(z)\Big|_{z=z_0} (= f(z_0))$ exists

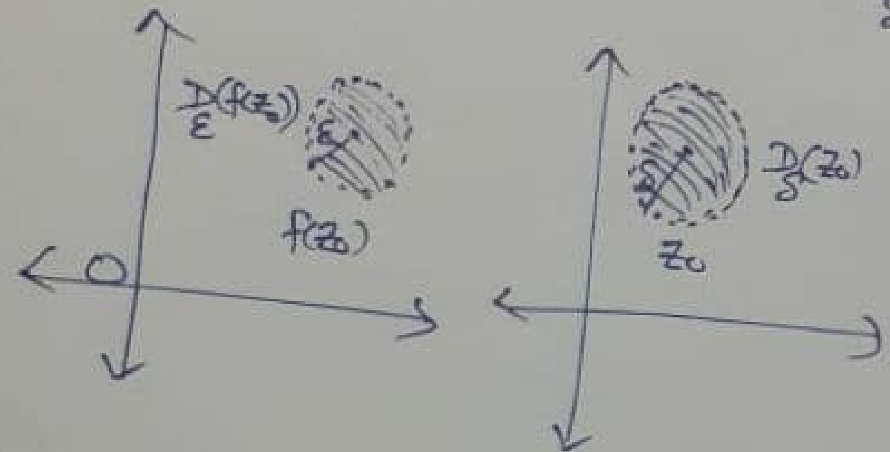


iii) Both are equal i.e. $\lim\limits_{z\to z_0} f(z) = f(z_0)$.

In terms of ε-δ: The complex function $f(z)$ is contin. at z_0 if for each $\varepsilon>0$ $\exists$ a $\delta>0$ s.that

$$|f(z)-f(z_0)|<\varepsilon \text{ whenever } |z-z_0|<\delta$$

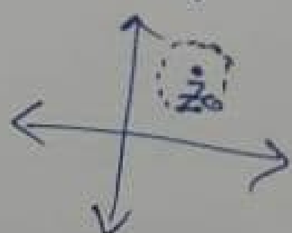
or that $f(z)\in D_\varepsilon(f(z_0))$ whenever $z\in D_\delta(z_0)$.



<u>Ex.</u> The function $f(z)=\bar{z}$ is contin. at every $z_0\in\mathbb{C}$.

As seen before, we have

$\lim_{z\to z_0} f(z)$ is $\bar{z}_0$.

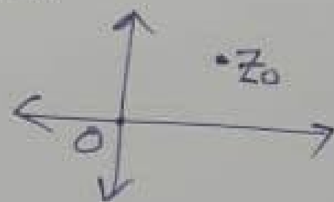


and evaluation $f(z)\Big|_{z=z_0}$ is obviously $\bar{z}_0$.

$\Rightarrow$ The two are equal. Hence f is contin.

<u>Remark:</u> Analogous to the case of limit, continuity of $f(z)$ can be determined by the continuity of its real and imaginary parts as revealed in the following theorem, which says that f is contin. iff its real and imaginary parts are contin.

Theorem 2: Let $f(z) = u(x,y) + i v(x,y)$ be defined in some nbd of z_0. Then f is contin. at $z_0 = x_0 + i y_0$ iff u and v are contin. at (x_0, y_0).



Proof: Follows from Theorem 1.

Ex 1. (Redone): Show that $\bar{z}$ is contin. at every point.

we have $f(z) = \bar{z}$ which can be written as $x - iy$, from where we get $u(x,y) = x$ and $v(x,y) = -y$.

Being real polynomials, both u & v are contin. everywhere.

So from Thm 2, it follows that f is contin. everywhere.

Ex 2 Check for contin. at $z = 1$ the function

$$f(z) = \begin{cases} \dfrac{z^3 - 1}{z - 1}, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases}$$

Sol^n: We have $f(z) = \dfrac{z^3 - 1}{z - 1} = \dfrac{(z-1)(z^2 + 1 + z)}{(z-1)} = z^2 + z + 1$ $\left(\because \text{ for limit } z \to 1,\ z \neq 1 \ \therefore z - 1 \neq 0 \right)$

$= (x + iy)^2 + (x + iy) + 1$

$= (x^2 - y^2 + x + 1) + i(2xy + y)$

$= u(x,y) + i v(x,y)$,

where $u(x,y) = x^2 - y^2 + x + 1$ and $v(x,y) = 2xy + y$

Now $\lim_{(x,y)\to(1,0)} u(x,y) = \lim_{(x,y)\to(1,0)} (x^2-y^2+x+1)$

$= 1-0+1+1 = 3$

and $u(x,y)\Big|_{(x,y)=(1,0)} = 3$

$\therefore \lim_{(x,y)\to(1,0)} u(x,y) = u(x,y)\Big|_{(x,y)=(1,0)}$

$\therefore u(x,y)$ is contin. at $(1,0)$. ✓

Next, $\lim_{(x,y)\to(1,0)} v(x,y) = \lim_{(x,y)\to(1,0)} (2xy+y)$

$= 2\cdot1\cdot0+0 = 0+0=0$

and $v(x,y)\Big|_{(x,y)=(1,0)} = 2\cdot1\cdot0+0=0$

$\Rightarrow v(x,y)$ is contin. at $(1,0)$.

By Thm 2, $f(z)$ is contin. at $z=1$. ✓

Q2. The above function for $z=i$?

$\lim_{(x,y)\to(0,1)} u(x,y) = \lim_{(x,y)\to(0,1)} (x^2-y^2+x+1)$

$= 0-1^2+0+1 = 0$

and $u(x,y)\Big|_{(x,y)=(0,1)} = ?$

$\therefore f(z) = 3$, when $|z|=1$

$\therefore u+iv = 3$, when $|z|=1$

$\therefore u(x,y)+iv(x,y) = 3$, when $z=i$

$\Rightarrow u(x,y) = 3$ when $z=i=(0,1)$

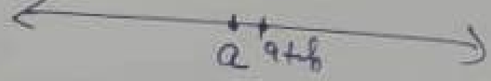
$u(x,y)\Big|_{z=i} = u(x,y)\Big|_{(x,y)=(0,1)} = 3$

$\therefore u(x,y)$ is not contin. at $z=i$

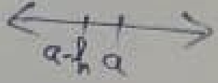
Thm 2 $\Rightarrow$ $f(z)$ is not contin. at $z=i$. ✓

Differentiability of $f(x)$: A function $f(x)$ is said to be differentiable at a point $x=a$ if (the limit) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

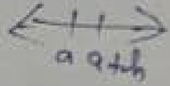
(or that $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists)



This exists iff $\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$ (called the L.H.D.) and



$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ (called the R.H.D.) both exist



and are equal. The common value is denoted by $f'(a)$ and is called the derivative of f at the point a.

Ex.1 $f(x)=x^2$ is differentiable everywhere.

Take a point x_0 , and compute LHD

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(\ -h)-f(\ )}{-h} = \lim_{h\to 0} \frac{(x_0-h)^2-x_0^2}{-h}$$

$$= \lim_{h\to 0} \frac{x_0^2+h^2-2x_0h-x_0^2}{-h} = \lim_{h\to 0} \frac{-h(2x_0-h)}{-h} = \lim_{h\to 0} (2x_0-h) = 2x_0-0 = 2x_0,$$

exists finitely everywhere. ✓

Similarly RHD comes out to be $2x_0$. ✓

LHD = RHD $\therefore f$ is diff. everywhere.

Ex.2 $f(x)=|x|$ is not diff. at 0.

$$= \begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

LHD comes out to be -1 while RHD turns out to be $+1$. ✓

Diff. of $u(x,y)$: The function $u(x,y)$ is said to be diff. at a point (a,b) if the differential du which is $u(a+h, b+k) - u(a,b)$ can be expressed as $Ah + Bk + h\,\phi(h,k) + k\,\psi(h,k)$ with the condition that $\phi(h,k) \to 0$, $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$, where A and B are defined as $A = u_x(x,y)\Big|_{(x,y)=(a,b)}$ that is $u_x(a,b)$

$$B = u_y(x,y)\Big|_{(x,y)=(a,b)} \quad \text{which is } u_y(a,b)$$

where we know the definitions of partial derivatives.

Example: check differentiability at $(0,0)$ of the function

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right) & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Sol: It can be seen that $f_x(0,0) = A = 0$ and $f_y(0,0) = B = 0$.

Now $df = f(0+h, 0+k) - f(0,0)$

$$= f(h,k) - f(0,0)$$

$$= h^2 \sin\left(\frac{1}{h}\right) + k^2 \sin\left(\frac{1}{k}\right) - 0,$$

which can be written as

$$= h\left(h \sin\left(\frac{1}{h}\right)\right) + k\left(k \sin\left(\frac{1}{k}\right)\right) + 0h + 0k$$

or as

$$= Ah + Bk + h\,\phi(h,k) + k\,\psi(h,k)$$

where $A = 0$, $B = 0 = f_y(0,0)$, $\phi(h,k) = h \sin\left(\frac{1}{h}\right)$
$= f_x(0,0)$

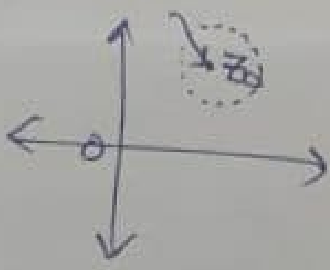
$$\psi(h,k) = k \sin\left(\frac{1}{k}\right)$$

where $\phi, \psi \to 0$ as $h, k \to 0$.

Hence, the function f is diff. at $(0,0)$. ✓

<u>Differentiability of $f(z)$</u>: Let $f(z)$ be a complex function of complex variable z which is defined in a nbd of a point z_0. Then the derivative of $f(z)$ at z_0, written as $f'(z_0)$, is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$



provided the limit exists. If it does, we say that f is differentiable at z_0.

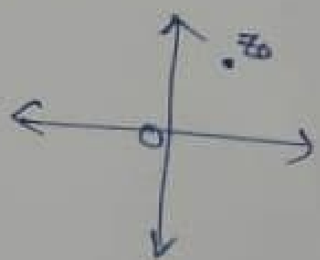
If we write $z - z_0 = \Delta z$, then we can express the above as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

<u>Ex.1</u> Check differentiability of the function $f(z) = z^2 + 1$.

Take an arbitrary point z_0 in the complex plane

Compute $\displaystyle \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$



$$= \lim_{\Delta z \to 0} \frac{[(z_0+\Delta z)^2 + 1] - [z_0^2 + 1]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^2 + \Delta z^2 + 2\Delta z\, z_0 + 1 - z_0^2 - 1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(\Delta z + 2z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \Delta z + 2z_0$$

$$= 0 + 2z_0$$

$$= 2z_0$$

$\therefore$ f is diff. everywhere.

$$= \frac{(z_0+\Delta z)\,(Re\, z_0 + Re\,\Delta z) - z_0\, Re\, z_0}{\Delta z}$$

$$= z_0 \frac{Re\,\Delta z}{\Delta z} + Re\, z_0 + Re\,\Delta z$$

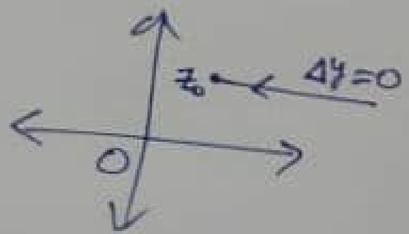
$$= (x_0+iy_0)\frac{\Delta x}{\Delta x + i\Delta y} + x_0 + \Delta x$$

Then

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[(x_0+iy_0)\frac{\Delta x}{\Delta x + i\Delta y} + x_0 + \Delta x \right]$$

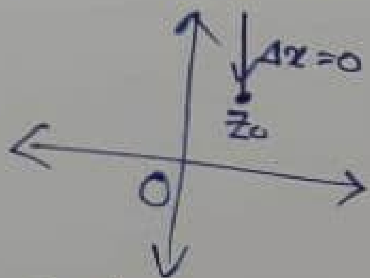
which is obtained as $(x_0+iy_0)\frac{\Delta x}{\Delta x + i0} + x_0 + \Delta x$ along a path parallel to x-axis in which case Δy will be zero,

which is $2x_0 + iy_0 + \Delta x$ --- (1),



and the above will be $(x_0+iy_0)\frac{0}{0+i\Delta y} + x_0 + 0$ along a path parallel to y-axis in which case Δx will be zero,

which is just x_0 --- (2).



Note that for $z_0 \neq (0,0)$, the two limits in (1) and (2) are not same. So limit does not exist. Hence f is not differentiable for such points.

Ex.4 Check diff. of the function $f(z) = \bar{z}$.

Ex. (Redone) Show that $f(z) = \bar{z}$ is not differentiable at $z=0$.

consider $\lim\limits_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim\limits_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim\limits_{\Delta z \to 0} \frac{f(\Delta z) - 0}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

consider the path $y = mx$, then we have $\Delta y = m\Delta x$

$$= \lim_{\Delta x \to 0} \frac{\Delta x - im\Delta x}{\Delta x + im\Delta x} = \lim_{\Delta x \to 0} \frac{1-im}{1+im}$$, which depends on m

$\Rightarrow$ the limit does not exist

$\Rightarrow$ $f(z) = \bar{z}$ is not differentiable at $z=0$. ✓

Q. check diff of $f(z)=|z|$ at the origin.

Sol^n: Consider

$$f'(z_0)=f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{z\to 0}\frac{f(z)}{z}=\lim_{z\to 0}\frac{|z|}{z}$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{x^2+y^2}}{x+iy}, \text{ choose } y=mx$$

$$\Rightarrow \lim_{x\to 0}\frac{\sqrt{x^2+m^2x^2}}{x+imx}=\lim_{x\to 0}\frac{x\sqrt{1+m^2}}{x(1+im)}=\pm\frac{\sqrt{1+m^2}}{1+im}, \text{ which depends on } m,$$

$\therefore$ $f'(0)$ does not exist $\therefore$ $|z|$ is not diff. at the origin.

Also, we can fit it into Thm I.

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{x^2+y^2}}{x+iy}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x-iy)\sqrt{x^2+y^2}}{x^2+y^2}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x-iy)}{\sqrt{x^2+y^2}}=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x}{\sqrt{x^2+y^2}}-i\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\left(u(x,y)+iv(x,y)\right)$$

$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists iff both $\lim_{\substack{x\to 0\\ y\to 0}}u(x,y)$ & $\lim_{\substack{x\to 0\\ y\to 0}}v(x,y)$ exist

Pick, say, $\lim_{\substack{x\to 0\\ y\to 0}}u(x,y)$, but it does not exist, since on choosing the path $y=mx$,

$$=\lim_{x\to 0}\frac{x}{\sqrt{x^2+m^2x^2}}=\lim_{x\to 0}\frac{1}{\sqrt{1+m^2}}=\frac{1}{\sqrt{1+m^2}}$$

which depends on m. ✓

$\Rightarrow$ f is not diff at 0. ✓